1. Find the first and second derivatives of y = f(x) at x =1.5 from the following data. Also find f″(x) at x = 3.5

x	1.5	2	2.5	3	3.5	4
y	3.375	7.0	13.625	24	38.875	59

Sol: The value x = 1.5 is beginning value of the table. So use Newton's forward difference formula to get the derivatives

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
1.5	3.375					
		3.625				
2	7.0		3.000			
		6.625		0.75		
2.5	13.625		3.75		0.0	
		10.375		0.75		0.0
3	24		4.50		0.0	
		14.875		0.75		
3.5	38.875		5.25			
		20.125				
4	59					

$x_0 = 1.5$. Hence u = 0.

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{\Delta^2 y_0}{2} + \frac{\Delta^3 y_0}{3} - \frac{\Delta^4 y_0}{4} + ...\right]$$

$$= \frac{1}{0.5}\left[3.625 - \frac{3}{2} + \frac{0.75}{3}\right] = 4.75$$

$$\left(\frac{d^2 y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - ......\right]$$

$$= \frac{1}{(0.5)^2}\left[3.00 - 0.75\right] = 9$$

x = 3.5 is nearer to the end value of the table. So, use Newton's Backward difference formula to get the derivative.

$$\frac{dy}{dx} = \frac{1}{h}\left[\nabla y_n + \frac{2v+1}{2!}\nabla^2 y_n + \frac{3v^2+6v+2}{3!}\nabla^3 y_n + ...\right]$$

$$v = \frac{x - x_n}{h} = \frac{3.5-4}{0.5} = -1$$

$$=\frac{1}{0.5}\left[20.125+\frac{2(-1)+1}{2!}(5.25)+\frac{3(-1)^2+6(-1)+2}{3!}(0.75)\right]=34.75$$

2. Find the values of cos30° and cos60° from the following table

x°	35	40	45	50	55
tanx°	0.7002	0.8391	1	1.1918	1.4281

Sol: 30° is nearer to the beginning value of the table

x°	y = tanx°	Δy	Δ^2y	Δ^3y	Δ^4y
35	0.7002				
		0.1389			
40	0.8391		0.0220		
		0.1609		0.0089	
45	1		0.0309		0.0227
		0.1918		0.0316	
50	1.1918		0.0445		
		0.2363			
55	1.4281				

$$\therefore\ \frac{dy}{dx}=\frac{1}{h}\left[\Delta y_0+\frac{2u-1}{2!}\Delta^2 y_0+\frac{3u^2-6u+2}{3!}\Delta^3 y_0+\frac{4u^3-18u^2+22u-6}{4!}\Delta^4 y_0+.......\right]$$

$$\frac{dy}{dx}=\frac{d}{dx}\tan x=\sec^2 x;\qquad u=\frac{x-x_0}{h}=\frac{30-35}{5}=-1$$

$$\sec^2 30=\frac{1}{5^{\circ}}\left[\begin{array}{l}0.1389+\frac{2(-1)-1}{2!}(0.0220)+\frac{3(-1)^2-6(-1)+2}{3!}(0.0089)+\\ \frac{4(-1)^3-18(-1)^2+22(-1)-6}{4!}(0.0227)\end{array}\right]=1.2883$$

Wkt $\sec^2 30° = \dfrac{1}{\cos^2 30}$. This implies $\cos^2 30°$ = 0.7762

Hence cos30° = 0.8810.

The value 60 is outside the given interval which is nearer to the end value of the table.
Therefore use Newton's backward difference interpolation formula,

$$\frac{dy}{dx}=\frac{1}{h}\left[\nabla y_n+\frac{2v+1}{2!}\nabla^2 y_n+\frac{3v^2+6v+2}{3!}\nabla^3 y_n+\frac{4v^3+18v^2+22v+6}{4!}\nabla^4 y_n+.......\right]$$

$$\frac{dy}{dx}=\frac{d}{dx}\tan x=\sec^2 x;\qquad v=\frac{x-x_n}{h}=\frac{60-55}{5}=1$$

$$\sec^2 60=\frac{1}{5}\left[0.2363+\frac{3}{2}(0.0445)+\frac{11}{6}(0.0316)+\frac{50}{24}(0.0227)\right]=3.8706$$

This implies $\cos^2 60^\circ = 0.258358$. Hence $\cos 60^\circ = 0.5083$. In this problem $h = 5^\circ$, convert in to radians. $5^\circ = 5 * (\pi/180)$ radians.

3. Find the gradient of the road at the middle point of the elevation above a datum line of seven points of road which are given below

x	0	300	600	900	1200	1500	1800
y	135	149	157	183	201	205	193

Sol: Forward difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
0	135y_{-3}						
		14					
300	149y_{-2}		-6				
		8		24			
600	157y_{-1}		18		-50		
		26		-26		70	
900	183y_0		-8		20		-86
		18		-6		-16	
1200	201y_1		-14		4		
		4		-2			
1500	205y_2		-16				
		-12					
1800	193 y_3						

Here $x_0 = 900$. We are using forward difference table. The upper most diagonal values are forward differences of y_{-3}. Stirling's central difference formula to get the derivative for equal intervals is used.

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\frac{1}{2}(\Delta y_0 - \Delta y_{-1}) - \frac{1}{12}(\Delta^3 y_{-1} + \Delta^3 y_{-2}) + \frac{1}{60}(\Delta^5 y_{-2} + \Delta^5 y_{-3})\right]$$

$h = 300$

$$\text{Gradient} = \frac{1}{300}\left[\frac{1}{2}(18-26) - \frac{1}{12}(-6-26) + \frac{1}{60}(-16+70)\right] = 0.085222$$

4. The table given below reveals the velocity 'v' of a body during the time 't' specified. Find its acceleration at t = 1.1.

t	1.0	1.1	1.2	1.3	1.4
v	43.1	47.7	52.1	56.4	60.8

Sol: Here t = 1.1 is nearer to the beginning value of the table . So use Newton's forward difference formula for equal intervals to get the derivative is used. h = 0.1

t	v	Δv	$\Delta^2 v$	$\Delta^3 v$	$\Delta^4 v$
1.0 t_0	43.1 v_0				
		4.6			
1.1	47.7		-0.2		
		4.4		0.1	
1.2	52.1		-0.1		0.1
		4.3		0.2	
1.3	56.4		0.1		
		4.4			
1.4	60.8				

Here the upper most diagonal values are forward differences of v_0. Wkt, rate of change of velocity is acceleration. So find the first derivative of v. By using Newton's forward difference formula for equal intervals,

$$A = \frac{dv}{dt} = \frac{1}{h}\left[\Delta v_0 + \frac{2u-1}{2!}\Delta^2 v_0 + \frac{3u^2-6u+2}{3!}\Delta^3 v_0 + ...\right] \text{ where } u = \frac{t-t_0}{h} = \frac{1.1-1.0}{0.1} = 1$$

$$= \frac{1}{0.1}\left[4.6 + \frac{1}{2!}(-0.2) + \frac{3-6+2}{3!}(0.1) + \frac{4-18+22-6}{4!}(0.1)\right] = 44.917$$

5. A rod is rotating in a plane. The following table gives the angle θ (in radians) through which the rod has turned for various values of time t(sec) . Calculate the angular velocity and angular acceleration of the rod at t = 0.6 sec.

t	0	0.2	0.4	0.6	0.8	1.0
θ	0	0.12	0.49	1.12	2.02	3.20

Sol:

	t	θ	$\Delta\theta$	$\Delta^2\theta$	$\Delta^3\theta$	$\Delta^4\theta$
-3	0	0	0.12			
				0.25		
-2	0.2	0.12	0.37		0.01	
				0.26		0
-1	0.4	0.49	0.63		0.01	
				0.27		0
0	0.6	1.12	0.90		0.01	
				0.28		
1	0.8	2.02	1.18			

2	1.0	3.20				

Here 0.6 is nearer to the middle value, so we use Stirling's central difference formula
Let $t_0 = 0.6$

$$\left(\frac{d\theta}{dt}\right)_{t=t_0} = \frac{1}{h}\left[\left[\frac{\Delta\theta_0 + \Delta\theta_{-1}}{2}\right] - \frac{1}{6}\left[\frac{\Delta^3\theta_{-1} + \Delta^3\theta_{-2}}{2}\right] + \frac{1}{30}\frac{\Delta^5\theta_{-2} + \Delta^5\theta_{-3}}{2} - \ldots\ldots\right]$$

Angular velocity = $\frac{1}{0.2}\left[\frac{0.90+0.63}{2} - \frac{1}{6}\frac{0.01+0.01}{2}\right] = 3.8167$

$$\left(\frac{d^2\theta}{dt^2}\right)_{t=t_0} = \frac{1}{h^2}\left[\Delta^2\theta_{-1} - \frac{1}{12}\Delta^4\theta_{-2} + \frac{1}{90}\Delta^6\theta_{-4} - \ldots\ldots\right]$$

Angular acceleration = $\frac{1}{(0.2)^2}\left[0.27\right] = 6.75$

This problem, we can apply Newton's backward difference interpolation formula for equal intervals.

6. Find f′(0) from the following data

x	-1	0	2	3
y	-2	-1	1	4

Sol: Here the values of x are not equally spaced. Hence use Lagrange's interpolation formula Let $x_0 = -1$; $x_1 = 0$; $x_2 = 2$ and $x_3 = 3$. The corresponding values of y are $y_0 = -2$; $y_1 = -1$; $y_2 = 1$ and $y_3 = 4$.

$$y(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1 + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3$$

$$y(x) = \frac{(x-0)(x-2)(x-3)}{(-1-0)(-1-2)(-1-3)}(-2) + \frac{(x+1)(x-2)(x-3)}{(0+1)(0-2)(0-3)}(-1) + \frac{(x+1)(x-0)(x-3)}{(2+1)(2-0)(2-3)}1 + \frac{(x+1)(x-0)(x-2)}{(3+1)(3-0)(3-2)}4$$

$$y(x) = \frac{x(x^2-5x+6)}{-12}(-2) + \frac{(x+1)(x^2-5x+6)}{6}(-1) + \frac{x(x^2-2x-3)}{-6} + \frac{x(x^2-x-2)}{12}4$$

$$y(x) = \frac{x^3-5x^2+6x}{6} + \frac{-x^3+4x^2-x-6}{6} + \frac{-x^3+2x^2+3x}{6} + \frac{2x^3-2x^2-4x}{6}$$

$$y(x) = f(x) = \frac{1}{6}(x^3 - x^2 + 4x - 6)$$

$f'(x) = \frac{1}{6}(3x^2 - 2x + 4)$

$f'(0) = \frac{4}{6} = \frac{2}{3}$.

7. Find $f'(5)$ from the following data

x	0	1	3	4
y	-12	0	6	12

Sol: Here the values of x are not equally spaced. Hence use Lagrange's interpolation formula Let $x_0 = 0$; $x_1 = 1$; $x_2 = 3$ and $x_3 = 4$. The corresponding values of y are $y_0 = -12$; $y_1 = 0$; $y_2 = 6$ and $y_3 = 12$.

$$y(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1 + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3$$

$$y(x) = \frac{(x-1)(x-3)(x-4)}{(0-1)(0-3)(0-4)}(-12) + \frac{(x-0)(x-3)(x-4)}{(1-0)(1-3)(1-4)}(0) + \frac{(x-0)(x-1)(x-4)}{(3-0)(3-1)(3-4)}6 + \frac{(x-0)(x-1)(x-3)}{(4-0)(4-1)(4-3)}12$$

$$y(x) = \frac{(x-1)\,(x^2-7x+12)}{-12}(-12) + 0 + \frac{x(x^2-5x+4)}{-6}6 + \frac{x(x^2-4x+3)}{12}12$$

$$y(x) = x^3 - 8x^2 + 19x - 12 - x^3 + 5x^2 - 4x + x^3 - 4x^2 + 3x$$

$y(x) = f(x) = x^3 - 7x^2 + 18x - 12$

Hence $f'(x) = 3x^2 - 14x + 18$

Therefore $f'(5) = 3*25 - 14*5 + 18 = 33$

8. Find the maximum and minimum values of f(x) from the following data

x	0	1	3	4
y = f(x)	-4	1	29	52

Sol: Here the values of x are not equally spaced. Hence use Lagrange's interpolation formula Let $x_0 = 0$; $x_1 = 1$; $x_2 = 3$ and $x_3 = 4$. The corresponding values of y are $y_0 = -4$; $y_1 = 1$; $y_2 = 29$ and $y_3 = 52$.

$$y(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1 + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

$$y(x) = \frac{(x-1)(x-3)(x-4)}{(0-1)(0-3)(0-4)}(-4) + \frac{(x-0)(x-3)(x-4)}{(1-0)(1-3)(1-4)}(1) + \frac{(x-0)(x-1)(x-4)}{(3-0)(3-1)(3-4)}29 + \frac{(x-0)(x-1)(x-3)}{(4-0)(4-1)(4-3)}52$$

$$y(x) = \frac{(x-1)\ (x^2-7x+12)}{-12}(-4) + \frac{x(x^2-7x+12)}{6} + \frac{x(x^2-5x+4)}{-6}29 + \frac{x(x^2-4x+3)}{12}52$$

$$y(x) = \frac{x^3-8x^2+19x-12}{3} + \frac{x^3-7x^2+12x}{6} + \frac{-29x^3+145x^2-116x}{6} + \frac{13x^3-52x^2+39x}{3}$$

$$y(x) = \frac{2x^3-16x^2+38x-24+x^3-7x^2+12x-29x^3+145x^2-116x+26x^3-104x^2+78x}{6}$$

$y(x) = f(x) = 3x^2 + 2x - 4$

$f'(x) = 6x + 2$

$f''(x) = 6$

For maximum or minimum, $f'(x) = 0$. This implies that $x = -1/3$

$f''(x) > 0$. Hence the function is minimum at $x = -1/3$.

Minimum value = $3(-1/3)^2 + 2(-1/3) – 4 = -4.33333$