1. Fit a cubic polynomial which takes the following values

x	0	1	2	3
y	1	2	1	10

Sol: Here values of x are equally spaced. h = 1. Since only four values of y are given. Using these four values of y we can compute the differences of order three. Here it is asked to fit a polynomial, so in this problem we can use either Newton's forward or backward difference formula.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	1			
		1		
1	2		-2	
		-1		12
2	1		10	
		9		
3	10			

Here h = 2. Use Newton's forward difference formula

$$y(x) = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \text{..........} \text{ where } u = \frac{x-x_0}{h}$$

$$u = \frac{x-0}{1} = x$$

$$y(x) = 1 + x + \frac{x(x-1)}{2!}(-2) + \frac{x(x-1)(x-2)}{3!}(12)$$

$$= 1 + x - x^2 + x + 2x^3 - 6x^2 + 4x$$

On simplification, we get $y(x) = 2x^3 - 7x^2 + 6x + 1$

Central difference formula for Interpolation

Newton's forward and backward difference Interpolation formula are not applicable to interpolate near the central value. To get more accurate results near the middle value of the table. we wii obtain a more suitable formula which utilizes differences close to the middle value of the table. Such formulae are known as central difference interpolation formulae.

Stirling's central difference formula

$$y(x) = y_0 + \frac{u}{1!}\left[\frac{\Delta y_0 + \Delta y_{-1}}{2}\right] + \frac{u^2}{2!}\Delta^2 y_{-1} + \frac{u(u^2-1^2)}{3!}\left[\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right] + \frac{u^2(u^2-1^2)}{4!}\Delta^4 y_{-2} + ...$$

where u = (x – x_0) / h

Note: To use this formula, we must have (-1 / 2) < u < (1 / 2).

1. Given the following table, find y(35) by using stirling's formula

x	20	30	40	50
y	512	439	346	243

Sol : Let us take x_0 = 30 as the origin

u = (x – x_0) / h = (35 – 30) / 10 = 5 / 10 = 0.5

x	u	y	Δy	Δ^2y	Δ^4y
20	-1	512			
			-73		
30	0	439		-20	
			-93		10
40	1	346		-10	
			-103		
50	2	243			

$$y(x) = y_0 + \frac{u}{1!}\left[\frac{\Delta y_0 + \Delta y_{-1}}{2}\right] + \frac{u^2}{2!}\Delta^2 y_{-1} + \frac{u(u^2-1^2)}{3!}\left[\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right] + \frac{u^2(u^2-1^2)}{4!}\Delta^4 y_{-2} + ...$$

y(35) = 439 + ((0.5) / 2) * (-93 – 73) + (0.25 * -20) / 2 + ((0.5) * (0.25 -1) / 3!) *

(10 + .,)

= 439 – 41.50 – 2.50 = 395

2. Using Stirling's formula, find y(1.22) from the following table

x	1.0	1.1	1.2	1.3	1.4
y	0.84147	0.89121	0.93204	0.96356	0.98545

Sol: Since we require to find the value of y at x = 1.22 which is nearer to the middle value of the table, take the origin at x = 1.2 and h = 0.1

u = (x – x_0) / h = (1.22 – 1.2) / 0.1 = 0.02 / 0.1 = 0.2

x	u	y	Δy	Δ^2y	Δ^4y	Δ^5y
1.0	-2	0.84147				
			0.04974			
1.1	-1	0.89121		-0.00891		
			0.04083		-0.00040	
1.2	0	0.93204		-0.00931		0.00008
			0.03152		-0.00032	
1.3	1	0.96356		-0.00963		
			0.02189			
1.4	2	0.98545				

By striling's formula

$$y(x) = y_0 + \frac{u}{1!}\left[\frac{\Delta y_0 + \Delta y_{-1}}{2}\right] + \frac{u^2}{2!}\Delta^2 y_{-1} + \frac{u(u^2-1^2)}{3!}\left[\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right] + \frac{u^2(u^2-1^2)}{4!}\Delta^4 y_{-2} + \dots\dots\dots$$

y(1.22) = 0.93204 + (0.2) ((0.04083 + 0.03152) / 2) + ((0.2)^2 / 2!) * (-0.00931) +

((0.2) * (0.04 – 1)) / 3! * ((-0.00040 – 0.00032) / 2) +

((0.04) * (0.04 – 1) / 4!) * 0.00008

= 0.93204 + 0.007235 – 0.0001862 + 0.00001152 – 0.000000128

= 0.939100192

Lagrange's formula

In cases, where the values of independent variable are not equally spaced and in cases when the differences of dependent variable are not small, ultimately, we will use Lagrange's Interpolation formula. Let y = f(x) be a function such that f(x) takes the values y_0, y_1, y_2, …., y_n corresponding to x = x_0, x_1, x_2, …., x_n. That is, $y_i = f(x_i)$,

i = 0, 1, 2, …., n

$$y(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1 + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3 + \dots\dots\dots$$

This formula can be applicable to equal interval also.

1.Using Lagrange's interpolation formula, find y(10) from the following table

x:	5	6	9	11
y:	12	13	14	16

Sol: Here the values of x are not equally spaced. To interpolate the value of y, use Lagrange's formula

$$y(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1 +$$
$$\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3 + ...$$

Let $x_0 = 5$; $x_1 = 6$; $x_2 = 9$ and $x_3 = 11$

$y_0 = 12$; $y_1 = 13$; $y_2 = 14$ and $y_3 = 16$

y(10) = [((10 – 6) (10 – 9) (10 – 11)) /((5 – 6) (5 – 9) (5 – 11))] * 12 +

[((10 – 5) (10 – 9) (10 – 11)) /((6 – 5) (6 – 9) (6 – 11))] * 13 +

[((10 – 5) (10 – 6) (10 – 11)) /((9 – 5) (9 – 6) (9 – 11))] * 14 +

[((10 – 5) (10 – 6) (10 – 9)) /((11 – 5) (11 – 6) (11 – 9))] * 16

= [(4 * 1 * -1) / (-1 * -4 * -6)] * 12 + [(5 * 1 * -1) / (1 * -3 * -5)] * 13 +

[(5 * 4 * -1) / (4 * 3 * -2)] * 14 + [(5 * 4 * 1) / (6 * 5 * 2)] * 16

= 14. 6666666

2.Using Lagrange's interpolation formula, find y(10) from the following table

x:	7	8	9	10
y:	3	1	1	9

Sol: Here the values of x are equally spaced. To interpolate the value of y, use Lagrange's formula

$$y(x)=\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0+\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1+$$
$$\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2+\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3+....$$

Let $x_0 = 7$; $x_1 = 8$; $x_2 = 9$ and $x_3 = 10$

$y_0 = 3$; $y_1 = 1$; $y_2 = 1$ and $y_3 = 9$

y(9.5) = [((9.5 – 8) (9.5 – 9) (9.5 – 10)) /((7 – 8) (7 – 9) (7 – 10))] * 3 +

[((9.5 – 7) (9.5 – 9) (9.5 – 10)) /((8 – 7) (8 – 9) (8 – 10))] * 1 +

[((9.5 – 7) (9.5 – 8) (9.5 – 10)) /((9 – 7) (9 – 8) (9 – 10))] * 1 +

[((9.5 – 7) (9.5 – 8) (9.5 – 9)) /((10 – 7) (10 – 8) (10 – 9))] * 9

= [(1.5 * 0.5 * -0.5) / (-1 * -2 * -3)] * 3 + [(2.5 * 0.5 * -0.5) / (1 * -1 * -2)] * 1 +

[(2.5 * 1.5 * -0.5) / (2 * 1 * -2)] * 1 + [(2.5 * 1.5 * 0.5) / (3 * 2 * 1)] * 9

= 3.625

3.Find a polynomial of degree three from

x	-1	0	2	3	
y	-8	3	1	12	and hencefind y(x = 1)

Sol: Here the values of x are not equally spaced. To interpolate the value of y, use Lagrange's formula

$$y(x)=\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0+\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1+$$
$$\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2+\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3+....$$

Let $x_0 = -1$; $x_1 = 0$; $x_2 = 2$ and $x_3 = 3$

$y_0 = -8$; $y_1 = 3$; $y_2 = 1$ and $y_3 = 12$

$$y(x) = [((x – 0) (x – 2) (x – 3)) /((-1 – 0) (-1 – 2) (-1 – 3))] * -8 +$$

$$[((x + 1) (x – 2) (x – 3)) /((0 + 1) (0 – 2) (0 – 3))] * 3 +$$

$$[((x + 1) (x - 0) (x – 3)) /((2 + 1) (2 – 0) (2 – 3))] * 1 +$$

$$[((x + 1) (x – 0) (x – 2)) /((3 + 1) (3 – 0) (3 – 2))] * 12$$

$$= [(x (x^2 – 5x + 6) / (-1 * -3 * -4)] * -8 + [(x + 1) (x^2 – 5x + 6) / (1 * -2 * - 3)] * 3 +$$

$$[(x (x^2 – 2x - 3) / (3 * 2 * -1)] * 1 + [(x (x^2 –x - 2) / (4 * 3 *1)] *12$$

$$y(x) = 2x^3 – 6x^2 + 3x + 3$$

put x = 1 on both sides, we get

$y(x = 1) = 2$