Romberg's method

This method is the modification of Trapezoidal rule. Suppose we evaluate I = $\int_{x_0}^{x_n} y\, dx$,

taking two different values of h, say h_1 and h_2 , then I= $I_2 + (I_2 - I_1) / 3$ ____________(I) where I_1 and I_2 are the values of I got by two different values of h, by Trapezoidal rule. By applying Trapezoidal rule many times, every times halving the value of h, we get a sequence of results A_1, A_2, A_3, …… We apply the formula given by (I) to each adjacent pairs and get the resultants B_1, B_2, B_3, ….(which are improved values). Again applying the formula (I) to each of pairs B_1, B_2, B_3, ….we get another sequence of better results C_1, C_2, C_3, …… Continuing in this way, we proceed until we get two successive values which are very close to each other. This systematic improvement of Richardson's method is called Romberg's method or Romberg integration.

1. Using Romberg's method, evaluate $\int_0^1 \frac{1}{1+x} dx$, correct to four decimal places. Hence evaluate $\log_e 2$.

   Sol : Given interval (a, b) = (0, 1) . This implies b – a = 1 – 0 = 1. Let h = 1/2

x	0	0.5	1
y = 1/(1+x)	1	0.6666	0.5

$\int_{x_0}^{x_n} y\, dx$ = (h/2) [($y_0 + y_n$) + 2($y_1 + y_2 + y_3 + \ldots\ldots + y_{n-1}$)].

$\int_0^1 \frac{1}{1+x} dx$ = (0.5/2) [($y_0 + y_2$) + 2(y_1)] = 0.25*((1+0.5) + (2*0.6666))

= 0.7083

Now, let h = 0.5/2 = 0.25

x	0	0.25	0.5	0.75	1
y = 1/(1+x)	1	0.8	0.6666	0.5714	0.5

$$\int_0^1 \frac{1}{1+x}dx = (0.5/2)\ [(y_0 + y_4) + 2(y_1 + y_2 + y_3)]$$

$= (0.25)\ [(1+0.5) + (2*(0.8 + 0.6666 + 0.5714))] = 0.6970$

Let h = 0.25/2 = 0.125

x	0	0.125	0.25	0.375	0.5	0.625	0.75	0.875	1
$y = \frac{1}{1+x}$	1	0.8889	0.8	0.7273	0.6666	0.6154	0.5714	0.5333	0.5

$$\int_0^1 \frac{1}{1+x}dx = (h/2)\ [(y_0 + y_8) + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7)]$$

$= (0.125/2)\ [(1+0.5) + 2(0.8889 + 0.8 + 0.7273 + 0.6666 + 0.6154 + 0.5174 + 0.5333)]$

$= 0.6941$

Therefore, the values of the given integral at different values of h are 0.7083, 0.6970 and 0.6941.

Let $I_1 = 0.7083$ and $I_2 = 0.6970$. Now $I = I_2 + (1/3)\ [I_2 - I_1]$

$= 0.6970 + (1/3)[0.6970 - 0.7083] = 0.6932$.

Let $I_1 = 0.6970$ and $I_2 = 0.6941$. Now $I = I_2 + (1/3)\ [I_2 - I_1]$

$= 0.6941 + (1/3)[0.6941 - 0.6970]$

$= 0.6931$. Now up to three decimals the values are same. Now, let us take $I_1 = 0.6932$ and $I_2 = 0.6931$, $I = I_2 + (1/3)\ [I_2 - I_1] = 0.6931 + (1/3)[0.6931 - 0.6932] = 0.6931$.

Hence $\int_0^1 \frac{1}{1+x}dx = 0.6931$.

By actual Integration, $\int_0^1 \frac{1}{1+x}dx = \left(\log(1+x)\right)_0^1 = \log 2 - \log 1 = \log 2$

Therefore, $\int_0^1 \frac{1}{1+x}dx = \log_{10} 2 = 0.6931$

2. Apply Romberg's method, to find $\int_0^1 \frac{1}{1+x^2}dx$, correct to four decimal places.

Sol : Given interval (a, b) = (0, 1) . This implies b – a = 1 – 0 = 1. Let h = 1/2

x	0	0.5	1

y = $1/(1+x^2)$	1	0.8	0.5

$$\int_{x_0}^{x_n} y\,dx = (h/2)\,[(y_0 + y_n) + 2(y_1 + y_2 + y_3 + \ldots\ldots + y_{n-1})].$$

$$\int_0^1 \frac{1}{1+x^2}\,dx = (0.5/2)\,[(y_0 + y_2) + 2(y_1)] = 0.25*((1+0.5) + (2*0.8))$$

$$= 0.775$$

Now, let h = 0.5/2 = 0.25

x	0	0.25	0.5	0.75	1
y = $1/(1+x^2)$	1	0.9412	0.8	0.64	0.5

$$\int_0^1 \frac{1}{1+x^2}\,dx = (0.5/2)\,[(y_0 + y_4) + 2(y_1 + y_2 + y_3)]$$

$$= (0.25)\,[(1+0.5) + (2*(0.9412+0.8+0.64))] = 0.7828$$

Let h = 0.25/2 = 0.125

x	0	0.125	0.25	0.375	0.5	0.625	0.75	0.875	1
$y = \frac{1}{1+x^2}$	1	0.9846	0.9412	0.8767	0.8	0.7191	0.64	0.5664	0.5

$$\int_0^1 \frac{1}{1+x^2}\,dx = (h/2)\,[(y_0 + y_8) + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7)]$$

$$= (0.125/2)\,[(1+0.5) + 2(0.9846+0.9412+0.8767+0.8+0.7191+0.64+0.5664)]$$

$$= 0.7848$$

Therefore, the values of the given integral at different values of h are 0.775, 0.7828 and 0.7848.

Let I_1 = 0.775 and I_2 = 0.7828. Now $I = I_2 + (1/3)\,[I_2 - I_1]$ = 0.7828 + (1/3)[0.7828 – 0.775]

= 0.7854.

Let I_1 = 0.7828 and I_2 = 0.7848. Now $I = I_2 + (1/3)\,[I_2 - I_1]$

$= 0.7848 + (1/3)[0.7848 - 0.7828]$

$= 0.7855$. Now up to three decimals the values are same. Now, let us take $I_1 = 0.7854$ and $I_2 = 0.7855$, $I = I_2 + (1/3)[I_2 - I_1] = 0.7855 + (1/3)[0.7855 - 0.7854] = 0.7855$.

Hence $\int_0^1 \frac{1}{1+x^2}dx = 0.7855$

3. Apply Romberg's method, evaluate $\int_1^2 \frac{1}{5+3x}dx$, compare with exact solution.

Sol : Given interval (a, b) = (1, 2) . This implies b – a = 2 – 1 = 1. Let h = 1/2

x	1	1.5	2
y = 1/(5+3x)	0.125	0.10526	0.09091

$\int_{x_0}^{x_n} y\,dx = (h/2)[(y_0 + y_n) + 2(y_1 + y_2 + y_3 + \ldots\ldots + y_{n-1})]$.

$\int_1^2 \frac{1}{5+3x}dx = (0.5/2)[(y_0 + y_2) + 2(y_1)] = 0.25*((0.125+0.09091) + (2*0.10526))$

$= 0.10661$

Now, let h = 0.5/2 = 0.25

x	1	1.25	1.5	1.75	2
y = 1/(5+3x)	0.125	0.11429	0.10526	0.09756	0.09091

$\int_0^1 \frac{1}{1+x^2}dx = (0.5/2)[(y_0 + y_4) + 2(y_1 + y_2 + y_3)]$

$= (0.25)[(0.125+0.09091) + (2*(0.11429+0.10526+0.09756))]$

$= 0.10627$

Let h = 0.25/2 = 0.125

x	1	1.125	1.25	1.375	1.5	1.625	1.75	1.875	2
$y = \frac{1}{5+3x}$	0.125	0.1194	0.11429	0.10959	0.10526	0.10127	0.09756	0.09412	0.09091

$$\int_0^1 \frac{1}{1+x^2}dx = (h/2)\ [(y_0 + y_8) + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7)]$$

= (0.125/2)[(0.125+0.09091)+2(0.1194+0.11429+0.10959+0.10526+ 0.10127+0.09756+0.09412)]

= 0.10618

Therefore, the values of the given integral at different values of h are 0.10661, 0.10627 and 0.10618.

Let I_1 = 0.10661 and I_2 = 0.10627. Now $I = I_2 + (1/3)\ [I_2 - I_1]$

= 0.10627 + (1/3)[0.10627 – 0.10661]

= 0.10616 = 0.1062 approximately

Let I_1 = 0.10627 and I_2 = 0.10618. Now $I = I_2 + (1/3)\ [I_2 - I_1]$

= 0.10618 + (1/3)[0.10618 – 0.10627]

= 0.10615 = 0.1062 approximately. Hence $\int_1^2 \frac{1}{5+3x} = 0.1062$.

By actual integration, $\int_1^2 \frac{1}{5+3x}dx = \frac{1}{3}[\log(5+3x)]_1^2$

= 1/3 (log 11- log 8) = 1/3 (log(11/8)) = 0.10615

4. The velocity of a particle which starts from rest is given by the following table

t (sec)	0	2	4	6	8	10	12	14	16	18	20
v(feet/sec)	0	16	29	40	46	51	32	18	8	3	0

Evaluate using Trapezoidal rule, the total distance travelled in 20 sec.

Sol : Here velocity given, it is asked to find the distance. Wkt, velocity = ds/dt

v dt = ds

Integrating both sides, ∫v dt = ∫ds. This implies s = ∫v dt

To find $\int_0^{20} v\,dt$

By Trapezoidal rule, $\int_0^{20} v\,dt = (h/2)\,[(v_0 + v_{10}) + 2(v_1 + v_2 + v_3 + \ldots\ldots + v_9)]$.

$= 2/2\,[(0+0) + 2(16 + 29 + 40 + 46 + 51 + 32 + 18 + 8 + 3)]$

$= 486$ feet

Hence, distance = 486 feet