20/1/16
Wednesday

Module 1: Fourier Series

In many engineering problems it is convinient to express a function in a series of sines and cosines in the form:

$$f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x \dots + b_1 \sin x + b_2 \sin 2x + \dots$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

Euler formulae of Fourier (Euler) constants:

The fourier series for the fn $f(x)$ defined in the interval $c < x < c+2\pi$ is given by:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where $a_0 = \frac{1}{\pi} \int_c^{c+2\pi} f(x)\,dx$

$$a_n = \frac{1}{\pi} \int_c^{c+2\pi} f(x) \cos nx\,dx$$

$$b_n = \frac{1}{\pi} \int_c^{c+2\pi} f(x) \sin nx\,dx$$

where a_0, a_n, b_n; $n = 1, 2, 3$ etc are fourier consts.

→ when $c = 0$, $0 < x < 2\pi$:

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx.$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx.$$

→ when $c = -\pi$, $-\pi < x < \pi$:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx.$$

case 1 : $f(x)$ is even $[f(-x) = f(x)]$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx. \quad = \underline{\underline{\frac{2}{\pi}\int_0^{\pi} f(x)\,dx.}}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \qquad \begin{bmatrix} \cos nx \text{ is even} \\ \cos(-\theta) = \cos\theta \end{bmatrix}$$

$$= \underline{\underline{\frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx.}}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \qquad \begin{bmatrix} \sin(-\theta) = -\sin\theta \\ \therefore \sin x \text{ is odd} \end{bmatrix}$$

$$= \underline{\underline{0}}$$

then, $$\underline{\underline{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx}}$$

case 2 : $f(x)$ is odd $[f(-x) = -f(x)]$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \underline{\underline{0}}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \qquad \begin{bmatrix} f(x) \text{ odd} \\ \cos nx \text{ even} \end{bmatrix}$$

$$= \underline{\underline{0}}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx. \quad \begin{bmatrix} f(x) \text{ odd} \\ \sin nx \text{ odd} \end{bmatrix}$$

$$= \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

then, $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$

Change of interval:

If $f(x)$ is in $c < x < c + 2l$.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$$

$$a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\,dx$$

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\frac{n\pi x}{l}\,dx$$

when $c=0$, $0<x<2l$:

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx.$$

when $c=-1$, $-l<x<l$

$$a_0 = \frac{1}{l}\int_{-l}^{+l} f(x)\,dx$$

$$a_n = \frac{1}{l}\int_{-l}^{+l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$b_n = \frac{1}{l}\int_{-l}^{+l} f(x)\sin\frac{n\pi x}{l}\,dx.$$

case 1. $f(x)$ is odd $[f(-x) = -f(+x)]$

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx \quad [f(x) \text{ odd}]$$

$$= 0$$

$$a_n = \frac{1}{l}\int_{-l}^{+l} f(x)\cos\frac{n\pi x}{l}\,dx. = 0$$

$$b_n = \frac{1}{l}\int_{-\pi}^{+\pi} f(x) \sin\frac{n\pi x}{l}\, dx$$

$$= \frac{2}{l}\int_{0}^{l} f(x) \sin\frac{n\pi x}{l}\, dx$$

then, $f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$

case 2 : $f(x)$ is even $[f(-x) = f(x)]$

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\, dx$$

$$= \frac{2}{l}\int_{0}^{l} f(x)\, dx$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x) \cos\frac{n\pi x}{l}\, dx$$

$$= \frac{2}{l}\int_{0}^{l} f(x) \cos\frac{n\pi x}{l}\, dx.$$

$$b_n = \frac{1}{l}\int_{-l}^{+l} f(x) \sin\frac{n\pi x}{l}\, dx = 0$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

Q. Find the fourier series in $f(x) = x - x^2$. in $-1 < x < 1$.

ans: $f(x) = x - x^2 \qquad -1 < x < 1$

$-l < x < l$

ie $l = 1$.

Fourier series,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$$

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Note : limit $0 \to \pi$

$\sin n\pi = 0$

$\cos n\pi = (-1)^n$

$\sin(2n+1)\pi/2 = (-1)^n$

$\cos(2n+1)\pi/2 = 0$

$\cos(2n+1)\pi = \cos(2n-1)\pi = -1$

$\cos 2n\pi = 1$

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\, dx$$

Q. Find the fourier series in $f(x) = e^{-x}$ in $-l < x < l$.

ans: Fourier series,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$$

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx$$

$$= \frac{1}{l}\int_{-l}^{l} e^{-x}\,dx.$$

$$= \frac{1}{l}\left[\frac{e^{-x}}{-1}\right]_{-l}^{l}$$

$$= -\frac{1}{l}\left[e^{-l} - e^{l}\right]$$

$$= \frac{2}{l}\left(\frac{e^{l} - e^{-l}}{2}\right)$$

$$= \frac{2}{l}\sinh l.$$

$$*\left[= (e^{l} - e^{-l}) \times \frac{2}{2} \qquad \left\{\frac{e^{l} - e^{-l}}{2} = \sinh l.\right\}\right.$$

$$\left. = 2\sinh l \right]*$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx.$$

$$= \frac{1}{l}\int_{-l}^{l} e^{-x}\cos\frac{n\pi x}{l}\,dx.$$

Note:

$$\int e^{ax}\cos(bx+c)\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\cos(bx+c) + b\sin(bx+c)\right]$$

$$\int e^{ax}\sin(bx+c)\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\sin(bx+c) - b\cos(bx+c)\right]$$

$$\left[a = -1,\ b = \frac{n\pi}{l}\right]$$

$$= \frac{1}{l}\left\{\frac{e^{-x}}{1+\left(\frac{n\pi}{l}\right)^2}\left[-\cos\frac{n\pi x}{l} + \frac{n\pi}{l}\sin\frac{n\pi x}{l}\right]\right\}_{-l}^{l}$$

$$= \frac{1}{l}\left\{\frac{1}{\frac{l^2+n^2\pi^2}{l^2}}\left[e^{-x}\left(-\cos\frac{n\pi x}{l} + \frac{n\pi}{l}\sin\frac{n\pi x}{l}\right)\right]_{-l}^{l}\right\}$$

$$= \frac{1}{l}\left\{\frac{l^2}{l^2+n^2\pi^2}\left[e^{-l}\left(-\cos\frac{n\pi l}{l} + \frac{n\pi}{l}\sin\frac{n\pi l}{l}\right) - e^{l}\left(-\cos\frac{n\pi(-l)}{l} + \frac{n\pi}{l}\sin\frac{n\pi(-l)}{l}\right)\right]\right\}$$

$$= \frac{1}{l} \left\{ \frac{l^2}{l^2+n^2\pi^2} \left[e^{-l} \left(-\cos n\pi + \frac{n\pi}{l} \sin n\pi \right) - e^{l} \left(\cos n\pi - \frac{n\pi}{l} \sin n\pi \right) \right] \right\}$$

$$= \frac{l}{l^2+n^2\pi^2} \left[-e^{-l}(-1)^n + e^{l}(-1)^n \right]$$

$$= \frac{l(-1)^n}{l^2+n^2\pi^2} \left[e^{l} - e^{-l} \right]$$

$$= \frac{l \cdot 2\sinh l \, (-1)^n}{l^2+n^2\pi^2}$$

$$b_n = \frac{1}{l} \int_{-l}^{l} f(x) \sin \frac{n\pi x}{l} \, dx$$

$$= \frac{1}{l} \int_{-l}^{l} e^{-x} \sin \frac{n\pi x}{l} \, dx$$

$$\left[a = -1,\ b = \frac{n\pi}{l} \right]$$

$$= \frac{1}{l} \left\{ \frac{e^{-x}}{1+\left(\frac{n\pi}{l}\right)^2} \left[-1 \cdot \sin \frac{n\pi x}{l} - \frac{n\pi}{l} \cos \frac{n\pi x}{l} \right] \right\}_{-l}^{l}$$

$$= \frac{1}{l} \left\{ \frac{l^2}{l^2+n^2\pi^2} \left[e^{l} \left(-\sin n\pi - \frac{n\pi}{l} \cos n\pi \right) - e^{l} \left(-\sin(n\pi) - \frac{n\pi}{l} \cos(-n\pi) \right) \right] \right\}$$

$$= \frac{l}{l^2+n^2\pi^2} \left[e^{-l} \times \frac{-n\pi}{l} (-1)^n + e^{l} \times \frac{-n\pi}{l} (-1)^n \right]$$

$$= \frac{n\pi}{l} (-1)^n \cdot \frac{l}{l^2+n^2\pi^2} \left(e^{l} - e^{-l} \right)$$

$$= (-1)^n \frac{n\pi}{l^2+n^2\pi^2} \, 2\sinh l$$

<u>Fourier series:</u>

$$e^{-x} = \frac{1}{l} \sinh l + \sum_{n=1}^{\infty} \left[\frac{l \cdot 2\sinh l \, (-1)^n}{l^2+n^2\pi^2} \cos \frac{n\pi x}{l} + \frac{(-1)^n n\pi \, 2\sinh}{l^2+n^2\pi^2} \sin \right.$$

HW
Q. Find the fourier series in $f(x) = 1 - x^2$ in $-1 < x < 1$.

UQ
ans: Fourier series,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right)$$

$$a_0 = \frac{1}{l} \int_{-l}^{l} f(x) \, dx$$

$$= \int_{-1}^{1} (1-x^2)\,dx$$

$$= \int_{-1}^{1} 1\,dx - \int_{-1}^{1} x^2\,dx$$

$$= [x]_{-1}^{1} - 2\int_{0}^{1} x^2\,dx$$

$$= -2\left[\frac{x^3}{3}\right]_0^1 + 2$$

$$= -2\left[\frac{1}{3} - 0\right] + 2$$

$$= \cancel{+1} - \frac{2}{3} + 2 = \frac{-2+6}{3} = \frac{4}{3}$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx.$$

$$= \int_{-1}^{1} (1-x^2)\cos n\pi x\,dx$$

$$= \int_{-1}^{1} \cos n\pi x\,dx - \int_{-1}^{1} x^2\cos n\pi x\,dx$$

$$= \cancel{2\int_0^{1} x\cos n\pi x\,dx} \quad \int_0$$

$$= \cancel{0} - 2\int_0^1 x^2\cos n\pi x\,dx + 2\int_0^1 \cos n\pi x\,dx$$

$$= -2\left[(x^2)\left(\frac{\sin n\pi x}{n\pi}\right) - (2x)\left(\frac{-\cos n\pi x}{(n\pi)^2}\right) + (2)\left(\frac{-\sin n\pi x}{(n\pi)^3}\right)\right]_0^1 + 2\left[\frac{\sin n\pi x}{n\pi}\right]_0^1$$

$$= -2\cdot\left[0 + \frac{2}{n^2\pi^2}\left((-1)^n - 0\right)\right] + 2[0]$$

$$= \frac{-4(-1)^n}{n^2\pi^2}$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \int_{-1}^{1} (1-x^2)\sin n\pi x\,dx$$

$$= \int_{-1}^{1} \sin n\pi x\,dx - \int_{-1}^{1} x^2\sin n\pi x\,dx.$$

$$= 0 - 0$$

$$= 0$$

Fourier series,

$$1-x^2 = \frac{4}{6} + \sum_{n=1}^{\infty}\left[\frac{-4(-1)^n}{n^2\pi^2} \times \cos n\pi x\right]$$

HW
UQ

Q. Find the fourier series, in $f(x) = x^2 - 2$ in $-2 < x < 2$.

ans: Fourier series;

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{l} + b_n\sin\frac{n\pi x}{l}\right)$$

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx$$

$$= \frac{1}{2}\int_{-2}^{2}(x^2-2)\,dx$$

$$= \frac{1}{2}\left\{\int_{-2}^{2} x^2\,dx - \int_{-2}^{2} 2\,dx\right\}$$

$$= \frac{1}{2}\left\{2\left[\frac{x^3}{3}\right]_0^2 - 2\left[x\right]_{-2}^{2}\right\}$$

$$= \frac{1}{2}\left[2\times\frac{8}{3} - 8\right]$$

$$= \frac{1}{2}\left[\frac{16}{3} - 8\right]$$

$$= \frac{1}{2}\times -\frac{8}{3}$$

$$= \underline{\underline{-\frac{4}{3}}}$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos n\pi x\,dx$$

$$= \frac{1}{2}\int_{-2}^{2}(x^2-2)\cos\frac{n\pi x}{2}\,dx$$

$$= \frac{1}{2}\left\{\int_{-2}^{2} x^2\cos\frac{n\pi x}{2}\,dx - 2\int_{-2}^{2}\cos\frac{n\pi x}{2}\,dx\right\}$$

$$= \frac{1}{2}\left\{2\int_{0}^{2} x^2\cos\frac{n\pi x}{2}\,dx - 4\int_{0}^{2}\cos\frac{n\pi x}{2}\,dx\right\}$$

$$= \frac{1}{2}\left\{2\left[(x^2)\left(\cancel{\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}}}^{\,0}\right) - (2x)\left(\frac{-\cos\frac{n\pi x}{2}}{\left(\frac{n\pi}{2}\right)^2}\right) + (2)\left(\cancel{\frac{-\sin\frac{n\pi x}{2}}{\left(\frac{n\pi}{2}\right)^3}}^{\,0}\right)\right]_0^2 - 4\left[\cancel{\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}}}^{\,0}\right]_0^2\right\}$$

$$= \frac{1}{\cancel{2}}\times\cancel{2}\left[\frac{4\times(-1)^n}{\left(\frac{n\pi}{2}\right)^2}\right]$$

$$= \frac{4(-1)^n\times 4}{n^2\pi^2}$$

$$= \underline{\underline{\frac{16(-1)^n}{n^2\pi^2}}}$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{2}\int_{-2}^{2}(x^2-2)\sin\frac{n\pi x}{2}\,dx$$

$$= \frac{1}{2}\left\{\int_{-2}^{2} x^2\sin\frac{n\pi x}{2}\,dx - 2\int_{-2}^{2}\sin\frac{n\pi x}{2}\,dx\right\}$$

$= 0 - 0$

$= \underline{\underline{0}}$

Fourier series;

$$x^2 - 2 = \frac{-4}{6} + \sum_{n=1}^{\infty}\left[\frac{16(-1)^n}{n^2\pi^2}\cos\frac{n\pi x}{2}\right]$$

UQ 2/16 Q

today

Expand $x\sin x$ as a fourier series in $-\pi < x < \pi$, deduce that $\frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} \ldots\ldots = \frac{\pi - 2}{4}$.

ans: $f(x) = x\sin x$ is even, $b_n = 0$.

Fourier series for $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

where, $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$

$= \frac{2}{\pi}\int_0^{\pi} x\sin x\,dx$

$= \frac{2}{\pi}\left[x\,{}^{-}\cos x - 1\cdot{}^{-}\sin x\right]_0^{\pi}$

$= \frac{2}{\pi}\left[-[\pi\cos\pi - 0] - [\sin\pi - \sin 0]\right]$

$= \frac{2}{\pi} \times \pi = \underline{\underline{2}}$

$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$

$= \frac{2}{\pi}\int_0^{\pi} x\sin x \cdot \cos nx\,dx$

$= \frac{2}{\pi}\int_0^{\pi} x\left[\tfrac{1}{2}(\sin(1+n)x + \sin(1-n)x)\right]dx$

$= \cancel{\frac{2}{\pi}} \times \frac{1}{\cancel{2}}\int_0^{\pi}[x\sin(n+1)x + x\sin(1-n)x]\,dx$

$= \frac{1}{\pi}\left\{\left[\frac{x\cdot{}^{-}\cos(n+1)x}{n+1} - \cancel{\frac{1\cdot{}^{-}\sin(n+1)x}{(n+1)^2}}^{\,0}\right] + \left[\frac{x\,{}^{-}\cos(1-n)x}{1-n} - \cancel{\frac{1\cdot{}^{-}\sin(1-n)x}{(1-n)^2}}^{\,0}\right]\right\}_0^{\pi}$

$= \frac{1}{\pi}\left\{\frac{-1}{n+1}[\pi\cos(n+1)\pi - 0] - \frac{1}{1-n}[\pi\cos(1-n)\pi - 0]\right\}$

$= \frac{1}{\pi}\left[\frac{-\pi}{n+1}(-1)^{n+1} + \frac{\pi}{n-1}(-1)^{n+1}\right]$

$= \frac{-(-1)^n(-1)^1}{n+1} + \frac{(-1)^n(-1)^{-1}}{n-1} = \frac{(-1)^n}{n+1} - \frac{(-1)^n}{n-1}$

$= (-1)^n\left[\frac{1}{n+1} - \frac{1}{n-1}\right]$

$= (-1)^n\left[\frac{n-1-n-1}{(n+1)(n-1)}\right]$

$$= \frac{-2(-1)^n}{n^2-1} \quad ; n \neq 1.$$

$$a_1 = \frac{2}{\pi}\int_0^{\pi} f(x)\cos x\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x.\sin x.\cos x\,dx \qquad [2\sin x\cos x = \sin 2x]$$

$$= \frac{1}{\pi}\int_0^{\pi} x.\sin 2x\,dx$$

$$= \frac{1}{\pi}\left[x.\frac{-\cos 2x}{2} - 1.\cancel{\frac{-\sin 2x}{2^2}}^{\,0} \right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[-\tfrac{1}{2}\left(\pi\cos 2\pi - 0\right)\right]$$

$$= -\tfrac{1}{2}$$

$$f(x) = 1 - \frac{1}{2}\cos x + \sum_{n=2}^{\infty} \frac{-2(-1)^n}{n^2-1}\cos nx.$$

$$= 1 - \tfrac{1}{2}\cos x - 2\left[\frac{1}{1\cdot 3}\cos 2x - \frac{\cos 3x}{2\cdot 4} + \frac{\cos 4x}{3\cdot 5} - \frac{\cos 5x}{4\cdot 6} + \frac{\cos 6x}{5\cdot 7}\ldots\right]$$

$x = \pi/2$.

$$\tfrac{\pi}{2}\sin\tfrac{\pi}{2} = 1 - 0 - 2\left[\frac{-1}{1\cdot 3} - 0 + \frac{1}{3\cdot 5} + 0 - \frac{1}{5\cdot 7}\ldots\right]$$

$$\tfrac{\pi}{2} - 1 = 2\left[\frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7}\ldots\right]$$

$$\frac{\pi - 2}{4} = \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7}\ldots\ldots$$

OB Q. Expand x^2 as fourier series in $-\pi < x < \pi$ and
HW hence S.T.

① $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

② $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} \ldots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}$.

③ $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$

ans: $f(x) = x^2$ is an even fn.

$\therefore b_n = 0$.

Fourier series for $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

where $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$

$$= \frac{2}{\pi}\int_0^{\pi} x^2\,dx$$

$$= \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi}$$

$$= \frac{2}{3\pi}\left[\pi^3 - 0\right]$$

$$= \frac{2\pi^3}{3\pi} = \frac{2\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x^2\cos nx\,dx$$

$$= \frac{2}{\pi}\left[x^2\cancel{\frac{\sin nx}{n}}^{0} - 2x\frac{-\cos nx}{n^2} + 2\cancel{\frac{-\sin nx}{n^3}}^{0}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{2x\cos nx}{n^2}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\times\frac{2}{n^2}\left[\pi(-1)^n\right]$$

$$= \frac{4}{n^2}(-1)^n$$

Fourier series

$$\cancel{x^2 = \frac{2\pi^2}{3\times 2} + \frac{4}{n^2}(-1)^n}$$

$$x^2 = \frac{2\pi^2}{3\times 2} + \sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^n\cos nx$$

$$= \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^n\cos nx$$

$$= \frac{\pi^2}{3} + 4\left[\frac{-1}{1^2}\cos x + \frac{1}{2^2}\cos 2x + \frac{-1}{3^2}\cos 3x\right.$$

$x = \pi$,

$$\pi^2 = \frac{\pi^2}{3} + 4\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2}\cdots\right]$$

$$\pi^2 - \frac{\pi^2}{3} = 4\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2}\cdots\right]$$

$$\frac{2\pi^2}{3} = 4\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2}\cdots\right]$$

$$\frac{2\pi^2}{12} = \left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2}\cdots\right]$$

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2}\cdots = \sum_{n=1}^{\infty}\frac{1}{n^2} \quad \text{--- (1)}$$

$x = 0$

$$0 = \frac{\pi^2}{3} + 4\left[\frac{-1}{1^2} + \frac{1}{2^2} + \frac{-1}{3^2}\right]$$

$$0 = \frac{\pi^2}{3} - 4\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} \cdots\right]$$

$$\frac{\pi^2}{3} = 4\left(\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} \cdots\right]$$

$$\frac{\pi^2}{12} = \left(\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} \cdots\right] = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} \quad \text{— ②}$$

① + ②.

$$\Rightarrow \frac{\pi^2}{6} + \frac{\pi^2}{12} = \left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} \cdots\right] + \left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} \cdots\right]$$

$$\Rightarrow \frac{3\pi^2}{12} = 2\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} \cdots\right]$$

$$\Rightarrow \frac{\pi^2}{8} = \left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right]$$

2/2/16 Tuesday

VQ. Q. Expand $f(x) = |\cos x|$ in $-\pi < x < \pi$.

ans. $|\cos x|$ is even, $\therefore b_n = 0$ and $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$.

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx.$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} \cos x\,dx + \int_{\pi/2}^{\pi} -\cos x\,dx\right]$$

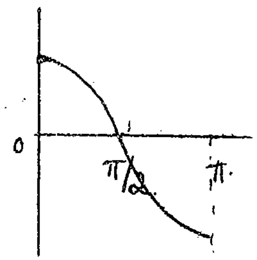
Note:

$|x| = x$ if $x > 0$

$= -x$ if $x < 0$.

$|\cos x| = \cos x$, $0 < x < \pi/2$

$= -\cos x$, $\pi/2 < x < \pi$.



$$= \frac{2}{\pi}\left[\left[\sin x\right]_0^{\pi/2} - \left[\sin x\right]_{\pi/2}^{\pi}\right]$$

$$= \frac{2}{\pi}\left[(1-0) - (0-1)\right] = 4/\pi$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx,$$

$$= 2/\pi\left[\int_0^{\pi/2} \cos x.\cos nx\,dx + \int_{\pi/2}^{\pi} -\cos x.\cos nx\,dx\right]$$

$$\boxed{2\cos A\cos B = \cos(A+B) + \cos(A-B)}$$

$$= \frac{1}{\pi}\left[\int_0^{\pi/2} (\cos(n+1)x + \cos(n-1)x)\,dx - \int_{\pi/2}^{\pi} (\cos(n+1)x + \cos(n-1)x)\,dx\right]$$

$$= \frac{1}{\pi}\left\{\left[\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1}\right]_0^{\pi/2} - \left[\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)}{n-1}\right]_{\pi/2}^{\pi}\right\}$$

$$= \frac{1}{\pi}\left\{\left[\frac{\sin(n+1)\pi/2}{n+1} - 0 + \frac{\sin(n-1)\pi/2}{n-1} - 0\right] - \left[0 - \frac{\sin(n+1)}{n+1}\pi/2 + 0 - \frac{\sin(n-1)\pi/2}{n-1}\right]\right\}$$

Note:

$\sin(n+1)0 = 0$ $\qquad \sin(\pi/2+\theta) = \cos\theta$

$\sin(n-1)0 = 0$ $\qquad \sin(\pi/2-\theta) = \cos\theta$

$\sin(n+1)\pi = 0$

$\sin(n-1)\pi = 0$

$\sin(n+1)\pi/2 = \sin(\pi/2 + n\pi/2) = \cos n\pi/2$

$\sin(n-1)\pi/2 = -\sin(\pi/2 - n\pi/2) = -\cos n\pi/2$

$$= \frac{1}{\pi}\left[\frac{\cos n\pi/2}{n+1} - \frac{\cos n\pi/2}{n-1} + \frac{\cos n\pi/2}{n+1} - \frac{\cos n\pi/2}{n-1}\right]$$

$$= \frac{2}{\pi}\cos n\pi/2\left[\frac{1}{n+1} - \frac{1}{n-1}\right]$$

$$= \frac{2}{\pi}\cos n\pi/2\left(\frac{n-1-(n+1)}{n^2-1}\right)$$

$$= \frac{2}{\pi}\cos n\pi/2\left[\frac{-2}{n^2-1}\right]$$

$$= \frac{-4}{\pi(n^2-1)}\cos n\pi/2 \qquad n \neq 1.$$

$$a_1 = \frac{2}{\pi}\int_0^{\pi} f(x)\cos x\, dx.$$

$$= \frac{2}{\pi}\int_0^{\pi} \cos^2 x\, dx.$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2}\cos^2 x\, dx + \int_{\pi/2}^{\pi} -\cos^2 x\, dx\right]$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2}\left(\frac{1+\cos 2x}{2}\right)dx - \int_{\pi/2}^{\pi}\frac{1+\cos 2x}{2}\, dx.\right]$$

$$= \frac{1}{\pi}\left\{\left[x + \frac{\sin 2x}{2}\right]_0^{\pi/2} - \left[x + \frac{\sin 2x}{2}\right]_{\pi/2}^{\pi}\right\}$$

$$= \frac{1}{\pi}\left[\pi/2 - 0 - (\pi - \pi/2)\right]$$

$$= \underline{\underline{0}}$$

Q. Expand $f(x) = |\cos x| = \frac{2}{\pi} + 0 + \sum_{n=2}^{\infty}\left(\frac{-4}{\pi(n^2-1)}\cos\frac{n\pi}{2}\cdot\cos nx\right)$

Q. Expand $f(x) = \sqrt{1-\cos x}$ in $0 < x < 2\pi$ hence deduce $\frac{1}{1\cdot3} + \frac{1}{3\cdot5} + \frac{1}{5\cdot7} + \cdots$

ans:

$$f(x) = \sqrt{1-\cos x}$$

$$= \sqrt{2\sin^2\frac{x}{2}} = \sqrt{2}\sin\frac{x}{2}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n \sin nx.$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi}\sqrt{2}\sin\frac{x}{2}\,dx.$$

$$= \frac{\sqrt{2}}{\pi}\left[\frac{-\cos x/2}{1/2}\right]_0^{2\pi}$$

$$= \frac{-2\sqrt{2}}{\pi}\left[\cos\pi - \cos 0\right]$$

$$= \frac{-2\sqrt{2}}{\pi}\left[-1-1\right]$$

$$= \frac{4\sqrt{2}}{\pi}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx.$$

$$= \frac{1}{\pi}\int_0^{2\pi}\sqrt{2}\sin\frac{x}{2}\cos nx\,dx.$$

$$= \frac{\sqrt{2}}{2\pi}\int_0^{2\pi}\left[\sin\left(n+\tfrac{1}{2}\right)x - \sin\left(n-\tfrac{1}{2}\right)x\right]dx.$$

$$\boxed{2\cos A\sin B = \sin(A+B) - \sin(A-B)}$$

$$= \frac{\sqrt{2}}{2\pi}\left[\frac{-\cos\left(n+\frac{1}{2}\right)x}{n+\frac{1}{2}} + \frac{\cos\left(n-\frac{1}{2}\right)x}{n-\frac{1}{2}}\right]_0^{2\pi}$$

$$= \frac{\sqrt{2}}{2\pi}\left[\frac{-\cos\left(\frac{2n+1}{2}\right)x}{\frac{2n+1}{2}} + \frac{\cos\left(\frac{2n-1}{2}\right)x}{\frac{2n-1}{2}}\right]_0^{2\pi}$$

$$= \frac{\sqrt{2}}{2\pi}\left[\frac{-2}{2n+1}\left(\cos(2n+1)\pi - 1\right) + \frac{2}{2n-1}\left(\cos(2n-1)\pi - 1\right)\right]$$

$$= \frac{\sqrt{2}}{2\pi}\left[\frac{-2}{2n+1}(-1-1) + \frac{2}{2n-1}(-1-1)\right]$$

$$= \frac{\sqrt{2}}{2\pi}\left[\frac{-2x-2}{2n+1} + \frac{2x-2}{2n-1}\right]$$

$$= \frac{\sqrt{2}}{2\pi}\left[\frac{4}{2n+1} - \frac{4}{2n-1}\right]$$

$$= \frac{4\times\sqrt{2}}{2\pi}\left(\frac{1}{2n+1} - \frac{1}{2n-1}\right)$$

$$= \frac{2\sqrt{2}}{\pi}\left[\frac{\cancel{2n}-1-(\cancel{2n}+1)}{(2n+1)(2n-1)}\right]$$

$$= \frac{2\sqrt{2}}{\pi}\left[\frac{-2}{4n^2-1}\right]$$

$$= \frac{-4\sqrt{2}}{\pi(4n^2-1)}$$

H.W.

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx.$$

$$= \frac{1}{\pi}\int_0^{2\pi} \sqrt{2}\sin\frac{x}{2}\sin nx.\,dx$$

$$= \frac{\sqrt{2}}{\pi}\int_0^{2\pi} \sin x/2 \sin nx\,dx$$

$$\boxed{-2\sin x\sin y = \cos(x+y) - \cos(x-y)}$$

$$= \frac{\sqrt{2}}{-2\pi}\int_0^{2\pi} -2\sin x/2 \sin nx\,dx$$

$$= \frac{-\sqrt{2}}{2\pi}\int_0^{2\pi}\left[\cos(1/2+n)x - \cos(1/2-n)x\right]dx$$

$$= \frac{-\sqrt{2}}{2\pi}\left[\frac{\sin(1/2+n)x}{1/2+n} - \frac{\sin(1/2-n)x}{1/2-n}\right]_0^{2\pi}$$

$$= \frac{-\sqrt{2}}{2\pi}\left[\frac{\sin\left(\frac{1+2n}{2}\right)x}{\frac{1+2n}{2}} - \frac{\sin\left(\frac{1-2n}{2}\right)x}{\frac{1-2n}{2}}\right]_0^{2\pi}$$

$$= \frac{-\sqrt{2}}{2\pi}\left[\frac{2}{2n+1}\underbrace{\sin(2n+1)\pi}_{0} + \frac{2}{2n-1}\underbrace{\sin(2n-1)\pi}_{0}\right]$$

$$= \underline{\underline{0}}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx$$

$$\sqrt{2}\sin\frac{x}{2} = \frac{4\sqrt{2}}{2\pi} + \sum_{n=1}^{\infty}\frac{-4\sqrt{2}}{\pi(4n^2-1)}\cos nx$$

$$\sqrt{2}\sin\frac{x}{2} = \frac{2\sqrt{2}}{\pi} + \frac{4\sqrt{2}}{\pi}\left[\frac{-1}{1\cdot 3}\cos x - \frac{1}{3\cdot 5}\cos 2x - \frac{1}{5\cdot 7}\cos 3x - \frac{1}{7\cdot 9}\cos 4x \dots\right.$$

$$\sqrt{2}\sin\frac{x}{2} - \frac{2\sqrt{2}}{\pi} = \frac{4\sqrt{2}}{\pi}\left[\frac{-1}{1\cdot 3}\cos x - \frac{1}{3\cdot 5}\cos 2x - \frac{1}{5\cdot 7}\cos 3x - \frac{1}{7\cdot 9}\cos 4x \dots\right.$$

$$\frac{\sqrt{2}\sin\frac{x}{2} - \frac{2\sqrt{2}}{\pi}}{\frac{4\sqrt{2}}{\pi}} = \left[\frac{-1}{1\cdot3}\cos x - \frac{1}{3\cdot5}\cos 2x - \frac{1}{5\cdot7}\cos 3x - \frac{1}{7\cdot9}\cos 4x \cdots\right]$$

$x = 0$

$$\frac{\sqrt{2}\sin 0 - \frac{2\sqrt{2}}{\pi}}{\frac{4\sqrt{2}}{\pi}} = \left[\frac{-1}{1\cdot3}\cos 0 - \frac{1}{3\cdot5}\cos 2\times0 - \frac{1}{5\cdot7}\cos 3\times0 - \frac{1}{7\cdot9}\cos 4\times0 \cdots\right]$$

$$\frac{0 - \frac{2\sqrt{2}}{\pi}}{\frac{4\sqrt{2}}{\pi}} = \left[\frac{-1}{1\cdot3} - \frac{1}{3\cdot5} - \frac{1}{5\cdot7} - \frac{1}{7\cdot9} \cdots\right]$$

$$\frac{-\cancel{2\sqrt{2}}/\cancel{\pi}}{2\,\cancel{4\sqrt{2}}/\cancel{\pi}} = \left[\frac{-1}{1\cdot3} - \frac{1}{3\cdot5} - \frac{1}{5\cdot7} - \frac{1}{7\cdot9} \cdots\right]$$

$$\frac{-1}{2} = \left[\frac{-1}{1\cdot3} - \frac{1}{3\cdot5} - \frac{1}{5\cdot7} - \frac{1}{7\cdot9} \cdots\right]$$

$$\frac{1}{1\cdot3} + \frac{1}{3\cdot5} + \frac{1}{5\cdot7} + \frac{1}{7\cdot9} \cdots = \frac{1}{2}$$

8/2/16 Monday → Fourier series for discontinuos fn:

Q. write the fourier expansion for;

$f(x) = -\pi, \quad -\pi < x < 0.$

$\quad = x, \quad 0 < x < \pi.$

and deduce $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \frac{\pi^2}{8}$.

ans: F.S is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx.$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx.$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\,dx + \int_{0}^{\pi} f(x)\,dx\right]$$

$$= \frac{1}{\pi}\left(\int_{-\pi}^{0} -\pi\,dx + \int_{0}^{\pi} x\,dx\right)$$

$$= \frac{1}{\pi}\left(-\pi\left[x\right]_{-\pi}^{0} + \left[\frac{x^2}{2}\right]_0^{\pi}\right)$$

$$= \frac{1}{\pi}\left[-\pi^2 + \frac{\pi^2}{2}\right] = \frac{1}{\pi}\left[\frac{-2\pi^2 + \pi^2}{2}\right]$$

$$= \frac{1}{2\pi} \times -\pi^{\cancel{2}} = \frac{-\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left(\int_{-\pi}^{0} -\pi\cos nx\,dx + \int_{0}^{\pi} x\cos nx\,dx\right)$$

$$= \frac{1}{\pi}\left(-\pi\left[\cancel{\frac{\sin nx}{n}}\right]_{-\pi}^{0} + \left[\cancel{\frac{x\sin nx}{n}} - \frac{-\cos nx}{n^2}\right]_0^{\pi}\right)$$

$$= \frac{1}{\pi}\left[\frac{\cos nx}{n^2}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left(\frac{\cos n\pi - \cos 0}{n^2}\right)$$

$$= \frac{1}{\pi n^2}\left((-1)^n - 1\right)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi\sin nx\,dx + \int_{0}^{\pi} x\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[-\pi\left(\frac{-\cos nx}{n}\right)_{-\pi}^{0} + \left(\frac{x\cos nx}{n} + \cancel{\frac{\sin nx}{n^2}}\right)_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n}[\cos 0 - \cos n\pi] + \frac{-1}{n}[\pi\cos n\pi - 0]\right]$$

$$= \frac{1}{\pi}\left(\frac{\pi}{n}(1-(-1)^n) - \frac{\pi(-1)^n}{n}\right)$$

$$= \frac{1}{n} - \frac{2}{\pi}(-1)^n$$

F.S.

$$f(x) = \frac{-\pi}{4} + \sum_{n=1}^{\infty}\frac{(-1)^n - 1}{\pi n^2}\cos nx + \sum_{n=1}^{\infty}\frac{1-2(-1)^n}{n}\sin nx$$

$$= \frac{-\pi}{4} + \left(\frac{-2}{\pi}\cos x + 3\sin x\right) + \left(0 - \frac{\sin 2x}{2}\right) + \left(\frac{-2\cos 3x}{\pi 3^2} + \frac{3\sin 3x}{3}\right) + \left(0 - \frac{\sin 4x}{4}\right) + \dots$$

Note:

If $x=c$ is a pt of discontinuity for the fn $f(x)$ then the sum of the fourier series on the RHS at that pt $= \frac{1}{2}\left[f(x-c) + f(x+c)\right]$

Here,

$x=0$ is a pt of discontinuity.

$f(0-0) = -\pi$ and $f(0+0) = 0$.

$$\therefore \frac{1}{2}\left[f(0-0) + f(0+0)\right] = -\frac{\pi}{4} - \frac{2}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2}\right)$$

$$\frac{-\pi+0}{2} = \frac{-\pi}{4} - \frac{2}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \dots\right)$$

$$-\frac{\pi}{2} + \frac{\pi}{4} = -\frac{2}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \dots\right)$$

$$\frac{-4\pi + 2\pi}{8} = \frac{-2}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \dots\right)$$

$$\frac{2\pi}{8} \times \frac{\pi}{2} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2}\dots$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$$

Q. Write the fourier expansion for,

$$f(x) = \pi x \quad 0 \le x \le 1$$
$$= \pi(2-x) \quad 1 \le x \le 2.$$

deduce $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} \dots = \frac{\pi^2}{8}$

ans. F.S:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

Here $2l = 2$

$$l = 2/2 = 1$$

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx$$
$$= \int_0^1 \pi x\,dx + \int_1^2 \pi(2-x)\,dx.$$
$$= \pi\left[\frac{x^2}{2}\right]_0^1 + \pi\left[2x - \frac{x^2}{2}\right]_1^2$$
$$= \pi \times 1/2 + \pi\left[(4-2) - \left(\frac{4}{2} - \frac{1}{2}\right)\right]$$
$$= \pi/2 + \pi\left[2 - \frac{3}{2}\right]$$
$$= \pi/2 + \pi \times 1/2 = \frac{2\pi}{2} = \pi$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x) \cos\frac{n\pi x}{l}\,dx.$$
$$= \int_0^1 \pi x \cos n\pi x\,dx + \int_1^2 \pi(2-x)\cos n\pi x\,dx$$
$$= \pi\left[\cancel{\frac{x \sin n\pi x}{n\pi}} - \frac{-\cos n\pi x}{(n\pi)^2}\right]_0^1 + \pi\left[(2-x)\cdot\frac{\sin n\pi x}{n\pi} - \frac{-1\cdot -\cos n\pi x}{(n\pi)^2}\right]_1^2$$
$$= \pi\left[\frac{1}{(n\pi)^2}\cdot((-1)^n - 1)\right] + \pi\left[\frac{-1}{(n\pi)^2}(1 - (-1)^n)\right]$$
$$= \frac{2}{n^2\pi}\left((-1)^n - 1\right)$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx.$$
$$= \left[\int_0^1 \pi x \sin n\pi x\,dx + \int_1^2 \pi(2-x)\sin n\pi x\,dx\right]$$
$$= \pi\left[x\cdot\frac{-\cos n\pi x}{n\pi} - 1\cdot\frac{-\sin n\pi x}{(n\pi)^2}\right]_0^1 +$$
$$\pi\left[(2-x)\frac{-\cos n\pi x}{n\pi} - \frac{-1\cdot -\sin n\pi x}{(n\pi)^2}\right]_1^2$$
$$= \pi\left[-\frac{1}{n\pi}((-1)^n - 0) + 0\right] + \pi\left[\frac{-1}{n\pi}\; 0 - (-1)^n - 0\right]$$
$$= 0$$

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi}\left((-1)^n - 1\right)\cos n\pi x.$$

$$= \frac{\pi}{2} + \frac{2}{\pi}\left[\frac{-2\cos n\pi x}{1^2} + 0 + \frac{-2\cos 3n\pi x}{3^2} + 0 \frac{-2\cos 5n\pi x}{5^2} + \cdots\right]$$

$x = 0$

$$\cancel{f(x)}\ \pi x = \frac{\pi}{2} + \frac{2}{\pi}\left[\frac{-2\cos 0}{1^2} - \frac{2\cos 0}{3^2} - \frac{2\cos 0}{5^2} + \cdots\right]$$

$$0 = \frac{\pi}{2} + \frac{2}{\pi} \times -2\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2}\right]$$

$$-\pi/2 = \frac{-4}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right]$$

$$\frac{-\pi}{2} \times \frac{\pi}{-4} = \left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right] = \frac{\pi^2}{8}$$

2/16. day →

Half range cosine series:

If $f(x)$ is defined in 0 to l, then the half range cosine series is:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

where $a_0 = \frac{2}{l}\int_0^l f(x)\,dx.$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx.$$

→ **Half range sine series:**

If $f(x)$ is a fn defined in $0 \le x \le l$, then the half range sine series is:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

where $b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx.$

Q. Obtain the half range sine series for e^x in $0 < x < 1$.

ans. Half range sine series for x in $0 < x < l$ is:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

where, $b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx.$

$$b_n = 2\int_0^1 e^x \sin n\pi x\,dx.$$

$$\left\{\int e^{ax}\sin(bx+c)\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\sin(bx+c) - b\cos(bx+c)\right]\right\}$$

$a = 1,\ b = n\pi,\ c = 0.$

$$b_n = 2\left[\frac{e^x}{1+n^2\pi^2}\left[\sin n\pi x - n\pi \cos n\pi x\right]\right]_0^1$$

$$= 2\left\{\frac{1}{1+n^2\pi^2}\left[e\left(0 - n\pi(-1)^n\right) - e^0\left(0 - n\pi\right)\right]\right\}$$

$$= \frac{2}{1+n^2\pi^2}\left[e \times -n\pi(-1)^n + n\pi\right]$$

$$= \frac{2n\pi}{1+n^2\pi^2}\left(1 - e(-1)^n\right)$$

H R sine series is :

$$e^x = \sum_{n=1}^{\infty} \frac{2n\pi}{1+n^2\pi^2}\left[1 - e(-1)^n\right] \sin n\pi x$$

Q. Expand $f(x) = \cos x$ as a half range sine series in $(0, \pi)$

Ans: Half range sine series in $(0, \pi)$ is :

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

$$= \sum_{n=1}^{\infty} b_n \sin nx \quad \text{bcoz } l = \pi$$

where $b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx$

$$b_n = \frac{2}{\pi}\int_0^{\pi} \cos x \sin nx \, dx$$

$$\boxed{2\cos A \sin B = \sin(A+B) - \sin(A-B)}$$

$$= \frac{1}{\pi}\int_0^{\pi}\left[\sin(1+n)x - \sin(1-n)x\right]dx.$$

$$= \frac{1}{\pi}\left[\frac{-\cos(1+n)x}{(1+n)} - \frac{-\cos(1-n)x}{(1-n)}\right]_0^{\pi}$$

Q. Expand $x\sin x$ in $(0, \pi)$ as a cosine series.

Ans: Half range cosine series in $(0, \pi)$ is :

$$f(x) = \cancel{\sum_{n=1}}\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx.$$

where $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\, dx$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cos nx \, dx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x\sin x \, dx$$

$$= \frac{2}{\pi}\left[x \times -\cos x - 1 \cdot -\sin x\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-x\cos x + \sin x\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left(-\pi\cos\pi + 0\right) + \left(\sin\pi - \sin 0\right)\right]$$

$$= \frac{2}{\pi}\left[1 + 0\right] = \frac{2}{\pi}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x\sin x\cos nx \, dx$$

$$\boxed{2\cos A\sin B = \sin(A+B) - \sin(A-B)}$$

$$= \frac{1}{\pi}\int_0^{\pi}\left(x\left[\sin(n+1)x - \sin(n-1)x\right]\right)dx$$

$$= \frac{1}{\pi}\left\{\int_0^{\pi} x\sin(n+1)x\,dx - \int_0^{\pi} x\sin(n-1)x\,dx\right\}$$

$$= \frac{1}{\pi}\left\{\left[x \cdot \frac{-\cos(n+1)x}{n+1} - 1 \cdot \frac{-\sin(n+1)x}{(n+1)^2}\right]_0^{\pi} - \left[x \cdot \frac{-\cos(n-1)x}{n-1} - 1 \cdot \frac{-\cos(n-1)x}{(n-1)^2}\right]_0^{\pi}\right\}$$

Q. Expand $(x-1)^2$ as a cosine series in $0 < x < 1$.
deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$.

ans: H.R cosine series is:

$$\cancel{f(x) = \sum_{n=1}^{\infty} b_n}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

where $a_0 = \frac{2}{l}\int_0^l f(x)\,dx$

$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx.$

$a_0 = \frac{2}{1}\int_0^1 (x-1)^2\,dx.$

$= 2\left[\frac{(x-1)^3}{3}\right]_0^1$

$= 2\left[0 - \left(\frac{(-1)^3}{3}\right)\right]$

$= \frac{2}{3}$

$a_n = \frac{2}{1}\int_0^1 (x-1)^2 \cos n\pi x\,dx$

$= 2\left[(x-1)^2 \cancel{\frac{\sin n\pi x}{n\pi}} - 2(x-1)\frac{\cos n\pi x}{(n\pi)^2} + 2\cancel{\frac{\sin n\pi x}{(n\pi)^3}}\right]_0^1$

$= 2\left[-\left(0 - -2\left(\frac{-\cos 0}{(n\pi)^2}\right)\right)\right]$

$= 2\left[-2\left(\frac{-1}{(n\pi)^2}\right)\right]$

$= \frac{4}{n^2\pi^2}$

H. R cosine series is ;

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}.$

$(x-1)^2 = \frac{2}{6} + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}\cos n\pi x.$

$(x-1)^2 = \frac{1}{3} + \left[\frac{4}{1^2\pi^2}\cos \pi x + \frac{4}{2^2\pi^2}\cos 2\pi x + \frac{4}{3^2\pi^2}\cos 3\pi x + \cdots\right]$

when $x = 1$,

$0 = \frac{1}{3} + \left[\frac{4}{\pi^2}\times -1 + \frac{4}{2^2\pi^2}\times 1 + \frac{4}{3^2\pi^2}\times -1 + \cdots\right]$

$0 = \frac{1}{3} + \left[\frac{-4}{\pi^2} + \frac{4}{2^2\pi^2} - \frac{4}{3^2\pi^2} + \cdots\right]$

$-\frac{1}{3} = \frac{4}{\pi^2}\left[\frac{-1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \cdots\right]$

$\frac{\pi^2}{12} = \left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right] \quad — (1)$

when $x=0$,

$$1 = \frac{1}{3} + \frac{4}{\pi^2}\left[\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\right]$$

$$1-\frac{1}{3} = \frac{4}{\pi^2}\left[\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\right]$$

$$\frac{\cancel{2}}{3}\times\frac{\pi^2}{\cancel{4}2} = \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots$$

$$\frac{\pi^2}{6} = \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots \quad \text{—(2)}$$

(1) + (2).

$$\Rightarrow \frac{\pi^2}{12}+\frac{\pi^2}{6} = \left[\frac{1}{1^2}-\cancel{\frac{1}{2^2}}+\frac{1}{3^2}-\cancel{\frac{1}{4^2}}\cdots\right] + \left[\frac{1}{1^2}+\cancel{\frac{1}{2^2}}+\frac{1}{3^2}+\cancel{\frac{1}{4^2}}\cdots\right]$$

$$\Rightarrow \frac{\pi^2+2\pi^2}{12} = 2\left[\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\right]$$

$$\Rightarrow \frac{\cancel{3}\pi^2}{\cancel{12}_4\times 2} = \left[\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\right]$$

$$\Rightarrow \frac{\pi^2}{8} = \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots$$

11/2/16 Thursday

Parseval's Thm for Fourier constants:

If $f(x)$ is defined in $(c, c+2l)$, the F.S for $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{l} + b_n\sin\frac{n\pi x}{l}\right)$

then $\frac{1}{l}\int_c^{c+2l}[f(x)]^2dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2+b_n^2)$

Particular cases:

1) If $c=0$, $(0, 2l)$

$$\frac{1}{l}\int_0^{2l}[f(x)]^2dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2+b_n^2)$$

2) If $c=-l$, $(-l, l)$

$$\frac{1}{l}\int_{-l}^{l}[f(x)]^2dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2+b_n^2)$$

3) If $f(x)$ is even,

$$\frac{2}{l}\int_0^{l}[f(x)]^2dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}a_n^2.$$

4) If $f(x)$ is odd,

$$\frac{2}{l}\int_0^{l}[f(x)]^2dx = \sum_{n=1}^{\infty}b_n^2.$$

5) <u>HRC</u>:

$$\frac{2}{l}\int_0^l [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2$$

6) <u>HRS</u>

$$\frac{2}{l}\int_0^l [f(x)]^2 dx = \sum_{n=1}^{\infty} b_n^2$$

↳ <u>Root mean square (rms value) of a fn</u>:

In interval (a,b).

$$[f(x)]_{rms} = \sqrt{\frac{\int_a^b [f(x)]^2 dx}{b-a}}$$

$$\left([f(x)]_{rms}\right)^2 = \frac{1}{b-a}\int_a^b [f(x)]^2 dx$$

$$= \frac{1}{2l}\int_c^{c+2l} [f(x)]^2 dx \quad \text{in } (c, c+2l)$$

Q. S.T the F.S for x in $0 < x < l$ is

$$x = \frac{2l}{\pi}\left[\sin\frac{\pi x}{l} - \frac{1}{2}\sin\frac{2\pi x}{l} + \frac{1}{3}\sin\frac{3\pi x}{l} \cdots\right]$$

deduce the value for $1/1^2 + 1/2^2 + 1/3^2 + \cdots$

ans: <u>HRS</u>:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

where $b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx.$

$$b_n = \frac{2}{l}\int_0^l x\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[x\cdot\frac{-\cos n\pi x}{l n\pi} - \frac{-\sin n\pi x}{l(n\pi)^2}\right]_0^l$$

$$= \frac{2}{l}\left[\frac{-l\cos n\pi l}{n\pi l} + \frac{\sin n\pi l}{(n\pi)^2 l}\right]$$

$$= \frac{-2l\cos \pi l}{n\pi l^2} + \cancel{\frac{2\sin n\pi l}{(n\pi)^2 l^2}}^{\,0}$$

$$= \frac{-2(-1)^n}{n\pi l} = \frac{-2l(-1)^n}{n\pi}$$

HRS series:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

$$= \sum_{n=1}^{\infty} \frac{-2l(-1)^n}{n\pi} \sin\frac{n\pi x}{l}$$

$$= \frac{2l}{\pi} \sum_{n=1}^{\infty} -\left[\frac{(-1)^n}{n} \times \sin\frac{n\pi x}{l}\right]$$

$$= \frac{2l}{\pi}\left\{-\left[\frac{(-1)}{1} \sin\frac{\pi x}{l}\right] + -\left[\frac{1}{2}\sin\frac{2\pi x}{l}\right] + -\left[\frac{(-1)}{3} \sin\frac{3\pi x}{l}\right] + -\left[\frac{1}{4}\sin\frac{4\pi x}{l}\right] \ldots\right\}$$

$$= \frac{2l}{\pi}\left\{\frac{\sin \pi x}{l} - \frac{\sin 2\pi x}{2l} + \frac{\sin 3\pi x}{3l} - \frac{\sin 4\pi x}{4l} + \ldots\right\}$$

$$= \frac{2l}{\pi}\left\{\frac{\sin\pi x}{l} - \frac{1}{2}\frac{\sin 2\pi x}{l} + \frac{1}{3}\frac{\sin 3\pi x}{l} - \frac{1}{4}\frac{\sin 4\pi x}{l} \ldots\right\}$$

By parsevall's thm,

$$\frac{2}{l}\int_0^l [f(x)]^2\, dx = \sum_{n=1}^{\infty} b_n^2$$

$$\frac{2}{l}\int_0^l (x)^2 dx = \sum_{n=1}^{\infty} \frac{4l^2}{n^2\pi^2}$$

$$\cancel{\frac{2}{l}}\left[\frac{x^3}{3}\right]_0^l = \cancel{\sum_{n=1}^{\infty}} \frac{\cancel{4}^{2} l^2}{\pi^2}\left[\sum_{n=1}^{\infty}\frac{1}{n^2}\right]$$

$$\frac{1}{\cancel{l}} \times \frac{l^{\cancel{3}\,2}}{3} = \frac{2l^2}{\pi^2}\left[\sum_{n=1}^{\infty}\frac{1}{n^2}\right]$$

$$\frac{l^2}{3} = \frac{2l^2}{\pi^2}\left[1/1^2 + 1/2^2 + 1/3^2 + \ldots\right]$$

$$\frac{\pi^2}{6} = 1/1^2 + 1/2^2 + 1/3^2 + \ldots$$

$$\left([f(x)]_{rms}\right)^2 = \frac{1}{l}\int_0^l [f(x)]^2 dx$$

$$= \frac{1}{l}\int_0^l x^2\, dx$$

$$= \frac{1}{l}\left[\frac{x^3}{3}\right]_0^l = \frac{1}{\cancel{l}} \times \frac{l^{\cancel{3}\,2}}{3}$$

$$= \frac{l^2}{3}$$

Q. Find the F.S for $y = x^2$ in $-\pi < x < \pi$ and deduce ~~the value for~~ $\frac{\pi^4}{90} = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \dots$

ans: x^2 is an even function.

$\therefore$ F.S for $y = x^2$.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx.$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x^2\,dx.$$

$$= \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\pi^3}{3}\right]$$

$$= \frac{2}{\cancel{\pi}} \times \frac{\pi^{\cancel{3}\,2}}{3}$$

$$= \frac{2\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\,dx$$

$$= \frac{2}{\pi}\left[\cancel{\frac{x^2 \sin nx}{n}}^{\,0} - \frac{2x\cos nx}{n^2} + \cancel{\frac{2\sin nx}{n^3}}^{\,0}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{2x\cos nx}{n^2}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{2\pi\cos n\pi}{n^2} - 0\right]$$

$$= \frac{2}{\pi}\left[\frac{2\pi(-1)^n}{n^2}\right]$$

$$= \frac{4\cancel{\pi}(-1)^n}{\cancel{\pi}n^2}$$

$$= \frac{4(-1)^n}{n^2}$$

F.S :

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$= \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}\cos nx$$

By parseval's thm,

$$a_0 = \frac{2\pi^2}{3}, \quad a_0^2 = \left(\frac{2\pi^2}{3}\right)^2 = \frac{4\pi^4}{9}$$

$$a_n = \frac{4(-1)^n}{n^2}, \quad a_n^2 = \frac{16(-1)^{2n}}{n^4}$$

$$\frac{2}{l}\int_0^l [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2$$

$$\frac{2}{\pi}\int_0^{\pi} (x^2)^2 dx = \frac{\frac{4\pi^4}{9}}{2} + \sum_{n=1}^{\infty} \frac{16(-1)^{2n}}{n^4}$$

$$\frac{2}{\pi}\int_0^{\pi} x^4 dx = \frac{4\pi^4}{18} + 16\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\frac{2}{\pi}\left[\frac{x^5}{5}\right]_0^{\pi} = \frac{2\pi^4}{9} + 16\left[\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots\right]$$

$$\frac{2}{\cancel{\pi}}\left[\frac{\pi^{\cancel{5}\,4}}{5}\right] = \frac{2\pi^4}{9} + 16\left[1/1^4 + 1/2^4 + 1/3^4 + \cdots\right]$$

$$\frac{\frac{2\pi^4}{5} - \frac{2\pi^4}{9}}{16} = 1/1^4 + 1/2^4 + 1/3^4 + \cdots$$

$$\frac{18\pi^4 - 10\pi^4}{45\times 16} = 1/1^4 + 1/2^4 + 1/3^4 + \cdots$$

$$\frac{8\pi^4}{720} = 1/1^4 + 1/2^4 + 1/3^4 + \cdots$$

$$\frac{\pi^4}{90} = 1 + 1/2^4 + 1/3^4 + \cdots$$

$$\left(\left[f(x)\right]_{rms}\right)^2 = \frac{2\pi^4}{5} \times \frac{1}{2}$$

$$= \frac{\pi^4}{5}$$

Q. Obtain the HRC & HRS to represent $f(x) = x - x^2$

HW in $(0,1)$, deduce the value of.

ⓐ $1/1^6 + 1/3^6 + 1/5^6 + \cdots$

ⓑ $1/1^4 + 1/2^4 + 1/3^4 + \cdots$

ans: <u>HRC series</u>:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

where $a_0 = \frac{2}{l}\int_0^l f(x)\,dx$.

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx.$$

$$a_0 = 2\int_0^1 x - x^2\,dx$$

$$= 2\left\{\left[\frac{x^2}{2}\right]_0^1 - \left[\frac{x^3}{3}\right]_0^1\right\}$$

$$= 2\left[\frac{1}{2} - \frac{1}{3}\right]$$

$$= 2\left[\frac{3-2}{6}\right]$$

$$= \cancel{2}\times\frac{1}{\cancel{6}_3} = \frac{1}{3}$$

$$a_n = 2\int_0^1 (x-x^2)\cos n\pi x \, dx$$

$$= 2\left\{\int_0^1 x\cos n\pi x\,dx - \int_0^1 x^2\cos n\pi x\,dx\right\}$$

$$= 2\left\{\left[x \times \cancel{\frac{\sin n\pi x}{n\pi}}^{0} - \frac{-\cos n\pi x}{(n\pi)^2}\right]_0^1 - \left[x^2\cancel{\frac{\sin n\pi x}{n\pi}}^{0} - 2x\frac{-\cos n\pi x}{(n\pi)^2} + \cancel{\frac{2\sin n\pi x}{(n\pi)^3}}^{0}\right]_0^1\right\}$$

$$= 2\left\{\left[\frac{(-1)^n}{(n\pi)^2} - \frac{1}{(n\pi)^2}\right] - \left[\frac{2(-1)^n}{(n\pi)^2}\right]\right\}$$

$$= 2\left[\frac{(-1)^n - 1 - 2(-1)^n}{(n\pi)^2}\right]$$

$$= 2\left[\frac{-(-1)^n - 1}{(n\pi)^2}\right]$$

$$= \frac{-2\left[(-1)^n + 1\right]}{(n\pi)^2}$$

<u>HRC series:</u>

$$x - x^2 = \frac{1}{6} + \sum_{n=1}^{\infty} \frac{-2\left[(-1)^n + 1\right]}{(n\pi)^2}\cos n\pi x.$$

By parseval's thm,

$$\frac{2}{l}\int_0^l (f(x))^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2$$

$$2\int_0^1 (x-x^2)^2\,dx = \frac{\left(\frac{1}{3}\right)^2}{2} + \sum_{n=1}^{\infty} \frac{4\left[(-1)^n+1\right]^2}{(n\pi)^4}$$

$$2\int_0^1 (x^2 - 2x^3 + x^4)\,dx = \frac{\left(\frac{1}{3}\right)^2}{2} + 4\left(0 + \frac{4}{(2\pi)^4} + 0 + \frac{4}{(4\pi)^4}\right.$$

$$2\left\{\left[\frac{x^3}{3}\right]_0^1 - 2\left[\frac{x^4}{4}\right]_0^1 + \left[\frac{x^5}{5}\right]_0^1\right\} = \frac{1}{18} + 4\left[\frac{4}{(2\pi)^4} + \frac{4}{(4\pi)^4} + \frac{4}{(6\pi)^4}\right.$$

$$2\left[\frac{1}{3} - 2\times\frac{1}{4} + \frac{1}{5}\right] = \frac{1}{18} + 16\left[\frac{1}{(2\pi)^4} + \frac{1}{(4\pi)^4} + \frac{1}{(6\pi)^4}\right.$$

$$\frac{\frac{2}{30} - \frac{1}{18}}{16} = \frac{1}{16\pi^4}\left[\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4}\cdots\right]$$

$$\frac{1}{1440}\times\frac{16\pi^4}{1} = \left[\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots\right]$$

$$\frac{\pi^4}{90} = \left[\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4}\cdots\right]$$

HRS series:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

where $b_n = \frac{2}{l}\int_0^l f(x) \sin\frac{n\pi x}{l}\, dx$

$$b_n = 2\int_0^1 (x - x^2) \sin n\pi x\, dx$$

$$= 2\left\{\int_0^1 x \sin n\pi x\, dx - \int_0^1 x^2 \sin n\pi x\, dx\right\}$$

$$= 2\left\{\left[x \frac{-\cos n\pi x}{n\pi} - \cancel{\frac{-\sin n\pi x}{(n\pi)^2}}\right]_0^1 - \left[x^2 \frac{-\cos n\pi x}{n\pi} - \cancel{2x \frac{-\sin n\pi x}{(n\pi)^2}} + \frac{2\cos n\pi x}{(n\pi)^3}\right]_0^1\right\}$$

$$= 2\left\{\left(\frac{-(-1)^n}{n\pi}\right) - \left[\left[\frac{-(-1)^n}{n\pi} - \frac{2(-1)^n}{(n\pi)^3}\right] - \left[\frac{-2\times 1}{(n\pi)^3}\right]\right]\right\}$$

$$= 2\left\{\frac{-(-1)^n}{n\pi} - \left[\frac{-(-1)^n}{n\pi} - \frac{2(-1)^n}{(n\pi)^3} + \frac{2}{(n\pi)^3}\right]\right\}$$

$$= 2\left[\cancel{\frac{-(-1)^n}{n\pi}} + \cancel{\frac{(-1)^n}{n\pi}} + \frac{2(-1)^n}{(n\pi)^3} - \frac{2}{(n\pi)^3}\right]$$

$$= 2\left[\frac{2}{(n\pi)^3}\left[(-1)^n - 1\right]\right]$$

$$= \frac{4}{(n\pi)^3}\left[(-1)^n - 1\right]$$

HRS series:

$$x - x^2 = \sum_{n=1}^{\infty} \frac{4}{(n\pi)^3}\left[(-1)^n - 1\right] \sin n\pi x.$$

By parseval's thm,

$$\frac{2}{l}\int_0^l \left[f(x)\right]^2 dx = \sum_{n=1}^{\infty} b_n^2.$$

$$2\int_0^1 (x - x^2)^2 dx = \sum_{n=1}^{\infty} \frac{16}{(n\pi)^6}\left[(-1)^n - 1\right]^2$$

$$2\int_0^1 (x^2 - 2x^3 + x^4)\, dx = \sum_{n=1}^{\infty} \frac{16}{(n\pi)^6}\left[(-1)^n - 1\right]^2$$

$$2\left\{\left[\frac{x^3}{3}\right]_0^1 - 2\left[\frac{x^4}{4}\right]_0^1 + \left[\frac{x^5}{5}\right]_0^1\right\} = 16\left[\frac{4}{\pi^6} + \frac{0}{(2\pi)^6} + \frac{4}{(3\pi)^6} + \frac{0}{(4\pi)^6} \cdots\right]$$

$$2\left[1/3 - 2\times 1/4 + 1/5\right] = 16\left[\frac{4}{(1\pi)^6} + \frac{4}{(3\pi)^6} + \frac{4}{(5\pi)^6} \cdots\right]$$

$$\frac{2}{30} \times \frac{1}{16} = 4\left[\frac{1}{1\pi^6} + \frac{1}{729\pi^6} + \frac{1}{15625\pi^6} \cdots\right]$$

$$\frac{1}{240} \times \frac{1}{4} = \frac{1}{\pi^6}\left[1/1^6 + 1/3^6 + 1/5^6 + \cdots\right]$$

$$\frac{1}{960} \times \pi^6 = 1/1^6 + 1/3^6 + 1/5^6 + \cdots$$

$$\frac{\pi^6}{960} = 1/1^6 + 1/3^6 + 1/5^6 + \cdots$$