Methods based on Finite Differences

Wkt, using relation between Shifting and differential operator, $E = e^{hD}$ ---------------------(I)

$hD = \log E = \log(1+\Delta)$(since $E = 1 + \Delta$)

$\log(1 + x) = x - (x^2/2) + (x^3/3) - (x^4/4) + \ldots..$

Therefore, $hD = \Delta - \Delta^2/2 + \Delta^3/3 - \ldots\ldots.$ ________________________(II)

$D = 1/h\,(\Delta - \Delta^2/2 + \Delta^3/3 - \ldots\ldots.)$

Wkt, $\nabla f(x) = f(x) - f(x-h)$. Also wkt $E^{-1} f(x) = f(x - h)$

$= (1 - E^{-1})f(x)$

$\nabla = 1 - E^{-1}$. This implies $E^{-1} = 1 - \nabla$

Taking log on both sides

$\log E = -\log(1 - \nabla)$

ie., $hD = \nabla + \nabla^2/2 + \nabla^3/3 + \ldots..$ ___________________________(III)

$D = 1/h\,(\nabla + \nabla^2/2 + \nabla^3/3 + \ldots..)$

From the relation between central difference operator and shifting operator,

$\delta = E^{1/2} - E^{-1/2}$

$= e^{hD/2} - e^{-hD/2}$ by (I)

$= 2\sinh(hD/2)$ (since $\sinh\theta = (e^{\theta} - e^{-\theta})/2$)

$\sinh\theta = (\theta/1!) - (\theta^3/3!) + (\theta^5/5!) - \ldots\ldots\ldots..$

Therefore $hD = \log E = 2(\sinh^{-1}(\delta/2))$

$= \delta - (1/2^2*3!)\,\delta^3 + \ldots\ldots$ ___________________________(IV)

$h^r D^r = (\Delta - \Delta^2/2 + \Delta^3/3 - \ldots\ldots.)^r$ by (II)

$= \Delta^r - rc_1\,\Delta^{r-1}(\Delta^2/2 - \Delta^3/3 + \ldots\ldots.) + \ldots\ldots\ldots..$

$= \Delta^r - (1/2)\,r(\Delta)^{r+1} + (1/24)\,r(3r + 5)\,\Delta^{r+2} - \ldots.$

Similarly $h^r D^r = (\nabla + \nabla^2/2 + \nabla^3/3 + \ldots..)^r$ by (III)

$$= \nabla^r + rc_1\,\nabla^{r-1}\left(\frac{\nabla^2}{2} + \frac{\nabla^3}{3} + \ldots..\right) + \ldots\ldots..$$

$$= \nabla^r + \frac{1}{2} r\,\nabla^{r+1} + \frac{1}{24} r\,(3r+5)\,\nabla^{r+2} + \ldots\ldots\ldots. \quad \text{and}$$

$h^r D^r = \delta^r - (1/24)\,r\,\delta^{(r+2)} + (1/5760)\,r\,(5r + 22)\,\delta^{(r+4)} - \ldots\ldots\ldots$by (IV)

In particular, differentiation methods for $r = 1, 2$ at $x = x_k$ becomes

$f'(x_k) = (1/h)\,(\Delta f_k - (1/2)\,\Delta^2 f_k + (1/3)\,\Delta^3 f_k - \ldots.)$

$= (1/h)\left(\nabla f_k + \frac{1}{2}\nabla^2 f_k + \frac{1}{3}\nabla^3 f_k + ...\right)$

$= (1/h)\ (\delta\ f_k – (1/24)\ \delta^3 f_k + ……….)$

$f''(x) = (1/h^2)\ (\Delta^2 f_k - \Delta^3 f_k + (11/12)\ \Delta^4 f_k - ……)$

$= (1/h^2)\left(\nabla^2 f_k + \nabla^3 f_k + \frac{11}{12}\nabla^4 f_k + ...\right)$

$= \{\ (1/h^2)\ (\delta^2 f_k - (1/12)\ \delta^4 f_k + (1/90)\ \delta^6 f_k - ……)$

Keeping only the first term each of the methods

$f_k' = (1/h)\ (f_{k+1} - f_k)$

$= (1/h)\ (f_k – f_{k-1})$

$= (1/h)\ (f_{k+(1/2)} – f_{k-(1/2)})$

Note : The averaging operator μ is defined by $\mu = (1/2)\ (E^{1/2} + E^{-1/2}) = \text{sqrt}(1 + \delta^2/4)$. This implies that $\mu / \text{sqrt}(1 + \delta^2/4) = 1$. ______________________________(V)

WKT $hD = 2(\sinh^{-1}(\delta/2)) = 1^* \ 2(\sinh^{-1}(\delta/2))$

$= \mu / \text{sqrt}(1 + \delta^2/4) * 2(\sinh^{-1}(\delta/2))$ by (V)

$hD = \mu\ (\delta – (1/3!)\ \delta^3 + (2^2/5!)\ \delta^5 - …..)$

$D = (1/h)\ \mu\ (\delta – (1/3!)\ \delta^3 + (2^2/5!)\ \delta^5 - …..)$

$f'(x_k) = (1/h)\ (\mu\delta f_k - (1/6)\ \mu\delta^3 f_k + (1/30)\ \mu\delta^5 f_k - …..)$