Poisson distribution

Poisson distribution is a discrete probability distribution. The distribution function of this distribution is $P(X=x)=\dfrac{e^{-\lambda}\lambda^{x}}{x!}$ *where* $x=0,1,2,3,....$ & λ being the parameter also called Probability mass function of Poisson distribution.

Poisson distribution is a legitimate probability distribution:

That is $\sum_{r=0}^{\infty} P(x=r)=\sum_{r=0}^{\infty}\dfrac{e^{-\lambda}\lambda^{r}}{r!}$

$$= e^{-\lambda}\sum_{r=0}^{\infty}\frac{\lambda^{r}}{r!}=e^{-\lambda}e^{\lambda}=1$$

Poisson distribution as a Limiting Form of Binomial distribution

1. n, the number of trials is indefinitely large, i.e., $n\to\infty$
2. p, the probability of success in each trial is very small, $p\to 0$.
3. np(=λ) is finite or $p=\lambda/n$ and $q=1-\lambda/n$, where λ is a positive real number.

Mean and variance of Poisson distribution:

Mean: E(X)

$$\bar{x}=\frac{\sum fx}{\sum f}=\sum\frac{e^{-\lambda}\lambda^{r}}{r!}$$

$$=e^{-\lambda}\lambda\sum\frac{\lambda^{r-1}}{(r-1)!}$$

$$=\lambda e^{-\lambda}e^{\lambda}=\lambda$$

Variance:

Variance of x = Var(x) = $E(x^{2})-(E(x))^{2}$

$$E\left(x^2\right)=\sum x_r^2 \frac{e^{-\lambda}\lambda^r}{r!}$$

$$= \sum_{r=0}^{\infty}\left(r\,(r-1)+r\right)\frac{e^{-\lambda}\lambda}{r!}$$

$$=\lambda^2 e^{-\lambda} \sum_{r=2}^{\infty} \frac{\lambda^{r-2}}{(r-2)!} + \lambda$$

$$= \lambda^2 e^{-\lambda} e^{\lambda} + \lambda = \lambda^2 + \lambda$$

Therefore Variance of x = Var(x) =

$$= E(x^2) - \left(E(x)\right)^2$$

$$= \lambda^2 + \lambda - \lambda^2 = \lambda$$

1. Fit a Poisson distribution for the following distribution:

x :	0	1	2	3	4	5
f :	142	156	69	27	5	400

Assume x follows the Poisson distribution.

$$\bar{x}=\frac{\sum fx}{\sum f} = \sum \frac{e^{-\lambda}\lambda^r}{r!}, \qquad r = 0,1,2,\ldots..\infty$$

x :	0	1	2	3	4	5	Total
f :	142	156	69	27	5	1	400
fx :	0	156	138	81	20	5	400

Mean $= \bar{x}=\frac{\sum fx}{\sum f} = \frac{400}{400} = 1 = \lambda$,

Theoretical frequencies are given by

$N\frac{e^{-\lambda}\lambda^r}{r!}$ where N = 400, obtained from the given distribution.

Therefore the theoretical frequencies are given by

$$400\ \frac{e^{-1}}{r!}, \qquad r = 0,1,2,\ldots.\infty$$

r = 0 gives $400\ \frac{e^{-1}}{0!} = 147.15$

r = 1 gives $400\ \frac{e^{-1}}{1!} = 147.15$

r = 2 gives $400\ \frac{e^{-1}}{2!} = 73.58$

r = 3 gives $400\ \frac{e^{-1}}{3!} = 24.53$

r = 4 gives $400\ \frac{e^{-1}}{4!} = 6.13$

r = 5 gives $400\ \frac{e^{-1}}{5\,!} = 1.23$

convert the theoretical frequencies to whole numbers, the theoretical frequencies are

x :	0	1	2	3	4	5	Total
Theoretical :	147.15	147.15	73.58	24.53	6.13	1.23	

Approximated theoretical frequencies are

x :	0	1	2	3	4	5	Total
Theoretical :	147	147	74	25	6	1	400

2. In a certain factory turning razor blades, there is a small probability of 1/500 for any blade to be defective. The blades are supplied in packets of 10. Use the Poisson distribution to calculate the approximate number of packets containing (i) no defective (ii) one defective and (iii) two defective blades in a consignment of 10,000 packets.

Here in this problem p=1/500. ; n = 10
np = 1/50 = 0.02 = $\bar{x}$

$$P(X=x) = \frac{e^{-\lambda}\lambda^{x}}{x!} = \frac{e^{-0.02}0.02^{x}}{x!}$$

(i) $P(X=0) = \dfrac{e^{-0.02}0.02^{0}}{0!} = 0.9802$

$\therefore$ The number of packets which are free from defectives blades
=0.9802 x 10000 = 9802

(ii) $P(X=1) = \dfrac{e^{-0.02}0.02^{1}}{1!} = 0.0196$

$\therefore$ The number of packets which there may be one defectives blade
=0.0196 x 10000 = 196

(iii) $P(X=2) = \dfrac{e^{-0.02}0.02^{2}}{2!} = 0.000196$

$\therefore$ The number of packets which there may be two defectives blades
=0.000196 x 10000 = 1.96$\approx$2

3. A car hire firm has two cars which it hires out daily. The number of demands for a car on each day is distributed as Poisson variate with mean λ = 1.5. Obtain the proportion of days on which (i) there was no demand and (ii) demand is refused;

Proportion of days for x demands for a care =

$$P(X=x) = \frac{e^{-1.5}1.5^{x}}{x!}$$

(i) Proportion of days on which there was no demand =

$$\text{P(X=0)} = \frac{e^{-1.5}1.5^0}{0!} = e^{-1.5} = 0.2231$$

(ii) the proportion of days on which the demand is refused=

$$P(x>2) = 1 - P(x \le 2) = P(x=0) + P(x=1) + P(x=2)$$

$$= e^{-1.5}\left(\frac{1.5^0}{0!} + \frac{1.5^1}{1!} + \frac{1.5^2}{2!}\right) = 0.1913$$

Exercise:

1. The number of accidents in a year to taxi-drivers in a city follows a Poisson distribution with mean equal to 3. Out of 1000 taxi drivers, approximately the number of drivers with (i) no accidents in a year; (ii) more than 3 accidents in a year.

   (Answer: 50 and 353)