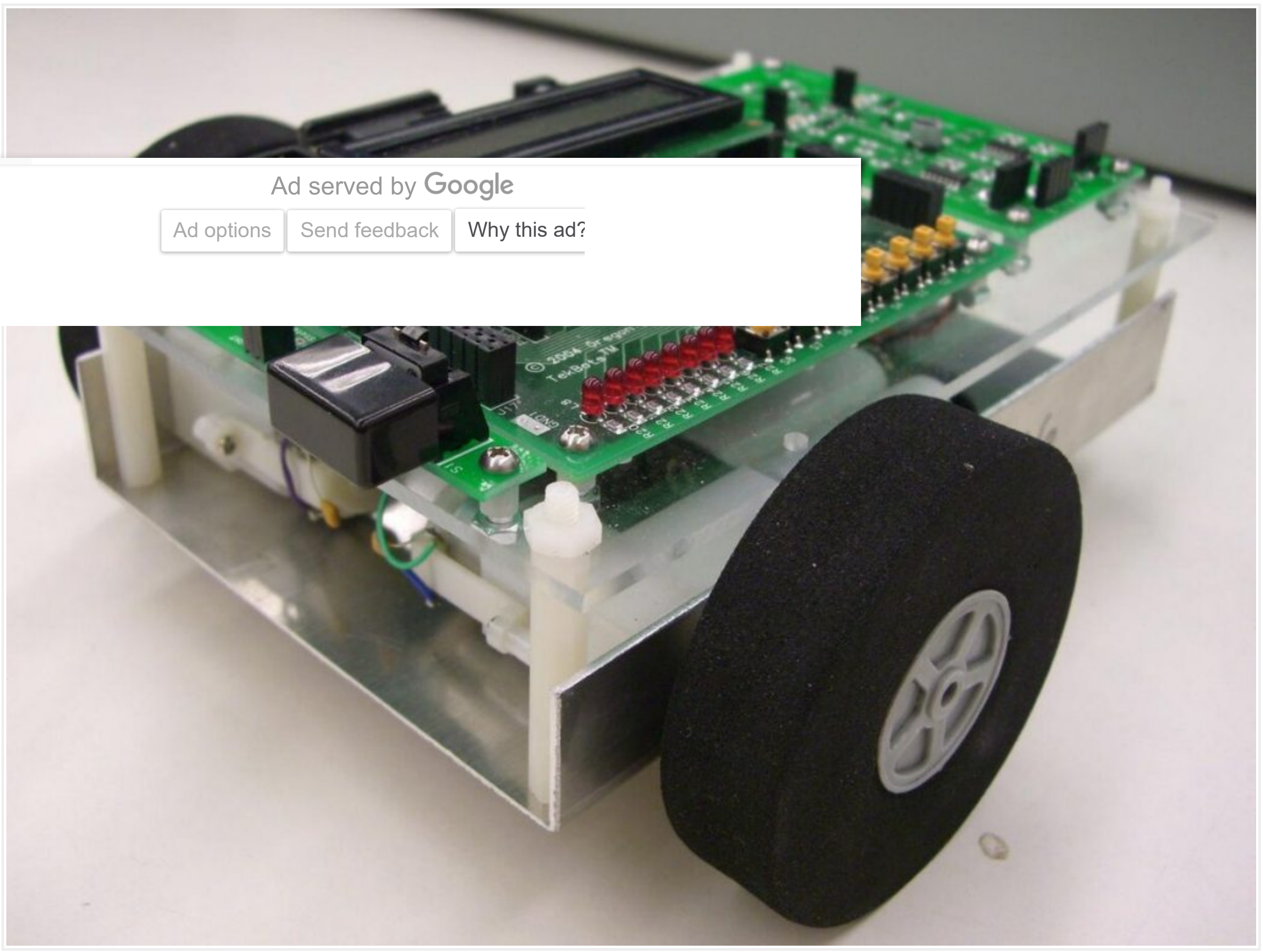
Automatic Addison

Build the Future

How To Derive the State Space Model for a Mobile Robot



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In this tutorial, we will learn how to create a **state space model** for a mobile robot. Specifically, we'll take a look at a type of mobile robot called a **differential drive robot**. A differential drive robot is a robot like the one on the cover photo of this post, as well as this one below. It is called differential drive because each wheel is driven independently from the other.

Prerequisites

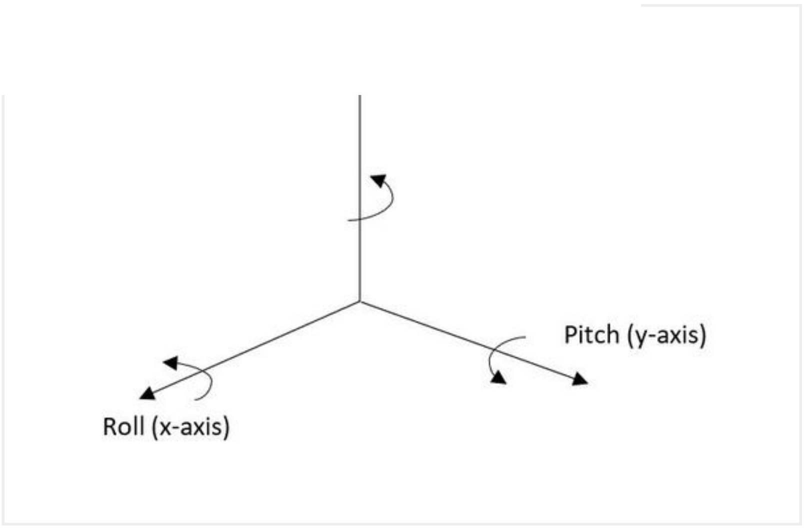
- Basic linear algebra
  - You understand what a matrix and a vector are.
  - You know how to multiply two matrices together.
- You know what a partial derivative is.
- Optional: Python 3.7 or higher (just needed for the last part of this tutorial)

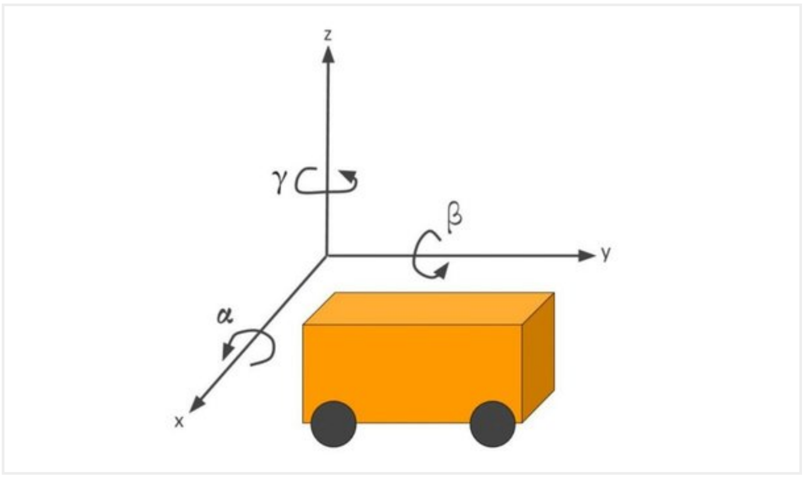
There are a lot of tutorials on YouTube if you don't have the prerequisites above (e.g. Khan Academy).

What is a State Space Model?

A **state space model** (often called the state transition model) is a mathematical equation that represents the motion of a robotic system from one timestep to the next. It shows how the current position (e.g. X, Y coordinate) and orientation (yaw (heading) angle γ) of the robot in the world is impacted by changes to the control inputs into the robot (e.g. velocity in meters per second...often represented by the variable v).

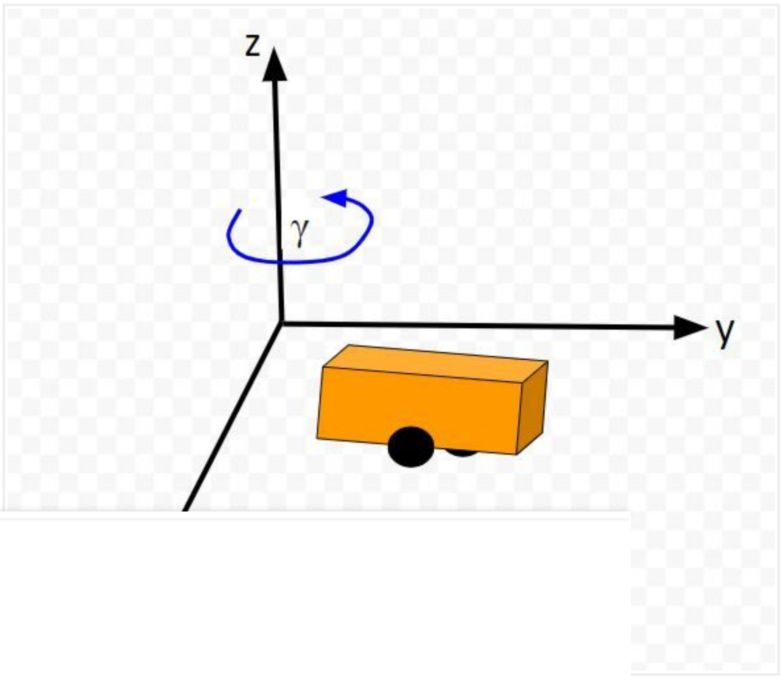
Note: In a real-world scenario, we would need to represent the world the robot is in (e.g. a room in your house, etc.) as an



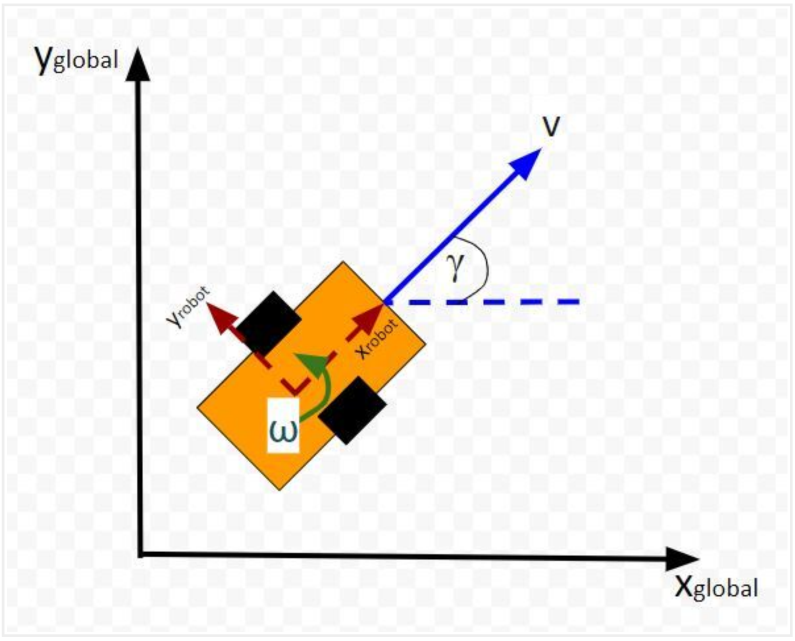


Deriving the Equations

Consider the mobile robot below. It is moving around on the flat x-y coordinate plane.



Let’s look at it from an aerial view.



What is the position and orientation of the robot in the world (i.e. global reference frame)? The position and orientation of a robot make up what we call the **state vector**.

$$StateVector = \begin{bmatrix} x_{t-1} \\ y_{t-1} \\ \gamma_{t-1} \end{bmatrix}$$

Note that x and y as well as the yaw angle γ are in the global reference frame.

The **yaw angle γ** describes the rotation around the z-axis (coming out of the page in the image above) in the counterclockwise direction (from the x axis). The units for the yaw angle are typically radians.

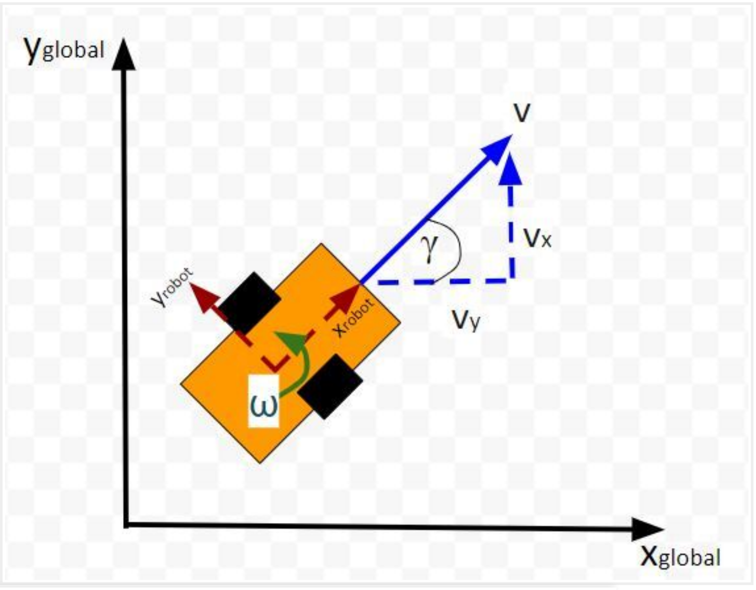
ω is the rotation rate (i.e. angular velocity or yaw rate) around the global z axis. Typically the units on this variable is radians per second.

The velocity of the robot v is in terms of the robot reference frame (labeled x_{robot} and y_{robot}).

In terms of the global reference frame we can break v down into two components:

- velocity in the x direction v_x
- velocity in the y direction v_y.

In most of my applications, I express the velocity in meters per second.



Therefore, with respect to the global reference frame, the robot’s motion equations are as follows:

linear velocity in the x direction = v_x = vcos(γ)

linear velocity in the y direction = v_y = vsin(γ)

angular velocity around the z axis = ω

Now let’s suppose at time t-1, the robot has the current state (x position, y position, yaw angle γ):

$$StateVector = \begin{bmatrix} x_{t-1} \\ y_{t-1} \\ \gamma_{t-1} \end{bmatrix}$$

We move forward one timestep **dt**. What is the state of the robot at time t? In other words, where is the robot location, and how is the robot oriented?

Remember that **distance = velocity * time**.

Therefore,

$$\begin{bmatrix} x_t \\ y_t \\ \gamma_t \end{bmatrix} = \begin{bmatrix} x_{t-1} + v_{t-1}\cos\gamma_{t-1} * dt \\ y_{t-1} + v_{t-1}\sin\gamma_{t-1} * dt \\ \gamma_{t-1} + \omega_{t-1} * dt \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

Converting the Equations to a State Space Model

We now know how to describe the motion of the robot mathematically. However the equation above is nonlinear. For some applications, like using the Kalman Filter (I'll cover Kalman Filters in a future post), we need to make our state space equations linear.

How do we take this equation that is nonlinear (note the cosines and sines)...

$$\begin{bmatrix} x_t \\ y_t \\ \gamma_t \end{bmatrix} = \begin{bmatrix} x_{t-1} + v_{t-1}\cos\gamma_{t-1} * dt \\ y_{t-1} + v_{t-1}\sin\gamma_{t-1} * dt \\ \gamma_{t-1} + \omega_{t-1} * dt \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

...and convert it into a form that looks like the equation of a line? In mathematical jargon, we want to make the equation above a linear discrete-time state space model of the following form.

$$\mathbf{x_t} = A_{t-1}\mathbf{x_{t-1}} + B_{t-1}\mathbf{u_{t-1}}$$

where:

$\mathbf{x_t}$ is our entire current state vector [x_t, y_t,γ_t] (i.e. note $\mathbf{x_t}$ in bold is the entire state vector...don't get that confused with x_t, which is just the x coordinate of the robot at time t)

d velocity, angular velocity].

I'll get to what the A and B matrices represent later in this post.

Let's write out the full form of the linear state space model equation:

$$\begin{bmatrix} x_t \\ y_t \\ \gamma_t \end{bmatrix} = A_{t-1}\begin{bmatrix} x_{t-1} \\ y_{t-1} \\ \gamma_{t-1} \end{bmatrix} + B_{t-1}\begin{bmatrix} v_{t-1} \\ \omega_{t-1} \end{bmatrix}$$

Let's go through this equation term by term to make sure we understand everything so we can convert this equation below into the form above:

$$\begin{bmatrix} x_t \\ y_t \\ \gamma_t \end{bmatrix} = \begin{bmatrix} x_{t-1} + v_{t-1}\cos\gamma_{t-1} * dt \\ y_{t-1} + v_{t-1}\sin\gamma_{t-1} * dt \\ \gamma_{t-1} + \omega_{t-1} * dt \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

How to Calculate the A Matrix

A is a matrix. The number of rows in the A matrix is equal to the number of states, and the number of columns in the A matrix is equal to the number of states. In this mobile robot example, we have three states.

The A matrix expresses **how the state of the system [x position,y position,yaw angle γ] changes from t-1 to t when no control command is executed** (i.e. when we don't give any speed (velocity) commands to the robot).

Typically a robot on wheels only drives when the wheels are commanded to turn. Therefore, for this case, A is the identity matrix (Note that A is sometimes F in the literature).

$$A_{t-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

In other applications, A might not be the identity matrix. An example is a helicopter. The state of a helicopter flying around in the air changes even if you give it no velocity command. Gravity will pull the helicopter downward to Earth regardless of what you do. It will eventually crash if you give it no velocity commands.

our mobile robot example, that is this

$$\begin{bmatrix} x_t \\ y_t \\ \gamma_t \end{bmatrix} = \begin{bmatrix} x_{t-1} + v_{t-1} \cos \gamma_{t-1} * dt \\ y_{t-1} + v_{t-1} \sin \gamma_{t-1} * dt \\ \gamma_{t-1} + \omega_{t-1} * dt \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

Our system is expressed as a nonlinear system now because the state is a function of cosines and sines (which are nonlinear trigonometric operations).

To get our state space model in this form...

$$\begin{bmatrix} x_t \\ y_t \\ \gamma_t \end{bmatrix} = A_{t-1} \begin{bmatrix} x_{t-1} \\ y_{t-1} \\ \gamma_{t-1} \end{bmatrix} + B_{t-1} \begin{bmatrix} v_{t-1} \\ \omega_{t-1} \end{bmatrix}$$

...we need to "linearize" the nonlinear equations. To do this, we need to calculate the Jacobian, which is nothing more than a fancy name for "matrix of partial derivatives."

Do you remember the equation for a straight line from grade school: **y=mx+b**? m is the slope. It is “change in y / change in x”.

The Jacobian is nothing more than a multivariable form of the slope m. It is “change in y1/x1, change in y2/x2, change in y3/x3, etc.…”

You can consider the Jacobian a slope on steroids. It represents how fast a group of variables (as opposed to just one variable…(e.g. m = rise/run = change in y/change in x) are changing with respect to another group of variables.

You start with this here:

$$\begin{bmatrix} x_t \\ y_t \\ \gamma_t \end{bmatrix} = \begin{bmatrix} x_{t-1} + v_{t-1}\cos\gamma_{t-1} * dt \\ y_{t-1} + v_{t-1}\sin\gamma_{t-1} * dt \\ \gamma_{t-1} + \omega_{t-1} * dt \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

You then calculate the partial derivative of the state vector at time t with respect to the states at time t-1. Don’t be scared at all the funny symbols inside this matrix A. When you’re calculating the Jacobian matrix, you calculate each partial derivative, one at a time.

$$A_{t-1} = \begin{bmatrix} \frac{\partial f_1}{\partial x_{t-1}} & \frac{\partial f_1}{\partial y_{t-1}} & \frac{\partial f_1}{\partial \gamma_{t-1}} \\ \frac{\partial f_2}{\partial x_{t-1}} & \frac{\partial f_2}{\partial y_{t-1}} & \frac{\partial f_2}{\partial \gamma_{t-1}} \\ \frac{\partial f_3}{\partial x_{t-1}} & \frac{\partial f_3}{\partial y_{t-1}} & \frac{\partial f_3}{\partial \gamma_{t-1}} \end{bmatrix}$$

You will have to calculate 9 partial derivatives in total for this example.

Again, if you are a bit rusty on calculating partial derivatives, there are some good tutorials online. We’ll do an example calculation now so you can see how this works.

Let’s calculate the first partial derivative in the upper left corner of the matrix.

$$\frac{\cos\gamma_{t-1} * dt)}{\partial x_{t-1}}$$

$$\frac{\partial(x_{t-1})}{\partial x_{t-1}} + \frac{\partial(v_{t-1}\cos\gamma_{t-1} * dt)}{\partial x_{t-1}} = 1 + 0 = 1$$

So, 1 goes in the upper-left corner of the matrix.

If you calculate all 9 partial derivatives above, you’ll get:

$$A_{t-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which is the identity matrix.

How to Calculate the B Matrix

The B matrix in our running example of a mobile robot is a 3×2 matrix.

The B matrix has the **same number of rows as the number of states and has the same number of columns as the number of control inputs**.

The control inputs in this example are the linear velocity (v) and the angular velocity around the z axis, ω (also known as the "yaw rate").

The B matrix expresses **how the state of the system (i.e. [x,y,γ]) changes from t-1 to t due to the control commands (i.e. control inputs v and ω)**

Since we're dealing with a robotic car here, we know that if we apply forward and angular velocity commands to the car, the car will move.

The equation for B is as follows. We need to calculate yet another Jacobian matrix. However, unlike our calculation for the A matrix, we need to compute the partial derivative of the state vector at time t with respect to the control inputs at time t-1.

$$B_{t-1} = \begin{bmatrix} \frac{\partial f_1}{\partial v_{t-1}} & \frac{\partial f_1}{\omega_{t-1}} \\ \frac{\partial f_2}{\partial v_{t-1}} & \frac{\partial f_2}{\omega_{t-1}} \\ \frac{\partial f_3}{\partial v_{t-1}} & \frac{\partial f_3}{\omega_{t-1}} \end{bmatrix}$$

Remember the equations for f_1, f_2, and f_3:

$$\begin{bmatrix} x_t \\ y_t \\ \gamma_t \end{bmatrix} = \begin{bmatrix} x_{t-1} + v_{t-1}\cos\gamma_{t-1} * dt \\ y_{t-1} + v_{t-1}\sin\gamma_{t-1} * dt \\ \gamma_{t-1} + \omega_{t-1} * dt \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

If you calculate the 6 partial derivatives, you'll get this for the B matrix below:

Putting It All Together

Ok, so how do we get from this:

$$\begin{bmatrix} x_t \\ y_t \\ \gamma_t \end{bmatrix} = \begin{bmatrix} x_{t-1} + v_{t-1}\cos\gamma_{t-1} * dt \\ y_{t-1} + v_{t-1}\sin\gamma_{t-1} * dt \\ \gamma_{t-1} + \omega_{t-1} * dt \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

Into this form:

$$\begin{bmatrix} x_t \\ y_t \\ \gamma_t \end{bmatrix} = A_{t-1} \begin{bmatrix} x_{t-1} \\ y_{t-1} \\ \gamma_{t-1} \end{bmatrix} + B_{t-1} \begin{bmatrix} v_{t-1} \\ \omega_{t-1} \end{bmatrix}$$

We now know the A and B matrices, so we can plug those in. The final state space model for our differential drive robot is as follows:

$$\begin{bmatrix} x_t \\ y_t \\ \gamma_t \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{t-1} \\ y_{t-1} \\ \gamma_{t-1} \end{bmatrix} + \begin{bmatrix} \cos\gamma_{t-1} * dt & 0 \\ \sin\gamma_{t-1} * dt & 0 \\ 0 & dt \end{bmatrix} \begin{bmatrix} v_{t-1} \\ \omega_{t-1} \end{bmatrix}$$

where v_{t-1} is the linear velocity of the robot in the robot's reference frame, and ω_{t-1} is the angular velocity in the robot's reference frame.

And there you have it. If you know the current position of the robot (x,y), the orientation of the robot (**yaw angle γ**), the linear velocity of the robot, the angular velocity of the robot, and the change in time from one timestep to the next, you can calculate the state of the robot at the next timestep.

Adding Process Noise

The world isn't perfect. Sometimes the robot might not act the way you would expect when you want it to execute a specific velocity command.

It is often common to add a noise term to the state space model to account for randomness in the world. The equation would then be:

$$\begin{bmatrix} x_t \\ y_t \\ \gamma_t \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{t-1} \\ y_{t-1} \\ \gamma_{t-1} \end{bmatrix} + \begin{bmatrix} \cos\gamma_{t-1} * dt & 0 \\ \sin\gamma_{t-1} * dt & 0 \\ 0 & dt \end{bmatrix} \begin{bmatrix} v_{t-1} \\ \omega_{t-1} \end{bmatrix} + \begin{bmatrix} noise_{t-1} \\ noise_{t-1} \\ noise_{t-1} \end{bmatrix}$$

Python Code Example for the State Space Model

ped in this tutorial. We will assume:

ar velocity of 0.05 radians per second.

The output will show the state of the robot at the next timestep, time t. I'll name the file **state_space_model.py**.

Make sure you have **NumPy** installed before you run the code. NumPy is a scientific computing library for Python.

If you're using Anaconda, you can type:

```
conda install numpy
```

Alternatively, you can type:

```
pip install numpy
```

Here is the code. You can copy and paste this code into your favorite IDE and then run it.

```python
import numpy as np

# Author: Addison Sears-Collins
# https://automaticaddison.com
# Description: A state space model for a differential drive mobile robot

# A matrix
# 3x3 matrix -> number of states x number of states matrix
# Expresses how the state of the system [x,y,yaw] changes
# from t-1 to t when no control command is executed.
# Typically a robot on wheels only drives when the wheels are commanded
# to turn.
# For this case, A is the identity matrix.
# A is sometimes F in the literature.
A_t_minus_1 = np.array([[1.0,  0,   0],
                        [  0,1.0,   0],
                        [  0,  0, 1.0]])

# The estimated state vector at time t-1 in the global
# reference frame
# [x_t_minus_1, y_t_minus_1, yaw_t_minus_1]
# [meters, meters, radians]
state_estimate_t_minus_1 = np.array([0.0,0.0,0.0])

# The control input vector at time t-1 in the global
# reference frame
# [v, yaw_rate]
# [meters/second, radians/second]
# In the literature, this is commonly u.
control_vector_t_minus_1 = np.array([4.5, 0.05])

# Noise applied to the forward kinematics (calculation
# of the estimated state at time t from the state
# transition model of the mobile robot). This is a vector
# with the number of elements equal to the number of states
process_noise_v_t_minus_1 = np.array([0.01,0.01,0.003])

yaw_angle = 0.0 # radians
delta_t = 1.0 # seconds

def getB(yaw,dt):
  """
  Calculates and returns the B matrix
  3x2 matix -> number of states x number of control inputs
  The control inputs are the forward speed and the
  rotation rate around the z axis from the x-axis in the
  counterclockwise direction.
  [v, yaw_rate]
  Expresses how the state of the system [x,y,yaw] changes
  from t-1 to t due to the control commands (i.e. control
  input).
  :param yaw: The yaw (rotation angle around the z axis) in rad
  :param dt: The change in time from time step t-1 to t in sec
    """
  B = np.array([[np.cos(yaw)*dt, 0],
                [np.sin(yaw)*dt, 0],
                [0, dt]])
  return B
```

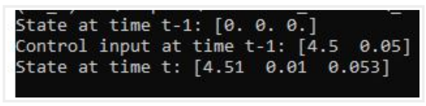
```
  print(f'State at time t-1: {state_estimate_t_minus_1}')
  print(f'Control input at time t-1: {control_vector_t_minus_1}')
  print(f'State at time t: {state_estimate_t}') # State after delta_t seconds

main()
```

Run the code:

```
python state_space_model.py
```

Here is the output:



```
State at time t-1: [0. 0. 0.]
Control input at time t-1: [4.5  0.05]
State at time t: [4.51  0.01  0.053]
```

And that’s it for the fundamentals of state space modeling. Once you know the physics of how a robotic system moves from one timestep to the next, you can create a mathematical model in state space form.

Keep building!

automaticaddison / December 10, 2020 / Robotics / ground, python