TECHNISCHE UNIVERSITÄT
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Nonlinear Finite Element Methods

Assignment

Submitted by

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Computational Materials Science

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1 Problem Description

The problem of creep of a thick-walled pipe under internal pressure p is considered. The pressure p increases linearly until its final value P_{max}, which reaches at time t_L and then hold until time t_f as shown in figure 2. Assuming a plane strain condition i.e. $\varepsilon_{zz} = 0$
Due to axisymmetric conditions, the only non vanishing equilibrium condition is

$$0 = \frac{\partial\left(r\sigma_{rr}\right)}{\partial r} - \sigma_{\phi\phi} \tag{1.1}$$

where σ_{rr} and $\sigma_{\phi\phi}$ are radial and hoop stresses (normal stress components) respectively with respect to polar coordinates.

Figure 1: Thick walled pipe

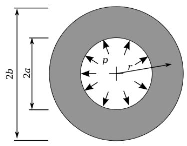
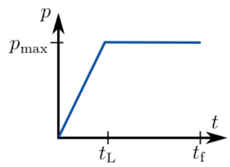


Figure 2: Load sequence



Equation 1.1 refers to the strong form of the equation. From 1.1 the reduced weak form will be

$$0 = \delta W = \int_a^b \underline{\delta\varepsilon}^{\mathrm{T}} \cdot \underline{\sigma} r \mathrm{d}r - \left[r\sigma_{rr}\delta u_r\right]_{r=a}^{b} \tag{1.2}$$

with stresses and strains given in Voigt notation

$$\underline{\sigma} = \begin{bmatrix} \sigma_{rr} \\ \sigma_{\phi\phi} \end{bmatrix}, \quad \underline{\delta\varepsilon} = \begin{bmatrix} \delta\varepsilon_{rr} = \frac{\partial \delta u_r}{\partial r} \\ \delta\varepsilon_{\phi\phi} = \frac{\delta u_r}{r} \end{bmatrix}, \quad \text{and similarly } \underline{\varepsilon} = \begin{bmatrix} \varepsilon_{rr} = \frac{\partial u_r}{\partial r} \\ \varepsilon_{\phi\phi} = \frac{u_r}{r} \end{bmatrix} \tag{1.3}$$

Here the displacement is only in radial direction, the only non vanishing component is $u_r(r)$. Hence the problem can be considered as 1D problem.
Boundary conditions for the problem shown in figure 1 are as follows

$$\sigma_{rr}(r = a) = -P$$
$$\sigma_{rr}(r = a) = 0$$

Linear visco-elastic behavior of the material is described by

$$\underline{\sigma} = \underline{C} \cdot \underline{\varepsilon} + \underline{\sigma}^{\mathrm{ov}} \tag{1.4}$$

$$\underline{\dot{\sigma}}^{\mathrm{ov}} = Q\,\mathrm{dev}(\dot{\varepsilon}) - \frac{1}{T}\underline{\sigma}^{\mathrm{ov}} \tag{1.5}$$

$$\underline{C} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu \\ \nu & 1-\nu \end{bmatrix}$$

E = Young's modulus
ν = Poisson's ratio
Q = modulus which governs over stress evolution ($\underline{\sigma}^{\mathrm{ov}}$)
T = time scale
Input parameters for this problem assigned for me are as follows:

var		E[MPa]	ν	Q[MPa]	T[s]	a[mm]	b[mm]	p_{max}[MPa]	t_L[s]	$t_f[s]$
0	EB	100000	0.30	50000	4	40	80	70	8	40

Table 1: Input parameters

2 Theory

2.1 Weak form

Rewriting equation 1.2 of weak form

$$0 = \delta W = \int_a^b \underline{\delta\varepsilon}^{\mathrm{T}} \cdot \underline{\sigma} r \mathrm{d}r - \left[r\sigma_{rr}\delta u_r \right]_{r=a}^{b} \tag{2.1}$$

Applying boundary conditions the equation results in

$$0 = \delta W = \int_a^b \underline{\delta\varepsilon}^{\mathrm{T}} \cdot \underline{\sigma} r \mathrm{d}r - pa\delta u_r|_{r=a} \tag{2.2}$$

2.2 Discretization

Discretizing the equation 2.2 using Galerkin's method. Choosing same ansatz function for displacement and virtual displacement.

$$\begin{aligned} u_j\left(x_k, t\right) &= \textstyle\sum_{I=1}^{n} N_I\left(x_k\right) u_{jI}(t) \quad \Rightarrow u_r\left(r, t\right) = \textstyle\sum_{I=1}^{n} N_I\left(r\right) u_{rI}(t) \\ \delta u_j\left(x_{k,t}\right) &= \textstyle\sum_{I=1}^{n} N_I\left(x_k\right) \delta u_{jI}(t) \quad \Rightarrow \delta u_r\left(r, t\right) = \textstyle\sum_{I=1}^{n} N_I\left(r\right) \delta u_{rI}(t) \end{aligned} \tag{2.3}$$

From equation 1.3 strain and virtual strain can be discretized as

$$\begin{aligned} \varepsilon_{rr} &= \frac{\partial u_r}{\partial r} = u_{r,r}(r,t) = \sum_{I=1}^{n} \frac{\partial N_I(r)}{\partial r} u_{rI}(t) \\ \varepsilon_{\phi\phi} &= \frac{u_r}{r} = \sum_{I=1}^{n} \frac{N_I(r)}{r} u_{rI}(t) \end{aligned} \tag{2.4}$$

Analogously for virtual strain

$$\begin{aligned} \delta\varepsilon_{rr} &= \sum_{I=1}^{n} \frac{\partial N_I(r)}{\partial r} \delta u_{rI}(t) \\ \delta\varepsilon_{\phi\phi} &= \sum_{I=1}^{n} \frac{N_I(r)}{r} \delta u_{rI}(t) \end{aligned} \tag{2.5}$$

Equations 2.3, 2.4 and 2.5 can be written in vectorial representation as

$$\begin{aligned} \underline{u} &= \underline{N}(r) \cdot \underline{\hat{u}} \\ \delta\underline{u} &= \underline{N}(r) \cdot \hat{\delta\underline{u}} \end{aligned} \tag{2.6}$$

$$\begin{aligned} \underline{\varepsilon} &= \underline{B} \cdot \underline{\hat{u}} \\ \delta\underline{\varepsilon} &= \underline{B} \cdot \hat{\delta\underline{u}} \end{aligned} \tag{2.7}$$

where,
$\underline{\mathrm{N}}$(r) is a ansatz function, which is generally taken as Lagrange polynomial.
$\underline{\mathrm{B}}$ is a parameter which relates strain and displacement
Substituting equations 2.6 and 2.7 in weak form (2.2) results in

$$\begin{aligned} \delta W &= \int_a^b \left(\underline{B} \cdot \delta\underline{\hat{u}}\right)^T \cdot \underline{\sigma} r dr - pa\delta u|_{r=a} = 0 \\ &= \delta\underline{\hat{u}} \int_a^b \underline{B} \cdot^T \cdot \underline{\sigma} r dr - pa\delta u|_{r=a} = 0 \\ &= \delta\underline{\hat{u}} \left(\int_a^b \underline{B}^T \cdot \underline{\sigma} r dr - pa\underline{X} \right) = 0 \qquad (2.8) \\ &= \delta\underline{\hat{u}} \left(F_{int} - F_{ext}\right) = 0 \qquad (2.9) \end{aligned}$$

where $\underline{X}$ = [1,0,0,....., 0] of size (1 x n), n is number of elements.

2.3 Isoparametric elements

Mapping physical element domain to simply shaped unit domain $\Omega_{\square}$

$$u_r = \sum_{I=1}^{n_e} N_I(\underline{\xi}) u^e_{rI}$$
$$r = \sum_{I=1}^{n_e} N_I(\underline{\xi}) r^e_I \tag{2.10}$$

where;
$N_I(\underline{\xi})$ is a shape function i terms of unit coordinates
r^e_I coordinates of the nodes
Since, here the problem considered as 1D, shape function is the form

$$u_r = \sum_{I=1}^{n_e} N_I(\underline{\xi}) u_{rI} \tag{2.11}$$

Shape function is considered as a Lagrange polynomial of 1 dimensional which is shown in figure 3

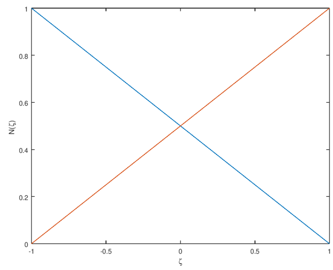


Figure 3: Shape function for 1D

$N_I(\underline{\xi}) = \left[\frac{1}{2}(1-\xi), \frac{1}{2}(1+\xi)\right]$
$\underline{\mathrm{B}}$ can be calculated using the relation 2.4 and 2.7

$$B = \begin{bmatrix} \frac{\partial [\underline{N}]}{\partial r} \\ \\ \frac{[\underline{N}]}{r} \end{bmatrix} = \begin{bmatrix} \frac{\partial N_1}{\partial r} & \frac{\partial N_2}{\partial r} \\ \\ \frac{N_1}{r} & \frac{N_2}{r} \end{bmatrix}$$

$$B = \begin{bmatrix} \frac{\partial N_1}{\partial r} & \frac{\partial N_2}{\partial r} \\ \\ \frac{N_1}{r^e_1 N_1 + r^e_2 N_2} & \frac{N_2}{r^e_1 N_1 + r^e_2 N_2} \end{bmatrix} \tag{2.12}$$

Using chain rule

$$\frac{\partial [\underline{N}]}{\partial r} = \frac{\partial [\underline{N}]}{\partial \xi} \cdot \frac{\partial \xi}{\partial r} = J^{-1} \frac{\partial [\underline{N}]}{\partial \xi}$$

J is the Jacobian matrix which is the partial derivative of radial coordinates to the elemental coordinates.

$$J = \frac{\partial r}{\partial \xi} = \sum_{I=1}^{n_e} \frac{\partial N_I(\underline{\xi})}{\partial r} r^e_I \tag{2.13}$$

Solving the equation 2.13 we get $J = \frac{h^e}{2}$, where h^e is the step size of element which increases satisfying geometric series.
Using equation 2.13, equation 2.12 can be written as

$$\underline{B} = \begin{bmatrix} \frac{-1}{h^e} & \frac{1}{h^e} \\ \\ \frac{1-\xi}{r^e_1(1-\xi)+r^e_2(1+\xi)} & \frac{1+\xi}{r^e_1(1-\xi)+r^e_2(1+\xi)} \end{bmatrix} \tag{2.14}$$

Discretizing the virtual work equation 2.8

$$\delta \underline{W} = \sum_{e=1} \delta \underline{W}^e - \underline{F}_{\text{ext}} | \delta u_r \tag{2.15}$$

$\underline{F}_{ext}$ is already solved for global coordinates, so no need to discretize it.
From 2.15

$$\begin{aligned}\delta \underline{W}^e &= \delta \underline{\hat{u}}^{e^T} \underline{F}^e_{int} \\ &= \delta \underline{\hat{u}}^{eT} \int_{\Omega^e} \underline{B}^T \cdot \underline{\sigma} r dr \\ &= \delta \underline{\hat{u}}^{eT} \int_{\Omega^e} \underline{B}^T \cdot \underline{\sigma} \cdot \underline{N} \cdot \underline{r}^e dr\end{aligned} \tag{2.16}$$

For converting from global coordinates integral to elemental coordinates integral we use det(J) factor i.e $\int_{\Omega^e} (\ ) dV = \int_{\Omega_\square} (\ ) det(\underline{J}) d\xi$
Equation 2.16 can be written as

$$\delta W^e = \delta \hat{u}^{eT} \int_{\Omega_\square} \underline{B}^T \cdot \underline{\sigma} \cdot \underline{N} \cdot \underline{r}^e J d\xi \tag{2.17}$$

$$F^e_{int} = \int_{\Omega_\square} B^T \cdot \sigma \cdot \underline{N} \cdot r^e J d\xi \tag{2.18}$$

In this case Jacobian is a scalar, so det(J) = J.
Considering equation 2.15

$$\begin{aligned}\delta W &= \sum_{e=1} \delta W^e - F_{\text{ext}} | \delta u_r \\ \delta W &= \sum_{e=1} \delta \hat{u}^{e^T} F^e_{int} - F_{\text{ext}} \cdot \delta \hat{u}\end{aligned} \tag{2.19}$$

2.4 Material routine

Using Euler backward method, solve for $\dot{\sigma}_{ov}$ from 1.5 results in

$$\sigma^{ov}_{m+1} = \frac{\sigma^{ov}_m + Qdev(\Delta\varepsilon)}{[1 + \frac{\Delta t}{T}]} \tag{2.20}$$

dev($\Delta\varepsilon$) is deviatoric part of $\Delta\varepsilon$ which is

$$\begin{aligned}\text{dev}(\Delta\varepsilon) &= \Delta\varepsilon - \frac{1}{3}[\text{tr}(\Delta\varepsilon)]I \\ \text{dev}(\Delta\varepsilon) &= \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix} \Delta\varepsilon\end{aligned} \tag{2.21}$$

Material tangent stiffness is obtained by differentiating (1.4) with $\Delta\varepsilon$

$$\begin{aligned}\frac{\delta\sigma_{m+1}}{\delta\varepsilon_{m+1}} &= \frac{\delta\sigma_{m+1}}{\delta\varepsilon_{m+1}} + \frac{\delta\sigma_{m+1}}{\delta\sigma^{ov}_{m+1}} \cdot \frac{\delta\sigma^{ov}_{m+1}}{\delta\varepsilon_{m+1}} \\ \frac{\delta\sigma_{m+1}}{\delta\varepsilon_{m+1}} &= \underline{C} \\ \frac{\delta\sigma_{m+1}}{\delta\sigma^{ov}_{m+1}} &= 1 \\ \frac{\delta\sigma^{ov}_{m+1}}{\delta\varepsilon_{m+1}} &= \frac{Q}{1 + \frac{\Delta t}{2T}} \left[I - \frac{(IXI)}{3} \right] \\ C_t &= C + \frac{1}{1 + \frac{\Delta t}{T}} \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix}\end{aligned} \tag{2.22}$$

C_t is the material tangent stiffness. Calculate over stress 2.20 and substitute in 1.4 and then substitute σ in (2.18)

$$
\begin{aligned}
F_{int}^e &= \int_{\Omega_\square} \underline{B}^T \cdot \underline{C}_t \cdot \underline{\varepsilon} \cdot \underline{N} \cdot r^e J d\xi \\
F_{int}^e &= \left(\int_{\Omega_\square} \underline{B}^T \cdot \underline{C}_t \cdot \underline{B} \cdot \underline{N} \cdot r^e J d\xi \right) \hat{u}^e \qquad (2.23) \\
F_{int}^e &= \underline{K}^e \cdot \hat{u}^e \\
\underline{K}^e &= \int_{\Omega_\square} \underline{B}^T \cdot \underline{C}_t \cdot \underline{B} \cdot \underline{N} \cdot r^e J d\xi \qquad (2.24)
\end{aligned}
$$

2.5 Numerical Integration

Using Guassian quadrature (numerical integration) with 1 Guass point per each element.

$$
\int_\Omega f(\xi) dV_\xi = \sum_{\alpha=1}^{n_e} \omega_\alpha f(\xi_\alpha)
$$

ω_α is the weight, which is equal to 2 in this problem
ξ_α is a sampling point, which is 0 for this problem

$$
K^e = 2\underline{B}(0)^T \cdot \underline{C}_t \cdot \underline{B}(0) \cdot \underline{N}(0) \cdot r^e J \qquad (2.25)
$$

2.6 Global coordinates

$\underline{\underline{A}}^e$, element connectivity matrix, converts all element equations like $\underline{F}_{int}^e$ and $\underline{K}^e$ to global using the following relations

$$
\begin{aligned}
\underline{\hat{u}}^e &= \underline{\underline{A}}^e \cdot \underline{\hat{u}} \\
\delta W &= \delta \hat{u}^T \sum_e \underline{\underline{A}}^T \cdot F_{int}^e - F_{ext} = 0 \\
F_{int} &= \sum_e \underline{\underline{A}}^{eT} \cdot \underline{F}_{int}^e \qquad (2.26) \\
K &= \sum_e \underline{\underline{A}}^{eT} \cdot \underline{K}^e \cdot \underline{\underline{A}}^e \qquad (2.27)
\end{aligned}
$$

Equating F_{int} and F_{ext} to solve for $\underline{\hat{u}}$
Since the problem is a nonlinear visco-elastic, used Newton-Raphson method for solving until the following convergence staisfies

$$
\|\hat{\mathbf{F}}_{\mathbf{ext}} \hat{-} \hat{\mathbf{F}}_{\mathbf{int}}\|_\infty < 0.005 \left\| \hat{\mathbf{F}}_{\text{int}} \right\|_\infty \qquad (2.28)
$$

$$
\|\boldsymbol{\Delta} \underline{\hat{\mathbf{u}}}_k\|_\infty < 0.005 \|\hat{\mathbf{u}}\|_\infty \qquad (2.29)
$$

with $\|\underline{\hat{o}}\|_\infty$denoting the infinity norm, i. e. the maximum component by amount of the column vector $\underline{\hat{o}}$

2.7 Analytical Solution

After solving the solution using finite element procedure, the linear solution is verified with the analytical solution for a linear elastic material considering Q = 0;

$$
u_r^{\text{elast}} = (1+\nu) \frac{p}{E} \frac{a^2}{b^2 - a^2} \left[(1-2\nu) r + \frac{b^2}{r} \right] \qquad (2.30)
$$

3 Structure of the program

The program is completely done in Octave. It is highly recommended to run the program in Octave. The program consists of total 7 files as follows:

- outcome.m
- meshGenerator.m
- Input_parameters.m
- processor.m
- element_routine.m
- material_routine.m
- analytical.m

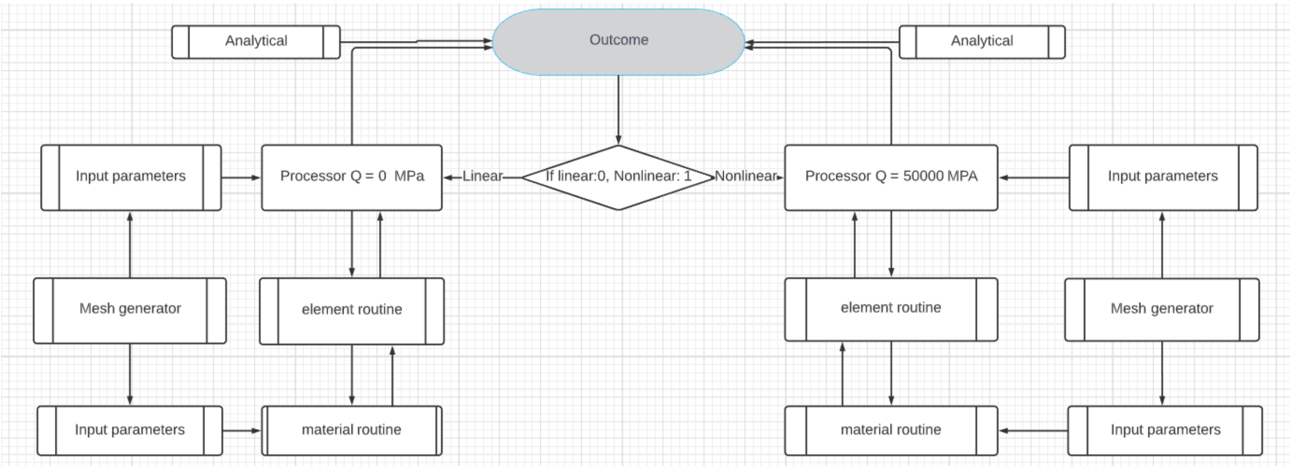


Figure 4: Flowchart of a program

The detailed description of a program is provided in the flowchart 4. I used meshGenerator.m file with little modifications as provided by Prof. Geralf Hütter.

4 User's manual

This manual is for individuals who compiles the program. This manual explains how to start and compile the program.
Initially all files must be saved in a single folder, in order to run without any errors. The program is completely done in Octave, it is highly recommended to run the program in Octave.

1. Start: Open outcome.m file, it is the main file which takes one input from user, i.e. is the problem linear or nonlinear? It then takes corresponding Q value based on the user input.
2. Inputs: Program gets all the inputs mentioned in table 1 ,except Q, along with number of elements, Guassian sampling points and weights from Input_parameters.m which is called in a program when it is necessary.
3. Outputs: The program generates three plots which are as follows
   - Radius vs Radial displacement, compared numerical solution with analytical solution
   - Radius vs Radial stress and Radius vs Hoop stress
   - Time vs Radial displacement for last node

5 Results

The problem is solved for 15 elements.

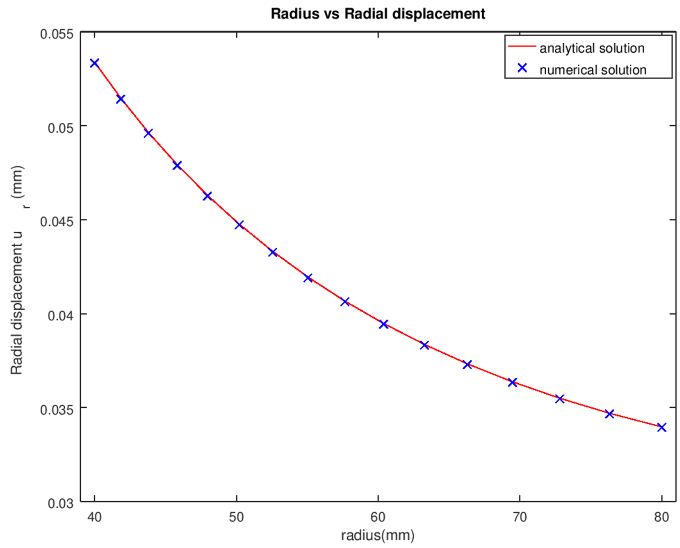


Figure 5: Comparison of numerical and analytical displacement values

From figure 5,numerical solution almost coincides with the analytical solution in terms of displacement.

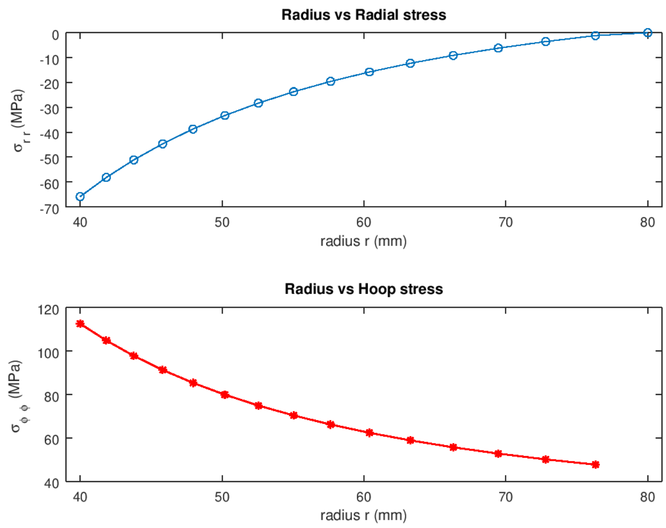


Figure 6: Radial and Hoop stress for corresponding nodes

Plots for stress distribution over the pipe is shown in figure 6

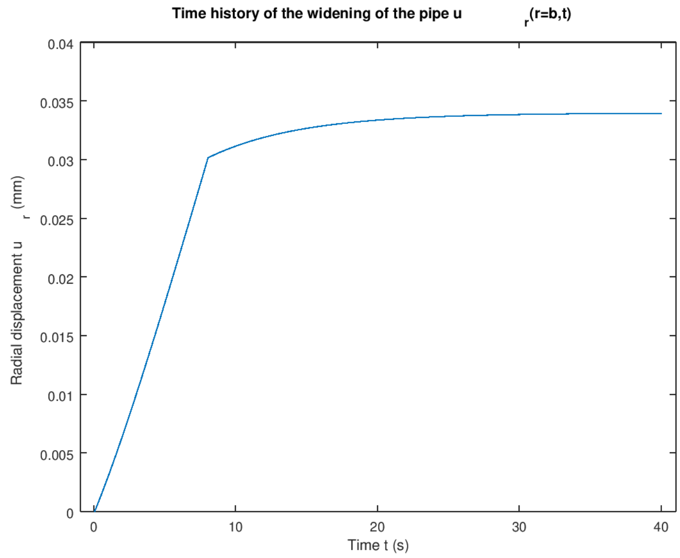


Figure 7: Time history of the widening of the pipe u_r(r=b,t) for t ϵ [0,t_f]

Time history of the widening of the pipe at last node is shown in figure 7. The time step in the program is considered as 0.05 sec based on the convergency and smoothness in the curve in figure 7

6 Verification

The table 2 summarizes the results for linear elastic case. As one can observe from the table 2 numerical value exactly matches with the analytical value of corresponding displacement.

Node	Radius [mm]	Numerical displacement[mm]	Analytical displacement[mm]	Radial stress [MPa]	Hoop stress [MPa]
1	40.00000	0.05339	0.05339	-65.88874	112.55540
2	41.84316	0.05147	0.05147	-58.18547	104.85213
3	43.77987	0.04966	0.04966	-51.11854	97.78521
4	45.81489	0.04793	0.04793	-44.63907	91.30574
5	47.95319	0.04630	0.04630	-38.70145	85.36812
6	50.20003	0.04476	0.04476	-33.26321	79.92988
7	52.56091	0.04331	0.04331	-28.28485	74.95151
8	55.04162	0.04195	0.04195	-23.72966	70.39632
9	57.64824	0.04067	0.04067	-19.56358	66.23025
10	60.38717	0.03948	0.03948	-15.75506	62.42173
11	63.26511	0.03836	0.03836	-12.27488	58.94155
12	66.28913	0.03733	0.03733	-9.09601	55.76268
13	69.46664	0.03637	0.03637	-6.19349	52.86016
14	72.80543	0.03550	0.03550	-3.54427	50.21094
15	76.31368	0.03470	0.03470	-1.12711	47.79378
16	80.00000	0.03397	0.03397	0.00000	

Table 2: Linear elastic problem

The table 3 summarizes the results for nonlinear visco-elastic case. As one can observe from the table 3 numerical solution relaxes towards the analytical solution (linear elastic) of corresponding displacement. Although in the figure 5 the plot of nonlinear displacement coincides with the analytical displacement, there is around 10^{-5} error.

Node	Radius [mm]	Numerical displacement[mm]	Analytical displacement[mm]	Radial stress [MPa]	Hoop stress [MPa]
1	40.00000	0.05333	0.05339	-65.88739	112.55405
2	41.84316	0.05142	0.05147	-58.18424	104.85090
3	43.77987	0.04960	0.04966	-51.11742	97.78408
4	45.81489	0.04788	0.04793	-44.63805	91.30471
5	47.95319	0.04625	0.04630	-38.70052	85.36718
6	50.20003	0.04472	0.04476	-33.26236	79.92902
7	52.56091	0.04327	0.04331	-28.28407	74.95073
8	55.04162	0.04191	0.04195	-23.72895	70.39561
9	57.64824	0.04063	0.04067	-19.56294	66.22960
10	60.38717	0.03944	0.03948	-15.75448	62.42113
11	63.26511	0.03832	0.03836	-12.27435	58.94100
12	66.28913	0.03729	0.03733	-9.09553	55.76218
13	69.46664	0.03634	0.03637	-6.19305	52.85971
14	72.80543	0.03547	0.03550	-3.54387	50.21053
15	76.31368	0.03467	0.03470	-1.12675	47.79341
16	80.00000	0.03394	0.03397	0.00000	

Table 3: Nonlinear visco-elastic problem