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An Improvement of Convergence Rate Estimates in the Lyapunov Theorem

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Let $X_1, X_2, \ldots, X_n$ be independent random variables defined on a probability space $(\Omega, \mathcal{A}, \mathrm{P})$ and satisfying the conditions

$$\mathrm{E}X_i = 0, \quad \mathrm{D}X_i = \sigma_i^2 > 0, \quad \mathrm{E}|X_i|^3 = \beta_i < \infty,$$

$$i = 1, 2, \ldots, n, \quad \sum_{i=1}^{n} \sigma_i^2 = 1.$$

The Berry–Esseen inequality gives an estimate for the rate of convergence of the distribution function F_n of the normalized sum $S_n = X_1 + X_2 + \ldots + X_n$ to the standard normal distribution function $\Phi(x)$, which has the form

$$\Delta_n \equiv \sup_{x \in \mathbb{R}} |F_n(x) - \Phi(x)| \le C_0 \sum_{i=1}^{n} \beta_i, \tag{1}$$

where C_0 is an absolute constant. It is known that this constant C_0 is bounded as

$$0.4097 \approx \frac{\sqrt{10} + 3}{6\sqrt{2\pi}} \le C_0 \le 0.5606$$

(the lower bound was obtained by Esseen [6] and the upper bound, by Tyurin [4]). Our purpose in this paper is to obtain sharpened upper bounds for the constant C_0. The method which we use is such that the obtained bounds monotonically increase in n; thus, we consider the absolute case (in which the bound for the constant is uniform in n) and the case of finite n separately.

Theorem 1. *For any* $n \ge 1$, *the constant* C_0 *in inequality* (1) *is estimated as*

$$C_0 \le 0.5600.$$

Theorem 2. *For* $1 \le n \le 10$, *the constant* $C(n)$ *in the inequality*

$$\Delta_n \le C(n) \sum_{i=1}^{n} \beta_i$$

is bounded by the values given in Table 1.

The estimate for $C(1)$ was obtained in [5] and is unimprovable. For $n \ge 11$, the constant $C(n)$ is estimated by using Theorem 1 as $\sup_{n \ge 11} C(n) \le 0.5600$.

The proofs of Theorems 1 and 2 are based on a method of Zolotarev improved by using Prawitz' smoothing inequality [7] and estimates for characteristic functions obtained in [2, 4]. We describe only the main ideas and state the corresponding assertions as lemmas.

We set

$$f(t) = \mathrm{E}e^{itS_n} = \prod_{j=1}^{n} f_j(t), \quad f_j(t) = \mathrm{E}e^{itX_j},$$

$$r(t) = \left|f(t) - e^{-t^2/2}\right|, \quad t \in \mathbb{R}.$$

Lemma 1 (see [7]). *For all* $t_0 \in (0, 1]$ *and* $T > 0$,

$$\Delta_n \le 2\int_0^{t_0} |K(t)| \cdot r(Tt)\,dt + 2\int_{t_0}^{1} |K(t)| \cdot |f(Tt)|\,dt$$

$$+ 2\int_0^{t_0} \left|K(t) - \frac{i}{2\pi t}\right| e^{-T^2t^2/2}\,dt + \frac{1}{\pi}\int_{t_0}^{\infty} e^{-T^2t^2/2}\frac{dt}{t},$$

where

Table 1

n	$C(n)$	n	$C(n)$
1	0.3704	6	0.5425
2	0.4857	7	0.5476
3	0.5111	8	0.5516
4	0.5259	9	0.5547
5	0.5356	10	0.5573

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$$K(t) = \frac{1}{2}(1-|t|) + \frac{i}{2}\left[(1-|t|)\cot\pi t + \frac{\operatorname{sgn} t}{\pi}\right],$$
$$-1 \le t \le 1.$$

Estimates for characteristic functions are given by the following lemmas. Let $\theta_0 \approx 3.995895$ be the unique root of the equation

$$\theta^2 + 2\theta\sin\theta + 6(\cos\theta - 1) = 0, \quad \pi \le \theta \le 2\pi,$$

$$\kappa \equiv \sup_{x>0} \frac{\left|\cos x - 1 + \frac{x^2}{2}\right|}{x^3} \approx 0.09916.$$

It can be shown that the maximum is attained at the point $x = \theta_0$. For $t \in \mathbb{R}$ and $\varepsilon > 0$, consider the function

$$\psi(t, \varepsilon) = \begin{cases} \frac{t^2}{2} - \kappa\varepsilon|t|^3, & \varepsilon|t| < \theta_0 \\ \frac{1-\cos(\varepsilon t)}{\varepsilon^2}, & \theta_0 \le \varepsilon|t| \le 2\pi \\ 0, & \varepsilon|t| > 2\pi. \end{cases}$$

It is easy to show that the function $\psi(t, \varepsilon)$ monotonically decreases in ε for each fixed $t \in \mathbb{R}$. We also introduce the notation

$$\ell_n = \sum_{i=1}^{n} \beta_i.$$

Lemma 2. *For any* $n \ge 1$ *and any* $t \in \mathbb{R}$,

$$|f(t)| \le \left[1 - \frac{2}{n}\psi(t, 2\ell_n)\right]^{n/2} \le \exp\{-\psi(t, 2\ell_n)\}.$$

The second (exponential) estimate for $|f(t)|$ was proved in [8]; the proof of the first (power) estimate is similar.

Lemma 3. *For any* $n \ge 1$ *and* $t \in \mathbb{R}$,

$$r(t) \le 2e^{-t^2/2}\int_0^{|t|}\sin\left(\frac{u\ell_n}{4} \wedge \frac{\pi}{2}\right)$$
$$\times ue^{u^2/2}\frac{\left[1 - \frac{2}{n}\left(\frac{u^2}{2} - 2\kappa\ell_n u^3\right)\right]^{n/2}}{\sqrt{1 - 2g(u\ell_n^{1/3} \wedge (6\kappa)^{-1})}}du,$$

$$r(t) \le 2e^{-t^2/2}\int_0^{|t|}\sin\left(\frac{u\ell_n}{4} \wedge \frac{\pi}{2}\right)$$
$$\times u\exp\{2\kappa\ell_n u^3 + g(u\ell_n^{1/3} \wedge (6\kappa)^{-1})\}du,$$

where $g(u) = \frac{u^2}{2} - 2\kappa u^3$ *for* $u \ge 0$; *moreover,* $g(u)$ *monotonically increases for* $0 \le u \le (6\kappa)^{-1}$.

The proof of this lemma uses estimates obtained in [2, 3].

Lemma 4 (see [1]). *For any distribution function* F *with mean* 0 *and variance* 1,

$$\sup_{x\in\mathbb{R}}|F(x) - \Phi(x)| \le \sup_{x>0}\left(\Phi(x) - \frac{x^2}{1+x^2}\right) = 0.54093\ldots.$$

Lemma 4 allows us not to consider the domain of values $\ell_n \ge \frac{0.541}{0.56} \approx 0.966$ in the proof of Theorem 1 and the domains $\ell_n \ge \frac{0.541}{C(n)}$ in the proof of Theorem 2.

Lemma 5 (see [9]). *If* $(1 - \max_{1\le k\le n}\sigma_k^2)^{-3/2}\ell_n \le 0.1$, *then*

$$\Delta_n \le 0.5151 \cdot \ell_n(1 - \max_{1\le k\le n}\sigma_k^2)^{-3/2}.$$

Since $\sigma_k^2 \le \beta_k^{3/2} \le \ell_n^{3/2}$ for all $0 \le k \le n$, it follows from Lemma 5 that $\Delta_n \le 0.5532\ell_n$ for $\ell_n \le 0.01$, which allows us not to consider the domain $\ell_n \le 0.01$ in the proof of Theorem 1.

Substituting estimates for $|f(t)|$ and $|r(t)|$ given by Lemmas 2 and 3 into the right-hand side of Prawitz' smoothing inequality from Lemma 1, we obtain a function $D(n, \ell, t_0, T)$ (or $D(\ell, t_0, T)$, if n-uniform estimates are used), which majorizes the uniform distance Δ_n for all $t_0 \in (0, 1]$, $T > 0$, and $\ell_n = \ell > 0$ (and $n \ge 1$). Observing that $\ell_n \ge \sum_{i=1}^{n}\sigma_i^3 \ge \frac{1}{\sqrt{n}}$ provided that $\sum_{i=1}^{n}\sigma_i^2 = 1$, we conclude that the constants $C(n)$ and C_0 can be sought in the form

$$C(n) = \max_{n^{-1/2} \le \ell \le 0.541/C(n)} \tilde{C}(n, \ell),$$

$$C_0 = \max_{0.01 \le \ell \le 0.97}\lim_{n\to\infty}\tilde{C}(n, \ell),$$

$$\tilde{C}(n, \ell) = \inf_{t_0, T}\frac{D(n, \ell, t_0, T)}{\ell}.$$

The monotonicity of the majorants for $|f(t)|$ and $r(t)$ with respect to ℓ_n implies that the function $\ell\tilde{C}(n, \ell)$ is monotonically nondecreasing with respect to $\ell > 0$ for each $n \ge 1$; therefore,

$$\tilde{C}(n, \ell) \le \tilde{C}(n, \ell_2)\frac{\ell_2}{\ell_1}, \quad \ell_1 \le \ell \le \ell_2, \quad n \ge 1,$$

and $\sup_{\ell}\tilde{C}(n, \ell)$ can be estimated by using values at finitely many points.

All computations were performed in the Matlab R2006b environment. It has turned out the function $\lim_{n\to\infty}\tilde{C}(n, \ell)$ attains its extremum value 0.55998 at $\ell \approx$

Table 2

n	ℓ	t_0	T
2	0.790	0.4313	3.8762
3	0.702	0.4052	4.3370
4	0.655	0.3898	4.6361
5	0.626	0.3804	4.8442
6	0.606	0.3732	4.9995
7	0.591	0.3677	5.1238
8	0.580	0.3638	5.2188
9	0.569	0.3591	5.3202
10	0.564	0.3577	5.3640

0.5085 ($t_0 \approx 0.4203$, $T \approx 5.9603$), which proves Theorem 1. For finite n, the extremum values of ℓ and the corresponding optimum values of the parameters t_0 and T are given in Table 2. The extremum values of $\tilde{C}(n, \ell)$ do not exceed those specified above, which proves Theorem 2.

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