Finite Differences

Let $y = f(x)$ be the given function of x. Let $y_0, y_1, y_2, \ldots, y_n$ be the values of y corresponding to the values $x_0, x_1, x_2, \ldots, x_n$. The values of y are called entries and the values of x are called arguments. Usually the arguments $x_0, x_1, x_2, \ldots, x_n$ are in general not equally spaced. If we subtract each value of y from the proceeding value (except y_0), we get $y_1 - y_0$, $y_2 - y_1$, $y_3 - y_2$, etc., The results obtained are known as first differences of y and it is denoted by Δ. Here Δ denotes an operation called forward difference operator. $\Delta y_0 = y_1 - y_0$; $\Delta y_1 = y_2 - y_1$; $\Delta y_2 = y_3 - y_2$; ...., $\Delta y_{n-1} = y_n - y_{n-1}$. For the purpose of our practical work, let us assume that the arguments are equally spaced. The arguments $x_0, x_1, x_2, \ldots, x_n$ can be taken as x_0, x_0+h, $x_0+2h, \ldots, x_0+nh$. Here 'h' is called interval of differencing.

Forward difference operator : Forward difference operator Δ is defined as

$\Delta f(x) = f(x + h) - f(x)$. Hence $\Delta f(x_0) = f(x_0 + h) - f(x_0)$ i.e., $\Delta y_0 = y_1 - y_0$.

$\Delta f(x_1) = f(x_1 + h) - f(x_1)$,i.e., $\Delta y_1 = y_2 - y_1$ Similarly $\Delta y_2 = y_3 - y_2$ and so on.,

Second forward difference is defined as $\Delta^2 f(x) = \Delta(\Delta f(x)) = \Delta(f(x + h) - f(x))$

$\Delta^2 y_0 = \Delta(\Delta y_0) = \Delta(y_1 - y_0) = \Delta y_1 - \Delta y_0$

$\Delta^2 y_1 = \Delta(\Delta y_1) = \Delta(y_2 - y_1) = \Delta y_2 - \Delta y_1$

$\Delta^2 y_2 = \Delta(\Delta y_2) = \Delta(y_3 - y_2) = \Delta y_3 - \Delta y_2$ and so on

$\Delta^2 y_{n-1} = \Delta(\Delta y_{n-1}) = \Delta(y_n - y_{n-1}) = \Delta y_n - \Delta y_{n-1}$

Third Forward difference

$\Delta^3 y_0 = \Delta(\Delta^2 y_0) = \Delta(\Delta y_1 - \Delta y_0) = \Delta^2 y_1 - \Delta^2 y_0$

$\Delta^3 y_1 = \Delta(\Delta^2 y_1) = \Delta(\Delta y_2 - \Delta y_1) = \Delta^2 y_2 - \Delta^2 y_1$

$\Delta^3 y_2 = \Delta(\Delta^2 y_2) = \Delta(\Delta y_3 - \Delta y_2) = \Delta^2 y_3 - \Delta^2 y_2$ and so on

$\Delta^3 y_{n-1} = \Delta(\Delta^2 y_{n-1}) = \Delta(\Delta y_n - \Delta y_{n-1}) = \Delta^2 y_n - \Delta^2 y_{n-1}$

Backward difference operator (∇)

Backward difference operator (∇) is defined as $\nabla f(x) = f(x) - f(x - h)$

$\nabla f(x_1) = f(x_1) - f(x_1 - h)$; i.e., $\nabla y_1 = y_1 - y_0$

$\nabla f(x_2) = f(x_2) - f(x_2 - h)$; i.e., $\nabla y_2 = y_2 - y_1$ Similarly $\nabla y_3 = y_3 - y_2$ and so on.

$\nabla f(x_n) = f(x_n) - f(x_n - h)$; i.e., $\nabla y_n = y_n - yn_{-1}$

Shifting (OR) Translation (OR) Displacement Operator (E)

Displacement operator E is defined as $Ef(x) = f(x + h)$ i.e., $Ey_x = y_{x+h}$

$E^2f(x) = E(Ef(x)) = Ef(x+h) = f(x + 2h)$ i.e., $E^2y_x = y_{x+2h}$

$E^3f(x) = E(E^2f(x)) = Ef(x + 2h) = f(x + 3h)$ i.e., $E^3y_x = y_{x+3h}$

In general $E^nf(x) = f(x+nh)$ i.e., $E^ny_x = y_{x+nh}$

Inverse operator E^{-1} is defined as $E^{-1}f(x) = f(x-h)$

$$E^{-r}f(x) = f(x - rh)$$

Central difference operator (δ)

Central difference operator δ is defined as $\delta f(x) = f\left(x+\frac{h}{2}\right) - f\left(x-\frac{h}{2}\right)$

i.e., $\delta y_x = y_{x+(h/2)} - y_{x-(h/2)}$

Averaging operator (μ)

Averaging operator μ is defined as $\mu f(x) = \frac{1}{2}\left[f\left(x+\frac{h}{2}\right) + f\left(x-\frac{h}{2}\right)\right]$

Unit operator (1)

Unit operator 1 is defined as $1.f(x) = f(x)$

Differential operator (D)

Differential operator D is defined as $Df(x) = f'(x)$

Properties

1. The operators Δ , ∇ , E, δ and μ are linear .

Forward difference operator Δ is linear i.e., $\Delta(af(x) \pm bg(x)) = a\Delta f(x) \pm b\, \Delta g(x)$, where 'a' and 'b' are constants, f(x) and g(x) are any two functions of x.

If a = b = 1 , then $\Delta(f(x) \pm g(x)) = \Delta f(x) \pm \Delta g(x)$

If b = 0 then $\Delta(af(x)) = a\Delta f(x)$

Relation between operators

1. Relation between forward difference operator (Δ) and shifting operator (E)

$\Delta f(x) = f(x + h) - f(x) = Ef(x) - f(x) = Ef(x) - 1.f(x) = (E - 1) f(x)$

Δ f(x) = (E – 1) f(x)

Δ = E – 1

This implies that E = 1 + Δ

2. Relation between backward difference operator (∇) and shifting operator (E)

$\nabla f(x) = f(x) - f(x-h) = f(x) - E^{-1}f(x) = 1.f(x) - E^{-1}f(x) = (1 - E^{-1})\, f(x)$

$\nabla f(x) = (1 - E^{-1})\, f(x)$

$\nabla = 1 - E^{-1}$

This implies that $E^{-1} = 1 - \nabla$

3. Relation between central difference operator (δ) and shifting operator (E)

$\delta f(x) = E^{(1/2)}f(x) - E^{(-1/2)}f(x) = (E^{(1/2)} - E^{(-1/2)})\, f(x)$

$\delta = E^{(1/2)} - E^{(-1/2)} = E^{\frac{1}{2}}(1 - E^{-1}) = E^{\frac{1}{2}}\nabla$

$\delta = E^{\frac{-1}{2}}(E - 1) = E^{\frac{-1}{2}}\Delta$

4. Relation between Averaging operator (μ) and shifting operator (E)

$\mu f(x) = \frac{1}{2}\left[E^{\frac{1}{2}} + E^{\frac{-1}{2}}\right] f(x)$

$\mu = \frac{1}{2}\left[E^{\frac{1}{2}} + E^{\frac{-1}{2}}\right]$

5. Relation between Differential operator (D) and shifting operator (E)

Wkt Ef(x) = f(x + h)

By Taylor's series E f(x) = $f(x) + \frac{h}{1!}f'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \ldots\ldots.$

$$= f(x) + \frac{h}{1!}Df(x) + \frac{h^2}{2!}D^2 f(x) + \frac{h^3}{3!}D^3 f(x) + \ldots\ldots\ldots\ldots$$

$$= \left[1 + \frac{h}{1!}D + \frac{h^2}{2!}D^2 + \frac{h^3}{3!}D^3 + \ldots\ldots..\right] f(x)$$

Since $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots\ldots..$

$Ef(x) = e^{hD} f(x)$

$E = e^{hD}$

$\text{Log } E = \log e^{hD}$

$hD = \log E = \log(1+\Delta)$ (since $E = 1 + \Delta$)

$$D = \frac{1}{h}\log(1+\Delta) = \frac{1}{h}\left[\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \ldots\ldots\ldots\right]$$

Results

1. If f(x) and g(x) are any two functions of x, then

   $\Delta(f(x)g(x)) = f(x+h)\,\Delta g(x) + g(x)\,\Delta f(x)$

2. If f(x) and g(x) are any two functions of x, then $\Delta\frac{f(x)}{g(x)} = \frac{g(x)\Delta f(x) - g(x)\Delta f(x)}{g(x+h)\,g(x)}$

3. The n^{th} forward differences of a n^{th} degree polynomial are constants.

   $\Delta(k) = k - k = 0$ where 'k' is constant

1. Find $\Delta^n(e^{ax+b})$

Sol : Wkt $\Delta f(x) = f(x+h) - f(x)$

$\Delta(e^{ax+b}) = e^{a(x+h)+b} - e^{ax+b}$

$= e^{ax+b}(e^{ah} - 1)$

$\Delta^2(e^{ax+b}) = \Delta(\Delta(e^{ax+b})) = \Delta(e^{ax+b}(e^{ah-1})) = e^{ah-1}\,\Delta(e^{ax+b}) = e^{ah-1}\,e^{ah-1}\,e^{ax+b}$

$= (e^{ah-1})^2 e^{ax+b}$

Similarly $\Delta^3(e^{ax+b}) = (e^{ah-1})^3 e^{ax+b}$

In general, $\Delta^n(e^{ax+b}) = (e^{ah-1})^n e^{ax+b}$