Sequential Probability Ratio Test

The Sequential Probability Ratio Test (SPRT) was developed by Abraham Wald more than a half century ago. It is widely used in quality control in manufacturing and detection of anomalies in medical trials.

SPRT Theory

SPRT was originally developed as an inspection tool to determine whether a given lot meets the production requirements. Basically, a sequential test is a method by which items are tested in sequence (one after another). The test results are reviewed after each test. Two tests of significance are applied to the data accumulated up to that time.

Concept of SPRT

Let's first use a simple example to explain the principal behind SPRT. Two vendors provide the same component to a company. Although the components from the two companies look exactly the same, their lifetime distributions are different. Components from vendor A have a mean life of $\mu_1 = 15$, and components made by vendor B have a mean life of $\mu_2 = 20$. An unlabeled box of components was received by the company. We want to determine if the components are from vendor A or from vendor B by conducting a test. The test should meet the following requirements:

- If the component is indeed from vendor A, the chance of making a wrong claim that it is from vendor B should be less than $\alpha_1 = 0.01$.
- If the component is indeed from vendor B, the chance of making a wrong claim that it is from vendor A should be less than $\alpha_2 = 0.05$.

Therefore, we need to conduct two statistical hypothesis tests. Since we know $\mu_2 > \mu_1$, the two tests are one-sided tests. The first test is for vendor A:

$$H_{0_1}: \quad \mu = \mu_1 \quad \text{at significance level of } \alpha_1$$

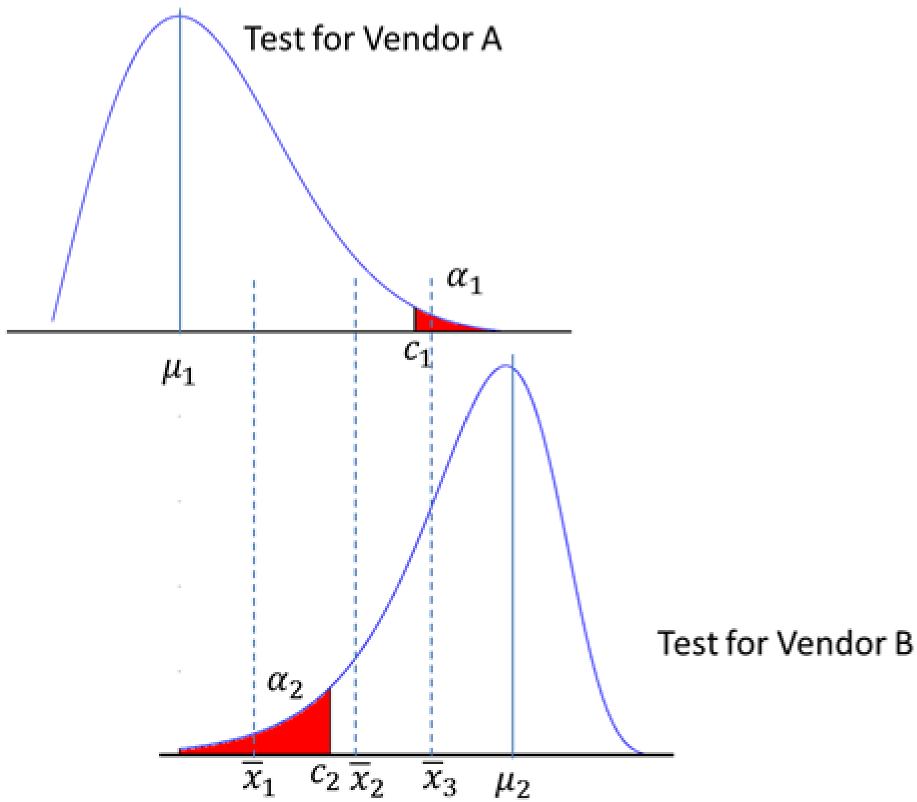
$$H_{A_1}: \quad \mu > \mu_1$$

- The second one is for vendor B:

$$H_{0_2}: \quad \mu = \mu_2 \quad \text{at significance level of } \alpha_2$$

$$H_{A_2}: \quad \mu < \mu_2$$

- These two separate hypothesis tests are shown graphically below:



- The top plot is for the first hypothesis test (vendor A). C_1 is the critical value at a significance level of α_1. If we take some samples and the sample mean is less than C_1, then we accept H_{0_1}, which is that the components are from vendor A. Otherwise, we accept H_{A_1}, that the components are not from vendor A.
- The bottom plot is for the second test (vendor B). C_2 is the critical value at a significance level of α_2. If a sample mean is greater than C_2, then we accept H_{0_2} that the component is from vendor B; otherwise, we accept H_{A_2}, that the component is not from vendor B.

When we take samples for the life test, the resulting sample mean has one of the following values:

- Assume a sample with mean $\bar{x}_1$ was drawn. For the test for vendor A, since it is less than C_1, we accept that $\mu = \mu_1$. For the test for vendor B, since it is less than C_2, we accept that $\mu < \mu_2$. The test is ended and we conclude that the component is from vendor A.
- Assume a sample with mean $\bar{x}_3$ was drawn. For the test for vendor A, since it is greater than C_1, we reject that $\mu = \mu_1$. For the test for vendor B, since it is greater than C_2, we accept that $\mu = \mu_2$. The test is ended and we conclude that the component is from vendor B.
- Assume a sample with mean $\bar{x}_2$ was drawn. For the test for vendor A, since it is less than C_1, we accept that $\mu = \mu_1$. For the test for vendor B, since it is greater than C_2, we accept

that $\mu = \mu_2$. We conclude the component is from both vendor A and vendor B, which is impossible. Therefore, the test is not ended and more samples are needed.

With more and more samples, the sample mean will be closer to the true population mean. The test will end with a conclusion either from vendor A or from vendor B. This is the principal behind a sequential test. A sequential probability ratio test is based on this idea.

Calculation of SPRT

Now assume the lifetime *t* of the component follows an exponential distribution. Let $\theta_A = \mu_A$ for vendor A and $\theta_B = \mu_B$ for vendor B. The probability density function (*pdf*) of the exponential distribution is:

$$f_\theta(t) = \frac{1}{\theta} e^{\left(-\frac{t}{\theta}\right)} \qquad \text{-------------------------------------(1)}$$

For an observed failure time *t*, if it is from vendor A, then the "probability" of observing it is:

$$\Lambda_{\theta_A} = f_{\theta_A}(t)\Delta t = \frac{1}{\theta_A} e^{\left(-\frac{t}{\theta_A}\right)} \Delta t \qquad \text{---(2)}$$

where Δt is a very small time duration around *t*.

If the observation is from vendor B, then the "probability" of observing it is:

$$\Lambda_{\theta_B} = f_{\theta_B}(t)\Delta t = \frac{1}{\theta_B} e^{\left(-\frac{t}{\theta_B}\right)} \Delta t \qquad \text{---(3)}$$

If the component is from vendor A, then Eqn. (2) will likely have a larger value than the one given in Eqn. (3), and vice versa.

The logarithm of the ratio of the above two probabilities is given by:

$$R=\ln\left(\frac{\Lambda_{\theta_B}}{\Lambda_{\theta_A}}\right)=\ln\left(\frac{\frac{1}{\theta_B}e^{\left(-\frac{t}{\theta_B}\right)}\Delta t}{\frac{1}{\theta_A}e^{\left(-\frac{t}{\theta_A}\right)}\Delta t}\right)=\left(-\frac{t}{\theta_B}+\frac{t}{\theta_A}\right)-\ln\left(\frac{\theta_B}{\theta_A}\right)$$

$$=\frac{\theta_B-\theta_A}{\theta_B\theta_A}t-\ln\left(\frac{\theta_B}{\theta_A}\right)$$

---------------------(4)

When there are more samples, the log-likelihood ratio becomes:

$$R=\ln\left(\frac{\Lambda_{\theta_B}}{\Lambda_{\theta_A}}\right)=\ln\left(\frac{\prod_{i=1}^{n}f_{\theta_B}(t_i)}{\prod_{i=1}^{n}f_{\theta_A}(t_i)}\right)=\frac{\theta_B-\theta_A}{\theta_B\theta_A}\sum_{i=1}^{n}t_i-n\ln\left(\frac{\theta_B}{\theta_A}\right)$$

--------------------(5)

If the ratio is greater than a critical value *U*, then the chance that the samples are from vendor B is much larger than the chance that the samples are from vendor A. We can conclude that the samples are from vendor B.

If the ratio is less than a critical value *L*, then the chance that the samples are from vendor A is much larger than the chance that the samples are from vendor B. We can conclude that the samples are from vendor A.

If the ratio is between *L* and *U*, then no conclusion can be made. More samples are needed. The decision is made based on the following formula:

$$L<R=\ln\left(\frac{\Lambda_{\theta_B}}{\Lambda_{\theta_A}}\right)<U$$

---(6)

But what are the values for *U* and *L*? *U* and *L* are determined based on the two significance levels α_1 and α_2. The significance level is also called a *Type I error*..

When the ratio is less than *L*, we accept vendor A:

$$\ln\left(\frac{\Lambda_{\theta_B}}{\Lambda_{\theta_A}}\right)<L\Rightarrow\frac{\Lambda_{\theta_B}}{\Lambda_{\theta_A}}<e^{L}$$

---(7)

When we accept vendor A, the probability of making the right decision (the component is from vendor A) should be greater than 1-α_1, as required by the hypothesis test. The probability of making the wrong decision (the component is actually from vendor B) should be less than α_2. Here α_1 is the Type I error α and α_2 is the Type II error β *for the hypothesis test for vendor A.*

Please note that Type I and Type II errors are related to a given statistical hypothesis test. Since SPRT combines two hypothesis tests together, it is very important to determine which one is the Type I error and which one is the Type II error.

When vendor A is accepted, based on the requirement for the Type I and Type II errors, we have:

$$\frac{\Lambda_{\theta_B}}{\Lambda_{\theta_A}} < \frac{\alpha_2}{\Lambda_{\theta_A}} < \frac{\alpha_2}{1-\alpha_1} \qquad \text{--(8)}$$

From Eqns. (7) and (8), we set:

$$e^L = \frac{\alpha_2}{1-\alpha_1} \Rightarrow L = \ln\left(\frac{\alpha_2}{1-\alpha_1}\right)$$

Similarly, when the ratio is larger than *U*, we accept vendor B:

$$\ln\left(\frac{\Lambda_{\theta_B}}{\Lambda_{\theta_A}}\right) > U \Rightarrow \frac{\Lambda_{\theta_B}}{\Lambda_{\theta_A}} > e^U$$

When we accept vendor B, the probability of making the right decision (the component is indeed from vendor B) should be greater than 1-α_2. The probability of making the wrong decision (the component is actually from vendor A) should be less than α_1. Here α_2 is the Type I error and α_1 is the Type II error for the *hypothesis test for vendor B*. Therefore, we have:

$$\frac{\Lambda_{\theta_B}}{\Lambda_{\theta_A}} > \frac{1-\alpha_2}{\Lambda_{\theta_A}} > \frac{1-\alpha_2}{\alpha_1}$$

From Eqns. (7) and (8), we can set:

$$e^U = \frac{1-\alpha_2}{\alpha_1} \Rightarrow U = \ln\left(\frac{1-\alpha_2}{\alpha_1}\right)$$

Combining all the above equations, we get the decision formula for SPRT as the follows:

$$\ln\left(\frac{\alpha_2}{1-\alpha_1}\right) < \frac{\theta_B-\theta_A}{\theta_B\theta_A}\sum_{i=1}^{n} t_i - n\ln\left(\frac{\theta_B}{\theta_A}\right) < \ln\left(\frac{1-\alpha_2}{\alpha_1}\right)$$

which is

$$\ln\left(\frac{\alpha_2}{1-\alpha_1}\right) + n\ln\left(\frac{\theta_B}{\theta_A}\right) < \frac{\theta_B-\theta_A}{\theta_B\theta_A}\sum_{i=1}^{n} t_i < \ln\left(\frac{1-\alpha_2}{\alpha_1}\right) + n\ln\left(\frac{\theta_B}{\theta_A}\right)$$

SPRT can be used for any distribution. The likelihood ratio can be calculated based on the assumed distribution.