More problems in Gauss Jacobi and Gauss Seidel method

1. Solve the system of equations

$4x_1 + x_2 + x_3 = 2$

$x_1 + 5x_2 + 2x_3 = -6$

$x_1 + 2x_2 + 3x_3 = -4$ using the Jacobi iteration method.

Use the initial approximations as

(i) $x_i = 0, i = 1, 2, 3,$

(ii) $x_1 = 0.5, x_2 = -0.5, x_3 = -0.5.$

Perform five iterations in each case.

Solution Note that the given system is diagonally dominant. Jacobi method gives the iterations

$x_1^{(k+1)} = 0.25\ [2 - (x_2^{(k)} + x_3^{(k)})]$

$x_2^{(k+1)} = 0.2\ [-6 - (x_1^{(k)} + 2x_3^{(k)})]$

$x_3^{(k+1)} = 0.33333\ [-4 - (x_1^{(k)} + 2x_2^{(k)})], k = 0, 1, ...$

We have the following results.

(i) $x_1(0) = 0, x_2(0) = 0, x_3(0) = 0.$

First iteration

$x_1^{(1)} = 0.25\ [2 - (x_2^{(0)} + x_3^{(0)})] = 0.5,$

$x_2^{(1)} = 0.2\ [-6 - (x_1^{(0)} + 2x_3^{(0)})] = -1.2,$

$x_3^{(1)} = 0.33333\ [-4 - (x_1^{(0)} + 2x_2^{(0)})] = -1.33333.$

Second iteration

$x_1^{(2)} = 0.25\ [2 - (x_2^{(1)} + x_3^{(1)})]$

$= 0.25\ [2 - (-1.2 - 1.33333)] = 1.13333,$

$x_2^{(2)} = 0.2\ [-6 - (x_1^{(1)} + 2x_3^{(1)})]$

$= 0.2\ [-6 - (0.5 + 2(-1.33333))] = -0.76668,$

$x_3^{(2)} = 0.33333\ [-4 - (x_1^{(1)} + 2x_2^{(1)})]$

$= 0.33333\ [-4 - (0.5 + 2(-1.2))] = -0.7.$

Third iteration

$x_1^{(3)} = 0.25\ [2 - (x_2^{(2)} + x_3^{(2)})]$

$= 0.25\ [2 - (-0.76668 - 0.7)] = 0.86667,$

$x_2^{(3)} = 0.2\ [-6 - (x_1^{(2)} + 2x_3^{(2)})]$

$= 0.2\ [-6 - (1.13333 + 2(-0.7))] = -1.14667,$

$x_3^{(3)} = 0.33333\ [-4 - (x_1^{(2)} + 2x_2^{(2)})]$

$= 0.33333\ [-4 - (1.13333 + 2(-0.76668))] = -1.19998.$

Fourth iteration

$x_1^{(4)} = 0.25\ [2 - (x_2^{(3)} + x_3^{(3)})]$

$= 0.25\ [2 - (-1.14667 - 1.19999)] = 1.08666,$

$x_2^{(4)} = 0.2\ [-6 - (x_1^{(3)} + 2x_3^{(3)})]$

$= 0.2\ [-6 - (0.86667 + 2(-1.19998))] = -0.89334,$

$x_3^{(4)} = 0.33333\ [-4 - (x_1^{(3)} + 2x_2^{(3)})]$

$= 0.33333\ [-4 - (0.86667 + 2(-1.14667))] = -0.85777.$

Fifth iteration

$x_1^{(5)} = 0.25\ [2 - (x_2^{(4)} + x_3^{(4)})]$

$= 0.25\ [2 - (-0.89334 - 0.85777)] = 0.93778,$

$x_2^{(5)} = 0.2\ [-6 - (x_1^{(4)} + 2x_3^{(4)})]$

$= 0.2\ [-6 - (1.08666 + 2(-0.85777))] = -1.07422,$

$x_3^{(5)} = 0.33333\ [-4 - (x_1^{(4)} + 2x_2^{(4)})]$

$= 0.33333\ [-4 - (1.08666 + 2(-0.89334))] = -1.09998.$

It is interesting to note that the iterations oscillate and converge to the exact solution

$x_1 = 1.0$, $x_2 = -1$, $x_3 = -1.0$.

(*ii*) $x_1^{(0)} = 0.5$, $x_2^{(0)} = -0.5$, $x_3^{(0)} = -0.5$.

First iteration

$x_1^{(1)} = 0.25\,[2 - (x_2^{(0)} + x_3^{(0)})] = 0.25\,[2 - (-0.5 - 0.5)] = 0.75,$

$x_2^{(1)} = 0.2\,[-6 - (x_1^{(0)} + 2x_3^{(0)})] = 0.2\,[-6 - (0.5 + 2(-0.5))] = -1.1,$

$x_3^{(1)} = 0.33333\,[-4 - (x_1^{(0)} + 2x_2^{(0)})]$

$= 0.33333\,[-4 - (0.5 + 2(-0.5))] = -1.16667.$

Second iteration

$x_1^{(2)} = 0.25\,[2 - (x_2^{(1)} + x_3^{(1)})]$

$= 0.25\,[2 - (-1.1 - 1.16667)] = 1.06667,$

$x_2^{(2)} = 0.2\,[-6 - (x_1^{(1)} + 2x_3^{(1)})]$

$= 0.2\,[-6 - (0.75 + 2(-1.16667))] = -0.88333,$

$x_3^{(2)} = 0.33333\,[-4 - (x_1^{(1)} + 2x_2^{(1)})]$

$= 0.33333\,[-4 - (0.75 + 2(-1.1))] = -0.84999.$

Third iteration

$x_1^{(3)} = 0.25\,[2 - (x_2^{(2)} + x_3^{(2)})]$

$= 0.25\,[2 - (-0.88333 - 0.84999)] = 0.93333,$

$x_2^{(3)} = 0.2\,[-6 - (x_1^{(2)} + 2x_3^{(2)})]$

$= 0.2\,[-6 - (1.06667 + 2(-0.84999))] = -1.07334,$

$x_3^{(3)} = 0.33333\,[-4 - (x_1^{(2)} + 2x_2^{(2)})]$

$= 0.33333\,[-4 - (1.06667 + 2(-0.88333))] = -1.09999.$

Fourth iteration

$x_1^{(4)} = 0.25\ [2 – (x_2^{(3)} + x_3^{(3)})]$

$= 0.25\ [2 – (– 1.07334 – 1.09999)] = 1.04333,$

$x_2^{(4)} = 0.2\ [– 6 – (x_1^{(3)} + 2x_3^{(3)})]$

$= 0.2\ [– 6 – (0.93333 + 2(– 1.09999))] = – 0.94667,$

$x_3^{(4)} = 0.33333\ [– 4 – (x_1^{(3)} + 2x_2^{(3)})]$

$= 0.33333\ [– 4 – (0.93333 + 2(– 1.07334))] = – 0.92887.$

Fifth iteration

$x_1^{(5)} = 0.25\ [2 – (x_2^{(4)} + x_3^{(4)})]$

$= 0.25\ [2 – (– 0.94667 – 0.92887)] = 0.96889,$

$x_2^{(5)} = 0.2\ [– 6 – (x_1^{(4)} + 2x_3^{(4)})]$

$= 0.2\ [– 6 – (1.04333 + 2(– 0.92887))] = – 1.03712,$

$x_3^{(5)} = 0.33333\ [– 4 – (x_1^{(4)} + 2x_2^{(4)})]$

$= 0.33333\ [– 4 – (1.04333 + 2(– 0.94667))] = – 1.04999.$

How do we find the initial approximations to start the iteration? If the system is diagonally dominant, then the iteration converges for any initial solution vector. If no suitable approximation is available, we can choose **x = 0,** that is $x_i = 0$ for all i. Then, the initial approximation becomes $x_i = b_i / a_{ii}$, for all i.

What is the disadvantage of the Gauss-Jacobi method? At any iteration step, the value of the first variable x_1 is obtained using the values of the previous iteration. The value of the second variable x_2 is also obtained using the values of the previous iteration, even though the updated value of x_1 is available. In general, at every stage in the iteration, values of the previous iteration are used even though the updated values of the previous variables are available. If we use the updated values of $x_1, x_2, \ldots, x_{i-1}$ in computing the value of the variable x_i, then we obtain a new method called Gauss-Seidel iteration method.

1. Find the solution of the system of equations

$$45x_1 + 2x_2 + 3x_3 = 58$$
$$-3x_1 + 22x_2 + 2x_3 = 47$$
$$5x_1 + x_2 + 20x_3 = 67$$

correct to three decimal places, using the Gauss-Seidel iteration method.

Solution The given system of equations is strongly diagonally dominant. Hence, we can expect fast convergence. Gauss-Seidel method gives the iteration

$$x_1^{(k+1)} = (1/45) * (58 - 2x_2^{(k)} - 3x_3^{(k)})$$
$$x_2^{(k+1)} = (1/22) * (47 + 3x_1^{(k+1)} - 2x_3^{(k)}),$$
$$x_3^{(k+1)} = (1/20) * (67 - 5x_1^{(k+1)} - x_2^{(k+1)}).$$

Starting with $x_1^{(0)} = 0$, $x_2^{(0)} = 0$, $x_3^{(0)} = 0$, we get the following results.

First iteration

$$x_1^{(1)} = (1/45) * (58 - 2x_2^{(0)} - 3x_3^{(0)}) = (1/45) * (58) = 1.28889,$$
$$x_2^{(1)} = (1/22) * (47 + 3x_1^{(1)} - 2x_3^{(0)}) = (1/22) * (47 + 3(1.28889) - 2(0))$$
$$= 2.31212,$$

$x_3^{(1)} = (1/20) * (67 – 5x_1^{(1)} – x_2^{(1)})$

$= (1/20) * (67 – 5(1.28889) – (2.31212)) = 2.91217.$

Second iteration

$x_1^{(2)} = (1/45) * (58 – 2x_2^{(1)} – 3x_3^{(1)})$

$= (1/45) * (58 – 2(2.31212) – 3(2.91217)) = 0.99198,$

$x_2^{(2)} = (1/22) * (47 + 3x_1^{(2)} – 2x_3^{(1)})$

$= 122(47 + 3(0.99198) – 2(2.91217)) = 2.00689,$

$x_3^{(2)} = (1/20) * (67 – 5x_1^{(2)} – x_2^{(2)})$

$= (1/20) * (67 – 5(0.99198) – (2.00689)) = 3.00166.$

Third iteration

$x_1^{(3)} = (1/45) * (58 – 2x_2^{(2)} – 3x_3^{(2)})$

$= (1/45) * (58 – 2(2.00689) – 3(3.00166) = 0.99958,$

$x_2^{(3)} = (1/22) * (47 + 3x_1^{(3)} – 2x_3^{(2)})$

$= (1/22) * (47 + 3(0.99958) – 2(3.00166)) = 1.99979,$

$x_3^{(3)} = (1/20) * (67 – 5x_1^{(3)} – x_2^{(3)})$

$= (1/20) * (67 – 5(0.99958) – (1.99979)) = 3.00012.$

Fourth iteration

$x_1^{(4)} = (1/45) * (58 – 2x_2^{(3)} – 3x_3^{(3)})$

$= (1/45) * (58 – 2(1.99979) – 3(3.00012)) = 1.00000,$

$x_2^{(4)} = (1/22) * (47 + 3x_1^{(4)} – 2x_3^{(3)})$

$= (1/22) * (47 + 3(1.00000) – 2(3.00012)) = 1.99999,$

$x_3^{(4)} = (1/20) * (67 – 5x_1^{(4)} – x_2^{(4)})$

$= (1/20) * (67 – 5(1.00000) – (1.99999)) = 3.00000.$

Rounding to three decimal places, we get $x_1 = 1.0$, $x_2 = 2.0$, $x_3 = 3.0$.

Example 1.23 Computationally show that Gauss-Seidel method applied to the system of equations

$3x_1 - 6x_2 + 2x_3 = 23$

$-4x_1 + x_2 - x_3 = -8$

$x_1 - 3x_2 + 7x_3 = 17$

diverges. Take the initial approximations as $x_1 = 0.9$, $x_2 = -3.1$, $x_3 = 0.9$. Interchange the first and second equations and solve the resulting system by the Gauss-Seidel method. Again take the initial approximations as $x_1 = 0.9$, $x_2 = -3.1$, $x_3 = 0.9$, and obtain the result correct to two decimal places. The exact solution is $x_1 = 1.0$, $x_2 = -3.0$, $x_3 = 1.0$.

Solution Note that the system of equations is not diagonally dominant. Gauss-Seidel method gives the iteration

$x_1^{(k+1)} = [23 + 6x_2^{(k)} - 2x_3^{(k)})]/3$

$x_2^{(k+1)} = [-8 + 4x_1^{(k+1)} + x_3^{(k)}]$

$x_3^{(k+1)} = [17 - x_1^{(k+1)} + 3x_2^{(k+1)}]/7.$

Starting with the initial approximations $x_1 = 0.9$, $x_2 = -3.1$, $x_3 = 0.9$, we obtain the following results.

First iteration

$x_1^{(1)} = (1/3) * [23 + 6x_2^{(0)} - 2x_3^{(0)}] = (1/3) * [23 + 6(-3.1) - 2(0.9)]$

$= 0.8667,$

$x_2^{(1)} = [-8 + 4x_1^{(1)} + x_3^{(0)}] = [-8 + 4(0.8667) + 0.9] = -3.6332,$

$x_3^{(1)} = (1/7) * [17 - x_1^{(1)} + 3x_2^{(1)}] = 17[17 - (0.8667) + 3(-3.6332)]$

$= 0.7477.$

Second iteration

$x_1^{(2)} = (1/3) * [23 + 6x_2^{(1)} - 2x_3^{(1)}] = 13[23 + 6(-3.6332) - 2(0.7477)]$

$= -0.0982,$

$x_2^{(2)} = [-8 + 4x_1^{(2)} + x_3^{(1)}] = [-8 + 4(-0.0982) + 0.7477]$

$= -7.6451,$

$x_3^{(2)} = (1/7) * [17 - x_1^{(2)} + 3x_2^{(2)}] = (1/7) * [17 + 0.0982 + 3(-7.6451)]$

$= -0.8339.$

Third iteration

$x_1^{(3)} = (1/3) * [23 + 6x_2^{(2)} - 2x_3^{(2)}]$

$= (1/3) * [23 + 6(-7.6451) - 2(-0.8339)] = -7.0676,$

$x_2^{(3)} = [-8 + 4x_1^{(3)} + x_3^{(2)}]$

$= [-8 + 4(-7.0676) - 0.8339] = -37.1043,$

$x_3^{(3)} = (1/7) * [17 - x_1^{(3)} + 3x_2^{(3)}]$

$= (1/7) * [17 + 7.0676 + 3(-37.1043)] = -12.4636.$

It can be observed that the iterations are diverging very fast.

Now, we exchange the first and second equations to obtain the system

$-4x_1 + x_2 - x_3 = -8$

$3x_1 - 6x_2 + 2x_3 = 23$

$x_1 - 3x_2 + 7x_3 = 17.$

The system of equations is now diagonally dominant. Gauss-Seidel method gives iteration

$x_1^{(k+1)} = [8 + x_2^{(k)} - x_3^{(k)}]/4$

$x_2^{(k+1)} = -[23 - 3x_1^{(k+1)} - 2x_3^{(k)}]/6$

$x_3^{(k+1)} = [17 - x_1^{(k+1)} + 3x_2^{(k+1)}]/7.$

Starting with the initial approximations $x_1 = 0.9$, $x_2 = -3.1$, $x_3 = 0.9$, we obtain the following results.

First iteration

$x_1^{(1)} = (1/4) * [8 + x_2^{(0)} - x_3^{(0)}] = (1/4) * [8 - 3.1 - 0.9] = 1.0,$

$x_2^{(1)} = -(1/6) * [23 - 3x_1^{(1)} - 2x_3^{(0)}] = -(1/6) * [23 - 3(1.0) - 2(0.9)]$

$= -3.0333,$

$x_3^{(1)} = (1/7) * [17 - x_1^{(1)} + 3x_2^{(1)}] = (1/7) * [17 - 1.0 + 3(-3.0333)]$

$= 0.9857.$

Second iteration

$x_1^{(2)} = (1/4) * [8 + x_2^{(1)} - x_3^{(1)}] = (1/4) * [8 - 3.0333 - 0.9857]$

$= 0.9953,$

$x_2^{(2)} = -(1/6) * [23 - 3x_1^{(2)} - 2x_3^{(1)}]$

$= -(1/6) * [23 - 3(0.9953) - 2(0.9857)] = -3.0071,$

$x_3^{(2)} = (1/7) * [17 - x_1^{(2)} + 3x_2^{(2)}] = (1/7) * [17 - 0.9953 + 3(-3.0071)]$

$= 0.9976.$

Third iteration

$x_1^{(3)} = (1/4) * [8 + x_2^{(2)} - x_3^{(2)}] =14[8 - 3.0071 - 0.9976] = 0.9988,$

$x_2^{(3)} = -(1/6) * [23 - 3x_1^{(3)} - 2x_3^{(2)}]$

$= -(1/6) * [23 - 3(0.9988) - 2(0.9976)] = -3.0014,$

$x_3^{(3)} = (1/7) * [17 - x_1^{(3)} + 3x_2^{(3)}] = (1/7) * [17 - 0.9988 + 3(-3.0014)]$

$= 0.9996.$

Fourth iteration

$x_1^{(4)} = (1/4) * [8 + x_2^{(3)} - x_3^{(3)}] = (1/4) * [8 - 3.0014 - 0.9996]$

$= 0.9998,$

$x_2^{(4)} = -(1/6) * [23 - 3x_1^{(4)} - 2x_3^{(3)}]$

$= -(1/6) * [23 - 3(0.9998) - 2(0.9996)] = -3.0002,$

$x_3^{(4)} = (1/7) * [17 - x_1^{(4)} + 3x_2^{(4)}] = (1/7) * [17 - 0.9998 + 3(-3.0002)]$

$= 0.9999.$

Since, all the errors in magnitude are less than 0.005, the required solution is

$x_1 = 0.9998$, $x_2 = -3.0002$, $x_3 = 0.9999$.

Rounding to two decimal places, we get $x_1 = 1.0$, $x_2 = -3.0$, $x_3 = 1.0$.