Name of the Course: NUMERICAL &STATISTICAL ANALYSIS
Topic: Numerical Solution of First order simultaneous differential equations(ODE) & Second order ODE (Ordinary Differential Equations)

Solving first order simultaneous ODE using Taylor's series method:

Algorithm for solving first order simultaneous ODE:

The differential equation can be written as

$$y' = \frac{dy}{dx} = f_1(x, y, z)$$

$$z' = \frac{dz}{dx} = f_2(x, y, z)$$

Here y and z are dependent variables and x is an independent variable. The initial conditions are $y(x_o) = y_o$ and $z(x_o) = z_0$.

Taylor's series of y at y_1 is given by

$$y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \frac{h^4}{4!} y_0^{iv} + \cdots \quad (1)$$

$$z_1 = z_0 + \frac{h}{1!} z_0' + \frac{h^2}{2!} z_0'' + \frac{h^3}{3!} z_0''' + \frac{h^4}{4!} z_0^{iv} + \cdots \quad (2)$$

Higher order derivatives of y y'', y''', y^{iv} and z'', z''', z^{iv} are determined because we need to get the values of the derivatives at x_o.

Substitute the values of y_0'', y_0''', y_0^{iv} and z_0'', z_0''', z_0^{iv} in the equations (1) and (2) respectively. Then calculate the value of y_1 and z_1. If y_2 and z_2 values are need to be determined, then the same algorithm is used. Instead of y_0 and z_0, y_1 and z_1 values are used.

Now we solve the simultaneous ODE using Taylor's method.

1. Solve: $\frac{dy}{dx} = z - x$ *and* $\frac{dz}{dx} = y + x$, with $y(0) = 1$ and $z(0) = 1$ by taking $h = 0.1$. Get $y(0.1)$ and $z(0.1)$ using Taylor's series method.

$$y' = z - x \qquad z' = y + x$$
$$y'' = z' - 1 \qquad z'' = y' + 1$$
$$y''' = z'' \qquad z''' = y''$$
$$y^{iv} = z''' \qquad z^{iv} = y'''$$

Here $z_0 = 1, y_0 = 1, x_0 = 0$

$$y'_0 = z_0 - x_0 = 1 \qquad z'_0 = y_0 + x_0 = 1$$
$$y''_0 = z'_0 - 1 = 0 \qquad z''_0 = y'_0 + 1 = 2$$
$$y'''_0 = z''_0 = 2 \qquad z'''_0 = y''_0 = 0$$
$$y^{iv} = z'''_0 = 0 \qquad z_0{}^{iv} = y'''_0 = 2$$

$$y(0.1) = y_1 = 1 + \frac{0.1}{1!}(1) + \frac{(0.1)^2}{2!}(0) + \frac{(0.1)^3}{3!}(2) + \frac{(0.1)^4}{4!}(0) = 1.1003$$
$$z(0.1) = z_1 = 1 + \frac{0.1}{1!}(1) + \frac{(0.1)^2}{2!}(2) + \frac{(0.1)^3}{3!}(0) + \frac{(0.1)^4}{4!}(2) = 1.1100$$

2. Find *y(0.1) , y(0.2), z(0.1), z(0.2)* using Taylor's series method given

$\frac{dy}{dx} = z + x$ *and* $\frac{dz}{dx} = x - y^2$, $y(0) = 2$ and $z(0) = 1$.

$x_0=0;\ x_1=0.1,\ x_2 = 0.2,\ h=x_2 - x_1 = 0.1, y_0=2, z_0=1.$

$$y' = z + x \qquad z' = x - y^2$$
$$y'' = z' + 1 \qquad z'' = 1 - 2yy'$$
$$y''' = z'' \qquad z''' = -2(y'^2 + yy')$$
$$y^{iv} = z''' \qquad z^{iv} = -2(3y'y'' + y\,y''')$$

$$y'_0 = z_0 + x_0 = 1 \qquad z'_0 = x_0 - y_0{}^2 = -4$$
$$y''_0 = z'_0 + 1 = -3 \qquad z''_0 = 1 - 2y_0 y'_0 = -3$$
$$y'''_0 = z''_0 = -3 \qquad z'''_0 = -2(y'_0{}^2 + y_0 y'_0) = 10$$
$$y_0{}^{iv} = z'''_0 = 10 \qquad z^{iv} = -2(3y'y'' + y\,y''') = -30$$

$$y(0.1)=y_1=2+\frac{0.1}{1!}(1)+\frac{(0.1)^2}{2!}(-3)+\frac{(0.1)^3}{3!}(-3)+\frac{(0.1)^4}{4!}(10)=2.0845$$

$$z(0.1)=z_1=1+\frac{0.1}{1!}(-4)+\frac{(0.1)^2}{2!}(-3)+\frac{(0.1)^3}{3!}(10)+\frac{(0.1)^4}{4!}(30)=0.5867$$

In the similar way y(0.2) and z(0.2) is calculated.

$$y_2=y_1+\frac{h}{1!}y_1'+\frac{h^2}{2!}y_1''+\frac{h^3}{3!}y_1'''+\frac{h^4}{4!}y_1^{iv}+\cdots$$

$$z_2=z_1+\frac{h}{1!}z_1'+\frac{h^2}{2!}z_1''+\frac{h^3}{3!}z_1'''+\frac{h^4}{4!}z_1^{iv}+\cdots$$

y_2= 2.1367 and z_2=0.1550

Solving Second order ODE using Taylor's series method:

Algorithm for solving Second order ODE:

$$y''=\frac{d^2y}{dx^2}=f_1(x,y,\frac{dy}{dx})$$

Together with initial conditions $y(x_0)=y_0$, $y'(x_0)=y_0'$.

$$y'=\frac{dy}{dx}=z=f_1(x,y,z)$$

$$y''=z'=\frac{d^2y}{dx^2}=f_2(x,y,z)$$

$$y_1=y_0+\frac{h}{1!}y_0'+\frac{h^2}{2!}y_0''+\frac{h^3}{3!}y_0'''+\frac{h^4}{4!}y_0^{iv}+\cdots$$

$$z_1=z_0+\frac{h}{1!}z_0'+\frac{h^2}{2!}z_0''+\frac{h^3}{3!}z_0'''+\frac{h^4}{4!}z_0^{iv}+\cdots$$

$$y_1 = y_0 + \frac{h}{1!} z_0 + \frac{h^2}{2!} z'_0 + \frac{h^3}{3!} z''_0 + \frac{h^4}{4!} z'''_0 + \cdots$$

Since $y_o' = z_0$, $y_o'' = z_0'$, $y_o''' = z_0''$, $y_o^{iv} = z_0'''$ substitute those values in y_1 above.

1. Evaluate the values of y(0.1) and y(0.2 when
   $$y'' - x(y')^2 + y^2 = 0$$
   Given y(0) = 1 and y'(0) = 0

Take z = y'

The equation reduces to

$$z' - xz^2 + y^2 = 0$$

or

$$z' = xz^2 - y^2$$

with the initial conditions

$x_0=0;\ x_1=0.1,\ h=x_2 - x_1 = 0.1, y_0=1, z_0=0.$

Here

$$z_1 = z_0 + \frac{h}{1!} z'_0 + \frac{h^2}{2!} z''_0 + \frac{h^3}{3!} z'''_0 + \frac{h^4}{4!} z_0^{iv} + \cdots$$

We get the derivatives of z.

$$z' = xz^2 - y^2$$
$$z'' = z^2 + 2xzz' - 2yy'$$
$$z''' = 2zz' + 2\left(xzz'' + xz' + xz'^2\right) - 2(y'^2 + yy')$$
$$\therefore z'_0 = x_0 z_0{}^2 - y_0{}^2 = -1$$
$$z''_0 = z_0{}^2 + 2x_0 z_0 z'_0 - 2y_0 y'_0 = 0$$
$$z'''_0 = 2z_0 z'_0 + 2\left(x_0 z_0 z''_0 + x_0 z'_0 + x_0 z'_0{}^2\right) - 2(y'_0{}^2 + y_0 y'_0) = 2$$

$$z(0.1)=z_1=0+\frac{0.1}{1!}(-1)+\frac{(0.1)^2}{2!}(0)+\frac{(0.1)^3}{3!}(2)+...=-0.0997$$

By Taylor series for y_1

$$y_1=y_0+\frac{h}{1!}y_0'+\frac{h^2}{2!}y_0''+\frac{h^3}{3!}y_0'''+\frac{h^4}{4!}y_0^{iv}+\cdots$$
$$=1+\frac{0.1}{1!}z_0+\frac{0.1^2}{2!}z_0'+\frac{0/1^3}{3!}z_0'''+...$$
$$=1+\frac{0.1}{1!}(0)+\frac{0.1^2}{2!}(-1)+\frac{0/1^3}{3!}(0)+...$$
$$=1-0.005=0.995$$

Similarly y(0.2)

$$y_2=y_1+\frac{h}{1!}y_1'+\frac{h^2}{2!}y_1''+\frac{h^3}{3!}y_1'''+\frac{h^4}{4!}y_1^{iv}+\cdots$$
$$=0.995+\frac{0.1}{1!}z_1+\frac{0.1^2}{2!}z_1'+\frac{0/1^3}{3!}z_1'''+...$$
$$z_1'=x_1z_1{}^2-y_1{}^2=(0.1)(-0.0997)^2-0.995^2=-0.9890$$
$$z_0''=-0.1687$$
$$=0.995+\frac{0.1}{1!}(-0.0997)+\frac{0.1^2}{2!}(-0.9890)+\frac{0.1^3}{3!}(-0.1687)=0.9801$$

$$y(0.1)=0.9950 \qquad \textit{and} \qquad y(0.2)=0.9801$$

2. Solve y''=y + xy' given y(0)=1, y'(0) = 0.And calculate y(0.1) using Taylors series. y(0.1)=1.005

Reference : Numerical methods , P.Kandasamy, K.Thilagavathy, K.Gunavathy, S.Chand & Company Ltd . Reprint 2008