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S4 CSE-A

MODULE 1

1.

FOURIER SERIES

Fourier series of a function is,

$$f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \cdots$$
$$+ b_1 \sin x + b_2 \sin 2x + \cdots$$

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)}$$

v.imp **Euler Formula**

[If $f(x)$ ~~is~~ in $c < x < c + 2\pi$ [0 to 2π
$-\pi$ to π]

then F.S for $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$

where $a_0 = \frac{1}{\pi} \int_c^{c+2\pi} f(x)\, dx.$

$$a_n = \frac{1}{\pi} \int_c^{c+2\pi} f(x) \cos nx \, dx$$

$$b_n = \frac{1}{\pi} \int_c^{c+2\pi} f(x) \sin nx \, dx.$$

Here a_0, a_n, b_n are called Euler formulas / Fourier constants

If $c = 0 \longrightarrow 0 < x < 2\pi$.

then $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$

where $a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) \, dx.$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx \, dx$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cdot \sin nx \, dx$$

If $c = -\pi \longrightarrow -\pi < x < \pi$

then $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$

where $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, dx$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx$$

2.

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx.$$

Case I.

If $f(x)$ is even $[f(-x) = f(x)]$.

then $a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) \, dx$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx \, dx$$ [$f(x)$ even, $\cos nx$ even]

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx$$ [$f(x)$ even, $\sin nx$ odd]

$= 0$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx.$$

even × even = even
even × odd = odd
odd × odd = even

Case II.

If $f(x)$ is odd $\left(f(-x) = -f(x)\right)$.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{f(x)\cos nx}_{\text{odd}}\,dx = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

$$\therefore f(x) = \sum_{n=1}^{n} b_n \sin nx$$

General Formula

Change of interval

[$f(x)$ is in $c < x < c+2l$

then F.S for $f(x)$,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$$

where $a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\,dx$

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x).\cos\frac{n\pi x}{l}\,dx$$

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x).\sin\frac{n\pi x}{l}\,dx$$]

If $c = 0$

$f(x)$ is in $0 < x < 2l$

then f.s for $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$

where $a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx.$$

If $c = -l$

$f(x)$ is in $-l < x < l$.

then f.s for $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l} \right)$

where, $a_0 = \frac{1}{l}\int_{-l}^{l} f(x)dx$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx.$$

Case I.

If $f(x)$ is even ie $f(-x) = f(x)$.

Then, $a_0 = \frac{2}{l}\int_{0}^{l} f(x)dx$

$$a_n = \frac{2}{l}\int_{0}^{l} f(x).\cos\frac{n\pi x}{l}\,dx$$

$\left(\begin{array}{l} f(x) \text{ even} \\ \cos\frac{n\pi x}{l} \text{ even} \end{array} \right)$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x).\sin\frac{n\pi x}{l}\,dx = 0 \quad \left[\begin{array}{l} f(x) \text{ even} \\ \sin\frac{n\pi x}{l} \text{ odd} \end{array} \right]$$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

Case II

If $f(x)$ is odd. ie, $f(-x) = -f(x)$.

Then $a_0 = \frac{1}{l}\int_{-l}^{l} f(x)dx = 0$.

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x).\cos\frac{n\pi x}{l}\,dx = 0$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x).\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\int_{0}^{l} f(x).\sin\frac{n\pi x}{l}\,dx.$$

$$\therefore f(x) = \sum_{n=1}^{n} b_n \sin\frac{n\pi x}{l}$$

Note

1. $\sin n\pi = 0 \quad \forall n$

2. $\cos n\pi = (-1)^n$ $\quad \left\{ \begin{array}{ll} \cos n\pi = 1 & \text{for } n = \text{even} \\ -1 & \text{for } n = \text{odd} \end{array} \right.$

3. $\sin (2n+1)\frac{\pi}{2} = (-1)^n$

4. $\cos (2n+1)\frac{\pi}{2} = 0.$

5. $$\int e^{ax} \sin(bx+c)\, dx = \frac{e^{ax}}{a^2+b^2}\left[a \sin(bx+c) - b\cos(bx+c)\right]$$

6. $$\int e^{ax} \cos(bx+c)\, dx = \frac{e^{ax}}{a^2+b^2}\left[a \cos(bx+c) + b\sin(bx+c)\right]$$

v.imp

Dirichlet's Conditions

Any function $f(x)$ can be developed as a Fourier series provided,

1. $f(x)$ is periodic, single valued and finite.

2. $f(x)$ has a finite no. of discontinouties in any one period.

3. $f(x)$ has atmost a finite no. of maxima and minima.

When these conditions are satisfied,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right]$$

and at a point of discontinouty, the sum of the series is equal to.

$$f(x) = \frac{1}{2}\left[\underbrace{f(x+0)}_{\text{Right hand limit}} + \underbrace{f(x-0)}_{\text{left hand limit}}\right]$$

1. Expand $x - x^2$ as a fourier series in $-1 < x < 1$.

F.S for $f(x)$ is, (F.S for $x-x^2$ in $(-l, l)$)

$$x - x^2 = f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right)$$

$$a_0 = \frac{1}{l} \int_{-l}^{l} f(x)\, dx \qquad l = 1$$

$$= \frac{1}{1} \int_{-1}^{1} (x - x^2)\, dx.$$

$$= \int_{-1}^{1} x\, dx - \int_{-1}^{1} x^2\, dx$$

$$= \left[\frac{x^2}{2}\right]_{-1}^{1} - \left[\frac{x^3}{3}\right]_{-1}^{1}$$

$$= \frac{1}{2}(1-1) - \frac{1}{3}(1 - -1)$$

$$= 0 - \frac{1}{3} \times 2$$

$$= -\frac{2}{3}$$

$$a_n = \frac{1}{l} \int_{-l}^{l} f(x) \cos \frac{n\pi x}{l}\, dx$$

$$= \frac{1}{1} \cdot \int_{-1}^{1} (x - x^2) \cos \frac{n\pi x}{l}\, dx$$

$$= \underbrace{\int_{-1}^{1} x \cdot \cos \frac{n\pi x}{l}\, dx}_{\text{odd} \times \text{even} = \text{odd}} - \underbrace{\int_{-1}^{1} x^2 \cdot \cos \frac{n\pi x}{l}\, dx}_{\text{even} \times \text{even} = \text{even}}$$

$$= 0 - 2 \times \int_{0}^{1} x^2 \cos \frac{n\pi x}{l}\, dx$$

$$= -2\int_{0}^{1} x^2 \cdot \cos n\pi x\, dx$$

$$b_n = \frac{1}{l} \int_{-l}^{l} f(x) \sin \frac{n\pi x}{l}\, dx$$

$$= \frac{1}{1} \cdot \int_{-1}^{1} (x - x^2) \sin \frac{n\pi x}{1}\, dx$$

$$= \underbrace{\int_{-1}^{1} x \cdot \sin \frac{n\pi x}{1}\, dx}_{\text{odd} \times \text{odd} = \text{even}} - \int_{-1}^{1} x^2 \cdot \sin n\pi x\, dx$$

$$= 2 \int_{0}^{1} x \sin n\pi x\, dx - 0$$

$$= 2 \int_{0}^{1} x \cdot \sin n\pi x\, dx$$

$$= 2 \left[x \cdot \frac{-\cos n\pi x}{n\pi} - \int 1 \cdot \frac{-\cos n\pi x}{n\pi} \right]_0^1$$

$$= 2\left[-x\cdot\frac{\cos n\pi x}{n\pi} + \frac{\sin n\pi x}{(n\pi)^2}\right]_0^1$$

$$= 2\left[-1.\frac{\cos n\pi}{n\pi} + \frac{\sin n\pi}{(n\pi)^2}\right]$$

$$= -\frac{2\cos n\pi}{n\pi} = \frac{-2(-1)^n}{n\pi}$$

$$a_n = -2\int_0^1 x^2 \cos n\pi x \, dx$$

$$= -2\left[x^2\cdot\frac{\sin n\pi x}{n\pi} - \int 2x\cdot\frac{\sin n\pi x}{n\pi}\right]_0^1$$

$$= -2\left[x^2\cdot\frac{\sin n\pi x}{n\pi} - \frac{2}{n\pi}\int x\cdot\sin n\pi x\right]_0^1$$

$$= -2\left[x^2\frac{\sin n\pi x}{n\pi} + \frac{2}{n\pi} \times \frac{\cos n\pi}{n\pi}\right]_0^1$$

$$= -2\left[\frac{\cancel{\sin n\pi}}{n\pi} + \frac{2}{(n\pi)^2}\cos n\pi\right]_0^1$$

$$= -\frac{4}{(n\pi)^2}\cdot\cos n\pi = \frac{-4(-1)^n}{n^2\pi^2}$$

2. Find the fourier expansion of e^{-x} in $-l<x<l$ in $-l<x<l$

F.S $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$

where, $a_0 = \frac{1}{l}\int_{-l}^{l} f(x)dx$

$$= \frac{1}{l}\int_{-l}^{l} e^{-x}\,dx.$$

$$= \frac{1}{l}\left[\frac{e^{-x}}{-1}\right]_{-l}^{l}$$

$$= -\frac{1}{l}\left(e^{-l} - e^{l}\right)$$

$$= \frac{2}{l}\left(\frac{e^{l} - e^{-l}}{2}\right)$$

$$= \frac{2}{l}\sinh l$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{l}\int_{-l}^{l} e^{-x}\cos\frac{n\pi x}{l}\,dx.$$

$a=-1$, $b=n\pi/l$, $c=0$

$$\int e^{ax}\cos(bx+c)\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\cos(bx+c) + b\sin(bx+c)\right]$$

$$= \frac{1}{l}\left[\frac{e^{-x}}{(-1)^2+(n\pi/l)^2}\left(-\cos\frac{n\pi x}{l}+\frac{n\pi}{l}\sin\frac{n\pi x}{l}\right)\right]_{-l}^{l}$$

$$= \frac{1}{l\left(1+\frac{n^2\pi^2}{l^2}\right)}\left[e^{-l}\left(-(-1)^n+\frac{n\pi}{l}\times 0\right) - e^{l}\left(-(-1)^n+\frac{n\pi}{l}\times 0\right)\right]$$

$$= \frac{l}{l^2+n^2\pi^2}\left[-e^{-l}\cdot(-1)^n + e^{l}(-1)^n\right]$$

$$= \frac{l\cdot(-1)^n}{l^2+n^2\pi^2}\left[e^{l}-e^{-l}\right]$$

$$= \frac{(-1)^n\cdot 2l}{l^2+n^2\pi^2}\cdot \sin hl$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx.$$

$$= \frac{1}{l}\int_{-l}^{l} e^{-x}\cdot\sin\frac{n\pi x}{l}\,dx.$$

$a=-1,\ b=\frac{n\pi}{l},\ c=0$

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$$\int e^{ax}\sin(bx+c)\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\sin(bx+c)-b\cos(bx+c)\right]$$

$$= \frac{1}{l}\left[\frac{e^{-x}}{(-1)^2+(n\pi/l)^2}\left(-\sin\frac{n\pi x}{l}-\frac{n\pi}{l}\cos\frac{n\pi x}{l}\right)\right]_{-l}^{l}$$

$$= \frac{l}{l^2+n^2\pi^2}\left[e^{-l}\left(0-\frac{n\pi}{l}\cdot(-1)^n\right) - e^{l}\left(0-\frac{n\pi}{l}\cdot(-1)^n\right)\right]$$

$$= \frac{\cancel{l}}{l^2+n^2\pi^2}\cdot\frac{n\pi}{\cancel{l}}\cdot(-1)^n\left[e^{l}-e^{-l}\right]$$

$$= \frac{2n\pi\cdot(-1)^n}{l^2+n^2\pi^2}\sin hl$$

$$\therefore e^{-x} = \frac{\sin hl}{l} + \sum_{n=1}^{\infty}\left[\frac{(-1)^n\cdot 2l}{l^2+n^2\pi^2}\sinh l\cdot\cos\frac{n\pi x}{l} + \frac{2n\pi\cdot(-1)^n}{l^2+n^2\pi^2}\sinh l\,\sin\frac{n\pi x}{l}\right]$$

3. Find the fourier expansion of $x \sin x$ in $-\pi < x < \pi$ and hence deduce the value of $\frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \cdots\cdots$

x = odd
$\sin x$ = odd
odd × odd = even

Since, $x \sin x$ is even.

$$F.S\ f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

where, $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx.$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cos nx\, dx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x \sin x\, dx$$

$\int uv\,dx$
$= uv_1 - u'v_{11} + u''v_{111} - \cdots$

$$= \frac{2}{\pi}\left[x \cdot -\cos x - \int -\cos x\, dx\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-x\cos x + \sin x\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\pi \cos\pi + \sin\pi - (0 + \sin 0)\right]$$

$$= \frac{2}{\pi}\left[\pi + 0 - 0\right]$$

$$= \frac{2}{\cancel{\pi}} \times \cancel{\pi}$$

$$= 2$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x \overset{B}{\sin x} \cdot \cos \overset{A}{nx}\, dx$$

$$\boxed{2\cos A \sin B = \sin(A+B) - \sin(A-B)}$$

$$= \frac{1}{\pi}\int_0^{\pi} x\left[\sin(n+1)x - \sin(n-1)x\right] dx$$

$$= \frac{1}{\pi}\left[x \cdot -\frac{\cos(n+1)x}{n+1} - 1 \cdot -\frac{\sin(n+1)x}{(n+1)^2} - \left\{x - \frac{\cos(n-1)x}{n-1} - 1 \cdot -\frac{\sin(n-1)x}{(n-1)^2}\right\}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{-1}{n+1}\cdot\left(\pi \cdot \overset{(-1)^{n+1}}{\cos(n+1)\pi} - 0\right) - 0 + \frac{1}{n-1}\cdot\left(\pi \cdot \overset{(-1)^{n-1}}{\cos(n-1)\pi} - 0\right) - 0\right]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^{n+2}\cdot\pi}{n+1} + \frac{(-1)^{n-1}\cdot\pi}{n-1}\right]$$

$$= (-1)^n\left[\frac{1}{n+1} - \frac{1}{n-1}\right]$$

$$= \frac{(-1)^n}{n^2-1}\left[(n-1)-(n+1)\right] = \frac{-2\cdot(-1)^n}{n^2-1}$$

if $n \neq 1$

If $n=1$,

ie, $$a_1 = \frac{2}{\pi}\int_0^{\pi} x\sin x\cos x\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x\cdot \underbrace{2\sin x\cdot\cos x}_{\sin 2x}\,dx.$$

$$= \frac{1}{\pi}\int_0^{\pi} x\sin 2x\,dx$$

$$= \frac{1}{\pi}\left[x\cdot\frac{-\cos 2x}{2} - 1\cdot\frac{-\sin 2x}{4}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\pi\cdot -\frac{\cos 2\pi}{2} + 1\cdot\frac{\cancel{\sin 2\pi}^{0}}{4} - 0\right]$$

$$= \frac{1}{\pi}\left[-\frac{1}{2}\times\pi\times 1 - 0\right] = \frac{1}{\pi}\times\frac{-\pi}{2} = \frac{-1}{2}$$

$$\therefore\ x\sin x = 1 - \frac{1}{2}\cos x + \sum_{n=2}^{\infty}\frac{-2\cdot(-1)^n}{n^2-1}\cos nx$$

6.

ie, $$x\sin x = 1 - \frac{1}{2}\cos x - 2\left(\frac{\cos 2x}{2^2-1} - \frac{\cos 3x}{3^2-1} + \frac{\cos 4x}{4^2-1} - \frac{\cos 5x}{5^2-1} + \cdots\right)$$

For checking x can take values like $x = 0, \pi/2, \pi$

Put $x = \pi/2$ $\therefore$ $x\sin x = \frac{\pi}{2}\cdot\sin\frac{\pi}{2} = \pi/2$

$$\Rightarrow \frac{\pi}{2} = 1 - \frac{1}{2}\cancel{\cos\pi/2}^{0} - 2\left(\frac{\cos\pi}{2^2-1} - \frac{\cancel{\cos 3\pi/2}^{0}}{3^2-1} + \frac{\cos 2\pi}{4^2-1} - \frac{\cancel{\cos 5\pi/2}^{0}}{5^2-1} + \cdots\right)$$

$$\frac{\pi}{2} = 1 - 2\left(\frac{-1}{3} + \frac{1}{15} + \frac{1}{35} + \cdots\right)$$

$$\frac{\pi}{2} - 1 = 2\left(\frac{1}{3} - \frac{1}{15} + \frac{1}{35} - \cdots\right)$$

$$\frac{\pi-2}{4} = \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \cdots$$

4) Limit applies

Expand $x \sin x$ in $0 < x < 2\pi$

F.S for $x \sin x = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$

where, $a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x)\, dx$

$$= \frac{1}{\pi} \cdot \int_0^{2\pi} x \sin x \, dx$$

$$= \frac{1}{\pi} \left[x \cdot -\cos x - 1 \cdot -\sin x \right]_0^{2\pi}$$

$$= \frac{1}{\pi} \left[-2\pi \cos 2\pi + \sin 2\pi - 0 \right]$$

$$= -\frac{2\pi}{\pi}$$

$$= \underline{\underline{-2}}$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx \, dx$$

$$= \frac{1}{\pi} \int_0^{2\pi} x \sin x \cos nx \, dx$$

$$= \frac{1}{2\pi} \int_0^{2\pi} x \cdot 2\cos nx \cdot \sin x \, dx.$$

$$= \frac{1}{2\pi} \int_0^{2\pi} x \cdot \left[\sin (n+1)x - \sin (n-1)x \right] dx.$$

$$= \frac{1}{2\pi} \left\{ x \cdot \frac{-\cos(n+1)x}{n+1} - 1 \cdot \frac{-\sin(n+1)x}{(n+1)^2} - \left\{ x \cdot \frac{-\cos(n-1)x}{n-1} - 1 \cdot \frac{-\sin(n-1)x}{(n-1)^2} \right\} \right\}_0^{2\pi}$$

$$= \frac{1}{2\pi} \left[-\frac{x \cdot \cos(n+1)x}{n+1} + \frac{\sin(n+1)x}{(n+1)^2} + \frac{x \cdot \cos(n-1)x}{n-1} - \frac{\sin(n-1)x}{(n-1)^2} \right]_0^{2\pi}$$

$$= \frac{1}{2\pi} \left[-\frac{1}{n+1} \cdot (-1)^{2n+2} \cdot 2\pi + \frac{1}{n-1} \cdot (-1)^{2n-2} \cdot 2\pi \right]$$

$$= \frac{1}{2\pi} \left[\frac{(-1)^{2n+3} \cdot 2\pi}{n+1} + \frac{(-1)^{2n-2} \cdot 2\pi}{n-1} \right]$$

$$= \frac{(-1)^{2n} \cdot 2\pi}{2\pi} \left[\frac{(-1)^3}{n+1} + \frac{1}{n-1} \right]$$

$$= -\frac{1}{n+1} + \frac{1}{n-1}$$

$$= \frac{1}{n-1} - \frac{1}{n+1}$$

$$= \frac{n+1-n+1}{n^2-1} = \frac{2}{n^2-1}, \quad n \neq 1$$

when $n=1$,

$$a_1 = \frac{1}{\pi}\int_0^{2\pi} x \sin x \cos x \, dx$$

$$= \frac{1}{2\pi} \cdot \int_0^{2\pi} x \cdot \underbrace{2\sin x \cos x}_{\sin 2x} \, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x \cdot \sin 2x \, dx$$

$$= \frac{1}{2\pi}\left[x \cdot \frac{-\cos 2x}{2} - 1 \cdot \frac{-\sin 2x}{4} \right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[-\cancel{2}\pi \cdot \frac{\overset{=1}{\cos 4\pi}}{\cancel{2}} + \frac{\cancel{\sin 4\pi}^{\,0}}{4} - 0 \right]$$

$$= \frac{1}{2\cancel{\pi}} \times -\cancel{\pi}$$

$$= \frac{-1}{2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \sin nx \, dx$$

why b_1

$$= \frac{1}{\pi} \cdot \int_0^{2\pi} x \sin x \cdot \sin nx \, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x \cdot 2\sin nx \cdot \sin x \, dx.$$

$A = nx, \; B = x.$

$2\sin A \sin B = \cos(A-B) - \cos(A+B)$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\left[\cos(n-1)x - \cos(n+1)x\right] dx$$

$$= \frac{1}{2\pi}\left[x \cdot \frac{\overset{0}{\sin (n-1)x}}{n-1} - 1 \cdot \frac{-\cos(n-1)x}{(n-1)^2} - \left\{ x \cdot \frac{\overset{0}{\cancel{\sin (n+1)x}}}{\cancel{n+1}} - 1 \cdot \frac{-\cos(n+1)x}{(n+1)^2} \right\} \right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\frac{1}{(n-1)^2} \cdot (-1)^{2n-2} - \frac{1}{(n+1)^2}(-1)^{2n+2} - \frac{1}{(n-1)^2} + \frac{1}{(n+1)^2} \right]$$

$$= \frac{1}{2\pi}\left[\frac{(-1)^{2n-2}}{(n-1)^2} + \frac{(-1)^{2n+3}}{(n+1)^2} - \frac{1}{(n-1)^2} + \frac{1}{(n+1)^2} \right]$$

$$= \frac{(-1)^{2n}}{2\pi}\left[\cancel{\frac{1}{(n-1)^2}} - \cancel{\frac{1}{(n+1)^2}} - \cancel{\frac{1}{(n-1)^2}} + \cancel{\frac{1}{(n+1)^2}} \right]$$

$$= \frac{1}{2\pi} \times 0$$

$= 0$ if $n \neq 1$

when $n=1$,

$$b_1 = \frac{1}{\pi}\int_0^{2\pi} x \sin x \cdot \sin x \, dx.$$

$$= \frac{1}{\pi}\cdot\int_0^{2\pi} x \cdot \sin^2 x \, dx. \qquad \left\{\sin^2 x = \frac{1-\cos 2x}{2}\right\}$$

$$= \frac{1}{2\pi}\cdot\int_0^{2\pi} x\,(1-\cos 2x)\,dx.$$

$$= \frac{1}{2\pi}\cdot\left[\int_0^{2\pi} x\,dx - \int_0^{2\pi} x\cos 2x\,dx\right]$$

$$= \frac{1}{2\pi}\left[\left[\frac{x^2}{2}\right]_0^{2\pi} - \left\{x \cdot \frac{\overset{0}{\cancel{\sin 2x}}}{2} - 1 \cdot -\frac{\cos 2x}{4}\right\}_0^{2\pi}\right]$$

$$= \frac{1}{2\pi}\left[\frac{4\pi^2}{2} - \left\{\frac{1}{4}\left(\cos 4\pi - \cos 0\right)\right\}\right]$$

$$= \frac{1}{2\pi}\left[2\pi^2 - \left\{\frac{1}{4}\,(\overset{0}{\cancel{1-1}})\right\}\right]$$

$$= \frac{1}{\cancel{2\pi}} \times \cancel{2}\cancel{\pi}^2.$$

$$= \pi$$

$$\therefore\ x\sin x = \frac{a_0}{2} + a_1\cos x + b_1\sin x + \sum_{n=2}^{\infty}\left(a_n\cos nx + b_n\sin x\right)$$

$$x\sin x = -1 - \frac{1}{2}\cos x + \pi\sin x + \sum_{n=2}^{\infty}\left(\frac{2}{n^2-1}\cdot\cos nx + 0\times\sin nx\right)$$

$$x\sin x = -1 - \frac{1}{2}\cos x + \pi\sin x + \sum_{n=2}^{\infty}\frac{2}{n^2-1}\cdot\cos nx$$

5. Expand $f(x)=x^2$ in $-\pi \le x \le \pi$ and hence show that,

(1) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$

(2) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{x^2} = \frac{\pi^2}{12}$

(3) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$

$f(x) = x^2$ is even function.

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x)\,dx$$

$$= \frac{2}{\pi} \int_0^{\pi} x^2\,dx$$

$$= \frac{2}{\pi} \left[\frac{x^3}{3}\right]_0^{\pi}$$

$$= \frac{2}{\cancel{\pi}} \times \frac{\pi^{\cancel{3}}}{3} = \frac{2\pi^2}{3}$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx\,dx$$

$$= \frac{2}{\pi} \int_0^{\pi} x^2 \cos nx\,dx$$

$$= \frac{2}{\pi} \left[x^2 \cdot \frac{\sin nx}{n} - 2x \cdot \frac{-\cos nx}{n^2} + 2 \cdot \frac{-\sin nx}{n^3} \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[\cancel{\frac{x^2 \sin nx}{n}}^{0} + 2x \frac{\cos nx}{n^2} - \cancel{\frac{2\sin nx}{n^3}}^{0} \right]_0^{\pi}$$

8.

$$= \frac{2}{\pi} \left[2 \times \pi \cdot \frac{\cos n\pi}{n^2} - 0 \right]$$

$$= \frac{2}{\cancel{\pi}} \times 2\cancel{\pi} \times \frac{(-1)^n}{n^2}$$

$$= \frac{4 \cdot (-1)^n}{n^2}$$

$$\therefore f(x) = x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4 \cdot (-1)^n}{n^2} \cdot \cos nx.$$

$$x^2 = \frac{\pi^2}{3} + 4 \left[-\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} - \frac{\cos 3x}{3^2} \cdots \right]$$

(1) when $x = \pi$

$$\pi^2 = \frac{\pi^2}{3} + 4 \left[-\frac{\cos \pi}{1^2} + \frac{\cos 2\pi}{2^2} - \frac{\cos 3\pi}{3^2} \cdots \right]$$

$$\pi^2 = \frac{\pi^2}{3} + 4 \left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \right]$$

$$\pi^2 - \frac{\pi^3}{3} = 4 \left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \right]$$

$$\frac{\cancel{2}\pi^2}{3\times\cancel{4}\,2} = \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots$$

$$\frac{\pi^2}{6} = \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots$$

(2) when $x=0$

$$0^2 = \frac{\pi^2}{3} + 4\left[-\frac{\cos 0}{1^2}+\frac{\cos 0}{2^2}-\frac{\cos 0}{3^2}+\cdots\right]$$

$$0 = \frac{\pi^2}{3} + 4\left[-\frac{1}{1^2}+\frac{1}{2^2}-\frac{1}{3^2}+\cdots\right]$$

$$0 = \frac{\pi^2}{3} - 4\left[\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\cdots\right]$$

$$4\left[\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\cdots\right] = \frac{\pi^2}{3}.$$

$$\therefore \frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\cdots = \frac{\pi^2}{12}$$

(3) Adding the above 2 series we get

$$2\left[\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\right] = \frac{\pi^2}{6}+\frac{\pi^2}{12}$$

$$\frac{1}{1^2}+\frac{1}{3^2}+\cdots = \frac{\cancel{18}^{\,3}\pi^2}{\cancel{6}\times\cancel{12}_{4}\times 2}$$

$$\therefore \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots = \frac{\pi^2}{8}$$

6. Expand $f(x) = |\cos x|$ in $-\pi < x < \pi$.

"Modulus of any function is even"

Eg: $|\cos x|$ is even

$|\sin x|$ is even

$f(x)$ is even.

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

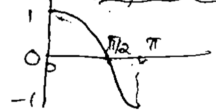
$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} |\cos x|\,dx.$$

$$\boxed{\begin{aligned} |\cos x| &= \cos x \text{ in } 0 < x < \pi/2 \\ &= -\cos x \text{ in } \pi/2 < x < \pi. \end{aligned}}$$

$$\boxed{\begin{aligned} |x| &= x \text{ if } x>0 \\ &= -x \text{ if } x<0 \end{aligned}}$$

graph of cos x



$$\therefore a_0 = \frac{2}{\pi}\left[\int_0^{\pi/2} \cos x\,dx + \int_{\pi/2}^{\pi} -\cos x\,dx\right]$$

$$= \frac{2}{\pi}\left[\left[\sin x\right]_0^{\pi/2} - \left[\sin x\right]_{\pi/2}^{\pi}\right]$$

$$= \frac{2}{\pi}\left[(1-0) - [0-1]\right]$$

$$= \frac{2}{\pi}(1 - -1)$$

$$= \frac{4}{\pi}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} |\cos x| \cos nx\,dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi/2} 2\cos x\cos nx\,dx + \int_{\pi/2}^{\pi} 2.^{-}\cos x\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\int_0^{\pi/2} 2\cos x\cos nx\,dx - \int_{\pi/2}^{\pi} 2\cos x\cos nx\,dx\right]$$

9.

$$= \frac{1}{\pi}\left[\int_0^{\pi/2}(\cos(n+1)x + \cos(n-1)x)\,dx - \int_{\pi/2}^{\pi}(\cos(n+1)x + \cos(n-1)x)\,dx\right]$$

$$= \frac{1}{\pi}\left[\left[\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1}\right]_0^{\pi/2} - \left[\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1}\right]_{\pi/2}^{\pi}\right]$$

$$\sin(n+1)\pi/2$$
$$= \sin\left(\frac{\pi}{2} + \frac{n\pi}{2}\right)$$
$$= \sin(\pi/2 + \theta)$$
$$= \cos\theta = \cos n\pi/2.$$
$$\sin(n-1)\pi/2$$
$$= -\sin(1-n)\pi/2$$
$$= -\sin(\pi/2 - n\pi/2)$$
$$= -\sin(\pi/2 - \theta)$$
$$= -\cos n\pi/2$$

$$= \frac{1}{\pi}\left[\frac{\sin(n+1)\pi/2}{n+1} + \frac{\sin(n-1)\pi/2}{n-1} - \left(\frac{1}{n+1}\{0 - \sin(n+1)\pi/2\} + \frac{1}{n-1}\{0 - \sin(n-1)\pi/2\}\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{\sin(n+1)\pi/2}{n+1} + \frac{\sin(n-1)\pi/2}{n-1} + \frac{\sin(n+1)\pi/2}{n+1} + \frac{\sin(n-1)\pi/2}{n-1}\right]$$

$$= \frac{2}{\pi}\left[\frac{\sin(n+1)\pi/2}{n+1} + \frac{\sin(n-1)\pi/2}{n-1}\right]$$

$$= \frac{2}{\pi}\left[\frac{\cos n\pi/2}{n+1} + \frac{^{-}\cos n\pi/2}{n-1}\right]$$

$$= \frac{2}{\pi}\left[\frac{\cos n\pi/2}{n+1} - \frac{\cos n\pi/2}{n-1}\right]$$

$$= \frac{2}{\pi}\cdot \cos n\pi/2 \left[\frac{1}{n+1} - \frac{1}{n-1}\right]$$

$$= \frac{2}{\pi} \cos n\pi/2 \cdot \frac{-2}{n^2-1}$$

$n-1-n-1$

$$= -\frac{4}{\pi(n^2-1)} \cdot \cos n\pi/2 \quad \text{if } n \neq 1$$

$$a_1 = \frac{2}{\pi}\left[\int_0^{\pi/2} \cos x \cdot \cos x\, dx + \int_{\pi/2}^{\pi} -\cos x \cdot \cos x\, dx\right]$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} \frac{1+\cos 2x}{2} dx - \int_{\pi/2}^{\pi} \frac{1+\cos 2x}{2} dx\right]$$

$$= \frac{1}{\pi}\left[\left[x + \frac{\sin 2x}{2}\right]_0^{\pi/2} - \left(x + \frac{\sin 2x}{2}\right)_{\pi/2}^{\pi}\right]$$

$$a_1 = \frac{1}{\pi}\left[\pi/2 - (\pi - \pi/2)\right]$$

$$= \frac{1}{\pi}\left[\pi/2 - \pi + \pi/2\right] = \frac{1}{\pi}\left[\pi - \pi\right] = 0$$

$$|\cos x| = \frac{2}{\pi} + \frac{4}{\pi}\left[\frac{\cos 2x}{3} - \frac{\cos 4x}{15} + \dots\right]$$

$f(x) = |\sin x|$, $-\pi < x < \pi$.

$f(x)$ is even.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{2}{\pi}\cdot\int_0^{\pi} f(x)\,dx$$

$|\sin x|$ is +ve in the interval 0 to π.

$$= \frac{2}{\pi}\int_0^{\pi} |\sin x|\,dx$$

$\sin x > 0$ in $0 < x < \pi$.

$$= \frac{2}{\pi}\int_0^{\pi} \sin x\,dx$$

7. Expand $f(x) = \sqrt{1-\cos x}$ in $0 < x < 2\pi$. and deduce $\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \dots$

$$f(x) = \sqrt{1-\cos x} = \sqrt{2\sin^2 x/2}$$

$$= \sqrt{2}\cdot \sin x/2$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right)$$

$$a_0 = \frac{1}{\pi}\cdot\int_0^{2\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} \sqrt{2} \sin x/2\, dx.$$

$$= \frac{\sqrt{2}}{\pi}\left[\frac{-\cos x/2}{1/2}\right]_0^{2\pi}$$

$$= \frac{-2\sqrt{2}}{\pi}\left[\cos \pi - \cos 0\right]$$

$$= -\frac{2\sqrt{2}}{\pi}\left[-1-1\right]$$

$$= \frac{4\sqrt{2}}{\pi}$$