1. Find $\Delta^n(\sin(ax + b))$

**Sol** : Wkt $\Delta f(x) = f(x + h) - f(x)$

$\Delta \text{Sin}(ax + b) = \text{Sin}(a(x + h) + b) - \sin(ax + b)$

$= \sin(ax + ah + b) - \sin(ax + b)$

Wkt $\sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$

$$\Delta \text{Sin}(ax + b) = 2\cos\left(\frac{ax+ah+b+ax+b}{2}\right)\sin\left(\frac{ax+ah+b-ax-b}{2}\right)$$

$$= 2\cos\left(ax+b+\frac{ah}{2}\right)\ \sin\left(\frac{ah}{2}\right)$$

Wkt $\sin\left(\frac{\pi}{2}+\theta\right) = \cos\theta$

$$\Delta \text{Sin}(ax + b) = 2\sin\left(ax+b+\frac{ah}{2}+\frac{\pi}{2}\right)\sin\left(\frac{ah}{2}\right)$$

$$\Delta^2 \text{Sin}(ax + b) = \Delta(\Delta \text{Sin}(ax + b))$$

$$= \Delta 2\sin\left(ax+b+\frac{ah}{2}+\frac{\pi}{2}\right)\sin\left(\frac{ah}{2}\right)$$

$$= 2\sin\left(\frac{ah}{2}\right)\Delta\left[\sin\left(ax+b+\frac{ah}{2}+\frac{\pi}{2}\right)\right]$$

$$= 2\sin\left(\frac{ah}{2}\right)2\cos\left(ax+b+\frac{ah}{2}+\frac{\pi}{2}+\frac{ah}{2}\right)\sin\left(\frac{ah}{2}\right)$$

$$= 2\sin\left(\frac{ah}{2}\right)2\sin\left(ax+b+2\frac{ah}{2}+\frac{\pi}{2}+\frac{\pi}{2}\right)\sin\frac{ah}{2}$$

$$\Delta^2\sin(ax+b) = 2^2\sin^2\left(\frac{ah}{2}\right)\sin\left(ax+b+2\left(\frac{ah}{2}+\frac{\pi}{2}\right)\right)$$

$$\textit{Similarly},\quad \Delta^3\sin(ax+b) = 2^3\sin^3\left(\frac{ah}{2}\right)\sin\left(ax+b+3\left(\frac{ah}{2}+\frac{\pi}{2}\right)\right)$$

$$\therefore \quad \Delta^n \sin(ax+b) = 2^n \sin^n\left(\frac{ah}{2}\right)\sin\left(ax+b+n\left(\frac{ah}{2}+\frac{\pi}{2}\right)\right)$$

2. Find $\Delta(\log(ax + b))$

**Sol** : Wkt $\Delta f(x) = f(x + h) – f(x)$

$\Delta(\log(ax + b) = \log(a(x+h) + b) – \log((ax + b))$

$= \log[(ax + ah + b) / (ax + b)]$

$= \log[1 + (ah / (ax + b))]$

$= \log [1 + (\Delta(ax + b) / (ax + b))]$

Since $\Delta(ax + b) = ax + ah + b – ax – b = ah$

Therefore $\Delta(\log(ax + b)) = \log [1 + (\Delta(ax + b) / (ax + b))]$.

3. Find $\Delta\tan^{-1}x$

**Sol** : Wkt $\Delta f(x) = f(x + h) – f(x)$

$\Delta\tan^{-1}x = \tan^{-1}(x + h) – \tan^{-1}x$

Wkt $\tan^{-1}A – \tan^{-1}B = \tan^{-1}[ (A – B) / (1 + AB)]$

$\Delta\tan^{-1}x = \tan^{-1}[(x + h – x) / (1 + x (x + h))]$

$= \tan^{-1}[h / (1 + x (x + h))]$.

4. Find $\Delta(e^{3x} \log 2x)$.

**Sol :** Let $f(x) = e^{3x}$ and $g(x) = \log 2x$

Wkt $\Delta(f(x) g(x)) = f(x + h) \Delta g(x) + g(x) \Delta f(x)$

$\Delta(e^{3x} \log 2x) = e^{3(x + h)} ( \Delta(\log 2x)) + \log 2x (\Delta(e^{3x}))$

$= e^{3x + 3h} (\log(2x + 2h) – \log 2x) + \log 2x (e^{3x + 3h} – e^{3x})$

$= e^{3x + 3h} (\log ((2x + 2h) / 2x)) + \log 2x (e^{3x}e^{3h} – e^{3x})$

$= e^{3x}e^{3h} (\log( 1 + (h/x)) + e^{3x}\log 2x (e^{3h} – 1)$

$= e^{3x} (e^{3h}(\log(1 + (\Delta x / x))) + \log 2x (e^{3h} – 1))$.

5. Find $\Delta(x \sin x)$

**Sol :** Let $f(x) = x$ and $g(x) = \sin x$

Wkt $\Delta(f(x) g(x)) = f(x + h) \Delta g(x) + g(x) \Delta f(x)$

$\Delta(x \sin x) = (x + h) \Delta(\sin x) + \sin x (\Delta x)$

$= (x + h)\,(\sin(x + h) – \sin x) + \sin x\,(x + h – x)$

$= (x + h)\,(2\cos((x+h + x) / 2)\,\sin((x + h – x ) / 2) + \sin x\; h$

$= (x + h)\; 2\cos( x + (h/2) )\,\sin (h /2) + h \sin x$

$= 2\sin(h /2)\,(x + h)\,\sin ((\pi/2) + x + (h/2)) + h \sin x$

$= 2\sin(h/2)\,(x + h)\,\sin(x + ((\pi + h)/2)) + h \sin x.$

6. Find $\Delta(2^x / x!)$

**Sol :** Let $f(x) = 2^x$ and $g(x) = x!$

Wkt $\Delta(f(x) / g(x)) = (g(x)\,\Delta f(x) - f(x)\,\Delta g(x)) / g(x)\,g(x + h)$

$\Delta f(x) = \Delta 2^x = 2^{x+h} – 2^x = 2^x(2^h – 1)$

$\Delta g(x) = \Delta x! = (x + h)! – x!$

$\Delta(2^x / x!) = (x!\; 2^x(2^h – 1) – 2^x[(x + h)! – x!] ) / x!\,(x + h)!$

7. Find $\Delta(x / \cos 2x)$

**Sol :** Let $f(x) = x$ and $g(x) = \cos 2x$

Wkt $\Delta(f(x) / g(x)) = (g(x)\,\Delta f(x) - f(x)\,\Delta g(x)) / g(x)\,g(x + h)$

$\Delta f(x) = \Delta x = x + h - x = h$

$\Delta\cos(ax + b) = \cos(ax + ah + b) – \cos(ax + b)$

$= - 2\sin((ax + ah + b + ax + b)/2)\,\sin((ax + ah + b - ax - b)/2)$

$= - 2\sin((2ax + ah + 2b )/2)\,\sin(ah /2)$

$= 2\sin(ah /2)\,\cos((\pi/2) + ax + b + (ah/2))$

Since $\cos((\pi/2) + \theta) = - \sin\theta$

Hence $\Delta g(x) = \Delta\cos 2x = 2\sin h\,\cos((\pi/2) + 2x + h)$

$\Delta(x / \cos 2x) = (\cos 2x\,.\,h – x.\,2\sin h\,\cos((\pi/2) + 2x + h) ) / \cos 2x\,\cos(2x + 2h)$

8. Prove that $E\nabla = \Delta = \nabla E$

**Proof : LHS :** $E\nabla$

$E\nabla f(x) = E[\nabla f(x)]$

$= E[f(x) – f(x – h)]$

$= f(x + h) – f(x – h + h)$

$= f(x + h) – f(x)$

$= \Delta f(x)$

$E\nabla f(x) = \Delta f(x)$

Hence $E\nabla = \Delta$

Similarly, $\nabla E$

$\nabla E f(x) = \nabla[Ef(x)] = \nabla[f(x+h)]$

$= f(x + h) - f(x + h - h)$

Since $\nabla f(x) = f(x) - f(x - h)$

$= f(x + h) - f(x) = \Delta f(x)$

$\nabla E \ f(x) = \Delta f(x)$

$\nabla E = \Delta$

Therefore, $E\nabla = \Delta = \nabla E$

9. Prove that $\delta E^{(1/2)} = \Delta$

**Proof : LHS :** $\delta E^{(1/2)}$

$\delta E^{(1/2)} f(x) = \delta[E^{(1/2)} f(x)]$

$= \delta[f(x + (1/2)h]$ , since $E^n f(x) = f(x + nh)$

$\delta f(x) = f(x + (h/2)) - f(x - (h/2))$

Hence, $\delta E^{(1/2)} f(x) = f(x + (h/2) + (h/2)) - f(x + (h/2) - (h/2))$

$= f(x + h) - f(x) = \Delta f(x)$

$\delta E^{(1/2)} f(x) = \Delta f(x)$

Hence, $\delta E^{(1/2)} = \Delta$

10. Prove that $hD = \log(1 + \Delta) = - \log(1 - \nabla) = \sinh^{-1}(\mu\delta)$

**Proof** : $Ef(x) = f(x+h) = f(x) + \frac{h}{1!} f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \ldots\ldots.$

$$= f(x) + \frac{h}{1!} Df(x) + \frac{h^2}{2!} D^2 f(x) + \frac{h^3}{3!} D^3 f(x) + \ldots\ldots\ldots\ldots$$

$$= \left[1 + \frac{h}{1!} D + \frac{h^2}{2!} D^2 + \frac{h^3}{3!} D^3 + \ldots\ldots..\right] f(x)$$

$$Since\ e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots\ldots\ldots$$

$Ef(x) = e^{hD} f(x)$

$E = e^{hD}$ ---------------------------------------------------(i)

Wkt, $\Delta f(x) = f(x + h) – f(x) = Ef(x) – f(x) = (E - 1) f(x)$

$\Delta = E – 1$

This implies $E = 1 + \Delta$

Hence $1 + \Delta = e^{hD}$

i.e., $\log(1 + \Delta) = hD$ ---------------------------------------(*)

Also, $\nabla f(x) = f(x) – f(x – h) = f(x) – E^{-1}f(x) = (1- E^{-1})f(x)$

$\nabla = 1 – E^{-1}$

$E^{-1} = 1 – \nabla$

From (i), $E^{-1} = e^{-hD}$

$1 – \nabla = e^{- hD}$

$\log( 1 – \nabla ) = - hD,$

$-\log( 1 – \nabla ) = hD$

$hD = -\log(1 – \nabla )$ ---------------------------------------------(**)

wkt, $\sinh\theta = (e^{\theta} – e^{-\theta}) / 2$

$\sinh(hD) = (e^{hD} – e^{-hD}) / 2 = (E – E^{-1}) / 2 = ((E^{(1/2)} + E^{(-1/2)})\ (E^{(1/2)} - E^{(-1/2)})) /2$

$= ( (E^{(1/2)} + E^{(-1/2)}) /2) (E^{(1/2)} - E^{(-1/2)}) = \mu\delta$ (by def)

$\text{Sinh}(hD) = \mu\delta$ ; Hence, $hD = \sinh^{-1}(\mu\delta)$ --------------------------(***)

From (*), (**) and (***) , $hD = \log(1 + \Delta) = - \log(1 - \nabla ) = \sinh^{-1}(\mu\delta)$

11. Prove that $1 + \mu^2\delta^2 = (1 + (\delta^2/2))^2$

**Proof : LHS :** $1 + \mu^2\delta^2 = 1 + ( (E^{(1/2)} + E^{(-1/2)}) /2)^2 (E^{(1/2)} - E^{(-1/2)})^2$

$= 1 + [(E^{(1/2)} + E^{(-1/2)}) (E^{(1/2)} - E^{(-1/2)})]^2/4$

$= 1 + [(E^{(1/2)})^2 + (E^{(-1/2)})^2] / 4$

$= 1 + (E – E^{-1})^2/4 = (4 + E^2 + (E^{-1})^2 – 2EE^{-1}) / 4$

$$= (E^2 + (E^{-1})^2 + 2) / 4 = (E + E^{-1})^2/4$$

**RHS:** $(1 + (\delta^2/2))^2 = (1 + (E^{(1/2)} - E^{(-1/2)})^2/2)^2$

$$= (1 + (E + E^{-1} - 2)/2)^2$$

$$= [(2 + E + E^{-1} - 2)/2]^2 = (E + E^{-1})^2/4$$

**LHS = RHS**

Hence, $1 + \mu^2\delta^2 = (1 + (\delta^2/2))^2$

**Exercise problems**

1. Prove that $E^{(1/2)} = \mu + (\delta/2)$
2. Prove that $E^{(-1/2)} = \mu - (\delta/2)$
3. Prove that $\mu\delta = ( (\Delta E^{-1}) / 2 ) + (\Delta/2)$
4. Prove that $\Delta = (\delta^2/2) + \delta\sqrt{(1 + (\delta^2/4)}$