(11)

Module 2

FOURIER TRANSFORMS

$$L(\sin at) = \int_0^\infty e^{-st} \sin at \, dt = \frac{a}{s^2+a^2}$$

$$L(\cos at) = \int_0^\infty e^{-st} \cos at \, dt = \frac{s}{s^2+a^2}$$

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi} \qquad \int_0^\infty \frac{\sin sx}{x} dx = \frac{\pi}{2}$$

definitions

✓imp

Fourier Integral Theorem

If $f(x)$ is a function satisfying Dirichlet's conditions in every interval however large then $f(x)$ can be represented as a fourier integral as,

$$f(x) = \frac{1}{\pi} \int_0^\infty \left[\int_{-\infty}^{\infty} f(t) \cos \lambda(t-x) \, dt \right] d\lambda$$

28.

(H)

Marye Vaughen S4 ESTA

✓imp

Fourier Sine Integral

If $f(x)$ is an odd function of x, then it can be represented as a fourier sine integral as,

$$f(x) = \frac{2}{\pi} \int_0^\infty \sin \lambda x \left(\int_0^\infty f(t) \cdot \sin \lambda t \, dt \right) d$$

✓imp

Fourier Cosine Integral

If $f(x)$ is an even function, then it can represented as a fourier cosine integ

$$f(x) = \frac{2}{\pi} \int_0^\infty \cos \lambda x \left(\int_0^\infty f(t) \cos \lambda t \, dt \right) d\lambda$$

Note.

At a point of discontinuity $x=a$, the fourier integral on the RHS $= \frac{1}{2}\left[f(x-a) + f(x+\right.$

1. Express $f(x) = 1 \; , \; |x| < 1$

$\qquad = 0 \; , \; |x| > 1.$

as a fourier integral . Hence evaluate

& ~~deduce~~ $\int_0^\infty \frac{\sin\lambda \cos\lambda x}{\lambda}\, dx$ and deduce

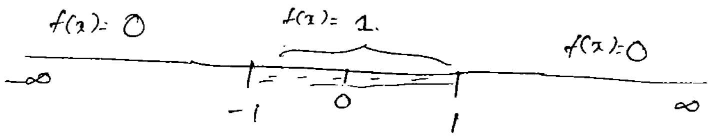
$$\boxed{\int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2}.}$$

Fourier integral formula is,

$$f(x) = \frac{1}{\pi} \cdot \int_0^\infty \left[\int_{-\infty}^\infty f(t) \cdot \cos\lambda(t-x)\, dt \right] d\lambda$$

$$|x| < 1$$

$$\Rightarrow \quad -1 < x < 1$$



$f(x) = 0, \quad -\infty < x < -1$

$\quad\;\; = 1, \quad -1 < x < 1$

$\quad\;\; = 0, \quad 1 < x < \infty$

$$\therefore f(x) = \frac{1}{\pi} \cdot \int_0^\infty \left[\int_{-\infty}^{-1} 0\, dt + \int_{-1}^{1} 1 \cdot \cos\lambda(t-x)\, dt + \int_1^\infty 0\, dt \right] d\lambda$$

$$= \frac{1}{\pi} \int_0^\infty \left[\int_{-1}^{1} \cos\lambda(t-x)\, dt \right] d\lambda$$

$$= \frac{1}{\pi} \cdot \int_0^\infty \left[\frac{\sin\lambda(t-x)}{\lambda} \right]_{-1}^{1} d\lambda$$

$$= \frac{1}{\pi} \cdot \int_0^\infty \frac{1}{\lambda} \left[\sin\lambda(1-x) - \sin\lambda(-1-x) \right] d\lambda$$

$$= \frac{1}{\pi} \cdot \int_0^\infty \frac{1}{\lambda} \; \sin(\overbrace{\lambda - \lambda x}^{C}) + \sin(\overbrace{\lambda + \lambda x}^{D})\, d\lambda$$

$$\sin C + \sin D = 2 \sin\frac{(C+D)}{2} \cdot \cos\left(\frac{C-D}{2}\right)$$

$$= \frac{1}{\pi} \int_0^\infty \frac{1}{\lambda} \cdot 2 \sin\lambda \cdot \cos\lambda x\, d\lambda$$

$$= \frac{2}{\pi} \cdot \int_0^\infty \frac{\sin\lambda \cdot \cos\lambda x}{\lambda}\, d\lambda$$

$$\frac{2}{\pi}\int_0^\infty \frac{\sin\lambda \cdot \cos\lambda x}{\lambda}\,d\lambda = f(x) \text{ in } (-\infty,\infty)$$
$$= 1 \quad \text{if } |x|<1$$
$$= 0 \quad \text{if } |x|>1$$

$$\int_0^\infty \frac{\sin\lambda \cos\lambda x}{\lambda}\,d\lambda = \frac{\pi}{2} \quad \text{if } |x|<1$$
$$= 0 \quad \text{if } |x|>1$$
$$= \frac{1}{2}\left(f(x+1)+f(x-1)\right)$$
$$= \frac{1}{2}\left(0+\pi/2\right)$$
$$= \frac{\pi}{4}$$

In a definite integral, the variable in an integral is immeterial.

$|x|=1$ is a discontinuity.

when $x = 0$, $f(x) = \pi/2$

$$\int_0^\infty \frac{\sin\lambda \cos 0}{\lambda}\,d\lambda = \frac{\pi}{2}$$

$$\int_0^\infty \frac{\sin\lambda}{\lambda}\,d\lambda = \frac{\pi}{2}$$

$$\therefore \int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$$

2. Find the fourier sine and cosine integrals of e^{-kx} and show that

$$\int_0^\infty \frac{\lambda \sin\lambda x}{k^2+\lambda^2}\,d\lambda = \frac{\pi}{2}\cdot e^{-kx} \quad \text{and}$$

$$\int_0^\infty \frac{\cos\lambda x}{k^2+\lambda^2}\,d\lambda = \frac{\pi}{2k}\cdot e^{-kx}.$$

Fourier sine integral, for e^{-kx}.

$$f(x) = \frac{2}{\pi}\cdot\int_0^\infty \sin\lambda x\left[\int_0^\infty f(t)\cdot\sin\lambda t\,dt\right]d\lambda$$

$$\therefore e^{-kx} = \frac{2}{\pi}\cdot\int_0^\infty \sin\lambda x\left(\int_0^\infty e^{-kt}\cdot\sin\lambda t\,dt\right)d\lambda$$

$s=k$, $a=\lambda$

$$\int_0^\infty e^{-st}\sin at\,dt = \frac{a}{s^2+a^2}$$

$$e^{-kx} = \frac{2}{\pi}\cdot\int_0^\infty \sin\lambda x\cdot\frac{\lambda}{k^2+\lambda^2}\,d\lambda$$

$$\therefore \int_0^\infty \frac{\lambda\sin\lambda x}{k^2+\lambda^2}\,d\lambda = \frac{\pi}{2}\cdot e^{-kx}$$

Fourier cosine integral for e^{-kx},

$$f(x) = \frac{2}{\pi} \cdot \int_0^\infty \cos \lambda x \left(\int_0^\infty f(t) \cdot \cos \lambda t \, dt \right) d\lambda$$

$$e^{-kx} = \frac{2}{\pi} \cdot \int_0^\infty \cos \lambda x \cdot \left(\int_0^\infty e^{-kt} \cos \lambda t \, dt \right) d\lambda.$$

$$L(\cos at) = \int_0^\infty e^{-st} \cdot \cos at \, dt = \frac{s}{s^2+a^2}.$$

here, $s = k$, $a = \lambda$.

$$\therefore e^{-kx} = \frac{2}{\pi} \cdot \int_0^\infty \cos \lambda x \cdot \frac{k}{k^2+\lambda^2} \, d\lambda.$$

$$\therefore \int_0^\infty \frac{\cos \lambda x}{k^2+\lambda^2} \, d\lambda = \frac{\pi}{2k} \cdot e^{-kx}$$

3. Using fourier integrals, show that

$$\int_0^\infty \frac{\sin \pi\lambda \cdot \sin \lambda x}{1-\lambda^2} \, d\lambda = \frac{\pi}{2} \sin x, \quad 0 \le x \le \pi$$

$$= 0, \quad x > \pi$$

$$f(x) = \frac{\pi}{2} \sin x, \quad 0 \le x \le \pi$$

$$= 0, \quad x > \pi.$$

Fourier sine integral,

$$f(x) = \frac{2}{\pi} \cdot \int_0^\infty \sin \lambda x \left(\int_0^\infty f(t) \cdot \sin \lambda t \, dt \right) d\lambda.$$

$$= \frac{2}{\pi} \cdot \int_0^\infty \sin \lambda x \cdot \left[\int_0^\pi \frac{\pi}{2} \sin t \cdot \sin \lambda t \, dt + \int_\pi^\infty 0 \cdot \sin \lambda t \, dt \right] d\lambda$$

$$= \cancel{\frac{2}{\pi}} \cdot \int_0^\infty \sin \lambda x \cdot \cancel{\frac{\pi}{2}} \cdot \left[\int_0^\pi \sin t \cdot \sin \lambda t \, dt \right] d\lambda$$

$$\therefore = \int_0^\infty \sin \lambda x \left[\int_0^\pi \sin t \cdot \sin \lambda t \, dt \right] d\lambda$$

$$\cancel{= \int_0^\infty \sin \lambda x \cdot \left[\frac{1}{2} \cdot \int_0^\pi 2 \sin \lambda t \sin t \, dt \right] d\lambda}$$

$$\cancel{= \frac{1}{2} \int_0^\infty \sin \lambda x \left[\int_0^\pi (\cos (\lambda-1)t - \cos (\lambda+1)t) \, dt \right] d\lambda}$$

$$\cancel{= \frac{1}{2} \cdot \int_0^\infty \sin \lambda x \left[\frac{\sin (\lambda-1)t}{\lambda-1} - \frac{\sin (\lambda+1)t}{\lambda+1} \right]_0^\pi d\lambda}$$

$$\cancel{\frac{1}{2}\cdot\int_0^\infty \sin\lambda x}$$

$$= \int_0^\infty \sin\lambda x\left[\frac{1}{2}\cdot\int_0^\pi 2\sin t\cdot \sin\lambda t\,dt\right]d\lambda$$

$$= \frac{1}{2}\cdot\int_0^\infty \sin\lambda x\left[\int_0^\pi (\cos(1-\lambda)t - \cos(1+\lambda)t)\,dt\right]d\lambda$$

$$= \frac{1}{2}\cdot\int_0^\infty \sin\lambda x\left[\frac{\sin(1-\lambda)t}{1-\lambda} - \frac{\sin(1+\lambda)t}{1+\lambda}\right]_0^\pi d\lambda$$

$$= \frac{1}{2}\cdot\int_0^\infty \sin\lambda x\left[\frac{\sin(\pi-\pi\lambda)}{1-\lambda} - \frac{\sin \pi+\pi\lambda}{1+\lambda}\right]d\lambda$$

$$= \frac{1}{2}\cdot\int_0^\infty \sin\lambda x\left[\frac{(1+\lambda)\sin(\pi-\pi\lambda) - (1-\lambda)\sin(\pi+\pi\lambda)}{1-\lambda^2}\right]d\lambda$$

$$= \frac{1}{2}\cdot\int_0^\infty \sin\lambda x.\left[\frac{(1+\lambda)\sin\pi\lambda - (1-\lambda)^{-}\sin\pi\lambda}{1-\lambda^2}\right]d\lambda$$

$$= \frac{1}{2}\cdot\int_0^\infty \sin\lambda x\left[\frac{\sin\pi\lambda(1+\lambda+1-\lambda)}{1-\lambda^2}\right]d\lambda$$

$$= \frac{1}{\cancel{2}}\times\cancel{2}\int_0^\infty \frac{\sin\lambda x.\ \sin\pi\lambda}{1-\lambda^2}\,d\lambda$$

$$= \int_0^\infty \frac{\sin\lambda x.\ \sin\pi\lambda}{1-\lambda^2}\,d\lambda$$

$$\therefore\ f(x) = \int_0^\infty \frac{\sin\lambda x.\ \sin\pi\lambda}{1-\lambda^2}\,d\lambda = \frac{\pi}{2}\sin x.\ ,\ 0 \le x \le \pi$$
$$= 0\ ,\quad x > \pi$$

4. Express $f(x) = 1$, $0 \le x \le \pi$
$= 0$, $x > \pi$ as a fourier sine integral, hence evaluate $\int_0^\infty \frac{(1-\cos\pi\lambda)\sin\lambda x}{\lambda}\,d\lambda$ and $\int_0^\infty \frac{(1-\cos\pi\lambda)\ \sin\pi\lambda}{\lambda}\,d\lambda$.

$f(x)$ as a sine integral,

$$f(x) = \frac{2}{\pi}.\int_0^\infty \sin\lambda x\left[\int_0^\infty f(t)\sin\lambda t\,dt\right]d\lambda$$

$$= \frac{2}{\pi}.\int_0^\infty \sin\lambda x\left[\int_0^\pi 1.\sin\lambda t\,dt + \int_\pi^\infty 0.\sin\lambda t\,dt\right]d\lambda$$

$$= \frac{2}{\pi}.\int_0^\infty \sin\lambda x.\left[\int_0^\pi \sin\lambda t\,dt\right]d\lambda.$$

$$= \frac{2}{\pi} \cdot \int_0^\infty \sin \lambda x \left[-\frac{\cos \lambda t}{\lambda} \right]_0^\pi d\lambda$$

$$= -\frac{2}{\pi} \cdot \int_0^\infty \frac{\sin \lambda x}{\lambda} \left(\cos \pi \lambda - \overset{1}{\cos 0} \right) d\lambda$$

$$= -\frac{2}{\pi} \cdot \int_0^\infty \frac{\sin \lambda x}{\lambda} \left(\cos \pi \lambda - 1 \right) d\lambda$$

$$= \frac{2}{\pi} \cdot \int_0^\infty \frac{(1 - \cos \pi \lambda) \sin \lambda x}{\lambda} d\lambda.$$

$$\therefore \int_0^\infty \frac{(1-\cos \pi \lambda) \sin \lambda x}{\lambda} d\lambda = \frac{\pi}{2} \cdot f(x)$$

$$\therefore \int_0^\infty \frac{(1-\cos \pi \lambda) \sin \lambda x}{\lambda} d\lambda = \frac{\pi}{2}, \quad 0 \le x \le \pi$$

$$= 0, \quad x > \pi$$

when $x = \pi$,

$f(x) = \pi/2$

$$\therefore \frac{\pi}{2} = \int_0^\infty \frac{(1-\cos \pi \lambda) \sin \pi \lambda}{\lambda} d\lambda$$

$$\therefore \int_0^\infty \frac{(1-\cos \pi \lambda) \sin \pi \lambda}{\lambda} d\lambda = \frac{\pi}{2}$$

$x = \pi$ is a point of discontinuity, hence

at $x = \pi$,

$$f(x) = \int_0^\infty \frac{(1-\cos \pi \lambda) \sin \lambda x}{\lambda} d\lambda = \frac{1}{2} \left[f(x-\pi) + f(x+\pi) \right]$$

$$= \frac{1}{2} \left[\frac{\pi}{2} + 0 \right]$$

$$\therefore \int_0^\infty \frac{(1-\cos \pi \lambda) \sin \pi \lambda}{\lambda} d\lambda = \frac{\pi}{4}$$

5. Find a fourier integral representation o.

$$f(x) = 0, \quad x < 0$$

$$= \frac{1}{2}, \quad x = 0$$

$$= e^{-x}, \quad x > 0$$

Deduce $\int_0^\infty \frac{1}{1+x^2} dx = \frac{\pi}{2}$.

$$f(x) = \frac{1}{\pi} \cdot \int_0^\infty \left[\int_{-\infty}^\infty f(t) \cos \lambda (t-x) dt \right] d\lambda$$

$$= \frac{1}{\pi} \int_0^\infty \left[\int_{-\infty}^0 0 \cdot \cos \lambda (t-x) dt + \int_0^\infty e^{-x} \cos \lambda (t-x) dt \right] d\lambda$$

$$= \frac{1}{\pi} \int_0^\infty \left[\int_0^\infty e^{-x} \cos \lambda(t-x)\, dt \right] d\lambda$$

$$= \frac{1}{\pi} \int_0^\infty \left[0 + \int_0^\infty e^{-x} \cos(\lambda t - \lambda x)\, dt \right] d\lambda$$

$a=-1,\ b=\lambda,\ c=-\lambda x$

$$= \frac{1}{\pi} \int_0^\infty \left[\frac{e^{-x}}{1^2+\lambda^2} \left\{ -\cos \lambda(t-x) + \lambda \sin\lambda(t-x) \right\} \right]_0^\infty d\lambda$$

$$= \frac{1}{\pi} \int_0^\infty \frac{-1}{1+\lambda^2} \left(-\cos \lambda x - \lambda \sin \lambda x \right) d\lambda$$

$$f(x) = \frac{1}{\pi} \int_0^\infty \frac{\lambda \sin \lambda x + \cos \lambda x}{1+\lambda^2}\, d\lambda$$

$$f(x) = \frac{1}{\pi} \int_0^\infty \frac{\lambda \sin \lambda x + \cos \lambda x}{1+\lambda^2}\, d\lambda$$

when $x = 0$, $f(x) = \frac{1}{2}$.

$$\therefore \frac{1}{2} = \frac{1}{\pi} \int_0^\infty \frac{0 + \cos 0}{1+\lambda^2}\, d\lambda$$

$$\Rightarrow \frac{1}{\pi} \int_0^\infty \frac{1}{1+\lambda^2}\, d\lambda = \frac{1}{2}.$$

$$\int_0^\infty \frac{1}{1+\lambda^2}\, d\lambda = \frac{\pi}{2}$$

$$\therefore \int_0^\infty \frac{1}{1+x^2}\, dx = \frac{\pi}{2}$$

6. Express $f(x) = a, \quad 0 \le x \le \pi$

$= 0, \quad x > \pi$ as a fourier sine integral. Evaluate $\int_0^\infty \frac{(1-\cos\pi\lambda)\sin\lambda x}{\lambda} d\lambda$.

$f(x)$ as a sine integral,

$$f(x) = \frac{2}{\pi} \cdot \int_0^\infty \sin\lambda x \cdot \left[\int_0^\infty f(t) \cdot \sin\lambda t \, dt\right] d\lambda$$

$$= \frac{2}{\pi} \cdot \int_0^\infty \sin\lambda x \left[\int_0^\pi a \sin\lambda t \, dt + \int_\pi^\infty 0 \cdot \sin\lambda t \, dt\right] d\lambda$$

$$= \frac{2}{\pi} \cdot \int_0^\infty \sin\lambda x \cdot \left[a \cdot \frac{-\cos\lambda t}{\lambda}\right]_0^\pi d\lambda$$

$$= \frac{2}{\pi} \times -\frac{a}{\lambda} \int_0^\infty \sin\lambda t \left[\cos\pi\lambda - \cos 0\right] d\lambda$$

$$= -\frac{2a}{\pi\lambda} \cdot \int_0^\infty \sin\lambda \cancel{x} (\cos\pi\lambda - 1) \, d\lambda$$

$$= \frac{-2a}{\pi\lambda} \int_0^\infty \sin\lambda \cancel{t} x (1 - \cos\pi\lambda) \, d\lambda$$

$$f(x) = \frac{2a}{\pi} \cdot \int_0^\infty \frac{(1-\cos\pi\lambda)\sin\lambda x}{\lambda} d\lambda$$

$$\therefore \int_0^\infty \frac{(1-\cos\pi\lambda)\sin\lambda x}{\lambda} d\lambda = f(x) \cdot \frac{\pi}{2a}.$$

$$\int_0^\infty \frac{(1-\cos\pi\lambda)\sin\lambda x}{\lambda} d\lambda = a \cdot \frac{\pi}{2a} = \frac{\pi}{2}$$

Fourier Transform

Fourier transform of a function $f(x)$ is denoted by,

$$F(f(x)) = F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \cdot e^{-isx} dx.$$

where, s is a parameter.

Corresponding inversion formula is,

$$f(x) = F^{-1}(F(s)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) \cdot e^{isx} ds$$

Fourier Sine Transform $\left[F_s(s)\right]$

$$F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cdot \sin sx \, dx$$

Corresponding inverse transform,

only imaginary part is taken

$$f(x) = F^{-1}\left(F_s(s)\right)$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_s(s) \cdot \sin sx \, ds$$

Fourier Cosine Transform $\left[F_c(s)\right]$

$$F_c(s) = \sqrt{\frac{2}{\pi}} \cdot \int_0^{\infty} f(x) \cos sx \, dx$$

corresponding inverse transform is,

$$f(x) = F^{-1}\left(F_c(s)\right) = \sqrt{\frac{2}{\pi}} \cdot \int_0^{\infty} F_c(s) \cdot \cos sx \, ds$$

Properties of Fourier Transform

1. **Linearity property** -

If $F(s)$ and $G(s)$ are fourier transform of $f(x)$ and $g(x)$ respectively, a and b are constants.

Then, $F\left(af(x) \pm bg(x)\right) = aF(f(x)) \pm b\,F(g(x))$

$$= aF(s) \pm b \cdot G(s)$$

$F(s) = F(f(x))$

2. **Shifting property** -

$$\boxed{\begin{aligned} F\left(f(x-a)\right) &= e^{-ias} F(s). \\ F\left(f(x+a)\right) &= e^{ias} F(s) \end{aligned}}$$

3. **Multiplication by x^n.**

$$F\left(x^n f(x)\right) = (i)^n \frac{d^n}{ds^n} F(s)$$

$$F\left(x f(x)\right) = i \cdot \frac{d}{ds} F(s)$$

Special cases

(i) $F_s\left(x f(x)\right) = -\frac{d}{ds} F_c(s)$

(ii) $F_c\left(x f(x)\right) = \frac{d}{ds} F_s(s)$

4. <u>Self - Reciprocal Property</u>-

A function $f(x)$ is said to be self - reciprocal if its fourier transform is obtained by just replacing the variable x by the variable s.

ie, $\boxed{F(f(x)) = F(s) = f(s)}$

1. Find the fourier transform of,

$$f(x) = 1-x^2 \quad , \quad |x|<1$$
$$= 0 \quad , \quad |x|>1$$

hence evaluate $\int_0^\infty \frac{x\cos x - \sin x}{x^3} \cdot \cos\frac{x}{2}\, dx$

$$F(f(x)) = F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-isx}\, dx.$$

$e^{-ia} = \cos a - i\sin a$

$e^{ia} = \cos a + i\sin a$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{1} 0 + \int_{-1}^{1} (1-x^2) e^{-isx}\, dx + \int_1^\infty 0 \right]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} \underset{\text{even}}{(1-x^2)} (\underset{\text{even}}{\cos sx} - i \underset{\text{odd}}{\sin sx})\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \cdot 2\int_0^1 (1-x^2)\cos sx\, dx + 0.$$

$$= \frac{2}{\sqrt{2\pi}} \cdot \left[\frac{\sin sx}{s} - \left\{ x^2 \frac{\sin sx}{s} - 2x \cdot \frac{-\cos sx}{s^2} + 2 \cdot -\frac{\sin sx}{s^3} \right\} \right]_0^1$$

$$= \frac{2}{\sqrt{2\pi}} \left(\frac{\sin sx}{s} - \frac{x^2 \sin sx}{s} - \frac{2\cos sx}{s^2} + \frac{2\sin sx}{s^3} \right]_0^1$$

$$= \frac{2}{\sqrt{2\pi}} \left(\cancel{\frac{\sin s}{s}} - \cancel{\frac{\sin s}{s}} - \frac{2(\cos s - 0)}{s^2} + \frac{2\sin s}{s^3} \right)$$

$$= \frac{2}{\sqrt{2\pi}} \left(\frac{2}{s^2} \left\{ \frac{\sin s}{s} - (\cos s) \right\} \right]$$

$$= \frac{4}{\sqrt{2\pi}} \left(\frac{\sin s - s\cos s}{s^3} \right] = F(s).$$

$$f(x) = F^{-1}(F(s)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s).\, e^{isx}\, ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{4}{\sqrt{2\pi}} \left(\frac{\sin s - s\cos s}{s^3}\right) e^{isx}\, ds$$

odd . odd = odd

$$= \frac{4}{2\pi} \int_{-\infty}^{\infty} \left(\frac{\overset{\text{odd}}{\sin s} - \overset{\text{odd}}{s\cos s}}{\underset{\text{odd}}{s^3}}\right)^{\text{even}} (\underset{\text{even}}{\cos sx} + i\underset{\text{odd}}{\sin sx})\, dx$$

$$= \frac{4}{2\pi} \cdot 2 \int_0^{\infty} \left(\frac{\sin s - s\cos s}{s^3}\right) . \cos sx\, ds + 0$$

$$f(x) = \frac{4}{\pi} \cdot \int_0^{\infty} \frac{\sin s - s\cos s}{s^3} . \cos sx\, ds$$

$$\therefore \int_0^{\infty} \frac{\sin s - s\cos s}{s^3} . \cos sx\, ds = \frac{\pi}{4} f(x)$$

$$= \frac{\pi}{4}(1-x^2), \quad |x|<1$$

$$= 0, \quad |x|>1$$

When $x = \frac{1}{2}$, the integral is,

$$f(x) = \frac{\pi}{4}\left(1-\left(\frac{1}{2}\right)^2\right) = \frac{\pi}{4}\left(1-\frac{1}{4}\right)$$

$$= \frac{\pi}{4} \times \frac{3}{4} = \underline{\underline{\frac{3\pi}{16}}}$$

$$\therefore \int_0^{\infty} \frac{\sin s - s\cos s}{s^3} . \cos\frac{s}{2}\, ds = \frac{\pi}{4}\left(1-\left(\frac{1}{2}\right)^2\right)$$

$$= \frac{3\pi}{16}$$

$$\Rightarrow \int_0^{\infty} \frac{s\cos s - \sin s}{s^3} . \cos\frac{s}{2}\, ds = \frac{-3\pi}{16}$$

$$\therefore \int_0^{\infty} \frac{x\cos x - \sin x}{x^3} . \cos\frac{x}{2}\, dx = \frac{-3\pi}{16}$$

---

8. Find the fourier transform of,

$e^{-|x|}$ and deduce $\int_0^{\infty} \frac{\cos xt}{1+t^2}\, dt = \frac{\pi}{2} e^{-x}$

$$F(f(x)) = F(s)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x).\, e^{-isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-|x|} . e^{-isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underset{\text{even}}{e^{-|x|}} (\underset{\text{even}}{\cos sx} - i\underset{\text{odd}}{\sin sx})\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \cdot 2 \int_0^\infty e^{-|x|} \cdot \cos sx\, dx + 0.$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^\infty e^{-x} \cdot \cos sx\, dx. \quad \text{in } (0,\infty) \quad |x| = x \quad x \text{ is +ve.}$$

$s = 1, \ a = s$

$$= \frac{2}{\sqrt{2\pi}} \cdot \frac{1}{1^2 + s^2}$$

$$\boxed{\begin{aligned} &\int_0^\infty e^{-st} \cdot \cos at\, dt \\ &= L(\cos at) \\ &= \frac{s}{s^2 + a^2} \end{aligned}}$$

$$= \frac{2}{\sqrt{2\pi}} \cdot \frac{1}{1+s^2}$$

Fourier inverse,

$$f(x) = F^{-1}(F(s))$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) \cdot e^{isx}\, ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{2}{\sqrt{2\pi}} \cdot \frac{1}{1+s^2} \cdot e^{isx}\, ds$$

$$= \frac{2}{2\pi} \int_{-\infty}^{\infty} \underbrace{\frac{1}{1+s^2}}_{\text{even}} (\underbrace{\cos sx}_{\text{even}} + i \underbrace{\sin sx}_{\text{odd}})\, ds$$

$$= \frac{2}{2\pi} \cdot 2 \int_0^\infty \frac{1}{1+s^2} \cos sx\, ds$$

$$e^{-|x|} = \cancel{4} \cdot \frac{2}{\pi} \cdot \int_0^\infty \frac{1}{1+s^2} \cos sx\, ds.$$

$$\therefore \int_0^\infty \frac{\cos sx}{1+s^2}\, ds = \frac{\pi}{2} \cdot e^{-|x|}.$$

ie, $\cancel{\int_0^\infty \cos xt}$

$$\text{ie, } \int_0^\infty \frac{\cos xs\, ds}{1+s^2} = \frac{\pi}{2} \cdot e^{-|x|}$$

Put $s = t$,

$$\therefore \int_0^\infty \frac{\cos xt}{1+t^2}\, dt = \frac{\pi}{2} \cdot e^{-x}$$

9. UQ Vimp Find the F.T of $e^{-ax^2}, a > 0$, prove that $f(x) = e^{-x^2/2}$ is self reciprocal under fourier transformation.

$$F(e^{-ax^2}) = F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ax^2} e^{-isx}\, dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(ax^2 + isx)}\, dx.$$

Consider $ax^2 + isx$.

$$(A+B)^2 = A^2 + 2AB + B^2$$

$A^2+2AB = ax^2+isx.$

$A^2 = ax^2$

$\Rightarrow A = \sqrt{a}\,x$

$2AB = isx.$

$\therefore B = \dfrac{is\cancel{x}}{2\sqrt{a}\,x} = \dfrac{is}{2\sqrt{a}}$

$B^2 = \dfrac{-s^2}{4a}$

$$\therefore F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(ax^2+isx)}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\left(ax^2+isx-\frac{s^2}{4a}+\frac{s^2}{4a}\right)}\,dx$$

$$= \frac{e^{-\frac{s^2}{4a}}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\left(\sqrt{a}x+\frac{is}{2\sqrt{a}}\right)^2}\,dx.$$

$$= \frac{e^{-\frac{s^2}{4a}}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-u^2}\frac{du}{\sqrt{a}}$$

put $\sqrt{a}\,x+\dfrac{is}{2\sqrt{a}} = u$

$\sqrt{a}\,dx = du$

$dx = \dfrac{du}{\sqrt{a}}$

$$= \frac{e^{-\frac{s^2}{4a}}}{\sqrt{2a\pi}}\cdot\sqrt{\pi}$$

$$= \frac{e^{-\frac{s^2}{4a}}}{\sqrt{2a}}$$

ie, $F\left(e^{-ax^2}\right) = \dfrac{e^{-\frac{s^2}{4a}}}{\sqrt{2a}}$

$(a = 1/2)$

$$F\left(e^{-x^2/2}\right) = \frac{e^{-\frac{s^2}{4\times\frac{1}{2}}}}{\sqrt{2\times\frac{1}{2}}} = e^{-\frac{s^2}{2}}$$

ie, $e^{-\frac{x^2}{2}}$ is self reciprocal.

$F(s) = F(f(x)) = f(s)$

Note

Vimp

10. Find the fourier cosine transform of $\dfrac{1}{1+x^2}$ hence deduce the fourier sine transform of $\dfrac{x}{1+x^2}$

Fourier cosine transform of $f(x)$,

$$F_c(s) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos sx\,dx.$$

Fourier sine transform of $f(x)$,

$$F_s(s) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x).\ \sin sx\,dx.$$

$$F_c\left(\frac{1}{1+x^2}\right) = F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\cos sx\,dx$$

$$= \sqrt{\frac{2}{\pi}} \cdot \int_0^\infty \frac{1}{1+x^2} \cos sx\,dx \text{ — I.}$$

$$\frac{dI}{ds} = \sqrt{\frac{2}{\pi}} \cdot \int_0^\infty \frac{-x \sin sx}{1+x^2}\,dx.$$

$$\frac{dI}{ds} = -\ F_s\left(\frac{x}{1+x^2}\right) \text{ — (1)}$$

$$\boxed{F_s\left(\frac{x}{1+x^2}\right) = \sqrt{\frac{2}{\pi}} \cdot \int_0^\infty \frac{x}{1+x^2} \sin sx\,dx}$$

$$\frac{dI}{ds} = -\sqrt{\frac{2}{\pi}} \cdot \int_0^\infty \frac{x^2}{x(1+x^2)} \sin sx\,dx$$

$$= -\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{1+x^2-1}{x(1+x^2)} \sin sx\,dx$$

$$= -\sqrt{\frac{2}{\pi}} \left[\underbrace{\int_0^\infty \frac{\sin sx}{x}\,dx}_{\pi/2} - \int_0^\infty \frac{\sin sx}{x(1+x^2)}\,dx \right]$$

$$\frac{dI}{ds} = -\sqrt{\frac{2}{\pi}} \cdot \times \frac{\pi}{2} + \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\sin sx}{x(1+x^2)}\,dx \text{ — (2)}$$

$$\frac{dI^2}{ds^2} = 0 + \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{x \cos sx}{x(1+x^2)}\,dx = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\cos sx}{1+x^2}\,dx$$

$$-\frac{d^2I}{ds^2} = I.$$

$$\frac{d^2I}{ds^2} - I = 0$$

$$\boxed{\begin{aligned}&(D^2-1)y = 0\\ &y = c_1e^x + c_2e^{-x}.\end{aligned}}$$

$$(D^2-1)\,I = 0$$

Auxillary eqn is $m^2 - 1 = 0$, $m = \pm 1$

$D = \frac{d}{ds}$

$y = I$

$x = s$

Solution is $I = c_1e^s + c_2e^{-s}$ — (3)

ie, $\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{1}{1+x^2} \cos sx\,dx = c_1e^s + c_2e^{-s}$

when $s=0 \Rightarrow \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{1}{1+x^2}\,dx = c_1 + c_2$

ie, $\sqrt{\frac{2}{\pi}} \left[\tan^{-1}x\right]_0^\infty = c_1 + c_2$

ie, $\sqrt{\frac{2}{\pi}} \left[\frac{\pi}{2} - 0\right] = c_1 + c_2$

$c_1 + c_2 = \sqrt{\frac{\pi}{2}}$ — (4)

from (3) $\frac{dI}{ds} = c_1e^s - c_2e^{-s}$

ie, $-\sqrt{\frac{\pi}{2}} + \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\sin sx}{x(1+x^2)}\,dx = c_1e^s - c_2e^{-s}$

$s = 0 \Rightarrow -\sqrt{\frac{\pi}{2}} + 0 = c_1 - c_2$ — (5)

$\circledast$ (4) is $c_1+c_2 = \sqrt{\frac{\pi}{2}}$

(5) is $c_1-c_2 = -\sqrt{\frac{\pi}{2}}$

$$\underline{c_1=0}$$

$$\underline{c_2=\sqrt{\frac{\pi}{2}}}$$

$$I = 0e^{s} + \sqrt{\frac{\pi}{2}}\, e^{-s}$$

ie, $\underline{F_c\left(\frac{1}{1+x^2}\right) = \sqrt{\frac{\pi}{2}}\; e^{-s}}$

$$\frac{dI}{ds} = 0e^{s} - \sqrt{\frac{\pi}{2}}\; e^{-s}$$

from (1) $\Rightarrow$ $F_S\left(\frac{x}{1+x^2}\right) = -\frac{dI}{ds} = \underline{\sqrt{\frac{\pi}{2}}\cdot e^{-s}}$

11. Find the Fourier transform of $x\cdot e^{-|x|}$ and deduce the value of

$$\int_0^\infty \frac{x \sin mx}{(1+x^2)^2}\, dx.$$

$$F(x f(x)) = i\frac{d}{ds}F(s)$$

$$= i\frac{d}{ds}\,F(f(x))$$

$$F(f(x)) = F\left(e^{-|x|}\right) = \sqrt{\frac{2}{\pi}}\cdot\frac{1}{1+s^2} \quad \text{(proved}$$

$$F(x f(x)) = F\left(x\cdot e^{-|x|}\right) = i\frac{d}{ds}\left(\sqrt{\frac{2}{\pi}}\cdot\frac{1}{1+s^2}\right)$$

$$= i\sqrt{\frac{2}{\pi}}\cdot\frac{-2s}{1+s^2}$$

$$= \underline{\frac{-2\sqrt{2}\,i}{\sqrt{\pi}}\cdot\frac{s}{(1+s^2)^2}}$$

By inverse formula,

$$x\cdot e^{-|x|} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F\left(xe^{-|x|}\right)e^{isx}\, ds$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{-2\sqrt{2}\,i}{\sqrt{\pi}}\cdot\frac{\overset{\text{odd}}{s}}{(1+s^2)^2}\left(\overset{\text{even}}{\cos sx}+\overset{\text{odd}}{i\sin sx}\right)$$

$$= \frac{-2i}{\pi}\left[2\cdot\int_0^\infty\frac{s}{(1+s^2)^2}\cdot i\sin sx\, ds\right)$$

$$= x\cdot e^{-|x|}$$

ie, $\frac{4}{\pi} \int_0^\infty \frac{s \sin sx \, ds}{(1+s^2)^2} = x \cdot e^{-|x|}$

$$\int_0^\infty \frac{s \sin sx}{(1+s^2)^2} \, ds = \frac{\pi}{4} x \cdot e^{-|x|}$$

$s = x$ and $x = m$

$$\int_0^\infty \frac{x \sin mx}{(1+x^2)^2} dx = \frac{\pi}{4} \, m e^{-m}$$

12. Fourier sine transform of $\frac{1}{x}$. in $(0,\infty)$, $|x| = x$

$$F_S(s) = F_S\left(\frac{1}{x}\right) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{1}{x} \sin sx \, dx$$

put $sx = t$

$s\,dx = dt$

$x = \frac{t}{s}$

$dx = \frac{dt}{s}$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\cancel{s}}{t} \sin t \frac{dt}{\cancel{s}}$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{1}{t} \sin t \, dt.$$

$$= \sqrt{\frac{2}{\pi}} \times \frac{\pi}{2}$$

Basic result $\boxed{\int_0^\infty \frac{\sin sx}{x} dx = \frac{\pi}{2}}$

13. U.Q. Find the Fourier cosine transform of e^{-x^2}.

$$F_C(s) = F_C\left(e^{-x^2}\right) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x^2} \cdot \cos sx \, dx = I$$

$$\frac{dI}{ds} = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x^2} \cdot -x \cdot \sin sx \, dx.$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{1}{2} \cdot \int_0^\infty \underbrace{-2x \, e^{-x^2}}_{v} \cdot \underbrace{\sin sx \, dx}_{u}$$

$\int 2x \cdot e^{-x^2} dx$

$u = -x^2$

$-2x\,dx = d$

$\int e^u du =$

$$= \frac{1}{\sqrt{2\pi}} \left[\sin sx \cdot e^{-x^2} \right]_0^\infty - \int_0^\infty s \cos sx \cdot e^{-x^2} dx.$$

$$= \frac{1}{\sqrt{2\pi}} \times 0 + -s \cdot \int_0^\infty e^{-x^2} \cos sx \, dx.$$

$$= -\frac{s}{2} \cdot \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x^2} \cos sx \, dx.$$

$$\frac{dI}{ds} = -\frac{s}{2} \cdot I.$$

$$\frac{dI}{I} = -\frac{s}{2}\, ds.$$

(variable separable form)

on integrating.

$$\int \frac{dI}{I} = -\int \frac{s}{2}\, ds$$

$$\log I = \frac{-1}{2} \cdot \frac{s^2}{2} + \underbrace{\log A}_{\text{const.}}$$

$$\log\left(\frac{I}{A}\right) = -\frac{s^2}{4}$$

$$\frac{I}{A} = e^{-s^2/4}$$

$$I = A \cdot e^{-s^2/4}$$

$$\sqrt{\frac{2}{\pi}} \cdot \int_0^\infty e^{-x^2} \cos sx\, dx = A \cdot e^{-s^2/4}$$

$$\boxed{\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}}$$

when $s=0$,

$$\sqrt{\frac{2}{\pi}} \cdot \int_0^\infty e^{-x^2} \cos 0\, dx = A \cdot e^0$$

$$\sqrt{\frac{2}{\pi}} \cdot \frac{\sqrt{\pi}}{2} = A \Longrightarrow A = \frac{1}{\sqrt{2}}.$$

$$\therefore F_c\left(e^{-x^2}\right) = \frac{1}{\sqrt{2}}\, e^{-s^2/4}$$

14. Find the function whose fourier sine transform is $\frac{e^{-as}}{s}$

v.imp

given $F_s(s) = \frac{e^{-as}}{s}$

let $f(x)$ is such that $F_s(f(x)) = F_s(s)$
$= \frac{e^{-as}}{s}$

Inverse formula,

$$f(x) = F^{-1}(F_s(s))$$

$$= \sqrt{\frac{2}{\pi}} \cdot \int_0^\infty F_s(s) \cdot \sin sx\, ds$$

$$= \sqrt{\frac{2}{\pi}} \cdot \int_0^\infty \frac{e^{-as}}{s} \cdot \sin sx\, ds.$$

$$\frac{d}{dx} f(x) = \sqrt{\frac{2}{\pi}} \cdot \int_0^\infty \frac{e^{-as}}{\cancel{s}} \cdot \cancel{s} \cdot \cos sx\, ds.$$

$$= \sqrt{\frac{2}{\pi}} \cdot \int_0^\infty e^{-as} \cos sx\, ds$$

$t = s$, $s = a$, $a = x$

$$\boxed{\int e^{-st} \cos at\, dt = \frac{s}{s^2+a^2}}$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{a}{a^2+x^2}$$

$$f(x) = \int \sqrt{\frac{2}{\pi}} \cdot \frac{a}{x^2+a^2}\, dx.$$

$$f(x) = \sqrt{\frac{2}{\pi}} \cdot \tan^{-1}\frac{x}{a} + c$$

$$\boxed{\int \frac{1}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)}$$

$x=0$,

$f(x)=0$.

$0 = 0 + C$

$C=0$

$$\therefore f(x) = \sqrt{\frac{2}{\pi}} \cdot \tan^{-1}\left(\frac{x}{a}\right)$$

15 Find the fourier sine transform of $\frac{e^{-ax}}{x}$.

and deduce $\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x}\, dx$.

$$F_S(s) = \sqrt{\frac{2}{\pi}} \cdot \int_0^\infty \frac{e^{-ax}}{x} \sin sx\, dx.$$

$$\frac{d}{ds} F_S(s) = \sqrt{\frac{2}{\pi}} \cdot \int_0^\infty \frac{e^{-ax}}{\cancel{x}} \cdot \cancel{x} \cdot \cos sx \cdot dx.$$

$$= \sqrt{\frac{2}{\pi}} \cdot \int_0^\infty e^{-ax} \cos sx\, dx.$$

$t=x$, $a=s$, $s=a$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{a}{a^2+s^2}$$

$$\frac{d}{ds} F_S(s) = \sqrt{\frac{2}{\pi}} \cdot \frac{a}{a^2+s^2}.$$

$$\therefore F_S(s) = \int \sqrt{\frac{2}{\pi}} \cdot \frac{a}{a^2+s^2}\, ds$$

$$F_S(s) = \sqrt{\frac{2}{\pi}} \tan^{-1}\left(\frac{s}{a}\right) + c$$

when $s=0$,

$F_S(s) = 0$

$0 = 0 + c \Rightarrow c = 0$

$$\therefore F_S(s) = \sqrt{\frac{2}{\pi}} \tan^{-1}\left(\frac{s}{a}\right)$$

$$\cancel{\sqrt{\frac{2}{\pi}}} \int_0^\infty \frac{e^{-ax}}{x} \sin sx\, dx = \cancel{\sqrt{\frac{2}{\pi}}} \cdot \tan^{-1}\left(\frac{s}{a}\right)$$

$$\int_0^\infty \frac{e^{-ax}}{x} \sin sx\, dx = \tan^{-1}\left(\frac{s}{a}\right).$$

16. Find the fourier transform of $\frac{x}{1+x^2}$ and show that $F_c\left(\frac{1}{1+x^2}\right) = F_s\left(\frac{x}{1+x^2}\right)$.

$$F_c\left(\frac{1}{1+x^2}\right) = \sqrt{\frac{\pi}{2}}\, e^{-s}$$

$$F_s\left(x f(x)\right) = \frac{-d}{ds}\left(F_c f(x)\right).$$

$$F_s\left(x \cdot \frac{x}{1+x^2}\right) = -\frac{d}{ds}\left(\sqrt{\frac{\pi}{2}} \cdot e^{-s}\right)$$

$$F(f(x)) = F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{x}{1+x^2} (\cos sx - i \sin sx)\, dx$$

odd, even, odd

$$= \frac{1}{\sqrt{2\pi}} \cdot 2 \int_0^{\infty} \frac{\cancel{x} - i \sin sx}{1+x^2}\, dx$$

$$= \frac{-2i}{\sqrt{2\pi}} \int_0^{\infty} \frac{x \sin sx}{1+x^2}\, dx.$$

$$= \frac{-i}{\sqrt{2\pi}} \int_0^{\infty} \underbrace{\frac{2x}{1+x^2}}_{u'} \cdot \underbrace{\sin sx}_{u}\, dx.$$

$$= \frac{-i}{\sqrt{2\pi}} \cdot \left[\sin sx \log(1+x^2)\right]_0^{\infty} -$$

$$f_c(p) = \int_0^{\infty} \frac{1}{1+x^2} \cos px\, dx = I \text{ —— } (1)$$

$$\frac{dI}{dp} = \int_0^{\infty} \frac{-x \sin px}{1+x^2}\, dx$$

$u = 1+x^2$

$du = 2x\, dx$

$\frac{1}{u}\, du$

$$= -\int_0^{\infty} \frac{(1+x^2-1)\sin px}{x(1+x^2)}\, dx$$

$$= \int_0^{\infty} \frac{\sin px}{x}\, dx + \int_0^{\infty} \frac{\sin px}{x(1+x^2)}\, dx$$

$\int s \cos sx \cdot \log(1+x^2)$

$$\frac{dI}{dp} = -\frac{\pi}{2} + \int_0^{\infty} \frac{\sin px}{x(1+x^2)}\, dx \text{ —— } (2)$$

Again, $\frac{d^2 I}{dp^2} = \int_0^{\infty} \frac{x \cos px}{x(1+x^2)}\, dx = \int_0^{\infty} \frac{\cos px}{1+x^2}\, dx = J$

from (1)

$$\Longrightarrow \frac{d^2 I}{dp^2} - I = 0 \text{ —— } (3)$$

Solution of (3) is $I = c_1 e^{p} + c_2 e^{-p}$ —— (4)

$$\frac{dI}{dp} = c_1 e^{p} - c_2 e^{-p}$$

when $p=0$ $\quad I = \int_0^\infty \frac{1}{1+x^2}\,dx = \frac{\pi}{2}$ and

$\frac{dI}{dp} = -\frac{\pi}{2}$, Applying to (4) and (5) we get

$C_1 + C_2 = \frac{\pi}{2}$ and $C_1 - C_2 = -\frac{\pi}{2}$ so that,

$C_1 = 0$, $C_2 = \frac{\pi}{2}$.

$\therefore$ from (4) $\quad I = \frac{\pi}{2} e^{-p}$

$$\Rightarrow \int_0^\infty \frac{\cos px}{1+x^2}\,dx = \frac{\pi}{2} e^{-p}.$$

$$\therefore \int_0^\infty \frac{-x\sin px}{1+x^2}\,dx = -\frac{\pi}{2} e^{-p} \Rightarrow$$

$$\int_0^x \frac{x\sin px}{1+x^2}\,dx = \frac{\pi}{2} e^{-p}$$

Integral Equation

It involves an integral and under that integral there is an unknown function $f(x)$.

It contain an unknow fn $f(x)$. Solution = finding $f(x)$ which

Application of fourier transform_

An integral equation is an equation in which an unknown function appears under the integral sign.

To solve it, compare the given integral eqn with the fourier transform or with the fourier sine or cosine transform.

1. $\int_0^\infty f(x).\cos\lambda x\,dx = e^{-\lambda}$.

$\lambda = s$

$$\int_0^\infty f(x).\cos sx\,dx = e^{-s}.$$

$$\sqrt{\frac{2}{\pi}}.\int_0^\infty f(x)\cos sx\,dx = \sqrt{\frac{2}{\pi}}.e^{-s}$$

$$F_c(f(x)) = F_c(s) = \sqrt{\frac{2}{\pi}}.\ e^{-s}$$

By inverse formula,

$$f(x) = \sqrt{\frac{2}{\pi}}.\int_0^\infty F_c(s)\cos sx\,ds$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}}\, e^{-s} \cos sx \, ds$$

$$= \frac{2}{\pi} \int_0^\infty e^{-s} \cos sx \, ds$$

$t = s,\ s = 1,\ a = x \quad \int_0^\infty e^{-st} \cos at \, dt = \frac{s}{s^2+a^2}$

$$= \frac{2}{\pi} \cdot \frac{1}{1^2+x^2}$$

$$f(x) = \frac{2}{\pi} \cdot \frac{1}{1+x^2}$$

2. $\int_0^\infty f(\alpha) \cdot \cos \alpha\theta \, d\theta = 1-\alpha, \quad 0 \le \alpha \le 1$

$= 0 \quad , \alpha > 1.$

and hence evaluate $\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt.$

$\theta = x.$

$\alpha = s.$

$$\int_0^\infty f(x) \cdot \cos sx \, dx = 1-s, \quad 0 \le s \le 1$$
$$= 0, \quad s > 1$$

$$\sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx = \sqrt{\frac{2}{\pi}} (1-s), \quad 0 \le s \le 1$$
$$0, \quad s > 1$$

$$F_c(f(x)) = F_c(s) = \sqrt{\frac{2}{\pi}} (1-s), \quad 0 \le s \le 1$$
$$= 0, \quad s > 1$$

By inverse formula,

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c(s) \cdot \cos sx \, ds$$

$$= \sqrt{\frac{2}{\pi}} \left[\int_0^1 \sqrt{\frac{2}{\pi}} (1-s) \cos sx \, ds + \int_1^\infty 0 \cdot \cos sx \, ds \right]$$

$$= \frac{2}{\pi} \int_0^1 (1-s) \cos sx \, ds.$$

$$= \frac{2}{\pi} \left[\frac{\sin sx}{x} - \left\{ s \cdot \frac{\sin sx}{x} - 1 \cdot \frac{-\cos sx}{x^2} \right\} \right]_0^1$$

$$= \frac{2}{\pi} \left[\cancel{\frac{\sin x}{x}} - \cancel{\frac{\sin x}{x}} - \frac{(\cos x - \cos 0)}{x^2} \right]$$

$$= \frac{2}{\pi} \left(\frac{1-\cos x}{x^2} \right)$$

$$\therefore f(\theta) = \frac{2(1-\cos\theta)}{\pi\theta^2}$$

$$F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx.$$

$$\cancel{= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{2(1-\cos\theta)}{\pi\theta^2} \cos}$$

$$\sqrt{\frac{2}{\pi}} \cdot \int_0^\infty \frac{2(1-\cos x)}{\pi x^2} \cdot \cos sx\,dx = \sqrt{\frac{2}{\pi}} \cdot (1-s), \quad 0 \le s \le 1$$
$$= 0, \quad s > 1$$

when $s=0$, $\sqrt{\frac{2}{\pi}} \cdot \int_0^\infty \frac{2(1-\cos x)}{\pi x^2}\,dx = \sqrt{\frac{2}{\pi}}$.

ie, $\frac{2}{\pi} \cdot \int_0^\infty \frac{2 \sin^2 \frac{x}{2}}{x^2}\,dx = 1.$

Put $x/2 = t$. $\quad x = 2t$. $\quad dx = 2\,dt$.

$\Longrightarrow \frac{2}{\pi} \int_0^\infty \frac{2 \sin^2 t}{(2t)^2} 2\,dt = 1.$

$\Longrightarrow \int_0^\infty \frac{\cancel{2} \sin^2 t}{\cancel{4}\,t^2} \cancel{2}\,dt = \frac{\pi}{2}.$

$\Longrightarrow \int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}$

---

3. H.W. Solve, $\int_0^\infty f(x) \sin tx\,dx = 1, \quad 0 \le t \le 1$
$$= 2, \quad 1 \le t < 2.$$
$$= 0, \quad t \ge 2$$

$t = s$,

$$\int_0^\infty f(x) \cdot \sin sx\,dx = 1, \quad 0 \le s \le 1$$
$$= 2, \quad 1 \le s < 2$$
$$= 0, \quad s \ge 2.$$

$$\sqrt{\frac{2}{\pi}} \cdot \int_0^\infty f(x) \sin sx\,dx = \sqrt{\frac{2}{\pi}}; \quad 0 \le s \le 1$$
$$= \frac{2\sqrt{2}}{\sqrt{\pi}}, \quad 1 \le s < 2$$
$$= 0, \quad s \ge 2$$

$$F_S(f(x)) = F_S(s) = \sqrt{\frac{2}{\pi}}, \quad 0 \le s \le 1$$
$$= 2\sqrt{\frac{2}{\pi}}, \quad 1 \le s < 2$$
$$= 0, \quad s \ge 2$$

By inverse formula,

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_S(s) \cdot \sin sx\,ds$$

$$= \sqrt{\frac{2}{\pi}} \left[\int_0^1 \sqrt{\frac{2}{\pi}} \sin sx\,ds + \int_1^2 2\sqrt{\frac{2}{\pi}} \sin sx\,ds + \int_2^\infty 0 \sin sx\,ds \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\sqrt{\frac{2}{\pi}} \left[\frac{-\cos sx}{x} \right]_0^1 + 2\sqrt{\frac{2}{\pi}} \left[\frac{-\cos sx}{x} \right]_1^2 \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[-\sqrt{2/\pi} \left(\frac{\cos x}{x} - \frac{1}{x} \right) + \right.$$
$$\left. - 2\sqrt{\frac{2}{\pi}} \times \frac{1}{x} \left(\cos 2x - \cos x \right) \right.$$

$$= \frac{2}{\pi}\left(\frac{-\cos x}{x} + \frac{1}{x} - \frac{2\cos 2x}{x} + \frac{2\cos x}{x}\right)$$

$$= \frac{2}{\pi}\left[\frac{-\cos x + 1 - 2\cos 2x + 2\cos x}{x}\right]$$

$$= \frac{2}{\pi}\left[\frac{1 - 2\cos 2x + \cos x}{x}\right]$$

$$\therefore f(x) = \frac{2 + 2\cos x - 4\cos 2x}{\pi x}$$

U.Q theorem only.

Convolution of functions

$f(x) = g(x)$

$$f(x) * g(x) = \int_{-\infty}^{\infty} f(t) \cdot g(x-t)\, dt$$

Convolution theorem

If $F(s)$ and $G(s)$ are F.T of $f(x)$ and $g(x)$

then $F\left(f(x) * g(x)\right) = \sqrt{2\pi} \cdot F(s) \cdot G(s)$.

4. Verify convolution theorem for $f(x) = g(x) = e^{-x^2}$

(Verify means evaluate 2 sides and prove equality of them)

Convolution of $f(x)$ and $g(x)$ is, 39.

$$f(x) * g(x) = \int_{-\infty}^{\infty} f(t) \cdot g(\cancel{x}-t)\, dt$$

$$= \int_{-\infty}^{\infty} e^{-t^2} \cdot e^{-(x-t)^2}\, dt$$

$$F\left(f(x) * g(x)\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) * g(x) \cdot e^{-isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} e^{-t^2} \cdot e^{-(x-t)^2}\, dt\right) e^{-isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-t^2} \cdot e^{-u^2} e^{-is(u+t)} \cdot dt\, du$$

x & t are independent variables.

$x - t = u$

$x = u + t$

$dx = du$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2} e^{-ist}\, dt + \int_{-\infty}^{\infty} e^{-u^2} \cdot e^{-isu}\, du.$$

$$= \sqrt{2\pi}\left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2} e^{-isx}\, dx\right] + \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2} \cdot e^{-isx}\, dx\right]$$

$$= \sqrt{2\pi} \cdot F(s) \cdot G(s)$$

$\therefore$ Convolution theorem is verified.

Parseval's Identities

for higher powers of sin & cos parseval's theorem is used.

Consider the functions $f(x)$ and $g(x)$ with fourier transforms $F(s)$ and $G(s)$, respectively, and $F_S(s)$, $F_C(s)$ and $G_S(s)$, $G_C(s)$ be the corresponding sine and cosine transforms of the functions.

Then, Parseval's identities are,

1. $\int_{-\infty}^{\infty} [f(x)]^2 dx = \int_{-\infty}^{\infty} [F(s)]^2 ds$

2. $\int_0^{\infty} [f(x)]^2 dx = \int_0^{\infty} [F_S(s)]^2 ds$

$= \int_0^{\infty} [F_C(s)]^2 ds$

3. $\int_0^{\infty} f(x) \cdot g(x)\, dx = \int_0^{\infty} F_S(s) \cdot G_S(s)\, ds$

$= \int_0^{\infty} F_C(s) \cdot G_C(s)\, ds.$

Basic results

① If $f(x) = e^{-ax}$

$F_S(s) = \sqrt{\frac{2}{\pi}}\ \frac{s}{s^2+a^2}$ $\qquad F_C(s) = \sqrt{\frac{2}{\pi}}\ \frac{a}{s^2+a^2}$

② If $f(x) = 1, \quad 0 < x < a$

$= 0, \quad x > a$

$F_C(s) = \sqrt{\frac{2}{\pi}} \cdot \frac{\sin as}{s} = F(s)$

$F_S(s) = \sqrt{\frac{2}{\pi}} \cdot \left(\frac{1-\cos as}{s}\right)$

5. Using Parseval's identities prove that,

$$\int_0^{\infty} \frac{dt}{(a^2+t^2)(b^2+t^2)} = \frac{\pi}{2ab\,(a+b)}$$

$(x=t)$

$f(t) = \frac{1}{a^2+t^2} \Rightarrow f(x) = \frac{1}{a^2+x^2}$

$g(t) = \frac{1}{b^2+t^2} \Longrightarrow g(x) = \frac{1}{b^2+t^2}$

$F_S(s) = F_S(f(t)) = \sqrt{\frac{2}{\pi}} \cdot \int_0^{\infty} f(t) \cdot \sin st\, dt$

$\int e^{-ax} \cos sx\, dx$ $\Longrightarrow F_S(f(x)) = \sqrt{\frac{2}{\pi}} \cdot \int_0^{\infty} f(x) \sin sx\, dx.$

$t = x,\ s = a,\ a = s.$

$= \sqrt{\frac{2}{\pi}} \cdot \int_0^{\infty} \frac{1}{a^2+x^2} \sin sx\, dx$

Let $f(x) = e^{-ax}$ and $g(x) = e^{-bx}$.

Then $F_C(s) = \sqrt{\frac{2}{\pi}} \cdot \frac{a}{a^2+s^2}$, $\quad G_C(s) = \sqrt{\frac{2}{\pi}} \cdot \frac{b}{b^2+s^2}$

Now using Parsevals identity for Fourier cosine transforms,

$$\int_0^\infty F_c(s)\cdot G_c(s)\, ds = \int_0^\infty f(x)\cdot g(x)\, dx.$$

$$\int_0^\infty \sqrt{\frac{2}{\pi}}\cdot \frac{a}{a^2+s^2}\cdot \sqrt{\frac{2}{\pi}}\cdot \frac{b}{b^2+s^2}\, ds = \int_0^\infty e^{-ax}\cdot e^{-bx}\, dx$$

$$\frac{2}{\pi}\cdot \int_0^\infty \frac{ab}{(a^2+s^2)(b^2+s^2)}\, ds = \int_0^\infty e^{-(a+b)x}\, dx$$

$$\frac{2ab}{\pi}\cdot \int_0^\infty \frac{ds}{(a^2+s^2)(b^2+s^2)} = \left[\frac{e^{-(a+b)x}}{-(a+b)}\right]_0^\infty$$

$$\frac{2ab}{\pi}\int_0^\infty \frac{ds}{(a^2+s^2)(b^2+s^2)} = \frac{-1}{a+b}\left[e^{-\infty} - e^{-0}\right]$$

$0 \quad -1 = -1$

$$\frac{2ab}{\pi}\cdot\int_0^\infty \frac{ds}{(a^2+b^2)(b^2+s^2)} = \frac{-1}{a+b}(0-1)$$

$$\frac{2ab}{\pi}\int_0^\infty \frac{ds}{(a^2+s^2)(b^2+s^2)} = \frac{1}{a+b}$$

$$\int_0^\infty \frac{ds}{(a^2+s^2)(b^2+s^2)} = \frac{\pi}{2ab(a+b)}$$

$$\therefore \int_0^\infty \frac{dt}{(a^2+t^2)(b^2+t^2)} = \frac{\pi}{2ab(a+b)}$$

---

6. Prove that (i) $\displaystyle\int_0^\infty \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{4a^3}$.

$\Rightarrow$ (ii) $\displaystyle\int_0^\infty \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{4a}$

(i) $f(x) = e^{-ax}$.

For exam derive $F_c(s)$.

$$F_c(s) = \sqrt{\frac{2}{\pi}}\cdot \frac{a}{s^2+a^2}.$$

By P.I,

$$\int_0^\infty [f(x)^2]\, dx = \int_0^\infty [F_c(s)]^2\, ds$$

$$\int_0^\infty (e^{-ax})^2\, dx = \int_0^\infty \left(\sqrt{\frac{2}{\pi}}\cdot \frac{a}{s^2+a^2}\right)^2 ds$$

$$\Rightarrow \int_0^\infty e^{-2ax}\, dx = \frac{2}{\pi}\int_0^\infty \frac{a^2}{(s^2+a^2)^2}\, ds.$$

$$\Rightarrow \left[\frac{e^{-2ax}}{-2a}\right]_0^\infty = \frac{2}{\pi}\int_0^\infty \frac{a^2}{(s^2+a^2)^2}\, ds.$$

$$\Rightarrow -\frac{1}{2a}[0-1] = \frac{2}{\pi}\cdot\int_0^\infty \frac{a^2}{(s^2+a^2)^2}\, ds$$

$$\Rightarrow \frac{1}{2a} = \frac{2}{\pi}\cdot\int_0^\infty \frac{a^2}{(s^2+a^2)^2}\, ds.$$

$$\therefore \int_0^\infty \frac{ds}{(s^2+a^2)^2} = \frac{1}{2a} \times \frac{\pi}{2a^2}$$

$$= \frac{\pi}{4a^3}$$

$$\therefore \int_0^\infty \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{4a^3}$$

(ii) $F_S(s) = \sqrt{\frac{2}{\pi}} \cdot \frac{s}{s^2+a^2}$

By P.I,

$$\int_0^\infty (f(x))^2 dx = \int_0^\infty \left[F_S(s)\right]^2 ds.$$

$$\Rightarrow \int_0^\infty \left(e^{-ax}\right)^2 dx = \int_0^\infty \left[\sqrt{\frac{2}{\pi}} \cdot \frac{s}{s^2+a^2}\right]^2 ds$$

$$\Rightarrow \int_0^\infty e^{-2ax}\, dx = \frac{2}{\pi} \int_0^\infty \frac{s^2}{(s^2+a^2)^2} ds.$$

$$\Rightarrow \left[\frac{e^{-2a}}{-2a}\right]_0^\infty = \frac{2}{\pi} \int_0^\infty \frac{s^2}{(s^2+a^2)^2} ds$$

$$\Rightarrow -\frac{1}{2a}(0-1) = \frac{2}{\pi} \int_0^\infty \frac{s^2}{(s^2+a^2)^2} ds$$

$$\Rightarrow \frac{1}{2a} = \frac{2}{\pi} \cdot \int_0^\infty \frac{x^2}{(x^2+a^2)^2} dx$$

$$\Rightarrow \int_0^\infty \frac{x^2}{(x^2+a^2)^2} dx = \frac{\pi}{4a}$$

7. Evaluate $\int_0^\infty \frac{\sin^2 ax}{x^2} dx.$

$f(x) = 1, \quad 0 < x < a$

$\cancel{f(x)} = 0, \quad x > a.$

$F_c(s) = \sqrt{\frac{2}{\pi}} \cdot \frac{\sin as}{s}$

$$\int_0^\infty [f(x)]^2 dx = \int_0^\infty [F_c(s)]^2 ds$$

$$\Rightarrow \int_0^a 1^2\, dx = \int_0^\infty \left(\sqrt{\frac{2}{\pi}} \cdot \frac{\sin as}{s}\right)^2 ds$$

$$\Rightarrow [x]_0^a = \frac{2}{\pi} \cdot \int_0^\infty \frac{\sin^2 as}{s^2} ds.$$

$$\Rightarrow a - 0 = \frac{2}{\pi} \cdot \int_0^\infty \frac{\sin^2 a\xi}{s^2} ds.$$

$$\Rightarrow a = \frac{2}{\pi} \cdot \int_0^\infty \frac{\sin^2 a\xi}{s^2} d\xi$$

$$\Rightarrow \int_0^\infty \frac{\sin^2 a\xi}{s^2} d\xi = \frac{\pi a}{2}$$

put $s = x$.

$$\therefore \int_0^\infty \frac{\sin^2 ax}{x^2}\,dx = \frac{\pi a}{2}$$

8. Prove that $\displaystyle\int_0^\infty \frac{\sin at}{t\,(a^2+t^2)}\,dt = \frac{\pi}{2}\cdot\left(\frac{1-e^{-a^2}}{a^2}\right).$

$$f(x) = 1, \quad 0 < x < a$$
$$= 0, \quad x > a$$

$$g(x) = e^{-ax}.$$

By P.I,

$$\int_0^\infty f(x)\cdot g(x)\,dx = \int_0^\infty F_c(s)\, G_c(s)\,ds.$$

$$\int_0^a 1\cdot e^{-ax}\,dx + \int_a^\infty 0\times e^{-ax}\,dx = \int_0^\infty \sqrt{\frac{2}{\pi}}\cdot\frac{\sin as}{s} \times \sqrt{\frac{2}{\pi}}\cdot\frac{a}{s^2+a^2}\,ds$$

$$\Rightarrow \int_0^a e^{-ax}\,dx = \frac{2}{\pi}\cdot\int_0^\infty \frac{\sin as}{s}\cdot\frac{a}{s^2+a^2}\,ds$$

$$\Rightarrow \left[\frac{e^{-ax}}{-a}\right]_0^a = \frac{2}{\pi}\cdot\int_0^\infty \frac{a\sin as}{s\,(a^2+s^2)}\,ds$$

$$\Rightarrow -\frac{1}{a}\left(e^{-a^2}-1\right) = \frac{2a}{\pi}\cdot\int_0^\infty \frac{\sin as}{s(a^2+s^2)}\,ds$$

$$\Rightarrow \frac{-e^{-a^2}+1}{a} = \frac{2a}{\pi}\int_0^\infty \frac{\sin as}{s(a^2+s^2)}\,ds$$

$$\Rightarrow \frac{1-e^{-a^2}}{a} = \frac{2a}{\pi}\int_0^\infty \frac{\sin as}{s(a^2+s^2)}\,ds$$

$$\Rightarrow \int_0^\infty \frac{\sin as}{s\,(a^2+s^2)}\,ds = \frac{\pi}{2a^2}\left(1-e^{-a^2}\right).$$

$$\Rightarrow \int_0^\infty \frac{\sin as}{s(a^2+s^2)}\,ds = \frac{\pi}{2}\left(\frac{1-e^{-a^2}}{a^2}\right)$$

Put $s = t$,

$$\therefore \int_0^\infty \frac{\sin at}{t\,(a^2+t^2)}\,dt = \frac{\pi}{2}\cdot\left(\frac{1-e^{-a^2}}{a^2}\right)$$

U.Q

9. Find the fourier transform of $f(x) = a-|x|, \ |x|<a$

$= 0 \ , \ \forall |x|>a$

deduce, $\int_0^\infty \left(\frac{\sin x}{x}\right)^4 dx$.

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \cdot e^{-isx} dx.$$

$\left(\frac{\sin x}{x}\right)^2 \rightarrow$ use fourier inverse

$\left(\frac{\sin x}{x}\right)^4 \rightarrow$ use parseval's identity.

$$F(f(x)) = F(s)$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{a} 0 \, dx + \int_{-a}^{a} a-|x| \cdot e^{-isx} dx + \int_{a}^{\infty} 0 \, dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} a-|x| \cdot e^{-isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} \overset{\text{even}}{(a-|x|)} \left(\overset{\text{even}}{\cos sx} - i \overset{\text{odd}}{\sin sx} \right) dx.$$

$$= \frac{2}{\sqrt{2\pi}} \cdot \cancel{\phi} \cdot \int_0^a (a-|x|) \cos sx \, dx.$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^a (a-x) \cos sx \, dx.$$

$$= \frac{2}{\sqrt{2\pi}} \left[a \cdot \frac{\sin sx}{s} - \left\{ x \frac{\sin sx}{s} - 1 \cdot \frac{-\cos sx}{s^2} \right\} \right]_0^a$$

$$= \frac{2}{\sqrt{2\pi}} \left[\frac{a \sin sx}{s} - \frac{x \cdot \sin sx}{s} - \frac{\cos sx}{s^2} \right]_0^a$$

$$= \frac{2}{\sqrt{2\pi}} \left[\cancel{\frac{a \sin as}{s}} - \cancel{\frac{a \sin as}{s}} - \frac{1}{s^2} (\cos as - 1) \right]$$

$$= \sqrt{\frac{2}{\pi}} \cdot \left(\frac{1-\cos as}{s^2} \right)$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{2 \sin^2\left(\frac{as}{2}\right)}{s^2}$$

By P.I,

$$\int_{-\infty}^{\infty} [f(x)]^2 dx = \int_{-\infty}^{\infty} [F(s)]^2 ds$$

$$\cancel{\int_{-\infty}^{\infty} \left(\sqrt{\frac{2}{\pi}} \cdot \frac{2 \sin^2\left(\frac{as}{2}\right)}{s^2} \right)^2}$$

$$\Rightarrow \int_{-a}^{a} (a-|x|)^2 dx = \int_{-\infty}^{\infty} \left[\sqrt{\frac{2}{\pi}} \cdot \frac{2 \sin^2\left(\frac{as}{2}\right)}{s^2} \right]^2 ds$$

$$\Rightarrow 2 \cdot \int_0^a (a-|x|)^2 dx = \frac{8}{\pi^2} \int_{-\infty}^{\infty} \frac{\sin^4\left(\frac{as}{2}\right)}{s^4} ds$$

$$\Rightarrow \cancel{2} \cdot \int_0^a (a-x)^2 dx = \frac{8}{\pi^2} \times \cancel{2} \cdot \int_0^{\infty} \frac{\sin^4\left(\frac{as}{2}\right)}{s^4} ds$$

$$\Rightarrow \cancel{2 \cdot \int_0^a}$$

$$\Rightarrow \left[\frac{(a-x)^3}{-3}\right]_0^a$$

$$= \frac{8}{\pi}\int_0^\infty \frac{\sin^4 t}{\left(\frac{2t}{a}\right)^4}\,\frac{2\,dt}{a}$$

$\frac{as}{2} = t$

$\Rightarrow \frac{a}{2}\,ds = dt$

$ds = \frac{2}{a}\cdot dt$

$\Rightarrow s = \frac{2t}{a}$

$$\Rightarrow -\frac{1}{3}\left(0 - a^3\right) = \frac{\cancel{8}}{\pi}\int_0^\infty \frac{\sin^4 t}{\cancel{16}\cdot t^4}\cdot a^{\cancel{4}3}\cdot\frac{\cancel{2}}{\cancel{a}}\,dt$$

$$\Rightarrow \frac{\cancel{a^3}}{3} = \frac{\cancel{a^3}}{\pi}\int_0^\infty \frac{\sin^4 t}{t^4}\,dt$$

$$\Rightarrow \int_0^\infty \frac{\sin^4 t}{t^4}\,dt = \frac{\pi}{3}$$

$$\Rightarrow \int_0^\infty \left(\frac{\sin x}{x}\right)^4 dt = \frac{\pi}{3}$$

Module 3

Partial Differential Equation

$z = f(x,y)$

$\frac{\partial z}{\partial x} = p$

$\frac{\partial z}{\partial y} = q$

$\frac{\partial^2 z}{\partial x^2} = r$

$\frac{\partial^2 z}{\partial x \partial y} = s$

$\frac{\partial^2 z}{\partial y^2} = t.$

* Formation of PDE

(I) Formation of PDE by eliminating arbitrary constants.

$z = ax + by + a^2 + b^2$

If no. of arbitrary constants = no. of independent variables then we get a PDE of degree 1.