1) If $\frac{dy}{dx} = \frac{x^2}{1+y^2}$, $y(0)=0$ find $y(0.25)$ and $y(0.5)$

Using Picard's method by considering 3 approximations.

$$y = y_0 + \int_{x_0}^{x} f(x,y)\,dx.$$

$$f(x,y) = \frac{x^2}{1+y^2}.$$

$$y^{(1)} = y_0 + \int_{x_0}^{x} f(x,y_0)\,dx.$$

$$= y_0 + \int_{0}^{x} \frac{x^2}{1+y_0^2}\,dx$$

$y(0)=0 \Rightarrow y_0 = 0 \Rightarrow y(x_0)=0$

$x_0 = 0$

$$y^{(1)} = 0 + \int_{0}^{x} \frac{x^2}{1+0}\,dx.$$

$$y^{(1)} = \frac{x^3}{3} \quad \text{— I}$$

$$y^{(1)} = \frac{x^3}{3}$$

$$y^{(2)} = y_0 + \int_{x_0}^{x} f(x, y^{(1)})\, dx$$

$$f(x, y^{(1)}) = \frac{x^2}{1+{y^{(1)}}^2} = \frac{x^2}{1+\left(\frac{x^3}{3}\right)^2}$$

$$= \frac{9x^2}{9+x^6}$$

$$= 0 + \int_0^x \frac{9x^2}{9+x^6}\, dx$$

$$x^3 = u$$

$$3x^2 dx = du$$

$$9\int \frac{du}{9+u^2}$$

$$= 3\tan^{-1}\left(\frac{x^3}{3}\right)$$

$$\frac{9}{3} \cdot \tan^{-1}\left(\frac{u}{3}\right)$$

$$= 3\tan^{-1}\left(\frac{x^3}{3}\right)$$

$$y^{(1)} = \frac{x^3}{3}$$

$$y^{(2)} = 3\tan^{-1}\left(\frac{x^3}{3}\right)$$

When $x = 0.25$ $\quad y^{(1)}(0.25) = \frac{(0.25)^3}{3} = \frac{0.015725}{3}$

$$= 0.005\,24$$

$$y^{(2)} = 3\tan^{-1}\left(\frac{x^3}{3}\right)$$

$$= 3\tan^{-1}(0.00524)$$

When $x = 0.5$

$$y^{(1)} = \frac{(0.5)^3}{3} = \frac{0.125}{3} = 0.041$$

$$y^{(2)} = 3\tan^{-1}\left(\frac{0.5^3}{3}\right) = 3\tan^{-1}(0.041)$$

2) Solve $\frac{dy}{dx} = x^2+y^2$, $y(0)=1$ by Picard's method.

$f(x,y) = x^2+y^2$ & $y(0)=1$

i.e., $y_0 = 1$

$x_0 = 0$

$$\frac{dy}{dx} = f(x,y)$$

$$y^{(1)} = y_0 + \int_{x_0}^{x} f(x, y_0)\, dx$$

$$y^{(1)} = 1 + \int_0^x (x^2+1)\, dx$$

$$= 1 + \left[\frac{x^3}{3} + x\right]_0^x = 1 + x + \frac{x^3}{3} \quad \text{— I}$$

$$y^{(2)} = y_0 + \int_{x_0}^{x} f(x, y^{(1)})\, dx$$

$$f(x, y^{(1)}) = x^2 + {y^{(1)}}^2$$

$$= x^2 + \left(1 + x + \frac{x^3}{3}\right)^2$$

$$= x^2 + 1 + x^2 + \frac{x^6}{9} + 2x + \frac{2x^4}{3} + \frac{2x^3}{3}$$

$$= 1 + 2x + 2x^2 + \frac{2x^3}{3} + \frac{2x^4}{3} + \frac{x^6}{9}$$

$$y^{(2)} = 1 + \int_0^x \left(1 + 2x + 2x^2 + \frac{2x^3}{3} + \frac{2x^4}{3} + \frac{x^6}{9}\right) dx.$$

$$= 1 + \left[x + x^2 + \frac{2x^3}{3} + \frac{2x^4}{12} + \frac{2x^5}{15} + \frac{x^7}{63}\right]$$

$$y^{(2)} = 1 + x + x^2 + \frac{2x^3}{3} + \frac{x^4}{6} + \frac{2x^5}{15} + \frac{x^7}{63}. \quad \text{---} \quad \text{II}.$$

3) Solve by Picard's method

$$y' + y = e^x \; ; \; y(0) = 0$$

$$y_0 = 0 \; ; x_0 = 0$$

$$y = y_0 + \int_{x_0}^{x} f(x, y)\, dx.$$

$$y^{(1)} = y_0 + \int_{x_0}^{x} f(x, y_0)\, dx.$$

$$f(x, y) = e^x - y.$$

$$y^{(1)} = y_0 + \int_{x_0}^{x} (e^x - y_0)\, dx.$$

$$= 0 + \int_0^x (e^x)\, dx = (e^x - 1)$$

$$y^{(2)} = y_0 + \int_{x_0}^{x} f(x, y^{(1)})\, dx$$

$$f(x, y^{(1)}) = e^x - y^{(1)}$$

$$= e^x - (e^x - 1) = 1$$

$$= 0 + \int_0^x 1\, dx.$$

$$= x.$$

$$f(x, y^{(2)}) = e^x - y^{(2)}$$

$$= e^x - x.$$

$$y^{(3)} = y_0 + \int_{x_0}^{x} f(x, y^{(2)})\, dx$$

$$= 0 + \int_0^x (e^x - x)\, dx.$$

$$= \left[e^x - \frac{x^2}{2}\right]_0^x$$

$$y^{(3)} = e^x - \frac{x^2}{2} - 1$$

$$y^{(4)} = y_0 + \int_{x_0}^{x} f(x, y^{(3)})\, dx.$$

$$= 0 + \int_0^x f(x, y^{(3)})\, dx$$

$$f(x, y^{(3)}) = e^x - y^{(3)}$$

$$= e^x - \left(e^x - \frac{x^2}{2} - 1\right)$$

$$= x^2/2 + 1.$$

$$y^{(4)} = 0 + \int_0^x \left(\frac{x^2}{2}+1\right) dx.$$

$$= \frac{x^3}{6} + x.$$

$$y^{(5)} = y_0 + \int_0^x f(x, y^{(4)}) dx.$$

$$f(x, y^{(4)}) = e^x - y^{(4)}$$

$$= e^x - \left(\frac{x^3}{6} + x\right)$$

$$= e^x - \frac{x^3}{6} - x.$$

$$y^{(5)} = 0 + \int_0^x \left(e^x - \frac{x^3}{6} - x\right) dx.$$

$$= \left[e^x - \frac{x^4}{24} - \frac{x^2}{2}\right]_0^x$$

$$= e^x - \frac{x^4}{24} - \frac{x^2}{2} - 1.$$

4) Solve by Picard's method

$$\frac{dy}{dx} = x + y^2 + 1 \qquad \text{Given } y(0) = 0 \qquad y_0 = 0, \; x_0 = 0$$

$$f(x,y) = x + y^2 + 1. \qquad y = y_0 + \int_{x_0}^{x} f(x,y)\,dx.$$

$$y^{(1)} = y_0 + \int_{x_0}^{x} f(x, y_0)\,dx$$

$$= 0 + \int_0^x (x+1)\,dx$$

$$= \frac{x^2}{2} + x. \quad \text{--- } \underline{\mathrm{I}}$$

Soln: $y = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{20}$.

$$y^{(2)} = y_0 + \int_{x_0}^{x} f(x, y^{(1)})\,dx$$

$$= 0 + \int_0^x$$

$$f(x, y^{(1)}) = x + y^{(1)2} + 1$$

$$= x + \left(\frac{x^2}{2} + x\right)^2 + 1$$

$$= 1 + x + x^2 + x^3 + \frac{x^4}{4}.$$

$$y^{(2)} = \int_0^x \left(1 + x + x^2 + x^3 + \frac{x^4}{4}\right) dx.$$

$$y^{(2)} = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{20} \quad \text{--- } \underline{\mathrm{II}}.$$