Numerical Integration

We know that, $\int_a^b y\,dx$ denotes the area enclosed by the curve y = f(x), the X axis, the ordinates between x = a and x = b. This integration is possible only if f(x) is known explicitly given and if it is integrable. The process of evaluating definite integral from a set of tabulated values of the integrand y = f(x) is called numerical integration. Numerical integration is stated as follows

Given a set of n + 1 paired values (x_i, y_i) i = 0, 1, 2, 3, ….., n of the function y = f(x)

x	x_0	x_1	x_2	x_3	..	..	..	..	x_n
y	y_0	y_1	y_2	y_3	..	..	..	..	y_n

It is required to compute $\int_{x_0}^{x_n} y\,dx$. This process when applied to a function of single variable is known as quadrature.

A general quadrature formula for equidistant ordinates (OR) Newton cote's quadrature formula

For equally spaced intervals, we have Newton's forward difference formula for interpolation as

$$y(x) = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u\,(u-1)}{2!}\Delta^2 y_0 + \frac{u\,(u-1)\,(u-2)}{3!}\Delta^3 y_0 + ……….\qquad \text{where} \qquad u = \frac{x-x_0}{h}$$

$$y(x) = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u^2-u}{2!}\Delta^2 y_0 + \frac{u^3-3u^2+2u}{3!}\Delta^3 y_0 + \frac{u^4-6u^3+11u^2-6u}{4!}\Delta^4 y_0 + ……….$$

Now, consider $\int_{x_0}^{x_n} y\,dx$. $\qquad u = \frac{x-x_0}{h}$

This implies $x = x_0 + uh$. Therefore, dx = hdu where h is interval of differencing.

When $x = x_0$; $u = (x_0 - x_0) / h = 0$

When $x = x_n$; $u = (x_n - x_0) / h = (x_0 + nh - x_0) / h = n$

$$\therefore \int_{x_0}^{x_n} y\,dx = \int_0^n y\,dx$$

$$= \int_0^n \left(y_0 + \frac{u}{1!}\Delta y_0 + \frac{u^2-u}{2!}\Delta^2 y_0 + \frac{u^3-3u^2+2u}{3!}\Delta^3 y_0 + \frac{u^4-6u^3+11u^2-6u}{4!}\Delta^4 y_0 + \dots\dots\dots\right) h\,du$$

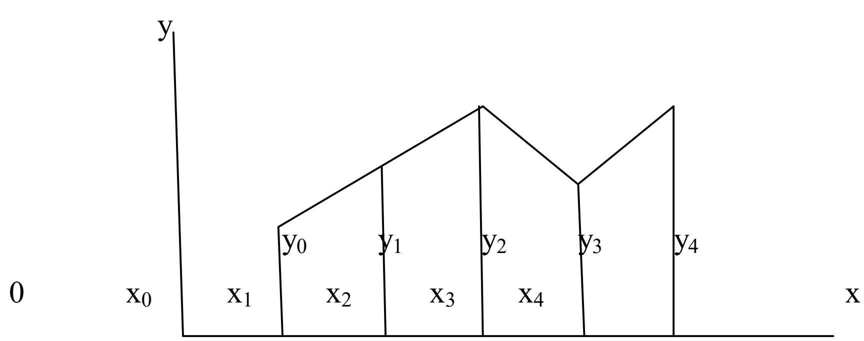
$$= h\left[uy_0 + \frac{u^2}{2}\Delta y_0 + \frac{1}{2!}\left(\frac{u^3}{3}-\frac{u^2}{2}\right)\Delta^2 y_0 + \frac{1}{3!}\left(\frac{u^4}{4}-u^3+u^2\right)\Delta^3 y_0 + \dots\dots\right]_0^n$$

$$\therefore \int_{x_0}^{x_n} y\,dx = h\left[ny_0 + \frac{n^2}{2}\Delta y_0 + \frac{1}{2}\left(\frac{n^3}{3}-\frac{n^2}{2}\right)\Delta^2 y_0 + \frac{1}{6}\left(\frac{n^4}{4}-n^3+n^2\right)\Delta^3 y_0 + \dots\dots\dots\right] \_\_\_\_\_\_\_\_\_\_ (\mathrm{I})$$

Equation (I) is known as Newton-cote's quadrature formula. From this general formula (I), we deduce the following quadrature rules by taking n = 1, 2 and 3 in (I). By putting n = I in (I), we obtain $\int_{x_0}^{x_n} y\,dx$ = (h/2) [(y_0 + y_n) + 2(y_1 + y_2 + y_3 + ……+ y_{n-1})]. This is known as Trapezoidal rule.

Geometrical Interpretation

Geometrically, if the ordered pairs (x_i, y_i) ;i = 0, 1, 2, …., n are plotted and if any two consecutive points are joined by straight lines, we get the figure as shown.



The area between f(x) , x axis and the ordinates x = x_0 and x = x_n is approximated to the sum of trapeziums as shown in figure.

Truncation error in Trapezoidal rule

Error $|E| < \frac{(b-a)h^2}{12} M$ where M is maximum value of $|y_0''|, |y_1''|, |y_2''|, \dots\dots$, if the interval is (a, b) and h = (b – a) / n. Hence the error in Trapezoidal rule is of order h^2.

Simpson's One-third rule

Setting n = 2 in quadrature formula, we have

$$\int_{x_0}^{x_n} y\,dx = (h/3)\,[(y_0 + y_n) + 2(y_2 + y_4 + y_6 + \ldots\ldots) + 4(y_1 + y_3 + y_5 + \ldots..)].$$

= (h / 3) [(sum of first and last ordinates) + 2(sum of odd ordinates) + 4(sum of even ordinates)]

This formula is known as Simpson's one third rule.

Truncation error in Simpson's one-third rule is $|E| < \frac{(b-a)h^4}{180} M$ where M is the numerically greater value of y_0'''', y_2'''', y_4'''', ……, y_{2n-2}'''', if the interval is (a, b) and h(2n) = b - a. Hence the error in Trapezoidal rule is of order h^4.

Simpson's Three-eighth rule

Putting n = 3 in Newton cote's formula, the formula obtained is known as Simpson's three eighth rule.

$$\int_{x_0}^{x_n} y\,dx = (3h/8)\,[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \ldots\ldots + y_{n-1}) + 2(y_3 + y_6 + y_9 + \ldots.. + y_{n-3})].$$

Note:

1. In Trapezoidal rule, y(x) is a linear function of x . The rule is the simplest one but it is least accurate. The accuracy of the result can be improves by increasing the number of intervals and decreasing the value of h.
2. In Simpson's one-third rule, y(x) is a polynomial of degree two. To apply this rule, n, the number of intervals must be even. That is the number of ordinates must be odd.
3. In Simpson's three-eighth rule, y(x) is a polynomial degree three. This rule is applicable only if , n, the number of intervals is a multiple of three.

Problems

1. Evaluate $\int_0^1 \frac{dx}{1+x^2}$ using Trapezoidal rule and hence find an approximate value of π.

Sol: Given interval (0, 1). Here h value is not given. Let us take h = 0.2

Wkt, n = (b – a) / h

x	0	0.2	0.4	0.6	0.8	1
y = 1 / (1 + x^2)	1	0.96154	0.86207	0.73529	0.60976	0.5

$$\int_{x_0}^{x_n} y\,dx = (h/2)\,[(y_0 + y_n) + 2(y_1 + y_2 + y_3 + \ldots\ldots + y_{n-1})].$$

$$\int_{0}^{1} y\,dx = (h/2)\,[(y_0 + y_5) + 2(y_1 + y_2 + y_3 + y_4)].$$

$$= (0.2 / 2)\,[(1 + 0.5) + 2(0.96154 + 0.86207 + 0.73529 + 0.60976)] = 0.\ 78732$$

By actual integration, $\int_0^1 \frac{dx}{1+x^2} = \left[\tan^{-1} x\right]_0^1 = \tan^{-1} 1 - \tan^{-1} 0 = \frac{\pi}{4} - 0 = \frac{\pi}{4}$

Hence, $\frac{\pi}{4} \cong 0.78732$. This implies π = 4 * 0.78732 = 3.14928.

2. Evaluate $\int_0^6 \frac{dx}{1+x}$ using Trapezoidal rule

Sol: Given interval (0, 6). Here h value is not given. Let us take h = 1

Wkt, n = (b – a) / h

x	0	1	2	3	4	5	6
y = 1/ (1 + x)	1	1/2	1/3	1/4	1/5	1/6	1/7

$$\int_{x_0}^{x_n} y\,dx = (h/2)\,[(y_0 + y_n) + 2(y_1 + y_2 + y_3 + \ldots\ldots + y_{n-1})].$$

$$\int_{0}^{6} y\,dx = (h/2)\,[(y_0 + y_6) + 2(y_1 + y_2 + y_3 + y_4 + y_5)].$$

$$= (1/2)\,[(1 + (1/7)) + 2((1/2) + (1/3) + (1/4) + (1/5) + (1/6))] = 2.\ 02142857.$$

3. The accelerations of a vehicle at nine timing instances from t = 0 to t = 40 with an interval of 5 are 40.0, 45.25, 48.5, 51.25, 54.35, 59.48, 61.5, 64.3 and 68.7 . Find the velocity at t = 40 using Trapezoidal rule.

Sol: Given h = 5.Let $x_0 = 0$; $x_1 = 5$; $x_2 = 10$; $x_3 = 15$; $x_4 = 20$; $x_5 = 25$; $x_6 = 30$;

x_7 = 35and x_8 = 40. The corresponding values of y are y_0 = 40 ; y_1 = 45.25 ;

y_2 = 48.5 ; y_3 = 51.25; y_4 = 54.35 ; y_5 = 59.48; y_6 = 61.5 ; y_7 = 64.3 and y_7 = 68.7

$$\int_{x_0}^{x_n} y\,dx = (h/2)\,[(y_0 + y_n) + 2(y_1 + y_2 + y_3 + \ldots\ldots + y_{n-1})].$$

$$\int_{0}^{40} y\,dx = (h/2)\,[(y_0 + y_8) + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7)].$$

$$= (5/2)\,[\,(40 + 68.7) + 2(45.25 + 48.5 + 51.25 + 54.35 +$$
$$59.48\ 61.5 + 64.3)] = 2194.9$$

4. Evaluate $\int_0^1 e^{-x^2}\,dx$ by dividing the range of integration in to four equal parts using Trapezoidal rule.

Sol: Given the number of intervals n = 4. Wkt h = (b – a) / n = (1 – 0) / 4 = 0.25

x	0	0.25	0.5	0.75	1
$y = e^{-x2}$	1	0.939413	0.7788	0.569782	0.367879

$$\int_{0}^{1} y\,dx = (h/2)\,[(y_0 + y_4) + 2(y_1 + y_2 + y_3)].$$

$$= (0.25 / 2)\,[\,(1 + 0.367879) + 2(0.939413 + 0.7788 + 0.569782)] = 0.742983625$$