Normal distribution

Normal distribution is continuous distribution of a variable with probability density function

$$f(x:\mu;\sigma)=\frac{1}{\sigma\sqrt{2\pi}}\,e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$ This is also known as Gaussian Probability distribution with two unknown parameters μ and σ.

Standardized normal variate:

Setting $z=\frac{x-\mu}{\sigma}=t$ gives the standardized normal variate.

Normal distribution is a legitimate distribution.

Let f(x) be normal distribution then

$$\int_{-\infty}^{\infty} f(x)\,dx=\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}\,dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2}\,dz \qquad \textit{where} \qquad z=\left(\frac{x-\mu}{\sigma}\right)$$

$$=\frac{2}{\sqrt{2\pi}}\int_{0}^{\infty} e^{-\frac{1}{2}z^2}\,dz \qquad \textit{as} \quad e^{-\frac{1}{2}z^2} \quad \textit{is even.}$$

$$=\frac{2}{\sqrt{\pi}}\int_{0}^{\infty} e^{-u^2}\,du \qquad \textit{where} \quad u=\frac{1}{\sqrt{2}}z$$

$$=\frac{2}{\sqrt{\pi}}.\frac{\sqrt{\pi}}{2}=1$$

Normal distribution as limiting form of binomial distribution:

When n is very large and neither p nor q is very small, the standard distribution can be regarded as the limiting form of the standardized binomial distribution.

Let $z=\frac{x-E(x)}{\sqrt{\text{var}(x)}}=\frac{x-np}{\sqrt{npq}}$, where np is the mean and $\sqrt{npq}$ is the standard deviation of the Binomial distribution.

When x=0, $z=\frac{0-np}{\sqrt{npq}}=\sqrt{\frac{np}{q}}$, when x=n $z=\frac{n-np}{\sqrt{npq}}=\frac{n\,(1-p)}{\sqrt{npq}}=\sqrt{\frac{np}{q}}$

As n becomes large, z various from - ∞ to ∞ and having the mean 0 and standard deviation as 1.

Properties of the Normal distribution:

1. If X follows N(μ,σ), then E(x) = μ and var(X) = σ^2.

$$E(X) = \int_{-\infty}^{\infty} x\ f(x)\ dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} x e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \left(\mu + \sqrt{2}\sigma\ z\right) e^{-\frac{1}{2}z^2} dz \qquad where \qquad z = \left(\frac{x-\mu}{\sigma}\right)$$

$$= \frac{\mu}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-z^2} dz + \sqrt{\frac{2}{\pi}}\ \sigma \int_{-\infty}^{\infty} z\ e^{-z^2} dz$$

$$= \frac{\mu}{\sqrt{\pi}} \sqrt{\pi} \quad (Since\ the\ second\ integrand\ is\ an\ odd\ function)$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2\ f(x)\ dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \left(\mu + \sqrt{2}\sigma\ z\right)^2 e^{-z^2} dz \qquad where \qquad z = \left(\frac{x-\mu}{\sigma}\right)$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \mu^2 e^{-z^2} dz + 2\sqrt{2}\mu\sigma \int_{-\infty}^{\infty} z e^{-z^2} dz + 2\sigma^2 \int_{-\infty}^{\infty} z^2 e^{-z^2} dz$$

$$= \mu^2 + 0 + \frac{2\sigma^2}{\sqrt{\pi}} \int_{0}^{\infty} z e^{-z^2}\ 2z\,dz \qquad (\because\ ze^{-z^2}\ is\ even)$$

$$= \mu^2 + \frac{2\sigma^2}{\sqrt{\pi}} \int_{0}^{\infty} u^{\frac{1}{2}} e^{-u} du \qquad \left(putting\ u = z^2\right)$$

$$= \mu^2 + \frac{2\sigma^2}{\sqrt{\pi}} \Gamma\left(\frac{3}{2}\right)$$

$$= \mu^2 + \frac{2\sigma^2}{\sqrt{\pi}} \frac{1}{2} \Gamma\left(\frac{1}{2}\right) \qquad \left(\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right)$$

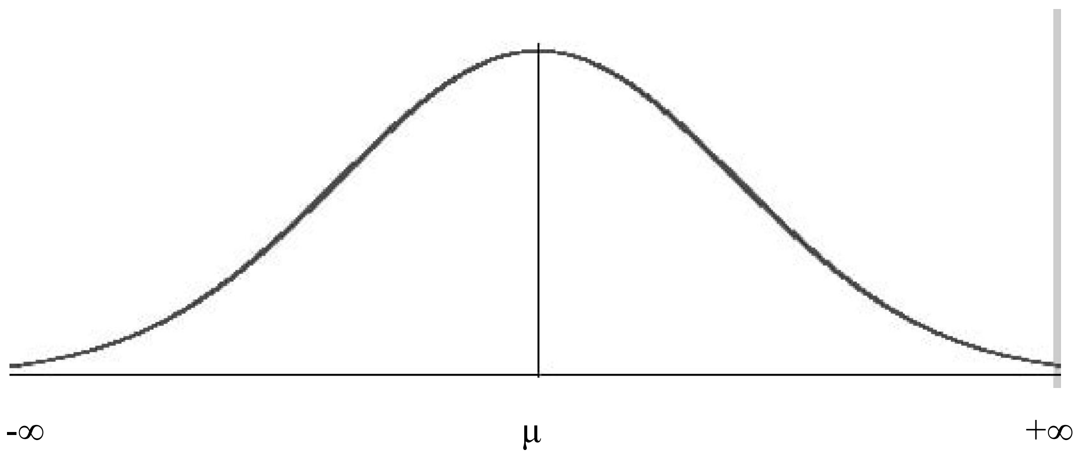
$$= \mu^2 + \sigma^2$$

$$\therefore \quad Var(X) = E(X^2) - \{E(X)\}^2 = \sigma^2$$

2. Median and mode of the normal distribution N(μ,σ):

   Mode = Median = μ

3. Normal distribution is a symmetrical distribution (bell – shaped curve)



1. In an engineering examination, a student is considered to have failed, secured second class, first class and distinction, according as he scores less than 45% between 45% and 60%, between 60% and 75% and above 75% respectively. In a particular year 10% of the students failed in the examination and 5% of the students get distinction. Find the percentage of students who have got first class and second class. (Assume Normal distribution of marks)

   Let X follows the percentage of marks scored by the students in the examination.

   Let X follow the distribution N(μ,σ):

   Given: $P(X<45)=0.10$ and $P(X>75) =0.05$

$$P\left(-\infty < X < \frac{45-\mu}{\sigma}\right) = 0.10$$

$$\text{i.e., } P\left(\frac{75-\mu}{\sigma} < \frac{X-\mu}{\sigma} < \infty\right) = 0.05$$

$$P\left(-\infty < z < \frac{45-\mu}{\sigma}\right) = 0.10$$

$$P\left(\frac{75-\mu}{\sigma} < z < \infty\right) = 0.05$$

$$P\left(0 < z < \frac{45-\mu}{\sigma}\right) = 0.5 - 0.10 = 0.4$$

$$P0\left(< z < \frac{75-\mu}{\sigma}\right) = 0.5 - 0.05 = 0.45$$

From the table of areas, we get

$$\frac{45-\mu}{\sigma} = 1.28 \quad and \quad \frac{75-\mu}{\sigma} = 1.64$$

$$45 = 1.28\sigma + \mu$$

$$75 = 1.64\sigma + \mu$$

Solving the above equations, we get

$\mu = 58.15$ and $\sigma = 10.28$

Therefore ,

Now P(a student gets first class)

$= P\ (60 < X < 75)$

$= P\left(\frac{60 - 58.15}{10.28} < Z < \frac{75 - 58.15}{10.28}\right)$

$= P(0.18 < Z < 1.64)$

$= P(0 < Z < 1.64) - P(0 < Z < 0.18)$

$= 0.4495 - 0.0714 = 0.3781$

$\therefore$ Percentage of students getting first class = 38 approximately.

And the percentage of students getting second class

= 100 – (sum of the percentages of students who have failed, got first class and got distinction)

= 100 – (10+38+5) = 47 approximately.

2. If X is a normal distribution with 30 and standard deviation 5. Find the probability that (i) 26 < x < 40 (ii) x > 45.

$$P(26 < X < 40) = P\left(\frac{26 - 30}{5} < Z < \frac{40 - 30}{5}\right)$$

$$= P\left(-\frac{4}{5} < Z < 2\right)$$

$$= P(0 < Z < 0.8) + P(0 < Z < 2)\ (by\ symmetry)$$

$$= 0.2881 + 0.4772 = 0.7653$$

$$P(X > 45) = P\left(Z > \frac{45 - 30}{5}\right) = P(Z > 3) = P(3 < Z < \infty)$$

$$= 0.5 - P(0 < Z < 3)$$

$$= 0.5 - 0.4987 = 0.0013$$

Exercises:

1. The marks obtained by the students in a certain subject are approximately normally distributed with mean 65 and standard deviation 5. What is the probability that a student would have scored above 75? Ans:0.0228

2. If the actual amount of instant coffee which a filling machine puts into '6-ounce' jars is a random variable having a normal distribution with SD = 0.05ounce and if only 3% of the jars are to contain less than 6 ounces of coffee, what must be the mean fill of these jars? Ans : $\mu = 6.094$ ounces