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## YOU'VE GOT A PROBLEM

Suppose that a bungee-jumping company hires you. You're given the task of predicting the velocity of a jumper (Fig. 1.1) as a function of time during the free-fall part of the jump. This information will be used as part of a larger analysis to determine the length and required strength of the bungee cord for jumpers of different mass.

You know from your studies of physics that the acceleration should be equal to the ratio of the force to the mass (Newton's second law). Based on this insight and your knowledge

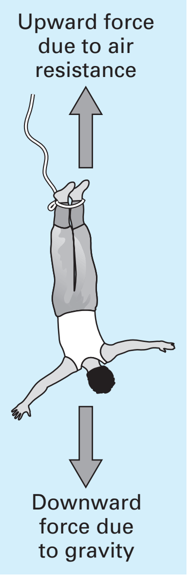



**FIGURE 1.1**
Forces acting on a free-falling bungee jumper.

of physics and fluid mechanics, you develop the following mathematical model for the rate of change of velocity with respect to time,

$$\frac{dv}{dt} = g - \frac{c_d}{m} v^2$$

where $v$ = downward vertical velocity (m/s), $t$ = time (s), $g$ = the acceleration due to gravity ($\cong 9.81\ \text{m/s}^2$), $c_d$ = a lumped drag coefficient (kg/m), and $m$ = the jumper's mass (kg). The drag coefficient is called "lumped" because its magnitude depends on factors such as the jumper's area and the fluid density (see Sec. 1.4).

Because this is a differential equation, you know that calculus might be used to obtain an analytical or exact solution for $v$ as a function of $t$. However, in the following pages, we will illustrate an alternative solution approach. This will involve developing a computer-oriented numerical or approximate solution.

Aside from showing you how the computer can be used to solve this particular problem, our more general objective will be to illustrate (*a*) what numerical methods are and (*b*) how they figure in engineering and scientific problem solving. In so doing, we will also show how mathematical models figure prominently in the way engineers and scientists use numerical methods in their work.

## 1.1 A SIMPLE MATHEMATICAL MODEL

A *mathematical model* can be broadly defined as a formulation or equation that expresses the essential features of a physical system or process in mathematical terms. In a very general sense, it can be represented as a functional relationship of the form

$$\begin{matrix}\text{Dependent} \\ \text{variable}\end{matrix} = f\left(\begin{matrix}\text{independent} \\ \text{variables}\end{matrix}, \text{parameters}, \begin{matrix}\text{forcing} \\ \text{functions}\end{matrix}\right) \tag{1.1}$$

where the *dependent variable* is a characteristic that typically reflects the behavior or state of the system; the *independent variables* are usually dimensions, such as time and space, along which the system's behavior is being determined; the *parameters* are reflective of the system's properties or composition; and the *forcing functions* are external influences acting upon it.

The actual mathematical expression of Eq. (1.1) can range from a simple algebraic relationship to large complicated sets of differential equations. For example, on the basis of his observations, Newton formulated his second law of motion, which states that the time rate of change of momentum of a body is equal to the resultant force acting on it. The mathematical expression, or model, of the second law is the well-known equation

$$F = ma \tag{1.2}$$

where $F$ is the net force acting on the body (N, or kg m/s$^2$), $m$ is the mass of the object (kg), and $a$ is its acceleration (m/s$^2$).

The second law can be recast in the format of Eq. (1.1) by merely dividing both sides by $m$ to give

$$a = \frac{F}{m} \tag{1.3}$$

where $a$ is the dependent variable reflecting the system's behavior, $F$ is the forcing function, and $m$ is a parameter. Note that for this simple case there is no independent variable because we are not yet predicting how acceleration varies in time or space.

Equation (1.3) has a number of characteristics that are typical of mathematical models of the physical world.

- It describes a natural process or system in mathematical terms.
- It represents an idealization and simplification of reality. That is, the model ignores negligible details of the natural process and focuses on its essential manifestations. Thus, the second law does not include the effects of relativity that are of minimal importance when applied to objects and forces that interact on or about the earth's surface at velocities and on scales visible to humans.
- Finally, it yields reproducible results and, consequently, can be used for predictive purposes. For example, if the force on an object and its mass are known, Eq. (1.3) can be used to compute acceleration.

Because of its simple algebraic form, the solution of Eq. (1.2) was obtained easily. However, other mathematical models of physical phenomena may be much more complex, and either cannot be solved exactly or require more sophisticated mathematical techniques than simple algebra for their solution. To illustrate a more complex model of this kind, Newton's second law can be used to determine the terminal velocity of a free-falling body near the earth's surface. Our falling body will be a bungee jumper (Fig. 1.1). For this case, a model can be derived by expressing the acceleration as the time rate of change of the velocity ($dv/dt$) and substituting it into Eq. (1.3) to yield

$$\frac{dv}{dt} = \frac{F}{m} \tag{1.4}$$

where $v$ is velocity (in meters per second). Thus, the rate of change of the velocity is equal to the net force acting on the body normalized to its mass. If the net force is positive, the object will accelerate. If it is negative, the object will decelerate. If the net force is zero, the object's velocity will remain at a constant level.

Next, we will express the net force in terms of measurable variables and parameters. For a body falling within the vicinity of the earth, the net force is composed of two opposing forces: the downward pull of gravity $F_D$ and the upward force of air resistance $F_U$ (Fig. 1.1):

$$F = F_D + F_U \tag{1.5}$$

If force in the downward direction is assigned a positive sign, the second law can be used to formulate the force due to gravity as

$$F_D = mg \tag{1.6}$$

where $g$ is the acceleration due to gravity (9.81 m/s$^2$).

Air resistance can be formulated in a variety of ways. Knowledge from the science of fluid mechanics suggests that a good first approximation would be to assume that it is proportional to the square of the velocity,

$$F_U = -c_d v^2 \tag{1.7}$$

where $c_d$ is a proportionality constant called the *lumped drag coefficient* (kg/m). Thus, the greater the fall velocity, the greater the upward force due to air resistance. The parameter $c_d$ accounts for properties of the falling object, such as shape or surface roughness, that affect air resistance. For the present case, $c_d$ might be a function of the type of clothing or the orientation used by the jumper during free fall.

The net force is the difference between the downward and upward force. Therefore, Eqs. (1.4) through (1.7) can be combined to yield

$$\frac{dv}{dt} = g - \frac{c_d}{m} v^2 \tag{1.8}$$

Equation (1.8) is a model that relates the acceleration of a falling object to the forces acting on it. It is a *differential equation* because it is written in terms of the differential rate of change $(dv/dt)$ of the variable that we are interested in predicting. However, in contrast to the solution of Newton's second law in Eq. (1.3), the exact solution of Eq. (1.8) for the velocity of the jumper cannot be obtained using simple algebraic manipulation. Rather, more advanced techniques such as those of calculus must be applied to obtain an exact or analytical solution. For example, if the jumper is initially at rest ($v = 0$ at $t = 0$), calculus can be used to solve Eq. (1.8) for

$$v(t) = \sqrt{\frac{gm}{c_d}} \tanh\left(\sqrt{\frac{gc_d}{m}}\,t\right) \tag{1.9}$$

where tanh is the hyperbolic tangent that can be either computed directly[1] or via the more elementary exponential function as in

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} \tag{1.10}$$

Note that Eq. (1.9) is cast in the general form of Eq. (1.1) where $v(t)$ is the dependent variable, $t$ is the independent variable, $c_d$ and $m$ are parameters, and $g$ is the forcing function.

## EXAMPLE 1.1 Analytical Solution to the Bungee Jumper Problem

**Problem Statement.** A bungee jumper with a mass of 68.1 kg leaps from a stationary hot air balloon. Use Eq. (1.9) to compute velocity for the first 12 s of free fall. Also determine the terminal velocity that will be attained for an infinitely long cord (or alternatively, the jumpmaster is having a particularly bad day!). Use a drag coefficient of 0.25 kg/m.

[1] MATLAB allows direct calculation of the hyperbolic tangent via the built-in function `tanh(x)`.

~~**Solution.** Inserting the parameters into Eq. (1.9) yields~~

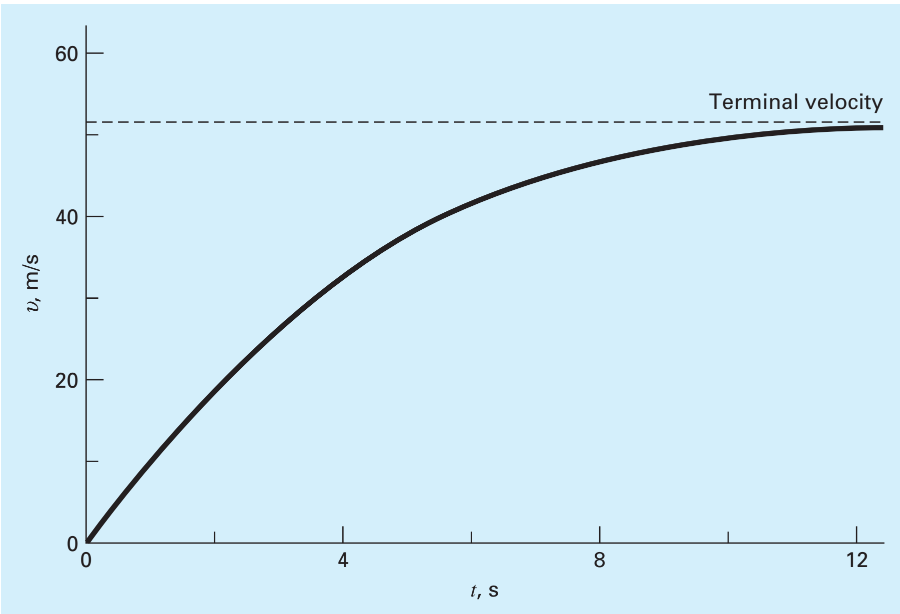
$$v(t) = \sqrt{\frac{9.81(68.1)}{0.25}} \tanh\left(\sqrt{\frac{9.81(0.25)}{68.1}}\, t\right) = 51.6938 \tanh(0.18977t)$$

which can be used to compute

| $t$, s | $v$, m/s |
|---|---|
| 0 | 0 |
| 2 | 18.7292 |
| 4 | 33.1118 |
| 6 | 42.0762 |
| 8 | 46.9575 |
| 10 | 49.4214 |
| 12 | 50.6175 |
| $\infty$ | 51.6938 |

According to the model, the jumper accelerates rapidly (Fig. 1.2). A velocity of 49.4214 m/s (about 110 mi/hr) is attained after 10 s. Note also that after a sufficiently long

**FIGURE 1.2**
The analytical solution for the bungee jumper problem as computed in Example 1.1. Velocity increases with time and asymptotically approaches a terminal velocity.



time, a constant velocity, called the *terminal velocity,* of 51.6983 m/s (115.6 mi/hr) is reached. This velocity is constant because, eventually, the force of gravity will be in balance with the air resistance. Thus, the net force is zero and acceleration has ceased.

Equation (1.9) is called an *analytical* or *closed-form solution* because it exactly satisfies the original differential equation. Unfortunately, there are many mathematical models that cannot be solved exactly. In many of these cases, the only alternative is to develop a numerical solution that approximates the exact solution.

*Numerical methods* are those in which the mathematical problem is reformulated so it can be solved by arithmetic operations. This can be illustrated for Eq. (1.8) by realizing that the time rate of change of velocity can be approximated by (Fig. 1.3):

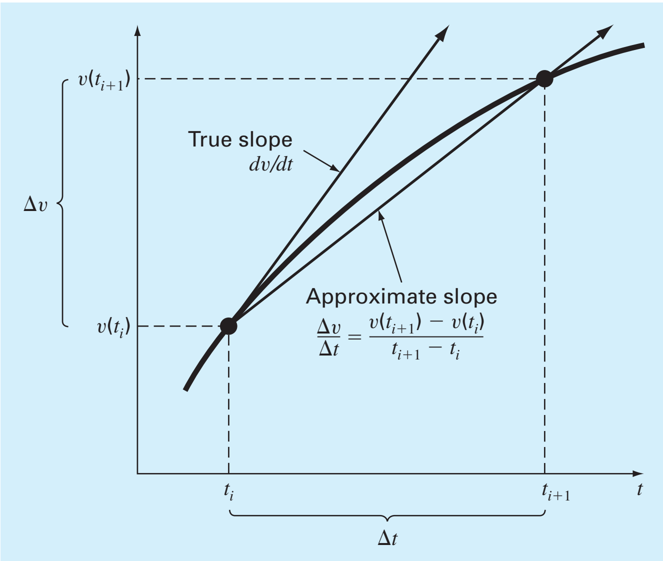
$$\frac{dv}{dt} \cong \frac{\Delta v}{\Delta t} = \frac{v(t_{i+1}) - v(t_i)}{t_{i+1} - t_i} \tag{1.11}$$

where $\Delta v$ and $\Delta t$ are differences in velocity and time computed over finite intervals, $v(t_i)$ is velocity at an initial time $t_i$, and $v(t_{i+1})$ is velocity at some later time $t_{i+1}$. Note that $dv/dt \cong \Delta v/\Delta t$ is approximate because $\Delta t$ is finite. Remember from calculus that

$$\frac{dv}{dt} = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t}$$

Equation (1.11) represents the reverse process.

**FIGURE 1.3**
The use of a finite difference to approximate the first derivative of $v$ with respect to $t$.



Equation (1.11) is called a *finite-difference approximation* of the derivative at time $t_i$. It can be substituted into Eq. (1.8) to give

$$\frac{v(t_{i+1}) - v(t_i)}{t_{i+1} - t_i} = g - \frac{c_d}{m} v(t_i)^2$$

This equation can then be rearranged to yield

$$v(t_{i+1}) = v(t_i) + \left[ g - \frac{c_d}{m} v(t_i)^2 \right] (t_{i+1} - t_i) \tag{1.12}$$

Notice that the term in brackets is the right-hand side of the differential equation itself [Eq. (1.8)]. That is, it provides a means to compute the rate of change or slope of $v$. Thus, the equation can be rewritten more concisely as

$$v_{i+1} = v_i + \frac{dv_i}{dt} \Delta t \tag{1.13}$$

where the nomenclature $v_i$ designates velocity at time $t_i$ and $\Delta t = t_{i+1} - t_i$.

We can now see that the differential equation has been transformed into an equation that can be used to determine the velocity algebraically at $t_{i+1}$ using the slope and previous values of $v$ and $t$. If you are given an initial value for velocity at some time $t_i$, you can easily compute velocity at a later time $t_{i+1}$. This new value of velocity at $t_{i+1}$ can in turn be employed to extend the computation to velocity at $t_{i+2}$ and so on. Thus at any time along the way,

New value = old value + slope × step size

This approach is formally called *Euler's method*. We'll discuss it in more detail when we turn to differential equations later in this book.

## EXAMPLE 1.2 Numerical Solution to the Bungee Jumper Problem

**Problem Statement.** Perform the same computation as in Example 1.1 but use Eq. (1.12) to compute velocity with Euler's method. Employ a step size of 2 s for the calculation.

**Solution.** At the start of the computation ($t_0 = 0$), the velocity of the jumper is zero. Using this information and the parameter values from Example 1.1, Eq. (1.12) can be used to compute velocity at $t_1 = 2$ s:

$$v = 0 + \left[ 9.81 - \frac{0.25}{68.1}(0)^2 \right] \times 2 = 19.62 \text{ m/s}$$

For the next interval (from $t = 2$ to 4 s), the computation is repeated, with the result

$$v = 19.62 + \left[ 9.81 - \frac{0.25}{68.1}(19.62)^2 \right] \times 2 = 36.4137 \text{ m/s}$$

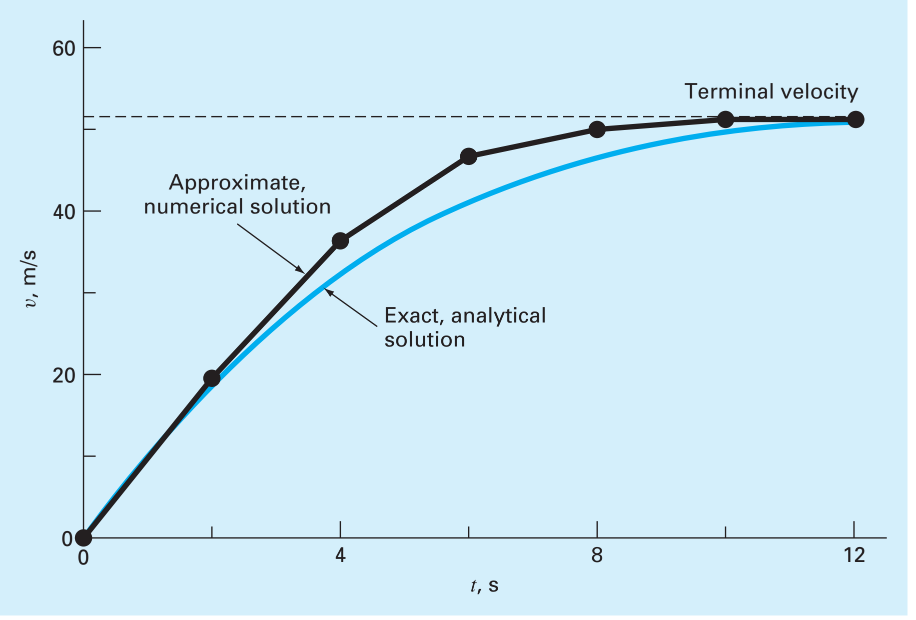



**FIGURE 1.4**
Comparison of the numerical and analytical solutions for the bungee jumper problem.

The calculation is continued in a similar fashion to obtain additional values:

| $t$, s | $v$, m/s |
|---|---|
| 0 | 0 |
| 2 | 19.6200 |
| 4 | 36.4137 |
| 6 | 46.2983 |
| 8 | 50.1802 |
| 10 | 51.3123 |
| 12 | 51.6008 |
| $\infty$ | 51.6938 |

The results are plotted in Fig. 1.4 along with the exact solution. We can see that the numerical method captures the essential features of the exact solution. However, because we have employed straight-line segments to approximate a continuously curving function, there is some discrepancy between the two results. One way to minimize such discrepancies is to use a smaller step size. For example, applying Eq. (1.12) at 1-s intervals results in a smaller error, as the straight-line segments track closer to the true solution. Using hand calculations, the effort associated with using smaller and smaller step sizes would make such numerical solutions impractical. However, with the aid of the computer, large numbers of calculations can be performed easily. Thus, you can accurately model the velocity of the jumper without having to solve the differential equation exactly.