Picard's Method:

Procedure for establishing the existence of a solution to a differential equation that passes through a point. This method of solving a differential equation approximately is one of successive approximation; that is, it is an iterative method in which the numerical results become more and more accurate, the more times it is used. It is used to convert the differential equation into an integral equation which involves integrals.

An approximate value of y (taken at first to be a constant) is substituted to the right hand side of the differential equation $\frac{dy}{dx} = f(x, y)$. Then this equation is integrated with respect to x as a second approximation, into which given numerical values are substituted and the result rounded off to a assigned number of decimal places or significant figures.

$$(y - y_0) = \int_{x_0}^{x} f(x, y)\,dx$$

$$i.e.,\ y = y_0 + \int_{x_0}^{x} f(x, y)\,dx$$

The iterative process is continued until two consecutive numerical solutions are the same when rounded off to the required number of decimal places.

After getting the first approximation $y^{(1)}$.

$$y^{(1)} = y_0 + \int_{x_0}^{x} f(x, y_0)\,dx$$

After getting the first approximation $y^{(1)}$ for y, use this value of $y^{(1)}$ in the place of y in *f(x,y)* of (1) and then integrate to get the second approximation $y^{(2)}$.

$$y^{(2)} = y_0 + \int_{x_0}^{x} f\left(x, y^{(1)}\right)dx$$

Proceeding in this way we get the n^{th} approximation of y as

$$y^{(n)} = y_0 + \int_{x_0}^{x} f\left(x, y^{(n-1)}\right)dx$$ which is the Picard's iterative formula.

The sequence $y^{(1)}, y^{(2)}, y^{(3)},\ldots,y^{(n)}$ should c0onverge to *y(x)*. Otherwise the process is not valid.

The condition for the convergence of the sequence are *f(x,y)* and $\frac{\partial f}{\partial y}$ are continuous.

1. Solve $y' = y - x^2$, $y(0) = 1$, by Picard's method upto the third approximation. Hence, find the value of *y(0.1), y(0.2)*.

$y' = y - x^2$

$\therefore y = y_0 + \int_{x_0}^{x} \left(y - x^2\right) dx$ (1)

$use \ \ y = y_0 = 1 \ on \ the \ RHS$

$y^{(1)} = y_0 + \int_{x_0}^{x} \left(1 - x^2\right) dx = 1 + x - \frac{x^3}{3}$ (2)

$use \ (2) \ again \, in \ (1)$

$y^{(2)} = 1 + \int_{0}^{x} \left(1 + x - \frac{x^3}{3} - x^2\right) dx = \left(1 + x + \frac{x^2}{2} - \frac{x^4}{12} - \frac{x^3}{3}\right)$ (3)

$use \ (3) \ again \, in \left(1\right),$

$y^{(3)} = 1 + \int_{0}^{x} \left(1 + x + \frac{x^2}{2} - \frac{x^4}{12} - \frac{x^3}{3} - x^2\right) dx = 1 + x + \frac{x^2}{2} - \frac{x^4}{12} - \frac{x^3}{6} - \frac{x^5}{60} + \ldots$ (4)

$Putting \ \ x = 0.1 \ in \ (4)$

$$y\ (0.1) = 1 + 0.1 + \frac{0.1^2}{2} - \frac{0.1^4}{12} - \frac{0.1^3}{6} - \frac{0.1^5}{60} + ..$$

$$= 1 + 0.1 + 0.005 - 0.0001666 - 0.00000833 - 0.000000166 = 1.1048249$$

$$y\ (0.2) = 1 + 0.2 + \frac{0.2^2}{2} - \frac{0.2^4}{12} - \frac{0.2^3}{6} - \frac{0.2^5}{60} + ..$$

$$= 1 + 0.2 + 0.02 - 0.000133333 - 0.000133333 - 0.000005333 = 1.218528$$

2. Solve $\frac{dy}{dx} = x + y$ given *y(0)=1*. Obtain the values of *y(0.1),y(0.2)* using Picard's method and check your answer with the exact solution.

$f\left(x, y\right) = x + y, \ x_0 = 0, y_0 = 1$

$\therefore y = y_0 + \int_{x_0}^{x} f\left(x, y\right) dx$

$y = 1 + \int_{0}^{x} f\left(x, y\right) dx$ (1)

$Putting \ y = y_0 \ on \ the \, R.H.S.,$

$y^{(1)} = 1 + \int_{x_0}^{x} f\left(x, 1\right) dx$

$= 1 + \int_{x_0}^{x} \left(x + 1\right) dx = 1 + x + \frac{x^2}{2}$ (2)

$Again \, use \ y = y^{(1)} on \, the \, R.H.S. \ on \left(1\right), we \ get$

$y^{(2)} = 1 + \int_{x_0}^{x} \left(x + 1 + x + \frac{x^2}{2}\right) dx = 1 + x + x^2 + \frac{x^3}{6}$ (3)

$Again \, use \ y = y^{(2)} on \, the \, R.H.S. \ on \left(1\right)$

$y^{(3)} = 1 + \int_{x_0}^{x} \left(x + 1 + x + x^2 + \frac{x^3}{6}\right) dx = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{24} + \ldots$

Setting $x = 0.1$

$$y(0.1) = 1 + 0.1 + 0.001 + \frac{(0.001)}{3} + \frac{(0.0001)}{24}$$

$$= 1 + 0.1 + 0.01 + 0.0003333 + 0.0000041$$

$$= 1.1103374$$

$$y(0.2) = 1 + 0.2 + (0.2)^2 + \frac{(0.2)^3}{3} + \frac{(0.2)^4}{24} = 1.242733$$

Integrating $\frac{d\,y}{d\,x} = x + y$, *we get* $y = 2e^x - x - 1$

$$\therefore y(0.1) = 2\,e^{0.1} - 0.1 - 1 = 1.11034184 \quad (\textit{actual value})$$

$$y(0.2) = 2\,e^{0.2} - 0.2 - 1 = 1.24280555$$