Muller Method

In this method, first we assume three approximate roots x_{i-1}, x_i and x_{i+1} of the equation $f(x) = 0$. The next better approximation x_{i+2} is obtained as the root of second degree polynomial equation $p(x) = 0$, where the second degree parabola $y = p(x)$ is assumed to pass through the points $\{x_{i-1}, f(x_{i-1})\}$, $\{x_i, f(x_i)\}$ and $\{x_{i+2}, f(x_{i+2})\}$.

Algorithm

1. Let x_0, x_1 and x_2 be the initial approximate roots of $f(x) = 0$. Compute $f(x_0)$, $f(x_1)$ and $f(x_2)$.

2. Compute $h_2 = x_2 - x_1$; $h_1 = x_1 - x_0$

   $f(x_2, x_1) = [f(x_2) - f(x_1)] / h_2$; $f(x_1, x_0) = [f(x_1) - f(x_0)] / h_1$

3. Set $k = 2$

   Compute $f(x_k, x_{k-1}, x_{k-2}) = [\, f(x_k, x_{k-1}) - f(x_{k-1}, x_{k-2})] / (h_k + h_{k-1})$

   $c_k = f(x_k, x_{k-1}) + h_k\, f(x_k, x_{k-1}, x_{k-2})$

$$h_{k+1} = \frac{-2f(x_k)}{c_k \pm \sqrt{c_k^{\,2} - 4f(x_k)\, f(x_k, x_{k-1}, x_{k-2})}}$$

   Choosing the sign so that the denominator is largest in magnitude

   Set $x_{k+1} = x_k + h_{k+1}$

4. Compute $f(x_{k+1})$ and $f(x_{k+1}, x_k) = \dfrac{f(x_{k+1}) - f(x_k)}{h_{k+1}}$

Set $k = k + 1$ and repeat steps (3) to (4) until we get the root with required degree of accuracy.

NOTE: This method converges for all initial approximations. If no better approximations are known, we choose $x_0 = -1$; $x_1 = 0$ and $x_2 = 1$.

1. Perform five iterations for the Muller method to find the root of the equation
$f(x) = \cos x - xe^x = 0$

Sol : Let $x_0 = -1$; $x_1 = 0$ and $x_2 = 1$

$h_2 = x_2 - x_1 = 1$; $h_1 = x_1 - x_0 = 1$

$f(x_2, x_1) = -3.1780$: $f(x_1, x_0) = 0.0918$

$f(x_2, x_1, x_0) = -1.6349$

$c_k = c_2 = f(x_2, x_1) + h_2\, f(x_2, x_1, x_0) = -4.8129$

$h_3 = -0.5584$; $\mathbf{x_3 = x_2 + h_3 = 0.4416}$

$c_3 = f(x_3, x_2) + h_3\, f(x_3, x_2, x_1)$

$f(x_3, x_2) = -4.2896$

$h_4 = 0.0710$; $\mathbf{x_4 = x_3 + h_4 = 0.5126}$

$c_4 = -2.8408 + (0.0710)*(-2.9725) = -3.0518$

$h_5 = 0.0051$

$\mathbf{x_5 = x_4 + h_5 = 0.5177.}$

2. Find the root of the equation $x^3 + x^2 - 1 = 0$ that lies between 0 and 1, correct to four places of decimals using Muller method.

Sol : Let $x_0 = 0$; $x_1 = 0.5$ and $x_2 = 1$

$h_2 = x_2 - x_1 = 0.5$; $h_1 = x_1 - x_0 = 0.5$

$f(x_3, x_2, x_1) = - 2.5172$; $c_3 = - 2.8834$

$f(x_2, x_1) = 3.25$: $f(x_1, x_0) = 0.75$

$f(x_2, x_1, x_0) = 2.5$

$c_k = c_2 = f(x_2, x_1) + h_2\, f(x_2, x_1, x_0) = 4.5$

$h_3 = - 0.2597$; $\mathbf{x_3 = x_2 + h_3 = 0.7403}$

$c_3 = f(x_3, x_2) + h_3\, f(x_3, x_2, x_1)$

$f(x_3, x_2) = 4.0286$

$f(x_3, x_2, x_1) = 3.2403$; $c_3 = 3.1872$

$h_4 = 0.0142$; $\mathbf{x_4 = x_3 + h_4 = 0.7546}$

$\mathbf{x_5 = x_6 = 0.7549}$

3. Use Muller method to find the root of $x^3 - 5x - 6 = 0$ that lies between 2 and 3.

Ans : 2.689

Birge-Vieta Method

In this method we seek to determine a real number p such that $(x - p)$ is a factor of the polynomial equation $p_n(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_n = 0$. If we divide the given equation by the factor (x-p) then we get a quotient Q and remainder R. The value of R depends upon P. We apply Newton-Raphson method and improve the initial approximation value. For the polynomial equations, the computation are systematized using synthetic division.

	a_0	a_1	a_2	a_3	. .	.	.	a_n
p	0	b_0p	b_1p	b_2p	. .	.	.	$b_{n-1}p$
	$a_0=b_0$	b_1	b_2	b_3	. .	.	.	b_n
	0	c_0p	c_1p	c_2p	. .	.	$b_{n-1}p$	
	$b_0=c_0$	c_1	c_2	c_3			c_{n-1}	

$P_{k+1} = p_k - (b_n / c_{n-1})$; $k = 0, 1, 2, 3, \ldots$

1. Use synthetic division and perform two iterations by Birge- Vieta method to find the smallest positive root of the equation $x^4 - 3x^3 + 3x^2 - 3x + 2 = 0$

Sol : Let $p_0 = 0.5$

	1	-3	3	-3	2
0.5		0.5	-1.25	0.875	-1.0625
	1	-2.5	1.75	-2.125	$0.9375 = b_4$
		0.5	-1.00	0.375	
	1	-2.0	0.75	$-1.750 = c_3$	

$P_1 = p_0 - (b_4 / c_3) = 0.5 + (0.9375 / 1.750) = 1.0356$

	1	-3	3	-3	2
1.0356		1.0356	-2.0343	1.0001	-2.0711

1	-1.9644	0.9657	-1.9999	-0.0711
	1.0356	-0.9619	0.0039	
1	-0.9288	0.0038	- 1.9960	

$P_2 = p_1 - (b_4 / c_3) = 1.0356 - (- 0.0711 / - 1.9960)$

$= 0.999979$

The exact root is 1.0

2.Using Birge-Vieta method find a real root correct to three decimal places of the equation $x^3-11x^2+32x-22=0$ with P = 0.5.

Sol : Using synthetic division

	1	-11	32	-22
0.5	0	0.5	-5.25	13.375
	1	-10.5	26.75	$-8.625= b_n$
0.5	0	0.5	-5	
	1	-9.5	$21.75= c_{n-1}$	

$p_1= 0.5-(-8.625)/(21.75)=0.89655$

Now we divide the given equation with x-0.89655.

1	-11	32	-22	
0.89655		0.89655	-9.058248	20.56842776
	1	-10.10345	22.941752	-1.4315722= b_n
		0.89655	-8.254446	
	1	-9.2609	14.687306= c_{n-1}	

p_2 = 0.89655-(-1.43115722)/(14.687306)=0.99402. Now we divide the given equation with x - 0.99402.

	1	-11	32	-22
0.99402		0.99402	-9.94614	21.921977
	1	-10.00598	22.05386	-0.078023= b_n
		0.99402	-8.958068	
	1	-9.01196	13.095792 = c_{n-1}	

P_3 = 0.99402-(-0.078023)/(13.095792)=0.999978.

Now we divide the given equation with x-0.999978.

	1	-11	32	-22
0.999978		0.999978	-9.999802	21.999714
	1	-10.000022	22.000198	-0.000286= b_n
		0.999978	-8.999845999	
	1	-9.000044	13.000352= c_{n-1}	

P_4 = 0.999978 - (-0.000286) / (13.000352)

= 0.99999.

Therefore, the root correct to three decimal places is 0.999.

Graeffe's Root Squaring method

This method is a direct method. This method is used to find the roots of a polynomial equation with real coefficients. An equation is of the form $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots\ldots + a_n = 0$ where a_i's ; $i = 0, 1, 2, \ldots\ldots, n$ are real. Wkt the n^{th} degree equation has 'n' only 'n' roots. Let $\alpha_1, \alpha_2, \alpha_3, \ldots\ldots, \alpha_n$ be roots of $f(x) = 0$. To find the roots of $f(x) = 0$ by this method, first to form the table.

	m	2^m	Coefficients							
Given equn	0	1	a_0	a_1	a_2	a_3	a_4	a_5	.	.
.			a_0^2	a_1^2	a_2^2	a_3^2	a_4^2	a_5^2	.	.
				$-2a_0a_2$	$-2a_1a_3$	$-2a_2a_4$	$-2a_3a_5$	$-2a_4a_6$	.	.
					$2a_0a_4$	$2a_1a_5$				
1^{st} Squaring	1	2	$a_0^2=b_0$	b_1	b_2	b_3	b_4	b_5	.	.
			b_0^2	b_1^2	b_2^2	b_3^2	b_4^2	b_5^2	.	.
				$-2b_0b_2$	$-2b_1b_3$	$-2b_2b_4$	$-2b_3b_5$	$-2b_4b_6$		.
					$2b_0b_4$	$2b_1b_5$	$2b_2b_6$			.
2^{nd} Squaring	2	4	$b_0^2 = c_0$	c_1	c_2	c_3	c_4	c_5 .		.
.										
			B_0	B_1	B_2	B_3	B_4	B_5		.

The second term is minus two times the product immediate neighboring coefficients a_{r-1} and a_{r+1} ; $r = 1, 2, 3, \ldots\ldots\ldots\ldots\ldots$. The third term is twice the product immediate neighboring coefficients a_{r-2} and a_{r+2} ; $r = 2, 3, 4,\ldots\ldots\ldots\ldots\ldots$ and so on. Here sign alternates. This process continued until there are no available coefficients to get the product term. The sum of these coefficients can be taken as some new coefficients namely $b_0, b_1, b_2, \ldots\ldots\ldots\ldots c_0, c_1, c_2,\ldots\ldots$
After a few squaring process, the new coefficients can be taken as $B_0, B_1, B_2, B_3, \ldots\ldots, Bn$.
Here there are three cases.

Case 1: The process of squaring is stopped when another process of squaring produces new coefficients, which are approximately the squares of the corresponding coefficients B_i's;

$i = 1,2,3,\ldots.$ $\left|R_i\right| = \left|\alpha_i\right|^{2^m} = \left|\frac{B_i}{B_{i-1}}\right|$ where $R_1, R_2, R_3,\ldots.., R_n$ are the roots of the new equation $B_0x^n + B_1x^{n-1} + B_2x^{n-2} + \ldots\ldots + B_n = 0$ which are 2^m th power of the roots of the given equation with sigh changed.

Case 2: After a few squaring process, if the magnitude of the coefficient B_i is half the square of the magnitude of corresponding coefficient in the previous equation, then this indicates that α_i is a double root. $\left|R_i^2\right| = \left|\alpha_i\right|^{2^{m+1}} = \left|\frac{B_{i+1}}{B_{i-1}}\right|$.

Case 3: If α_k and α_{k+1} are two complex conjugate roots, then this would make the coefficients of x^{n-k} in the successive squaring to fluctuate both in magnitude and sign. $\beta_k^{2(2^m)} = \left|\frac{B_{k+1}}{B_{k-1}}\right|$. If the equation possesses only two complex roots namely p ± iq. Wkt sum of the roots = $-a_1/a_0$. This gives the value of p. Since $|\beta_k|^2 = p^2 + q^2$ and $|\beta_k|$ is known already, q is known from this relation..

1. Solve by Graeffe's method $2x^3 + x^2 - 2x - 1 = 0$ (4 squarings)

Sol:

	m	**2^m**	**coefficients**			
Given equn	0	1	2	1	-2	-1
			4	1 -2*2*-2	4 -2*1*-1	1
1^{st} squaring	1	2	4	9	6	1
			16	81 -48	36 -18	1
2^{nd} squaring	2	4	16	33	18	1
			256	1089 -576	324 -66	1
3^{rd} squaring	3	8	256	513	258	1
			65536	233169 -132096	66564 -1026	1
4^{th} squaring	4	16	65536	131073	65538	1
			B_0	B_1	B_2	B_3

$$|R_i| = |\alpha_i|^{2^m} = \left|\frac{B_i}{B_{i-1}}\right|$$

$$|R_1| = |\alpha_1|^{2^4} = \left|\frac{B_1}{B_0}\right| = \left|\frac{131073}{65536}\right|$$

$$|\alpha_1| = \left|\frac{131073}{65536}\right|^{\frac{1}{16}} = 1.0442 \cong 1$$

$$|R_2| = |\alpha_2|^{2^4} = \left|\frac{B_2}{B_1}\right| = \left|\frac{65538}{131073}\right|$$

$$|\alpha_2| = \left|\frac{65538}{131073}\right|^{\frac{1}{16}} = 0.9576 \cong 1$$

$$|R_3| = |\alpha_3|^{2^4} = \left|\frac{B_3}{B_2}\right| = \left|\frac{1}{65538}\right|$$

$$|\alpha_3| = \left|\frac{1}{65538}\right|^{\frac{1}{16}} = 0.49999.$$

Now we have to find the sign of the roots. i.e., α_1 is a root or - α_1 is a root . i.e., by verifying $f(\alpha_1) = 0$ or $f(-\alpha_1) = 0$. Using Descarte's rule of signs, we can find the sign of the roots.

Descarte's Rule of signs: 1. An equation f(x) = 0 cannot have more number of positive roots then there are changes of signs in terms of the polynomial f(x).

2. An equation f(x) = 0 cannot have more number of negative roots then there are changes of signs in terms of the polynomial f(-x).

Using Descarte's rule, the given equation has one sign change in terms of f(x) and two sign changes in terms of f(-x) and so the given equation has one positive root and two negative roots. The roots of the given equation are 1, -1 and 0.5.

2. Solve $x^3 - x^2 - x - 2 = 0$ by Graeffe's root squaring method.

Sol:

	m	**2^m**		**coefficients**		
Given equn	0	1	1	-1	-1	-2
			1	1 -2*1*-1	1 -2*-1*-2	4
1^{st} squaring	1	2	1	3	-3	4
			1	9 6	9 -24	16
2^{nd} squaring	2	4	1	15	-15	16
			1	225 30	225 -480	256
3^{rd} squaring	3	8	1	255	-255	256
			1	65025 510	65025 -130560	65536
4^{th} squaring	4	16	1	65535	-65535	65536
			1	$4.2948\text{x}10^9$ 131070	$4.2948\text{x}10^9$ $-8.5898\text{x}10^9$	$4.2949\text{x}10^9$
5^{th} squaring	5	32	1	$4.2949\text{x}10^9$	$-4.2949\text{x}10^9$	$4.2949\text{x}10^9$
			B_0	B_1	B_2	B_3

Here, the coefficients of x in the successive squaring to fluctuate both in magnitude and sign. This indicates that, the root is a complex root. Here the coefficient B_1 is approximately equal to previous square value. Hence $\left|R_1\right| = \left|\alpha_1\right|^{2^5} = \left|\frac{B_1}{B_0}\right| = \left|\frac{4.2949x10^9}{1}\right|$

$$\left|\alpha_1\right| = \left|4.2949\right|^{\frac{1}{32}} \cong 2$$

$$\beta_k^{\,2(2^m)} = \left|\frac{B_{k+1}}{B_{k-1}}\right| \qquad \beta_2^{\,2(2^5)} = \left|\frac{B_3}{B_1}\right| = \left|\frac{4.2949x10^9}{4.2949x10^9}\right| \Rightarrow \beta_2 = 1$$

If p + iq is one root then p – iq is also a root. Wkt sum of the roots = $-a_1/a_0$.

So α_1 + p + iq + p – iq = -(-1) / 1 = 1. i.e., α_1 + 2p = 1 or p = - 0.5 (since α_1 = 2)

Since (p + iq) (p – iq) = $p^2 + q^2 = \beta_2^2$. i.e., $(-0.5)^2 + q^2 = 1$. This implies that q = 0.866. Therefore the roots of the equation are 2, -0.5+0.866 and -0.5-0.866.

1. Perform two iterations with Muller method for the equation $\log_{10}x - x + 3 = 0$; $x_0 = 1/4$; $x_1 = 1/2$ and $x_2 = 1$.

Sol : Let $x_0 = 0.25$; $x_1 = 0.5$ and $x_2 = 1$

$h_2 = x_2 - x_1 = 0.5$; $h_1 = x_1 - x_0 = 0.25$

$f(x_2, x_1) = [f(x_2) - f(x_1)] / h_2$; $f(x_1, x_0) = [f(x_1) - f(x_0)] / h_1$

$f(x_2, x_1) = - 0.39794$: $f(x_1, x_0) = 0.20412$

$f(x_2, x_1, x_0) = [\, f(x_2, x_1) - f(x_1, x_0)] / (h_2 + h_1)$

$f(x_2, x_1, x_0) = -0.802747$

$c_2 = f(x_2, x_1) + h_2\, f(x_2, x_1, x_0) = - 0.799313$

$h_{2+1} = -2\, f(x_2) / (c_2 \pm \sqrt{(c_2^2 - 4\, f(x_2)\, f(x_2, x_1, x_0)}\,)$

$h_3 = 1.157225$; $\mathbf{x_3 = x_2 + h_3 = 2.157225}$

$c_3 = f(x_3, x_2) + h_3\, f(x_3, x_2, x_1)$

$f(x_3, x_2) = - 0.711469$

$f(x_3, x_2, x_1) = -0.189189$; $c_3 = - 0.930404$

$h_4 = 1.043339$; $\mathbf{x_4 = x_3 + h_4 = 3.200564}$

2. Use the Birge - vieta method to find a real root correct to three decimals of the equation $x_5 - x + 1 = 0$; $p_0 = - 1.5$

Sol : Wkt $p_{k+1} = p_k - (b_n / c_{n-1})$; $k = 0, 1, 2, \ldots\ldots$

First iteration $p_0 = -1.5.$

$$\begin{array}{r|rrrrrr}
-1.5 & 1 & 0 & 0 & 0 & -1 & 1 \\
 & & -1.5 & 2.25 & -3.375 & 5.0625 & -6.0938 \\
\hline
 & 1 & -1.5 & 2.25 & -3.375 & 4.0625 & -5.0938 \\
 & & -1.5 & 4.5 & -10.125 & 20.25 & \\
\hline
 & 1 & -3 & 6.75 & -13.5 & 24.3125 &
\end{array}$$

$$p_1 = -1.5 + \frac{5.0938}{24.3125} = -1.2905,$$

Second iteration $p_1 = -1.2905.$

$$\begin{array}{r|rrrrrr}
-1.2905 & 1 & 0 & 0 & 0 & -1 & 1 \\
 & & -1.2905 & 1.6654 & -2.1492 & 2.7735 & -2.2887 \\
\hline
 & 1 & -1.2905 & 1.6654 & -2.1492 & 1.7735 & -1.2887 \\
 & & -1.2905 & 3.3308 & -6.4476 & 11.0941 & \\
\hline
 & 1 & -2.5810 & 4.9962 & -8.5968 & 12.8676 &
\end{array}$$

$$p_2 = -1.2905 + \frac{1.2887}{12.8676} = -1.1903.$$

Third iteration $p_2 = -1.1903$

$$
\begin{array}{r|cccccc}
-1.1903 & 1 & 0 & 0 & 0 & -1 & 1 \\
 & & -1.1903 & 1.4168 & -1.6864 & 2.0073 & -1.1990 \\
\hline
 & 1 & -1.1903 & 1.4168 & -1.6864 & 1.0073 & -0.1990 \\
 & & -1.1903 & 2.8336 & -5.0593 & 8.0294 & \\
\cline{2-6}
 & 1 & -2.3806 & 4.2504 & -6.7457 & 8.0367 &
\end{array}
$$

$$p_3 = -1.1903 + \frac{0.1990}{9.0367} = -1.1683.$$

Fourth iteration $p_3 = -1.1683.$

$$
\begin{array}{r|cccccc}
-1.1683 & 1 & 0 & 0 & 0 & -1 & 1 \\
 & & -1.1683 & 1.3649 & -1.5946 & 1.8630 & -1.0082 \\
\hline
 & 1 & -1.1683 & 1.3649 & -1.5946 & 0.8630 & -0.0082 \\
 & & -1.1683 & 2.7298 & -4.7838 & 7.4519 & \\
\cline{2-6}
 & 1 & -2.3366 & 4.0947 & -6.3784 & 9.3149 &
\end{array}
$$

$$p_3 = -1.1683 + \frac{0.0082}{8.3149} = -1.1673.$$

The root correct to three decimals is – 1.167.

Deflated polynomial

– 1.167	1	0	0	0	–1	1
		– 1.167	1.3619	– 1.5893	1.8547	
	1	– 1.167	1.3619	– 1.5893	0.8547	

The deflated polynomial is given by

$$x^4 - 1.167x^3 + 1.3619x^2 - 1.5893x + 0.8547 = 0.$$

3. Apply Graeffe's root squaring method to find the roots of

the equation $x^3 - 2x + 2 = 0$

Sol :

m	2^m				
0	1	1	0	-2	2
		1	0 4	4 0	4
1	2	1	4	4	4
		1	16 -8	16 -32	16
2	4	1	8	-16	16

		1	64	256	256
			32	– 256	
3	8	1	96	0	256
		1	9216	0	65536
			0	– 49152	
4	16	$1 = B_0$	$9216 = B_1$	$-49152 = B_2$	$65536 = B_3$

Since B_2 is alternately positive and negative, we have a pair of complex roots based on B_1, B_2, B_3.

One real root is $|\xi_1|^{16} = 9216$ or $|\xi_1| = 1.7692$. On substituting into the given polynomial, we find that root must be negative. Hence, one real is $\xi_1 = -1.7692$.

To find the pair of complex roots $p \pm iq$, we have

$$|\beta|^{32} = \left|\frac{B_3}{B_1}\right| \quad \text{or} \quad \beta = 1.0632 = \sqrt{p^2 + q^2}.$$

Also, $\quad \xi_1 + 2p = 0 \quad \text{or} \quad p = 0.8846,$

$$q^2 = \beta^2 - p^2 \quad \text{or} \quad q = 0.5898.$$

Hence, roots are $0.8846 \pm 0.5898i$.

1. Perform two iterations with the Miller method for the equation $x^3 - (1/2) = 0$; $x_0 = 0$; $x_1 = 1$; $x_2 = (1/2)$

   Ans : $x_3 = 0.7676$; $x_4 = 0.7929$

2. Use the Birge-vieta method find a real root correct to three decimals of the equation

   $x^6 - x^4 - x^3 - 1 = 0$; $p_0 = 1.5$

   Ans : 1.404

3. Find to two decimals the real and complex roots of the equation $x^5 = 3x - 1$ using Birge-vieta method.

   Ans : 0.33, 1.21, -1.39 , -0.08 + 1.33i and -0.08 – 1.33i