Interpolation

The process of computing the intermediate values of y, from the given set of tabular values of the function is known as Interpolation. Let $y = f(x)$ be the given function of x. Here x is known as argument and y is known as entry. In general the arguments are equally spaced. For our convenience the arguments $x_0, x_1, x_2, \ldots, x_n$ can be taken as $x_0, x_0 + h, x_0 + 2h, \ldots, x_0 + nh$. Here 'h' is known as interval of differencing. y_0 means $f(x_0)$; y_1 means $f(x_1)$ etc., and y_n is $f(x_n)$. We also know, the forward difference operator Δ. $\Delta f(x) = f(x + h) - f(x)$. Given the tabular values of the function $y = f(x)$

x	x_0	x_1	x_2	x_3	..	..	..	..	x_n
y	y_0	y_1	y_2	y_3	..	..	..	..	y_n

If we require to compute y_i i.e.,$y(x = x_i)$ from the given set of tabular values where $x_0 < x_i < x_n$ is known as interpolation. Similarly, if we compute the values of y, outside the given interval is known as extrapolation. But in general, the word interpolation is used for both the cases.

Forward difference table

Wkt $\Delta f(x) = f(x + h) - f(x)$

$\Delta y_0 = y_1 - y_0$; $\Delta y_1 = y_2 - y_1$; $\Delta y_2 = y_3 - y_2$ and so on

$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$; $\Delta^2 y_1 = \Delta y_2 - \Delta y_1$; $\Delta^2 y_2 = \Delta y_3 - \Delta y_2$ and so on

$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$; $\Delta^3 y_1 = \Delta^2 y_2 - \Delta^2 y_1$; $\Delta^3 y_2 = \Delta^2 y_3 - \Delta^2 y_2$ and so on

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
x_0	y_0						
		Δy_0					
x_1	y_1		$\Delta^2 y_0$				
		Δy_1		$\Delta^3 y_0$			
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$		
		Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$	
x_3	y_3		$\Delta^2 y_2$		$\Delta^4 y_1$		$\Delta^6 y_0$
		Δy_3		$\Delta^3 y_2$		$\Delta^5 y_1$	
x_4	y_4		$\Delta^2 y_3$		$\Delta^4 y_2$		
		Δy_4		$\Delta^3 y_3$			
x_5	y_5		$\Delta^2 y_4$				
		Δy_5					
x_6	y_6						

Forward difference table is used to interpolate the values of y nearer to the beginning value of the table. In calculation, we use the uppermost diagonal values in the forward difference table.

Similarly,

Backward difference table

Wkt $\nabla f(x) = f(x) - f(x-h)$

$\nabla y_1 = y_1 - y_0$; $\nabla y_2 = y_2 - y_1$; $\nabla y_3 = y_3 - y_2$ and so on

$\nabla^2 y_1 = \nabla y_1 - \nabla y_0$; $\nabla^2 y_2 = \nabla y_2 - \nabla y_1$; $\nabla^2 y_3 = \nabla y_3 - \nabla y_2$ and so on

$\nabla^3 y_1 = \nabla^2 y_1 - \nabla^2 y_0$; $\nabla^3 y_2 = \nabla^2 y_2 - \nabla^2 y_1$; $\nabla^3 y_3 = \nabla^2 y_3 - \nabla^2 y_2$ and so on

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$	$\nabla^5 y$	$\nabla^6 y$
x_0	y_0						
		∇y_1					
x_1	y_1		$\nabla^2 y_2$				
		∇y_2		$\nabla^3 y_3$			
x_2	y_2		$\nabla^2 y_3$		$\nabla^4 y_4$		
		∇y_3		$\nabla^3 y_4$		$\nabla^5 y_5$	
x_3	y_3		$\nabla^2 y_4$		$\nabla^4 y_5$		$\nabla^6 y_6$
		∇y_4		$\nabla^3 y_5$		$\nabla^5 y_6$	
x_4	y_4		$\nabla^2 y_5$		$\nabla^4 y_6$		
		∇y_5		$\nabla^3 y_6$			
x_5	y_5		$\nabla^2 y_6$				
		∇y_6					
x_6	y_6						

Backward difference table is used to interpolate the values of y nearer to the end value of the table. In calculation, we use the lowermost diagonal values in the backward difference table.

Newton's forward difference formula for equal intervals

$$y(x) = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \dots\dots\dots \text{ where } u = \frac{x-x_0}{h}$$

This formula is used to interpolate (or extrapolate) the values of y nearer to beginning value of the table.

Newton's backward difference formula for equal intervals

$$y(x) = y_n + \frac{v}{1!}\nabla y_n + \frac{v(v+1)}{2!}\nabla^2 y_n + \frac{v(v+1)(v+2)}{3!}\nabla^3 y_n + \dots\dots\dots \text{ where } v = \frac{x-x_n}{h}$$

This formula is used to interpolate (or extrapolate) the values of y nearer to end value of the table.

Problem

1.Estimate the production for the year 2004 and 2006 from the following data

Year	2001	2002	2003	2004	2005	2006	2007
Production	200	220	260	-------	350	-------	430

Sol: Let us take year as x and production as y. Since only five values of y are given, using these values we can compute only up to fourth order differences. Hence fifth forward difference of y is zero. So, $\Delta^5 y_k = 0$.

Wkt, $\Delta = E - 1$. Hence $(E-1)^5 y_k = 0$. ________________(*)

If we put k = 0 then $(E-1)^5 y_0 = 0$

$(E^5 - 5E^4 + 10E^3 - 10E^2 + 5E - 1)\ y_0 = 0$

$E^5y_0 - 5E^4y_0 + 10E^3y_0 - 10E^2y_0 + 5Ey_0 - y_0 = 0$

Wkt, $E^n y_x = y_{x+nh}$

$y_5 - 5y_4 + 10y_3 - 10y_2 + 5y_1 - y_0 = 0$

$y_5 - 5(350)\ y_4 + 10y_3 - 10(260) + 5(220) - 200 = 0$

i.e., $y_5 + 10y_3 = 3450$ ______________________(I)

put k = 1 in (*)

$(E^5 - 5E^4 + 10E^3 - 10E^2 + 5E - 1)y_1 = 0$

$y_6 - 5y_5 + 10y_4 - 10y_3 + 5y_2 - y_1 = 0$

$430 - 5y_5 + 10(350) - 10y_3 + 5(260) - 220 = 0$

$5y_5 + 10y_3 = 5010$ ______________________(II)

Solving (I) and (II), we get $y_3 = 306$ and $y_5 = 390$

2. Find the value of y at x = 21 and x = 28 from the following data

x	20	23	26	29
y	0.342	0.3907	0.4384	0.4848

Sol: Here the values of x are equally spaced. x = 21 is nearer to the beginning value of the table. Hence use Newton's forward difference formula for interpolation for equal intervals .

We shall first form the difference table

x	y	Δy	Δ^2y	Δ^3y
20 23 26 29	0.342 0.3907 0.4384 0.4848	0.0487 0.0477 0.0464	-0.001 -0.0013	-0.0003

First to find the value of y when x = 21. Here h = 3

$$y(x) = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + ... \text{ where } u = \frac{x - x_0}{h}$$

$$u = \frac{21-20}{3} = 0.3333$$

$$y(21) = 0.342 + (0.3333)(0.0487) + \frac{(03333)(0.3333-1)}{2}(-0.001) + \frac{(0.3333)(0.3333-1)(0.3333-2)}{6}(-0.0003)$$

$= 0.3583$

Next, to find the value of y when x = 28. x = 28 is nearer to the end value of the table. So use Newton's backward difference formula for interpolation for equal intervals

$$y(x) = y_n + \frac{v}{1!}\nabla y_n + \frac{v(v+1)}{2!}\nabla^2 y_n + \frac{v(v+1)(v+2)}{3!}\nabla^3 y_n + ... \text{ where } v = \frac{x - x_n}{h}$$

$$v = \frac{28-29}{3} = -0.3333$$

y(28) = 0.4848 – 0.3333 (0.0464) +((-0.3333)(-0.3333 + 1)(-0.0013)) / 2 +

((-0.3333)(-0.3333 + 1) (-0.3333 + 2) (-0.0003)) / 6

= 0.4695.

3. Fit a polynomial of degree four using the following data

x	2	4	6	8	10
y	0	0	1	0	0

Sol: Here values of x are equally spaced. h = 2. Since only five values of y are given. Using these five values of y we can compute the differences of order four. Here it is asked to fit a polynomial, so in this problem we can use either Newton's forward or backward difference formula.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
2	0				
		0			
4	0		1		
		1		-3	
6	1		-2		6
		-1		3	
8	0		1		
		0			
10	0				

Here h = 2. Use Newton's forward difference formula

$$y(x) = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \text{..........} \text{ where } u = \frac{x-x_0}{h}$$

$$u = \frac{x-2}{2}$$

$$y(x) = 0+0+\left(\frac{x-2}{2}\right)\left(\frac{x-2}{2}-1\right)\frac{1}{2}+\left(\frac{x-2}{2}\right)\left(\frac{x-2}{2}-1\right)\left(\frac{x-2}{2}-2\right)\left(\frac{-3}{6}\right)+$$
$$\left(\frac{x-2}{2}\right)\left(\frac{x-2}{2}-1\right)\left(\frac{x-2}{2}-2\right)\left(\frac{x-2}{2}-3\right)\left(\frac{6}{24}\right)$$

On simplification, we get $y(x) = (x^4 - 24x^3 + 196x^2 - 624x + 640) / 64$