PCA EXAMPLE

1. Given the following data, use PCA to reduce the dimensions from 2 to 1

Feature	**Example 1**	**Example 2**	**Example 3**	**Example 4**
X	4	8	13	7
Y	11	4	5	14

Solution:

Step 1: Data set

No. of features n=2

No. Of samples N=4

Step 2: computation of mean of variables

$$\bar{x} = \frac{4+8+13+7}{4} = 8$$

$$\bar{y} = \frac{11+4+5+14}{4} = 8.5$$

Step 3: Computation of covariance matrix ordered pairs are

$$\begin{bmatrix} \text{cov}(x,x) & \text{cov}(x,y) \\ \text{cov}(y,x) & \text{cov}(y,y) \end{bmatrix}$$

If there are n variables, then n^2 Ordered pairs

Cov (x, y) = $\frac{1}{N-1}\sum(x-\bar{x})(y-\bar{y})$

Cov(x,x) = $\frac{1}{4-1}$ $[(4-8)^2+(8-8)^2+(13-8)^2+(7-8)^2] = 14$

Cov (x,y) = $\frac{1}{4-1}$ $[(4-8)(11-8.5)+(8-8)(4-8.5)+(13-8)(5-8.5)+(7-8)(14-8.5)] = -11$

Cov (y,x) = -11

Cov (y, y) = $\frac{1}{N-1}\sum(y-\bar{y})^2$

$= \frac{1}{4-1}[(11-8.5)^2+ (4-8.5)^2+ (5-8.5)^2+ (14-8.5)^2] = 23$

Cov matrix $S = \begin{bmatrix} 14 & -11 \\ -11 & 23 \end{bmatrix}$

Step 4: Eigen values and eigen vectors & Normalised Eigen vectors

A be square matrix of order n . Suppose there exist a non zero column matrix X and a scalar λ such that Ax= λx . Then λ is called the Eigen value of A and X is called the Eigen vector of A corresponding to the eigen value λ.

Now Ax = λX

(A- λI)X = 0. The solution of the characteristic equation | A- λI|= 0 is known as Eigen value of the matrix A. For each eigen value, solving the equations
(A- λI)X = 0 we get eigen vectors.

Characteristic eqn | S- λI|=0

$| S- \lambda I|= \lambda^2-37\lambda+201=0$

λ= 30.3849 & 6.6151

Here, U be the eigen vector corresponding to the eigen value λ.

We know that (S- λI)U=0

Let λ_1 = 30-.3849

$(S- \lambda I)U_I = 0$

$$\left[\begin{pmatrix} 14 & -11 \\ -11 & 23 \end{pmatrix} - \begin{pmatrix} 30.3849 & 0 \\ 0 & 30.3849 \end{pmatrix}\right]\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0$$

$$U_1 = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{bmatrix} 11 \\ -16.3849 \end{bmatrix}$$

Normalised Eigen vector of $U_1 = U_1^N = \begin{bmatrix} \dfrac{11}{\sqrt{11^2 + (-16.3849)^2}} \\ \dfrac{-16.3849}{\sqrt{11^2 + (-16.3849)^2}} \end{bmatrix}$

$e_1 = \begin{pmatrix} 0.5774 \\ -0.8303 \end{pmatrix}$

Let λ_2=6.6151

$(S- \lambda_2)U_2 = 0$

$$\left(\begin{bmatrix} 14 & -11 \\ -11 & 23 \end{bmatrix} - \begin{bmatrix} 6.6151 & 0 \\ 0 & 6.6151 \end{bmatrix} \right) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0$$

$U_2 = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{bmatrix} 16.3849 \\ 11 \end{bmatrix}$

Normalised Eigen vector of $U_2 = U_2^N = \begin{bmatrix} \dfrac{16.3849}{\sqrt{11^2 + (16.3849)^2}} \\ \dfrac{11}{\sqrt{11^2 + (16.3849)^2}} \end{bmatrix}$

$e_2 = \begin{pmatrix} 0.8303 \\ 0.5774 \end{pmatrix}$

Step 5 : Derive new data set (λ 1 > λ2 ; 30.849 > 6.6151)

First Principal	Ex.1	Ex.2	Ex.3	Ex.4
Component	p_{11}	p_{12}	p_{13}	p_{14}

$p_{11} = e_1^T \begin{bmatrix} 4-8 \\ 11-8.5 \end{bmatrix} = \begin{bmatrix} 0.5574 & -0.8303 \end{bmatrix} \begin{bmatrix} -4 \\ 2.5 \end{bmatrix}$ = - 4.0352

$p_{12} = e_1^T \begin{bmatrix} 8-8 \\ 4-8.5 \end{bmatrix} = \begin{bmatrix} 0.5574 & -0.8303 \end{bmatrix} \begin{bmatrix} 0 \\ -4.5 \end{bmatrix}$ = -3.7361

$p_{13} = e_1^T \begin{bmatrix} 13-8 \\ 5-8.5 \end{bmatrix} = 5.6928$

$p_{14} = e_1^T \begin{bmatrix} 7-8 \\ 14-8.5 \end{bmatrix} = -5.1238$

PC1	-4.3052	3.7361	5.6928	-5.1238

Example 2

In the following table, there are 4 features and a total of 5 training samples

f_1	f_2	f_3	f_4
1	2	3	4
5	5	6	7
1	4	2	3
5	3	2	1
8	1	2	2

$\sigma^2 = [(1-4)^2 + (5-4)^2 + (1-4)^2 + (5-4)^2 + (8-4)^2]/4$

$= (9+1+9+1+16)/4$

$= 36/4 = 9$

Number of features = N=4

Number of Samples = n = 5

mean and Standard deviation for each feature

$x_{new} = (x-\mu)/\sigma$

$\sigma^2 = \sum \frac{(x-\bar{x})^2}{n-1}$

	f_1	f_2	f_3	f_4
μ	4	3	3	3.4
σ	3	1.58114	1.73205	2.30217

f_1	f_2	f_3	f_4
-1	-0.6325	0	0.26062
0.3333	1.2649	1.7321	1.56374
-1	0.6325	-0.5774	-0.1738
0.3333	0	-0.5774	-1.0425
1.3333	-1.2649	-0.5774	-0.6081

Covariance matrix

$$\begin{array}{ccccc} & f1 & f2 & f3 & f4 \\ f1 & Var(f_1) & Co\,\mathrm{var}(f_1, f_2) & Co\,\mathrm{var}(f_1, f_3) & Co\,\mathrm{var}(f_1, f_4) \\ f2 & Co\,\mathrm{var}(f_2, f_1) & Var(f_2) & Co\,\mathrm{var}(f_2, f_3) & Co\,\mathrm{var}(f_2, f_4) \\ f3 & Co\,\mathrm{var}(f_3, f_1) & Co\,\mathrm{var}(f_3, f_2) & Var(f_3) & Co\,\mathrm{var}(f_3, f_4) \\ f4 & Co\,\mathrm{var}(f_4, f_1) & Co\,\mathrm{var}(f_{4,}\ f_2) & Co\,\mathrm{var}(f_{4,}\ f_3) & Var(f_4) \end{array}$$

μ=0

σ=0

covar(x,y)= = $\frac{1}{N-1}\sum\left(x-\bar{x}\right)(y—\bar{y})$

var f_1 = $(-1.00)^2 + (0.33)^2 + (-1.00)^2 + (0.33)^2 + (1.33-0)^2/5$

=0.8

$$Co\,\mathrm{var}\ matrix\ = S = \begin{array}{c|cccc|} & f_1 & f_2 & f_3 & f_4 \\ f_1 & 0.8 & -0.25298 & 0.0385 & -0.1448 \\ f_2 & -0.25298 & 0.8 & 0.51121 & 0.4945 \\ f_3 & 0.0385 & 0.51121 & 0.8 & 0.75236 \\ f_4 & -0.1448 & 0.4945 & 0.75236 & 0.8 \end{array}$$

Now to find the Eigen values & Eigeen vectors of the above matrix

Consider (S- λI)v=0. The characteristic equation is | S- λI |=0

Solving this equation, we get the eigen values are 2.51579, 1.06529 , 0.39389 & 0.02503

For each eigen value, there is an eigen vector. $(S- \lambda I)v=0$

Solving the simultaneous equations, either using Cramer's rule or any other method

When $\lambda_1 = 2.51579$

$$e_1 = \begin{bmatrix} 0.16196 \\ -0.52405 \\ -0.58590 \\ -0.59655 \end{bmatrix}$$

When $\lambda_2 = 1.06529$

$$e_2 = \begin{bmatrix} -0.91701 \\ -0.20692 \\ -0.32054 \\ -0.11594 \end{bmatrix}$$

Similarly, $\lambda_3 = 0.39389$

$$e_3 = \begin{bmatrix} 0.30707 \\ -0.81732 \\ 0.18825 \\ 0.44973 \end{bmatrix} \text{ and}$$

$\lambda_4 = 0.02503$

$$e_4 = \begin{bmatrix} 0.19616 \\ 0.120615 \\ -0.72009 \\ 0.65455 \end{bmatrix}$$

sort $\lambda 1 > \lambda 2 > \lambda 3 > \lambda 4$

Now pick K eigen values. Here k = 2

2 top eigen values are λ_1 and λ_2 and the corresponding eigen vectors are

$$\begin{array}{cc} e_1 & e_2 \\ \end{array}$$

$$\begin{bmatrix} 0.16196 & -0.91706 \\ -0.52405 & 0.20692 \\ -0.58589 & -0.32054 \\ -0.59655 & -0.11594 \end{bmatrix}$$

Transform the original matrix in to

(Feature matrix) x (top k eigen vectors) = transformed data

$$\begin{bmatrix} -1 & -0.63246 & 0 & 0.26062 \\ 0.3333 & 1.2649 & 1.7321 & 1.5637 \\ -1 & 0.6325 & -0.5774 & -0.1738 \\ 0.3333 & 0 & -0.5774 & -1.0425 \\ 1.3333 & -1.2649 & -0.5774 & -0.6081 \end{bmatrix} \begin{bmatrix} 0.16196 & -0.91706 \\ -0.52405 & 0.20692 \\ -0.58589 & -0.32054 \\ -0.59655 & -0.11594 \end{bmatrix}$$

$$= \begin{bmatrix} 0.014003 & 0.755575 \\ -2.55653 & -0.78043 \\ -0.05148 & 1.253135 \\ 1.014150 & 0.000239 \\ 1.57986 & -1.228917 \end{bmatrix}$$