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Review

QCD (Quantum chromodynamics) at low energies

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Abstract

The modern status of basic low energy QCD parameters is reviewed. It is demonstrated that the recent data allow one to determine the light quark mass ratios with an accuracy of 10–15%. The general analysis of vacuum condensates in QCD is presented, including those induced by external fields. The QCD coupling constant $\alpha_s(m_\tau^2)$ is found from the τ-lepton hadronic decay rate. The contour improved perturbation theory includes the terms up to α_s^4. The influence of instantons on $\alpha_s(m_\tau^2)$ determination is estimated. $V-A$ spectral functions of the τ-decay are used for construction of the $V-A$ polarization operator $\Pi_{V-A}(s)$ in the complex s-plane. The operator product expansion (OPE) is used up to dimension $D = 10$ and the sum rules along the rays in the complex s-plane are constructed. This makes it possible to separate the contributions of operators of different dimensions. The best values of the quark condensate and $\alpha_s\langle 0|\bar{q}q|0\rangle^2$ are found. The value of the quark condensate is confirmed by considering the sum rules for baryon masses. The value of the gluon condensate is found in four ways: by considering the $V + A$ polarization operator based on the τ-decay data and by studying the sum rules for polarization operator momenta in charmonia in the vector, pseudoscalar and axial channels. All of these determinations are in agreement and result in $\langle(\alpha_s/\pi)G^2\rangle = 0.005 \pm 0.004$ GeV^4. Valence quark distributions in the proton are calculated in QCD using the OPE in the proton current virtuality. The quark distributions agree with those found from the deep inelastic scattering data. The same value of the gluon condensate is favoured.

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Contents

1. Introduction

Nowadays, it is reliably established that the true (microscopic) theory of strong interaction is quantum chromodynamics (QCD), the nonabelian gauge theory of interacting quarks and gluons. The main confirmation of QCD comes from considering the processes at high energies and high momentum transfers, where, because of asymptotic freedom, high precision of the theoretical calculation is achieved and comparison with experiment confirms QCD with a very good accuracy. In the domain of low energies and momentum transfers (by such a domain in this paper I mean the domain of momentum transfers $Q^2 \sim 1$–$5\ \text{GeV}^2$) the situation is more complicated: the QCD coupling constant α_s is large, $\alpha_s \sim 0.5$–0.3 and many-loop perturbative calculations are necessary. Unlike in quantum electrodynamics (QED), the vacuum in QCD has a nontrivial structure: due to nonperturbative effects, nonzero fluctuations of gluonic and quark fields persist in the QCD vacuum. The nontrivial vacuum structure of QCD manifests itself in the presence of vacuum condensates, analogous to those in condensed matter physics (for instance, spontaneous magnetization). Therefore, α_s corrections and nonperturbative effects must be correctly accounted for in QCD calculations in this domain.

At lower energies and $Q^2 \lesssim 1\ \text{GeV}^2$, analytical QCD calculations are not reliable. The useful methods are: the chiral effective theory, lattice calculation and various model approaches. It is, however, very desirable to have a matching of all these approaches with

QCD calculations at Q^2 about 1 GeV^2. To achieve this, knowledge of low energy QCD parameters is necessary.

In order to fix the notation I present here the form of the QCD Lagrangian:

$$L = i \sum_q \bar{\psi}_q^a (\nabla_\mu \gamma_\mu + i m_q) \psi_q^a - \frac{1}{4} G_{\mu\nu}^n G_{\mu\nu}^n, \tag{1}$$

where

$$\nabla_\mu = \partial_\mu - ig \frac{\lambda^n}{2} A_\mu^n$$
$$G_{\mu\nu}^n = \partial_\mu A_\mu^n - \partial_\nu A_\mu^n + g f^{nml} A_\mu^m A_\nu^l. \tag{2}$$

ψ_q^a and A_μ^n are quark and gluon fields, $a = 1, 2, 3$; $n, m, l = 1, 2, \ldots, 8$ are colour indices, λ^n and f^{nml} are Gell-Mann matrices and f-symbols, m_q are bare (current) quark masses, $q = u, d, s, c, \ldots$.

Vacuum condensates are very important in the elucidation of the QCD structure and in the description of hadron properties at low energies. Condensates, particularly quark and gluonic ones, were investigated starting in the 1970s. Here, first, one should note the QCD sum rule method of Shifman, Vainshtein and Zakharov [1], which was based on the idea of the leading role of condensates in the calculation of masses of the low lying hadronic states. In the papers of the 1970s to 1980s it was assumed that the perturbative interaction constant is comparatively small (e.g., $\alpha_s(1\ \text{GeV}) \approx 0.3$), so that it is enough to restrict to the first-order terms in α_s and sometimes even disregard perturbative effects in the region of masses larger than 1 GeV. At present it is clear that α_s is considerably larger ($\alpha_s(1\ \text{GeV}) \sim 0.5$). In a number of cases there appeared the results of perturbative calculations to orders α_s^2 and α_s^3. New, more precise experimental data at low energies had been obtained.

This review presents the modern status of QCD at low energies. In Section 2 the values of light quark masses are discussed. Section 3 contains the definition of condensates and the description of their general properties. In Section 4 the QCD coupling constant α_s is determined from the data on hadronic τ-decay and their evolution with Q^2 (in the four-loop approximation) is given. In Section 5 quark and gluon condensates are found from the τ-decay data on $V - A$, $V + A$ and V correlators. The sum rules for the nucleon mass with account taken of α_s corrections are analysed in Section 6 and it is shown that they are well satisfied at the same value of the quark condensate, which was found from the τ-decay data. There are various methods of gluon condensate determination: (a) from $V + A$ correlators; (b) from charmonium sum rules are considered in Section 7. The QCD sum rules for valence quark distributions in nucleons are presented in Section 8; valence u and d quark distributions at low Q^2 were found and the restrictions on condensates were obtained. Finally, Section 9 summarizes the state of the art of low energy QCD.

2. The masses of light quarks

The u, d, s quark masses were first estimated by Gasser and Leutwyler about 30 years ago: it was demonstrated that $m_u, m_d \sim 5$ MeV and $m_s \sim 100$ MeV [2,3]. In 1977

Weinberg [4], using partial conservation of axial current and the Dashen theorem [5] to account for electromagnetic self-energies of mesons, proved that the ratios m_u/m_d and m_s/m_d may be expressed through K and π masses:

$$\frac{m_u}{m_d} = \frac{m^2_{K^+} - m^2_{K^0} + 2m^2_{\pi^0} - m^2_{\pi^+}}{m^2_{K^0} - m^2_{K^+} + m^2_{\pi^+}} \tag{3}$$

$$\frac{m_s}{m_d} = \frac{m^2_{K^0} + m^2_{K^+} + m^2_{\pi^+}}{m^2_{K^0} - m^2_{K^+} + m^2_{\pi^+}}. \tag{4}$$

Numerically, (3) and (4) are equal:

$$\frac{m_u}{m_d} = 0.56 \qquad \frac{m_s}{m_d} = 20.1. \tag{5}$$

On the basis of consideration of mass splitting in the baryon octet Weinberg assumed that $m_s = 150$ MeV at the scale of about 1 GeV. Then

$$m_u = 4.2\ \text{MeV}, \qquad m_d = 7.5\ \text{MeV}, \qquad m_s = 150\ \text{MeV} \tag{6}$$

at 1 GeV. The large m_s/m_d ratio explains the large mass splitting in the pseudoscalar meson octet. For $m^2_{K^+}/m^2_{\pi^+}$ we have $[\bar{m} = (m_u + m_d)/2]$

$$\frac{m^2_{K^+}}{m^2_{\pi^+}} = \frac{m_s + \bar{m}}{2\bar{m}} = 13 \tag{7}$$

in perfect agreement with experiment. The ratio m^2_η/m^2_π expressed in terms of quark mass ratios is also in good agreement with experiment.

The ratios (3) and (4) were obtained in the first order in quark masses. Therefore, their accuracy is of the order of accuracy of SU(3) symmetry, i.e. about 20%.

In [6] it was demonstrated that there is a relation valid in the second order in quark masses:

$$\left(\frac{m_u}{m_d}\right)^2 + \frac{1}{Q^2}\left(\frac{m_s}{m_d}\right)^2 = 1. \tag{8}$$

Using the Dashen theorem for electromagnetic self-energies of π- and K-mesons, one may express Q as

$$Q^2_D = \frac{(m^2_{K^0} + m^2_{K^+} - m^2_{\pi^+} + m^2_{\pi^0})(m^2_{K^0} + m^2_{K^+} - m^2_{\pi^+} - m^2_{\pi^0})}{4m^2_{\pi^0}(m^2_{K^0} - m^2_{K^+} + m^2_{\pi^+} - m^2_{\pi^0})}. \tag{9}$$

Numerically, Q_D is equal: $Q_D = 24.2$. However, the Dashen theorem is valid in the first order in quark masses. The electromagnetic mass difference of K-mesons calculated in [7] by using the Cottingham formula and in [8] by the large N_c approach increased $\Delta m_k = (M_{K^+} - M_{K^0})_{e.m.}$ from the Dashen value $\Delta m_k = 1.27$ MeV to $\Delta m_k = 2.6$ MeV and, correspondingly, decreased Q to $Q = 22.0 \pm 0.6$. The other way to find Q is from $\eta \to \pi^+\pi^-\pi^0$ decay, using the chiral effective theory. Unfortunately, the next to leading corrections are large in this approach [9], which makes the accuracy of the results uncertain. It was found from the $\eta \to \pi^+\pi^-\pi^0$ decay data with account taken of interaction in the final state that $Q = 22.4 \pm 0.9$ [10], $Q = 22.7 \pm 0.8$ [11] and $Q = 22.8 \pm 0.4$ [12] (the latter from the Dalitz plot). So, the final conclusion is that Q is in the interval $21.5 < Q < 23.5$. (It must be mentioned that the experiment where $\Gamma(\eta \to 2\gamma)$ and, consequently, $\Gamma(\eta \to \pi^+\pi^-\pi^0)$ were measured using the Primakoff effect is absent from the last data release of the Particle Data Group [13], while it persisted in the previous ones. See [14] for a review.) The ratio $\gamma = m_u/m_d$ can also be found from the ratio of $\psi' \to (J/\psi)\eta$ and $\psi' \to (J/\psi)\pi^0$ decays [15,16]. In [15] it was proved that

$$r = \frac{\Gamma(\psi' \to J/\psi + \pi^0)}{\Gamma(\psi' \to J/\psi + \eta)} = 3\left(\frac{1-\gamma}{1+\gamma}\right)^2 \left(\frac{m_\pi}{m_\eta}\right)^4 \left(\frac{p_\pi}{p_\eta}\right)^3 \tag{10}$$

where p_π and p_η are the pion and η momenta in the ψ' rest frame. Eq. (10) is valid in the first order in quark mass. The Particle Data Group data release [13] gives

$$r_{\text{exp}} = (3.04 \pm 0.71) \times 10^{-2}. \tag{11}$$

In the recent CLEO Collaboration experiment [17] it was found that $r_{\text{exp}} = (4.01 \pm 0.45) \times 10^{-2}$. Averaging these two experimental values, assuming the theoretical uncertainty in (10) as 30% and adding in quadrature the theoretical and experimental errors, we get from (10)

$$\gamma = \frac{m_u}{m_d} = 0.407 \pm 0.060. \tag{12}$$

A value close to that of (12) was found recently in [18]. The substitution of (12) into (8) with account taken of the above-mentioned uncertainty of Q results in

$$\frac{m_s}{m_d} = 20.8 \pm 1.3. \tag{13}$$

The value in (12) is slightly lower; thus the lowest order result (5) and (13) agrees with it. The values of (12) and (13) are in agreement with recent lattice calculations [19].

The calculation of absolute values of quark masses is a more subtle problem. First of all, the masses are scale dependent. In perturbation theory their scale dependence is given by the renormalization group equation

$$\frac{\mathrm{d}m(\mu)}{m(\mu)} = -\gamma[\alpha_s(\mu)]\frac{\mathrm{d}\mu^2}{\mu^2} = -\sum_{r=1}^{\infty}\gamma_r a^r(\mu^2)\frac{\mathrm{d}\mu^2}{\mu^2}. \tag{14}$$

In (14), $a = \alpha_s/\pi$, $\gamma_1 = 1$, $\gamma_2 = 91/24$, $\gamma_3 = 12.48$ for three flavours in the $\overline{MS}$ scheme [20]. In the first order in α_s it follows from (14) that

$$\frac{m(Q^2)}{m(\mu^2)} = \left[\frac{\alpha_s(\mu^2)}{\alpha_s(Q^2)}\right]^{\gamma_m}, \tag{15}$$

where $\gamma_m = -4/9$ is the quark mass anomalous dimension. There is no good convergence of the series (14) below $\mu = 2$ GeV ($\alpha_s(2\text{ GeV}) = 0.31$). The recent calculations of m_s using QCD sum rules [21], from the τ-decay data [22,23] and for a lattice [19,24], are in not very good agreement with one another. The mean value estimated in [13] is $m_s(2\text{ GeV}) \approx 105$ MeV with an accuracy of about 20%. By taking $m_s(1\text{ GeV})/m_s(2\text{ GeV}) = 1.35$ we have then $m_s(1\text{ GeV}) \approx 142$ MeV and, according to (12) and (13), $m_d(1\text{ GeV}) =$

6.8 MeV, $m_u(1\text{ GeV}) = 2.8$ MeV. The difference $m_d - m_u$ is equal to $m_d - m_u = 4.0 \pm 1.0$ MeV. This value agrees with one found using QCD sum rules from the baryon octet mass splitting [25] and D and D^* isospin mass differences [26], $m_d - m_u = 3 \pm 1$ MeV. For $m_u + m_d$ we have $m_u + m_d = 9.6 \pm 2.5$ MeV in comparison with $m_u + m_d = 12.8 \pm 2.5$ MeV found in [27]. For completeness I present here also the value of $m_c(m_c)$ (see below, Section 7.1):

$$m_c(m_c) = 1.275 \pm 0.015\,\text{GeV}. \tag{16}$$

3. Condensates

3.1. General properties

In QCD (or in a more general case, in quantum field theory), by “condensates” one means the vacuum mean values $\langle 0|O_i|0\rangle$ of the local (i.e. taken at a single point of space–time) operators $O_i(x)$, which arise due to nonperturbative effects. The latter point is very important and needs clarification. When determining vacuum condensates one implies averaging only over nonperturbative fluctuations. If for some operator O_i a nonzero vacuum mean value appears also in the perturbation theory, it should not be taken into account in determination of the condensate—in other words, when determining condensates the perturbative vacuum mean values should be subtracted in calculations of the vacuum averages. One more specification is necessary. The perturbation theory series in QCD are asymptotic series. So, vacuum mean operator values may appear due to some summing of asymptotic series. The vacuum mean values of such kinds are commonly referred to as vacuum condensates.

In quantum field theory it is assumed that vacuum correlators $\Pi_{AB}(x, y)$ in coordinate space of any two local operators $A(x)$, $B(y)$

$$\Pi_{AB}(x, y) = \langle 0|T\{A(x), B(y)\}|0\rangle$$

at spacelike $(x-y)^2 \le 0$, and small $x-y$ $(x-y \to 0)$, may be represented as an operator product expansion (OPE) series:

$$\Pi_{AB}(x-y) = \sum_i a_i(x-y)\langle 0|O_i(0)|0\rangle,$$

where $a_i(x-y)$ are called the coefficient functions and are given by perturbation theory. (Strict proofs of this statement were obtained only in perturbation theory and for some models.) Here, again one must take care to separate perturbative and nonperturbative parts in the definition of condensates. The perturbation expansion for $a_i(x-y)$ is an asymptotic series and the terms which arise by summing of such series may be interpreted as contributions of higher dimension operators. $a_i(x-y)$ may be infrared divergent. This is a signal of the appearance of an additional condensate in the OPE. Also, the OPE for $\Pi_{AB}(x-y)$ is probably an asymptotic series. In order to avoid all these problems in practical calculations, it is necessary to require a good convergence of the OPE and perturbation series in the domain of interest.

Separation of perturbative and nonperturbative contributions into vacuum mean values has some arbitrariness. Usually [28,29], this arbitrariness is avoided by introducing some normalization point μ^2 ($\mu^2 \sim 1$ GeV^2). Integration over momenta of virtual quarks and gluons in the region below μ^2 is related to condensates, above μ^2, to perturbative theory. In such a formulation, condensates depend on the normalization point μ: $\langle 0|O_i|0\rangle = \langle 0|O_i|0\rangle_\mu$. Other methods for determination of condensates are also possible (see below Section 5.2).

In perturbation theory, there appear corrections to condensates as a series in the coupling constant $\alpha_s(\mu)$:

$$\langle 0|O_i|0\rangle_Q = \langle 0|O_i|0\rangle_\mu \sum_{n=0}^{\infty} C_n^{(i)}(Q,\mu)\alpha_s^n(\mu). \tag{17}$$

The running coupling constant α_s in the right-hand part of (17) is normalized at the point μ. The left-hand part of (17) represents the value of the condensate normalized at the point Q. Coefficients $C_n^{(i)}(Q,\mu)$ may have logarithms $\ln Q^2/\mu^2$ in powers up to n for $C_n^{(i)}$. Summing the terms with the highest powers of logarithms leads to the appearance of the so-called anomalous dimension of operators, so in general form this can be written as

$$\langle 0|O_i|0\rangle_Q = \langle 0|O_i|0\rangle_\mu \left(\frac{\alpha_s(\mu)}{\alpha_s(Q)}\right)^{\gamma_i} \sum_{n=0}^{\infty} c_n^{(i)}(Q,\mu)\alpha_s^n(\mu), \tag{18}$$

where γ_i are anomalous dimensions (numbers), and $c_n^{(i)}$ already have no leading logarithms. If there exist several operators of the given (canonical) dimension, then their mixing is possible in perturbation theory. Then the relations (17) and (18) become matrix ones.

In their physical properties, condensates in QCD have much in common with condensates appearing in condensed matter physics: such as superfluid liquid (Bose condensate) in liquid ^{4}He, Cooper pair condensate in superconductors, spontaneous magnetization in magnets etc. That is why, analogously to effects in the physics of condensed matter, it can be expected that if one considers QCD at finite temperature T, with T increasing to some $T = T_c$ there will be phase transition and condensates (or a part of them) will be destroyed. In particular, such a phenomenon must hold for condensates responsible for spontaneous symmetry breaking—at $T = T_c$ they should vanish and symmetry must be restored. (In principle, certainly, QCD may have a few phase transitions.)

Condensates in QCD are divided into two types: chirality conserving and chirality violating. As was demonstrated in the previous section, the masses of light quarks u, d, s in the QCD Lagrangian are small compared with the characteristic scale of hadronic masses $M \sim 1$ GeV. In neglecting light quark masses the QCD Lagrangian becomes chiral invariant: left-hand and right-hand (in chirality) light quarks do not interact with each other; both vector and axial currents are conserved (except for the flavour-singlet axial current, the nonconservation of which is due to an anomaly). The accuracy of light quark mass neglect corresponds to the accuracy of isotopic symmetry, i.e. a few per cent in the case of u and d quarks, and the accuracy of SU(3) symmetry, i.e. 10–15% in the case of s quarks. In the case of condensates violating chiral symmetry, perturbative vacuum mean values are proportional to light quark masses and are zero within $m_u = m_d = m_s = 0$. So, such condensates are determined in the theory much better than those conserving chirality and, in principle, may be found experimentally with a higher accuracy.

Among chiral symmetry violating condensates, of the most importance is the quark condensate $\langle 0|\bar{q}q|0\rangle$ ($q = u, d$ are the fields of u and d quarks). $\langle 0|\bar{q}q|0\rangle$ may be written in the form

$$\langle 0|\bar{q}q|0\rangle = \langle 0|\bar{q}_L q_R + \bar{q}_R q_L|0\rangle \tag{19}$$

where q_L, q_R are the fields of left-hand and right-hand (in chirality) quarks. As follows from (19), a nonzero value of a quark condensate means the transition of left-hand quark fields into right-hand ones and it not having a small value would mean chiral symmetry violation in QCD. (If chiral symmetry is not violated spontaneously, then at small m_u, m_d, $\langle 0|\bar{q}q|0\rangle \sim m_u, m_d$). By virtue of isotopic invariance,

$$\langle 0|\bar{u}u|0\rangle = \langle 0|\bar{d}d|0\rangle. \tag{20}$$

For a quark condensate there holds the Gell-Mann–Oakes–Renner relation [30]

$$\langle 0|\bar{q}q|0\rangle = -\frac{1}{2}\frac{m_\pi^2 f_\pi^2}{m_u + m_d}. \tag{21}$$

Here m_π, f_π are the mass and the constant of π^+-meson decay ($m_\pi = 140$ MeV, $f_\pi = 131$ MeV); m_u and m_d are the masses of u and d quarks. Relation (21) is obtained in the first order in m_u, m_d, m_s (for its derivation see, e.g., [31]). To estimate the value of the quark condensate one may use the values of quark masses $m_u + m_d = 9.6$ MeV, presented in Section 2. Substituting these values into (21) we get

$$\langle 0|\bar{q}q|0\rangle = -(260 \text{ MeV})^3. \tag{22}$$

The value of (6) has a characteristic hadronic scale. This shows that chiral symmetry, which is fulfilled with a good accuracy in the light quark Lagrangian ($m_u, m_d/M \sim 0.01$), is spontaneously violated in the hadronic state spectrum.

Another argument in favour of spontaneous violation of chiral symmetry in QCD is provided by the existence of massive baryons. Indeed, in the chiral symmetrical theory all fermionic states should be either massless or parity degenerate. Obviously, baryons, in particular nuclei, do not possess this property. It can be shown [32,31] that both these phenomena—the presence of the chiral symmetry violating quark condensate and the existence of massive baryons—are closely connected with each other. According to the Goldstone theorem, the spontaneous symmetry violation leads to the appearance of massless particles in the physical state spectrum—of Goldstone bosons. In QCD, Goldstone bosons can be identified with a π-meson triplet within $m_u, m_d \to 0$, $m_s \neq 0$ (SU(2) symmetry) or with an octet of pseudoscalar mesons (π, K, η) within the limit m_u, m_d, $m_s \to 0$ (SU(3) symmetry). The presence of Goldstone bosons in QCD makes it possible to formulate the low energy chiral effective theory of strong interactions (see reviews [33,34,31]).

A quark condensate may be considered as an order parameter in QCD corresponding to spontaneous violation of the chiral symmetry. At the temperature of restoration of the chiral symmetry $T = T_c$ it must vanish. The investigation of the temperature dependence of the quark condensate in the chiral effective theory [35] shows that $\langle 0|\bar{q}q|0\rangle$ vanishes at $T = T_c \approx 150$–200 MeV. Similar indications were obtained also in the lattice calculations [36].

Thus, the quark condensate: (1) has the lowest dimensions ($d = 3$) as compared with other condensates in QCD; (2) determines masses of usual (nonstrange) baryons; (3) is the order parameter in the phase transition between the phases of violated and restored chiral symmetry. These three facts determine its important role in low energy hadronic physics.

Let us estimate the accuracy of the numerical value of (22). The quark condensate, like quark masses, depends on the normalization point and has anomalous dimensions equal to $\gamma_{\bar{q}q} = -\gamma_m = \frac{4}{9}$. In (22) the normalization point μ was taken as $\mu \simeq 1$ GeV. The Gell-Mann–Oakes–Renner relation is derived up to correction terms linear in quark masses. In the chiral effective theory it is possible to estimate the correction terms and, thereby, the accuracy of Eq. (21) appears to be of order 10%. The accuracy of the value taken above, $m_u + m_d = 9.6$ MeV, which enters (21) seems to be of order 20%. The value of the quark condensate may also be found from the sum rules for proton mass (see Section 6) as well as from structure functions at τ-decay (Section 5). The quark condensate of strange quarks is somewhat different from $\langle 0|\bar{u}u|0\rangle$. In [32] the following was obtained:

$$\langle 0|\bar{s}s|0\rangle / \langle 0|\bar{u}u|0\rangle = 0.8 \pm 0.1. \tag{23}$$

The next in dimension ($d = 5$) condensate which violates chiral symmetry is the quark gluonic one:

$$g\left\langle 0\left|\bar{q}\sigma_{\mu\nu}\frac{\lambda^n}{2}G^n_{\mu\nu}q\right|0\right\rangle \equiv m_0^2\langle 0|\bar{q}q|0\rangle. \tag{24}$$

Here $G^n_{\mu\nu}$ is the gluonic field strength tensor, λ^n are the Gell-Mann matrices, $\sigma_{\mu\nu} = (i/2)(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)$. The value of the parameter m_0^2 was found in [37] from the sum rules for baryonic resonances:

$$m_0^2 = 0.8 \pm 0.2\ \mathrm{GeV}^2. \tag{25}$$

The same value of m_0^2 was found from the analysis of B-mesons using QCD sum rules [38], close to the (25) value of $m_0^2 = 1.0\ \mathrm{GeV}^2$ calculated within the model of field correlators [39]. The anomalous dimension of the operator in (24) is small [40]. Therefore the anomalous dimension of m_0^2 is approximately equal to $\gamma_m = -4/9$.

Consider now condensates conserving chirality. A fundamental role here is played by the gluonic condensate of the lowest dimension:

$$\left\langle 0\left|\frac{\alpha_s}{\pi}G^n_{\mu\nu}G^n_{\mu\nu}\right|0\right\rangle. \tag{26}$$

Since the gluonic condensate is proportional to the vacuum mean value of the trace of the energy–momentum tensor $\theta_{\mu\nu}$, its anomalous dimension is zero. The existence of a gluonic condensate was first indicated by Shifman, Vainshtein and Zakharov [1]. They also obtained its numerical value from the sum rules for charmonium:

$$\left\langle 0\left|\frac{\alpha_s}{\pi}G^n_{\mu\nu}G^n_{\mu\nu}\right|0\right\rangle = 0.012\ \mathrm{GeV}^4. \tag{27}$$

As was shown by the same authors, the nonzero and positive value of the gluonic condensate means that the vacuum energy is negative in QCD: the vacuum energy density in QCD is given by $\varepsilon = -(9/32)\langle 0|(\alpha_s/\pi)G^2|0\rangle$. The persistence of the quark field in the vacuum destroys (or suppresses) the condensate. Therefore, if a quark is embedded into the vacuum, this results in its excitation, i.e., in an increase of energy. Thereby, it become possible to explain the bag model in QCD: in the domain around the quark there appears an excess of energy, which is treated as the energy density B in the bag model (although the magnitude of B probably does not agree with the value of ε which follows from (27)). In [1] perturbative effects were taken into account only in the order α_s, the value for α_s being taken about two times smaller than the modern one. Later many attempts were made to determine the value of the gluonic condensate by studying various processes and by applying various methods. But the results of different approaches were inconsistent with each other and with (27) and sometimes the difference was even very large—the values of the condensate appeared to be a few times larger. All of this requires one to reanalyse the methods of $\langle 0|\frac{\alpha_s}{\pi}G^2|0\rangle$ determination on the basis of modern values of α_s, which will be done in Sections 7 and 8.

The $d = 6$ gluonic condensate is of the form

$$g^3 f^{abc}\langle 0|G^a_{\mu\nu}G^b_{\nu\lambda}G^c_{\lambda\mu}|0\rangle \tag{28}$$

(f^{abc} are structure constants of the SU(3) group). There are no reliable methods for determining it from experimental data. There is only an estimate [41] which follows from the model of a diluted instanton gas:

$$g^3 f^{abc}\langle 0|G^a_{\mu\nu}G^b_{\nu\lambda}G^c_{\lambda\mu}|0\rangle = \frac{4}{5}(12\pi^2)\frac{1}{\rho_c^2}\left\langle 0\left|\frac{\alpha_s}{\pi}G^2_{\mu\nu}\right|0\right\rangle, \tag{29}$$

where ρ_c is the instanton effective radius in the given model (for estimation one may take $\rho_c \sim (1/3 - 1/2)\, fm$).

The general form of $d = 6$ condensates built from quark fields is

$$\alpha_s\langle 0|\bar{q}_i O_\alpha q_i \cdot \bar{q}_k O_\alpha q_k|0\rangle \tag{30}$$

where q_i, q_k are quark fields of u, d, s quarks, O_α are Dirac and $SU(3)$ matrices. Following [1], Eq. (30) is usually factorized: in the sum over intermediate states in all channels (i.e., if necessary, after Fierz transformation), only the vacuum state is taken into account. The accuracy of such an approximation is $\sim 1/N_c^2$, where N_c is the number of colours, i.e. ∼10%. After factorization Eq. (30) reduces to

$$\alpha_s\langle 0|\bar{q}q|0\rangle^2, \tag{31}$$

if $q = u, d$. The anomalous dimension of (31) is 1/9 and it can be approximately set to zero. And finally, $d = 8$ quark condensates assuming factorization reduce to

$$\alpha_s\langle 0|\bar{q}q|0\rangle \cdot m_0^2\langle 0|\bar{q}q|0\rangle. \tag{32}$$

(The notation of (24) is used.) It should be noted, however, that the factorization procedure in the $d = 8$ condensate case is not quite certain. For this reason, it is necessary to require their contribution to be small.

There are few gluon and quark–gluon condensates of dimension 8. (The full list of them is given in [42].) As a rule, the factorization hypothesis is used for their calculation. The other way to estimate the values of these condensates is to use the dilute instanton gas model. However, the latter, for some condensates, gives results (at accepted values of instanton gas model parameters) one order of magnitude larger than those from the factorization method. Arguments were presented [43] indicating that the instanton gas model overestimates the values of $d = 8$ gluon condensates. Therefore, the estimates based on the factorization hypothesis are more reliable here.

The violation of the factorization hypothesis is stronger for higher dimension condensates. So, this hypothesis may be used only for their estimations by order of magnitude.

3.2. Condensates induced by external fields

The meaning of such condensates can be easily understood by comparing with analogous phenomena in the physics of condensed matter. If the above-considered condensates can be compared, for instance with ferromagnets, where magnetization is present even in the absence of an external magnetic field, condensates induced by an external field are similar to diamagnets or paramagnets. Consider the case of the constant external electromagnetic field $F_{\mu\nu}$. In its presence there appears a condensate induced by an external field (in the linear approximation in $F_{\mu\nu}$):

$$\langle 0|\bar{q}\sigma_{\mu\nu}q|0\rangle_F = e_q \chi F_{\mu\nu}\langle 0|\bar{q}q|0\rangle. \tag{33}$$

As was shown in Ref. [44], to a good approximation $\langle 0|\bar{q}\sigma_{\mu\nu}q|0\rangle_F$ is proportional to e_q—the charge of the quark q. As a result, the field vacuum expectation value, $\langle 0|\bar{q}\sigma_{\mu\nu}q|0\rangle_F$, violates chiral symmetry. So, it is natural to separate $\langle 0|\bar{q}q|0\rangle$ as a factor in Eq. (33). The universal quark flavour independent quantity χ is called the magnetic susceptibility of the quark condensate. Its numerical value was found in [45] using a special sum rule:

$$\chi = -(5.7 \pm 0.6)\ \mathrm{GeV}^2. \tag{34}$$

Another example is the external constant axial isovector field A_μ, the interaction of which with light quarks is described by the Lagrangian

$$L' = (\bar{u}\gamma_\mu\gamma_5 u - \bar{d}\gamma_\mu\gamma_5 d)A_\mu. \tag{35}$$

In the presence of this field there appear, induced by it, condensates:

$$\langle 0|\bar{u}\gamma_\mu\gamma_5 u|0\rangle_A = -\langle 0|\bar{d}\gamma_\mu\gamma_5 d|0\rangle_A = f_\pi^2 A_\mu \tag{36}$$

where $f_\pi = 131$ MeV is the constant of $\pi \to \mu\nu$ decay. The right-hand part of Eq. (36) is obtained assuming $m_u, m_d \to 0$, $m_\pi^2 \to 0$ and follows directly from consideration of the polarization operator of axial currents $\Pi^A_{\mu\nu}(q)$ in the limit $q \to 0$, when because of axial current conservation the nonzero contribution to $\Pi^A_{\mu\nu}(q)_{q\to 0}$ emerges only from the one-pion intermediate state. Eq. (36) was used to calculate the axial coupling constant in β-decay, g_A [46]. A relation analogous to (36) holds in the case of the octet axial field. Of special interest is the condensate induced by the singlet (in flavours) constant axial field

$$\langle 0|j^{(0)}_{\mu 5}|0\rangle = 3 f_0^2 A^{(0)}_\mu \tag{37}$$

$$j^{(0)}_{\mu 5} = \bar{u}\gamma_\mu\gamma_5 u + \bar{d}\gamma_\mu\gamma_5 d + \bar{s}\gamma_\mu\gamma_5 s \tag{38}$$

and the Lagrangian of the interaction with the external field has the form

$$L' = j^{(0)}_{\mu 5} A^{(0)}_{\mu}. \tag{39}$$

Constant f_0 cannot be calculated by the method used when deriving Eq. (36), since the singlet axial current is not conserved because of an anomaly and the singlet pseudoscalar meson η' is not a Goldstone one. The constant f_0^2 is proportional to the topological susceptibility of the vacuum [47]:

$$f_0^2 = \frac{4}{3} N_f^2 \chi'(0), \tag{40}$$

where N_f is the number of light quarks, $N_f = 3$, and the topological susceptibility of the vacuum $\chi(q^2)$ is defined as

$$\chi(q^2) = i \int \mathrm{d}^4 x \mathrm{e}^{iqx} \langle 0 | T\, Q_5(x), Q_5(0) | 0 \rangle \tag{41}$$

$$Q_5(x) = \frac{\alpha_s}{8\pi} G^n_{\mu\nu}(x) \tilde{G}^n_{\mu\nu}(x), \tag{42}$$

where $\tilde{G}^n_{\mu\nu}$ is dual to $G^n_{\mu\nu}$: $\tilde{G}^n_{\mu\nu} = (1/2)\varepsilon_{\mu\nu\lambda\sigma} G^n_{\lambda\sigma}$. Using the QCD sum rule, one may relate f_0^2 with the part of the proton spin Σ carried by quarks in polarized ep (or μp) scattering [47]. The value of f_0^2 was found from the self-consistency condition of the sum rule obtained (or from the experimental value of Σ):

$$f_0^2 = (2.8 \pm 0.7) \cdot 10^{-2}\ \mathrm{GeV}^2. \tag{43}$$

The value of the derivative at $q^2 = 0$ of the vacuum topological susceptibility $\chi'(0)$ related to it (more precisely, its nonperturbative part) is equal to

$$\chi'(0) = (2.3 \pm 0.6) \cdot 10^{-3}\ \mathrm{GeV}^2. \tag{44}$$

The value $\chi'(0)$ is of essential interest for studying properties of the vacuum in QCD.

4. Test of QCD at low energies on the basis of τ-decay data

4.1. Determination of $\alpha_s(m_\tau^2)$

Collaborations ALEPH [48], OPAL [49] and CLEO [50] measured with a good accuracy the relative probability of hadronic decays of τ-leptons, $R_\tau = B(\tau \to \nu_\tau + hadrons)/B(\tau \to \nu_\tau e \overline{\nu}_e)$, and the vector V and axial A spectral functions. Below I present the results of the theoretical analysis of these data on the basis of the operator product expansion (OPE) in QCD [51,52] (see also [53,54]). In the perturbation theory series the terms up to α_s^4 will be taken into account, in OPE, the operators up to dimension 8. I restrict myself to the case of total hadronic strangeness equal to zero.

Consider the polarization operator of hadronic currents

$$\Pi^J_{\mu\nu} = i \int \mathrm{e}^{iqx} \langle T\, J_\mu(x) J_\nu(0)^\dagger \rangle \mathrm{d}x = (q_\mu q_\nu - \delta_{\mu\nu} q^2) \Pi^{(1)}_J(q^2) + q_\mu q_\nu \Pi^{(0)}_J(q^2),$$
$$\text{where} \quad J = V, A; \qquad V_\mu = \bar{u}\gamma_\mu d, \qquad A_\mu = \bar{u}\gamma_\mu\gamma_5 d. \tag{45}$$

The spectral functions measured in τ-decay are imaginary parts of $\Pi_J^{(1)}(s)$ and $\Pi_J^{(0)}(s)$, $s = q^2$:

$$v_1/a_1(s) = 2\pi \operatorname{Im} \Pi_{V/A}^{(1)}(s + i0), \qquad a_0(s) = 2\pi \operatorname{Im} \Pi_A^{(0)}(s + i0). \tag{46}$$

Functions $\Pi_V^{(1)}(q^2)$ and $\Pi_A^{(0)}(q^2)$ are analytical functions in the q^2 complex plane with a cut along the right-hand semiaxis starting from $4m_\pi^2$ for $\Pi_V^{(1)}(q^2)$ and $9m_\pi^2$ for $\Pi_A^{(0)}(q^2)$. Function $\Pi_A^{(1)}(q^2)$ has kinematical pole at $q^2 = 0$, since the physical combination which has no singularities is $\delta_{\mu\nu} q^2 \Pi_A^{(1)}(q^2)$. Because of axial current conservation in the limit of massless quarks this kinematical pole is related to the one-pion state contribution to $\Pi_A(q)$, which has the form [51]

$$\Pi_{\mu\nu}^A(q)_\pi = -\frac{f_\pi^2}{q^2}(q_\mu q_\nu - \delta_{\mu\nu} q^2) - \frac{m_\pi^2}{q^2} q_\mu q_\nu \frac{f_\pi^2}{q^2 - m_\pi^2}. \tag{47}$$

The chiral symmetry violation may result in corrections of order $f_\pi^2(m_\pi^2/m_\rho^2)$ in $\Pi_A^{(1)}(q^2)$ (m_ρ is the characteristic hadronic mass), i.e., in the theoretical uncertainty in the magnitude of the residue of the kinematical pole in $\Pi_A^{(1)}(q^2)$ of order $\Delta f_\pi^2/f_\pi^2 \sim m_\pi^2/m_\rho^2$.

Consider first the ratio of the total probability of hadronic decays of τ-leptons into states with zero strangeness to the probability of $\tau \to \nu_\tau e \overline{\nu}_e$. This ratio is given by the equality [55]

$$\begin{aligned} R_{\tau,V+A} &= \frac{B(\tau \to \nu_\tau + hadrons_{S=0})}{B(\tau \to \nu_\tau e \bar{\nu}_e)} \\ &= 6|V_{ud}|^2 S_{EW} \int_0^{m_\tau^2} \frac{\mathrm{d}s}{m_\tau^2}\left(1 - \frac{s}{m_\tau^2}\right)^2 \\ &\quad \times \left[\left(1 + 2\frac{s}{m_\tau^2}\right)(v_1 + a_1 + a_0)(s) - 2\frac{s}{m_\tau^2} a_0(s)\right] \end{aligned} \tag{48}$$

where $|V_{ud}| = 0.9735 \pm 0.0008$ is the matrix element of the Kobayashi–Maskawa matrix, $S_{EW} = 1.0194 \pm 0.0040$ is the electroweak correction [56]. Only the one-pion state is in practice contributing to the last term in [58] and it appears to be small:

$$\Delta R_\tau^{(0)} = -24\pi^2 \frac{f_\pi^2 m_\pi^2}{m_\tau^4} = -0.008. \tag{49}$$

Write

$$\omega(s) \equiv v_1 + a_1 + a_0 = 2\pi \operatorname{Im} [\Pi_V^{(1)}(s) + \Pi_A^{(1)}(s) + \Pi_A^{(0)}(s)] \equiv 2\pi \operatorname{Im} \Pi(s). \tag{50}$$

As follows from Eq. (47), $\Pi(s)$ has no kinematical pole, only a right-hand cut. It is convenient to transform the integral in Eq. (48) into that over the circle of radius m_τ^2 in the complex s-plane [57–59]:

$$R_{\tau,V+A} = 6\pi i |V_{ud}|^2 S_{EW} \oint_{|s|=m_\tau^2} \frac{\mathrm{d}s}{m_\tau^2}\left(1 - \frac{s}{m_\tau^2}\right)^2 \left(1 + 2\frac{s}{m_\tau^2}\right) \Pi(s) + \Delta R_\tau^{(0)}. \tag{51}$$

Eq. (51) allows one to express $R_{\tau,V+A}$ in terms of $\Pi(s)$ at large $|s| = m_\tau^2$, where perturbative theory and the OPE are valid.

Calculate first the perturbative contribution to Eq. (51). To this end, use the Adler function $D(Q^2)$:

$$D(Q^2) \equiv -2\pi^2 \frac{\mathrm{d}\Pi(Q^2)}{\mathrm{d}\ln Q^2} = \sum_{n\geq 0} K_n a^n, \qquad a \equiv \frac{\alpha_s}{\pi}, \qquad Q^2 \equiv -s, \tag{52}$$

the perturbative expansion of which is known up to terms $\sim\alpha_s^4$. In the $\overline{MS}$ regularization scheme $K_0 = K_1 = 1$, $K_2 = 1.64$ [60], $K_3 = 6.37$ [61] for three flavours and for K_4 there are the estimates $K_4 = 25 \pm 25$ [62] and $K_4 = 27 \pm 16$ [63]. The renormalization group equation yields

$$\frac{\mathrm{d}a}{\mathrm{d}\ln Q^2} = -\beta(a) = -\sum_{n\geq 0} \beta_n a^{n+2} \tag{53}$$

$$\ln \frac{Q^2}{\mu^2} = -\int_{a(\mu^2)}^{a(Q^2)} \frac{\mathrm{d}a}{\beta(a)}, \tag{54}$$

in the $\overline{MS}$ scheme for three flavours $\beta_0 = 9/4$, $\beta_1 = 4$, $\beta_2 = 10.06$, $\beta_3 = 47.23$ [64–66]. Integrating over Eq. (52) and using Eq. (53) we get

$$\Pi(Q^2) = \frac{1}{2\pi^2} \int_{a(\mu^2)}^{a(Q^2)} D(a) \frac{\mathrm{d}a}{\beta(a)}. \tag{55}$$

Put $\mu^2 = m_\tau^2$ and choose some (arbitrary) value $a(m_\tau^2)$. With the help of Eq. (54) one may determine then $a(Q^2)$ for any Q^2 and by analytical continuation for any s in the complex plane. Then, calculating (55) find $\Pi(s)$ in the whole complex plane. Substitution of $\Pi(s)$ into Eq. (51) determines R_τ for the given $a(m_\tau^2)$ up to power corrections. Thereby, knowing R_τ from experiment it is possible to find the $a(m_\tau^2)$ corresponding to it. Note that with such an approach there is no need to expand the denominator in Eqs. (54) and (55) in the inverse powers of $\ln Q^2/\mu^2$. Advantages of transformation of the integral to over the real axis (48) in the contour integral are the following. It can be expected that the applicability region of the theory presented as perturbation theory (PT) + operator product expansion (OPE) in the complex s-plane is off the dashed region in Fig. 1. It is evident that at positive and comparatively small s PT + OPE does not work.

As is well known, in perturbation theory, in the expansion over the powers of inverse $\ln Q^2$, in the first order in $1/\ln Q^2$ the running coupling constant $\alpha_s(Q^2)$ has an unphysical pole at some $Q^2 = Q_0^2$. If $\beta(a)$ is kept in the denominator in (54), then in the n-loop approximation ($n > 1$) a branch cut with a singularity $\sim(1 - Q^2/Q_0^2)^{-1/n}$ appears instead of pole. The position of the singularity is given by

$$\ln \frac{Q_0^2}{\mu^2} = -\int_{a(\mu^2)}^{\infty} \frac{\mathrm{d}a}{\beta(a)}. \tag{56}$$

Near the singularity the last term in the expansion of $\beta(a)$ dominates and gives the before-mentioned behaviour. Since the singularity became weaker, one may expect a better convergence of series, which would allow one to go to lower Q^2.

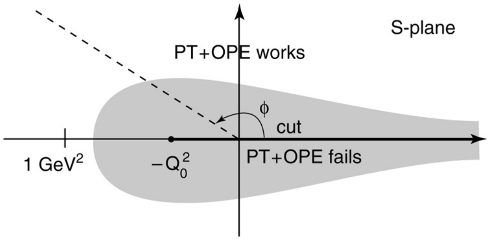

Fig. 1. The applicability region of PT and OPE in the complex plane s. In the dashed region, PT + OPE does not work.

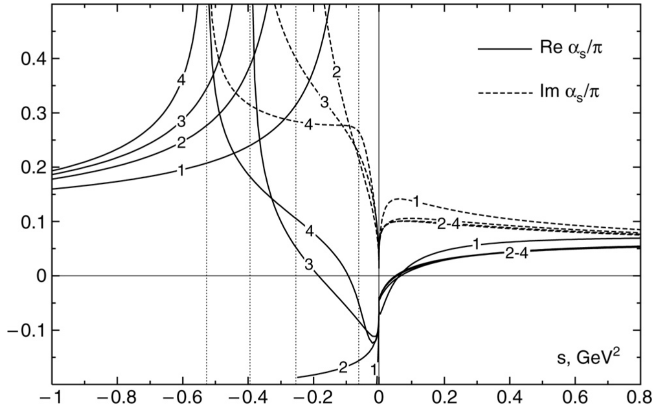

Fig. 2. Real and imaginary parts of $\alpha_{\overline{\rm MS}}(s)/\pi$ as an exact numerical solution of the RG equation (54) on real axes for different numbers of loops. The initial condition is chosen as $\alpha_s = 0.355$ at $s = -m_\tau^2$, $N_f = 3$. Vertical dotted lines display the position of the unphysical singularity at $s = -Q_0^2$ for each approximation ($4 \to 1$ from left to right).

The real and imaginary parts of $\alpha_s(s)/\pi$, obtained as numerical solutions of Eq. (54) for various numbers of loops, are plotted in Fig. 2 as functions of $s = -Q^2$. The τ-lepton mass was chosen as the normalization point, $\mu^2 = m_\tau^2$, and $\alpha_s(m_\tau^2) = 0.355$ was put in. As is seen from Fig. 2, at negative s the perturbation theory converges at $s < -1\,\mathrm{GeV}^2$ and in order to have a good precision of the results four-loop calculations are necessary. At positive s, especially for $\mathrm{Im}\,(\alpha_s/\pi)$, the convergence of the series is much better. This comes from the fact that in the chosen integral form of the renormalization group equation (54) the expansion over $\pi/\ln(Q^2/\Lambda^2)$ is avoided, this expansion not being a small parameter at intermediate Q^2. (The systematical method of analytical continuation from the spacelike to the timelike region with summation of π^2 terms was suggested in [67] and developed in [68].) For instance, in the next to leading order

$$2\pi\,\mathrm{Im}\,\Pi(s+i0) = 1 + \frac{1}{\pi\beta_0}\left[\frac{\pi}{2} - \mathrm{arctg}\left(\frac{1}{\pi}\ln\frac{s}{\Lambda^2}\right)\right] \tag{57}$$

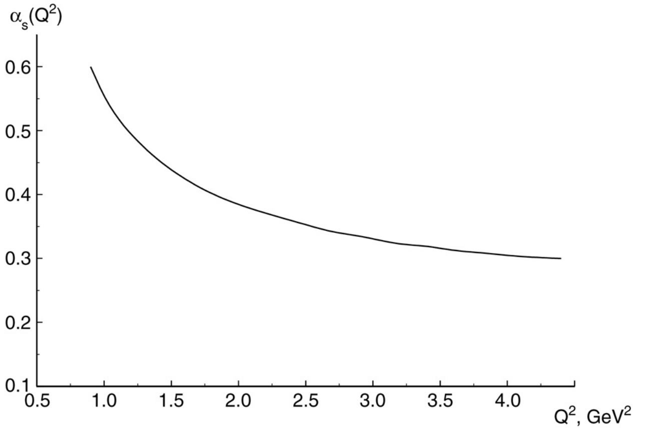

Fig. 3. $\alpha_s(Q^2)$ normalized at m_τ^2; $\alpha_s(m_\tau^2) = 0.33$.

instead of

$$2\pi \,\mathrm{Im}\, \Pi(s + i0) = 1 + \frac{1}{\beta_0 \ln(s/\Lambda^2)}, \tag{58}$$

which would follow in the case of small $\pi/\ln(s/\Lambda^2)$.

$\alpha_s(Q^2)$ at $Q^2 > 0$ in the low Q^2 region ($0.8 < Q^2 < 5\ \mathrm{GeV}^2$) is plotted in Fig. 3. (Four loops are taken into account; $\alpha_s(m_\tau^2)$ is put equal to $\alpha_s(m_\tau^2) = 0.33$. As follows from the τ-decay rate, $\alpha_s(m_\tau^2) = 0.352 \pm 0.020$ and the value of one standard deviation below the mean one is favoured by low energy sum rules.)

Integration over the contour allows one to avoid the dashed region in Fig. 1 (except for the vicinity of the positive semiaxis, the contribution of which is suppressed by the factor $(1 - \frac{s}{m_\tau^2})^2$ in Eq. (51)), i.e. to work in the applicability region of PT + OPE.

The OPE terms, i.e., power corrections to the polarization operator, are given by the formula [1]

$$\begin{aligned}\Pi(s)_{nonpert} &= \sum_{n\geq 2} \frac{\langle O_{2n}\rangle}{(-s)^n}\left(1 + c_n \frac{\alpha_s}{\pi}\right)\\ &= \frac{\alpha_s}{6\pi\, Q^4}\langle 0|G^a_{\mu\nu}G^a_{\mu\nu}|0\rangle\left(1 + \frac{7}{6}\frac{\alpha_s}{\pi}\right) + \frac{4}{Q^4}(m_u + m_d)\,\langle 0|\bar{q}q|0\rangle\\ &\quad + \frac{128}{81\, Q^6}\pi\alpha_s\langle 0|\bar{q}q|0\rangle^2_\mu\left[1 + \left(\frac{29}{24} + \frac{17}{18}\ln\frac{Q^2}{\mu^2}\right)\frac{\alpha_s}{\pi}\right] + \frac{\langle O_8\rangle}{Q^8}\end{aligned} \tag{59}$$

(α_s-corrections to the first and third terms in Eq. (59) were calculated in [69] and [70], respectively). Contributions of terms proportional to m_u^2, m_d^2 are neglected. When calculating the $d = 6$ term, the factorization hypothesis was used. The gluon condensate of dimension $d = 6$, $g^3\langle 0|G^3|0\rangle$ (28), does not contribute to the polarization operator (59). This is a consequence of the general theorem, proved by Dubovikov and Smilga [71], that in the case of self-dual gluonic fields there are no contributions of gluon condensates of dimensions higher than $d = 4$ to vector and axial current polarization operators. Since the vacuum expectation value of the G^3 operator does not vanish for self-dual gluonic fields, this means the vanishing of the coefficient in front of the $g^3\langle 0|G^3|0\rangle$ condensate in (59). The same argument applies to dimension 8 gluon operators g^4G^4 with the exception of some of them, like $g^4[\,G^n_{\mu\alpha}G^n_{\mu\beta} - (1/4)\delta_{\alpha\beta}G^n_{\mu\nu}G^n_{\mu\nu}\,]^2$, which have zero expectation values in any self-dual field. But the latter are suppressed by a small factor $1/4\pi^2$ arising from loop integration in comparison with the tree diagram, corresponding to the $d = 8$ four-quark condensate $\langle O_8\rangle \sim \langle \bar{q}Gq \cdot \bar{q}q\rangle$ contribution. The contribution from this condensate may be estimated as $|\langle O_8\rangle| < 10^{-3}$ GeV [52] (see below, Section 5.1) and appears to be negligibly small. The $d = 8$ two-quark–two-gluon operator $O_8' \sim g^2 D\bar{q}GGq$ is nonfactorizable, its vacuum mean value is suppressed by $1/N_c$ and one may assume that its contribution to (59) is also small. It can be readily seen that $d = 4$ condensates (up to small α_s corrections) give no contribution to the integral-over-contour equation (51). $R_{\tau,V+A}$ may be represented as

$$R_{\tau,V+A} = 3|V_{ud}|^2 S_{EW}\left(1 + \delta'_{em} + \delta^{(0)} + \delta^{(6)}_{V+A}\right) + \Delta R^{(0)} = 3.486 \pm 0.016 \qquad (60)$$

where $\delta'_{em} = (5/12\pi)\alpha_{em}(m_\tau^2) = 0.001$ is the electromagnetic correction [72], $\delta^{(6)}_{A+V} = -(3.3 \pm 1.1) \times 10^{-3}$ is the contribution of the $d = 6$ condensate (see below) and $\delta^{(0)}$ is the PT correction. The right-hand part presents the experimental value obtained as a difference between the total probability of hadronic decays $R_\tau = 3.647 \pm 0.014$ [73] and the probability of decays in states with the strangeness $S = -1$, $R_{\tau,s} = 0.161 \pm 0.007$ [74, 75]. For perturbative correction, it follows from Eq. (60) that

$$\delta^{(0)} = 0.208 \pm 0.006. \qquad (61)$$

From (61), employing the above-described method, the constant $\alpha_s(m_\tau^2)$ was found [52]:

$$\alpha_s(m_\tau^2) = 0.352 \pm 0.020. \qquad (62)$$

The calculation was made with account taken of the terms $\sim\alpha_\tau^4$; the theoretical error was assumed to be equal to the last term contribution. Possibly the error is underestimated (by $\sim$0.010), since the theoretical and experimental errors were added in quadrature. The value $\alpha_s(m_\tau^2)$ (62) corresponds to

$$\alpha_s(m_z^2) = 0.121 \pm 0.002. \qquad (63)$$

This value is in agreement with a recent determination [76] of $\alpha_s(m_z^2)$ from the whole set of data:

$$\alpha_s(m_z^2) = 0.1182 \pm 0.0027. \qquad (64)$$

4.2. *Instanton corrections*

Some nonperturbative features of QCD may be described in the so-called instanton gas model (see [77] for an extensive review and the collection of related papers in [79]). That is, one computes the correlators in the $SU(2)$ instanton field embedded in the $SU(3)$ colour group. In particular, the two-point correlator of the vector currents was computed long ago [80]. In addition to the usual tree-level correlator $\sim \ln Q^2$, it has a correction which depends on the instanton position and radius ρ. In the instanton gas model these parameters are integrated out. The radius is averaged over some concentration $n(\rho)$, for which some model is used. Concerning the two-point correlator of charged axial currents, the only difference from the vector case is that the term with zero modes must be taken with opposite sign. In the coordinate representation the answer can be expressed in terms of elementary functions; see [80]. An attempt to compare the instanton correlators with the ALEPH data in the coordinate space was made in Ref. [81].

We shall work in momentum space. Here the instanton correction to the spin-J parts $\Pi^{(J)}$ of the correlator (45) can be written in the following form:

$$\Pi^{(1)}_{V,\mathrm{inst}}(q^2) = \int_0^\infty \mathrm{d}\rho\, n(\rho)\left[-\frac{4}{3q^4} + \sqrt{\pi}\rho^4 G^{30}_{13}\left(-\rho^2 q^2 \,\middle|\, \begin{matrix}1/2\\ 0,0,-2\end{matrix}\right)\right]$$
$$\Pi^{(0)}_{A,\mathrm{inst}}(q^2) = \int_0^\infty \mathrm{d}\rho\, n(\rho)\left[-\frac{4}{q^4} - \frac{4\rho^2}{q^2} K_1^2\left(\rho\sqrt{-q^2}\right)\right]$$
$$\Pi^{(1)}_{A,\mathrm{inst}}(q^2) = \Pi^{(1)}_{V,\mathrm{inst}}(q^2) - \Pi^{(0)}_{A,\mathrm{inst}}(q^2), \qquad \Pi^{(0)}_{V,\mathrm{inst}}(q^2) = 0. \tag{65}$$

Here K_1 is a modified Bessel function; $G^{pq}_{mn}(z|\ldots)$ is the Meijer function. Definitions, properties and approximations of Meijer functions can be found in, for instance, [82]. In particular, the function in (65) can be written as the following series:

$$\sqrt{\pi} G^{30}_{13}\left(z \,\middle|\, \begin{matrix}1/2\\ 0,0,-2\end{matrix}\right) = \frac{4}{3z^2} - \frac{2}{z} + \frac{1}{2\sqrt{\pi}}\sum_{k=0}^{\infty} z^k \frac{\Gamma(k+1/2)}{\Gamma^2(k+1)\,\Gamma(k+3)}$$
$$\times \Big\{[\ln z + \psi(k+1/2) - 2\,\psi(k+1) - \psi(k+3)\,]^2$$
$$+ \psi'(k+1/2) - 2\,\psi'(k+1) - \psi'(k+3)\Big\} \tag{66}$$

where $\psi(z) = \Gamma'(z)/\Gamma(z)$. For large $|z|$ one can obtain its approximation by the saddle-point method:

$$G^{30}_{13}\left(z \,\middle|\, \begin{matrix}1/2\\ 0,0,-2\end{matrix}\right) \approx \sqrt{\pi} z^{-3/2} \mathrm{e}^{-2\sqrt{z}}, \qquad |z| \gg 1. \tag{67}$$

The formulae (65) should be treated in the following way. One adds Π_{inst} to the usual polarization operator with perturbative and OPE terms. But the terms $\sim 1/q^4$ must be absorbed by the operator O_4 in Eq. (65), since the gluonic condensate $\langle G^2\rangle >$ is averaged over all field configurations, including the instanton one. Notice the negative sign before $1/q^4$ in Eq. (64). This arises because the negative contribution of the quark condensate $\langle m\bar{q}q\rangle$ in the instanton field exceeds the positive contribution of the gluonic condensate $\langle G^2\rangle$. In the real world $\langle m\bar{q}q\rangle$ is negligible.

The correlators (65) possess appropriate analytical properties; they have a cut along the positive real axes:

$$\mathrm{Im}\,\Pi^{(1)}_{V,\mathrm{inst}}(q^2+i0)=\int_0^\infty \mathrm{d}\rho\, n(\rho)\,\pi^{3/2}\rho^4 G^{20}_{13}\left(\rho^2 q^2\left|\begin{matrix}1/2\\0,0,-2\end{matrix}\right.\right) \tag{68}$$

$$\mathrm{Im}\,\Pi^{(0)}_{A,\mathrm{inst}}(q^2+i0)=-\int_0^\infty \mathrm{d}\rho\, n(\rho)\,\frac{2\pi^2\rho^2}{q^2} J_1\left(\rho\sqrt{q^2}\right) N_1\left(\rho\sqrt{q^2}\right). \tag{69}$$

We shall consider below the instanton gas model. It is a model with fixed instanton radius:

$$n(\rho)=n_0\,\delta(\rho-\rho_0). \tag{70}$$

In [77] the following was estimated:

$$\rho_0\approx 1/3\,\mathrm{fm}\approx 1.5\text{–}2.0\,\mathrm{GeV}^{-1},\qquad n_0\approx 1\,\mathrm{fm}^{-4}\approx(1.0\text{–}1.5)\times10^{-3}\,\mathrm{GeV}^4. \tag{71}$$

In fact, it was the instanton liquid model, with account taken of the instanton self-interaction, that was mainly considered in [77], but the arguments from which the estimations (71) follow refer also to the instanton gas model. In this case, the value of n_0 (71) should be considered as an upper limit (see also [78]).

Now we consider the instanton contribution to the τ-decay branching ratio. Since the instanton correlator (65) has a $1/q^2$ singular term in the expansion near 0 (see Eq. (66)), the integrals must be taken over the circle, as in (51). In the instanton model the function $a_0(s)$ differs from the experimental δ-function, which gives a small correction. So we shall ignore the last term in (48) and consider the integral with $\Pi^{(1)}_{V+A}+\Pi^{(0)}_A$ in (51). The instanton correction to the τ-decay branching ratio can be brought into the following form:

$$\begin{aligned}\delta_{\mathrm{inst}}&=-48\,\pi^{5/2}\int_0^\infty \mathrm{d}\rho\, n(\rho)\,\rho^4\, G^{20}_{13}\left(\rho^2 m_\tau^2\left|\begin{matrix}1/2\\0,-1,-4\end{matrix}\right.\right)\\ &\approx\frac{48\pi^2 n_0}{\rho_0^2 m_\tau^6}\,\sin(2\rho_0 m_\tau).\end{aligned} \tag{72}$$

Since the parameters (71) are determined quite approximately, we may explore the dependence of δ_{inst} on them. δ_{inst} versus ρ_0 for fixed $n_0=1.5\times10^{-3}$ GeV^4 is shown in Fig. 4.

As seen from Fig. 4(a) the instanton correction to the hadronic τ-decay is extremely small except for the unreliably low value of the instanton radius $\rho_0<1.5\,\mathrm{GeV}^{-1}$. At the favourable value [77] $\rho_0=1.7\,\mathrm{GeV}^{-1}$ the instanton correction to R_τ is almost exactly zero. This confirms the calculations of $\alpha_s(m_\tau^2)$ (Section 4.1), where the instanton corrections were not taken into account.

Eq. (72) can be used in another way. That is, the τ mass can be considered as the free parameter s_0. The dependence of the fractional corrections $\delta^{(0)}$ and $\delta^{(0)}_{0.330}+\delta_{\mathrm{inst}}$ on s_0 is shown in Fig. 4(b). The result strongly depends on the instanton radius and rather essentially on the density n_0. For $\rho_0=1.7\,\mathrm{GeV}^{-1}$ and $n_0=1\,\mathrm{fm}^{-4}$, the instanton curve is outside the errors already at $s_0\sim 2$ GeV^2, where the perturbation theory is expected to work.

We came to the conclusion that in the case of variable τ mass the instanton contribution becomes large at $s_0<2$ GeV^2. That means that $R_{\tau,V+A}(s_0)$ given by (51) cannot be

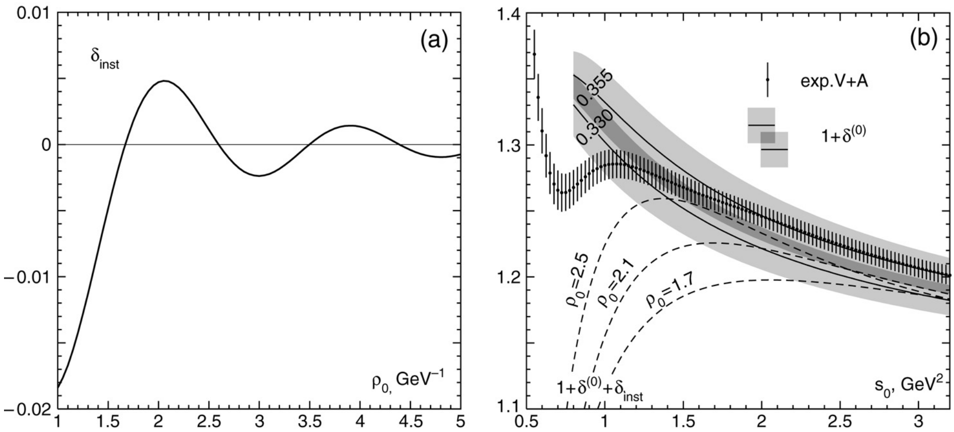

Fig. 4. The instanton correction to the τ-decay ratio versus ρ_0 (a) and "versus τ mass" (b) for $n_0 = 1.5 \times 10^{-3}\,\text{GeV}^4$. The thin solid lines in (b) are the values of $1 + \delta^{(0)}(s_0)$, where $\delta^{(0)}(s_0)$ are perturbative corrections, calculated as described in Section 4.1. The upper curve corresponds to $\alpha_s(m_\tau^2) = 0.355$, the lower one, to $\alpha_s(m_\tau^2) = 0.330$. The shadowed regions represent the uncertainties in perturbative calculations; the dark shadowed band is their overlap. The dashed lines are $1 + \delta^{(0)}(s) + \delta_{\text{inst}}$; $\delta^{(0)}(s)$ corresponds to $\alpha_s(m_\tau^2) = 0.330$.

represented by PT + OPE at $s_0 < 2\ \text{GeV}^2$ and the results obtained in this way are not reliable.

4.3. Comparison with other approaches

There are many calculations of $\alpha_s(m_\tau^2)$ from the total τ-decay rate, using the same idea as was used above—the contour improved fixed order perturbation theory [53,55, 57–59,73]. (For more recent ones, see [23,54].) The results of these calculations coincide with those presented above in the limit of errors and give $\alpha_s(m_\tau^2) = 0.33$–$0.35$. From these values, by using the renormalization group, one can find $\alpha_s(m_Z^2) = 0.118$–0.121 in agreement with $\alpha_s(m_Z^2)$ determinations from other processes (see [13,76]).

Previously only one renormalization scheme was considered—the $\overline{MS}$ scheme. In the BLM renormalization scheme [83], which has some advantages from the point of view of perturbative pomeron theory [84], the result is $\alpha_s(m_\tau^2) = 0.621 \pm 0.008$ [85], corresponding in the framework of the BLM scheme to the same value of $\alpha_s(m_Z^2) = 0.117$–0.122. At low scales, however, the $\alpha_s(Q^2)$ behaviour is essentially different from that presented in Fig. 3.

A few words about α_s calculations in analytical QCD (see [86] and references herein) are in order. According to this theory the coupling constant $\alpha_s(Q^2)$ is calculated by means of the renormalization group in the spacelike region $Q^2 > 0$. Then, by analytical continuation to $s = -Q^2 > 0$, $\text{Im}\,\alpha_s(s)$ was found on the right semiaxes. It was assumed that $\alpha_s(s)$ is an analytical function in the complex s-plane with a cut along the right semiaxes, $0 \leq s \leq \infty$. The analytical $\alpha_s(s)_{an}$ is then defined in the whole s-plane by the dispersion relation. Such an $\alpha_s(s)_{an}$ has no unphysical singularities. Let us calculate $\alpha_s(m_\tau^2)_{an}$ using the same experimental data as before, i.e. $\delta^{(0)}$ given by Eq. (61).

In the analytical QCD the contour integral (51) is equal to the integral of Im $\Pi(s)$ over real positive axes. (In the previous calculation the integral was running from $s = -Q_0^2$ to m_τ^2.) Qualitatively, it leads to a much smaller R_τ in the analytical QCD than in the conventional approach with the same $\alpha_s(m_\tau^2)$; or vice versa, it is necessary to have much larger $\alpha_s(m_\tau^2)_{an}$ in order to get the experimental R_τ. The calculation of the integral (51) with $\Pi(s)$ expressed through $\alpha_s(s)_{anal}$ shows that the experimental R_τ results in $\alpha_s(m_z^2) = 0.141 \pm 0.004$ in contradiction with other data. (In [87] an attempt was made to get an agreement of analytical QCD with common values of $\alpha_s(m_z^2)$. With this goal, the constituent quark model with a specific quark–antiquark potential was used in the domain of low and intermediate s. Evidently, such approach cannot be considered as an α_s determination in QCD: in this approach QCD is modified on a large circle in the complex plane of radius $|s| = m_\tau^2$ in contradiction to the basic assumption of α_s calculation from the hadronic τ-decay rate.)

5. Determination of condensates from spectral functions of τ-decay

5.1. Determination of the quark condensate from the $V - A$ spectral function

In order to determine the quark condensate from τ-decay data it is convenient to consider the difference $V-A$ of polarization operators $\Pi_V^{(1)} - \Pi_A^{(1)}$, where the contribution of perturbative terms is absent. $\Pi_V^{(1)}(s) - \Pi_A^{(1)}(s)$ is represented by the OPE:

$$\Pi_V^{(1)}(s) - \Pi_A^{(1)}(s) = \sum_{D\geq 4} \frac{O_D^{V-A}}{(-s)^{D/2}} \left(1 + c_D \frac{\alpha_s(s)}{\pi}\right). \tag{73}$$

The gluonic condensate contribution drops out in the $V - A$ difference and only the following condensates up to $D = 10$ remain:

$$O_4^{V-A} = 2\,(m_u + m_d)\,\langle 0|\bar{q}q|0\rangle = -f_\pi^2 m_\pi^2 \;[1] \tag{74}$$

$$\begin{aligned} O_6^{V-A} &= 2\pi\alpha_s \langle 0|(\bar{u}\gamma_\mu\lambda^a d)(\bar{d}\gamma_\mu\lambda^a u) - (\bar{u}\gamma_5\gamma_\mu\lambda^a d)(\bar{d}\gamma_5\gamma_\mu\lambda^a u)|0\rangle \\ &= -\frac{64\pi\alpha_s}{9}\langle 0|\bar{q}q|0\rangle^2 \;[1] \end{aligned} \tag{75}$$

$$O_8^{V-A} = -8\pi\alpha_s\, m_0^2 \langle 0|\bar{q}q|0\rangle^2, \;[88,51]^1 \tag{76}$$

$$O_{10}^{V-A} = -\frac{8}{9}\pi\alpha_s \langle 0|\bar{q}q|0\rangle^2 \left[\frac{50}{9} m_0^4 + 32\pi^2 \langle 0|\frac{\alpha_s}{\pi} G^2|0\rangle\right] \;[89] \tag{77}$$

where m_0^2 is determined in Eq. (24). In the right-hand sides of (75)–(77) the factorization hypothesis was used. For the O_6 operator it is expected [1] that the accuracy of the factorization hypothesis is of order $1/N_c^2 \sim 10\%$, where $N_c = 3$ is the number of colours. For operators of dimensions $d \geq 8$ the factorization procedure is not unique. (But, as a rule, the differences arising are not very large—for the $d = 8$ operator entering Eq. (73) it is about 20%). The accuracy of the factorization hypothesis becomes worse with increase

[1] There was a sign error in the contribution of O_8 in [51].

of the operator dimensions: for O_8^{V-A} it is worse than for O_6^{V-A} and for O_{10}^{V-A} it is worse than for O_8^{V-A}.

Operators O_4 and O_6 have approximately zero anomalous dimensions; the O_8 anomalous dimension is equal to 11/27. Calculations of the coefficients in front of α_s in Eq. (73) gave $c_4 = 4/3$ [90] and $c_6 = 89/48$ [91]. (For O_4 the α_s^2 correction is known [90]: $(59/6)(\alpha_s/\pi)^2$.) The α_s corrections to O_8^{V-A} are unknown—they are included in the value of m_0^2 which is not certainly known; α_s corrections to O_{10} are unknown also. (In this section indices $V - A$ will be omitted and O_D will mean condensates with α_s corrections included.)

Our aim is to compare the OPE theoretical predictions with the experimental data on $V - A$ structure functions measured in τ-decay and with the help of such a comparison to determine the magnitude of the most important condensate O_6. The condensate O_4 is small and is known with a good accuracy:

$$O_4 = -0.5 \times 10^{-3}\ \text{GeV}^4. \tag{78}$$

We put $m_0^2 = 0.8\ \text{GeV}^2$ and in the analysis of the data the values of the condensates O_8 and O_{10} are taken to be equal to

$$O_8 = -2.8 \times 10^{-3}\ \text{GeV}^8 \tag{79}$$

$$O_{10} = -2.6 \times 10^{-3}\ \text{GeV}^{10} \tag{80}$$

and their dependence on Q^2, arising from anomalous dimensions, is neglected.

In the calculation of the numerical values of (78) and (79), it was assumed that $a_{\bar{q}q}(1\ \text{GeV}^2) \equiv -(2\pi)^2\langle 0|\bar{q}q|0\rangle_{1\ \text{GeV}} = 0.65\ \text{GeV}^3$, $\langle 0|(\alpha_s/\pi)G^2|0\rangle = 0.005\ \text{GeV}^4$—see below, Eqs. (87) and (117).

As was shown in [51], the dimension $d = 8$ four-quark operators for vector and axial currents are of opposite sign and equal in absolute value up to terms of order $1/N_c^2$: $O_8^V = -O_8^A(1 + O(N_c^{-2}))$. (The exact value of the N_c^{-2} correction is uncertain—it depends on the factorization procedure.) So, for O_8^{V+A} we have from (79) the estimation $|O_8^{V+A}| < 10^{-3}\ \text{GeV}^8$, which was used in calculating $\Pi(s)_{nonpert}$, Eq. (59).

For $\Pi_V^{(1)}(s) - \Pi_A^{(1)}(s)$, the subtractionless dispersion relation is valid:

$$\Pi_V^{(1)}(s) - \Pi_A^{(1)}(s) = \frac{1}{2\pi^2}\int_0^\infty \frac{v_1(t) - a_1(t)}{t - s}\,\mathrm{d}t + \frac{f_\pi^2}{s} \tag{81}$$

(The last term in the right-hand part is the kinematic pole contribution.) The experimental data for $v_1(s) - a_1(s)$ are presented in Fig. 5.

In order to improve the convergence of OPE series as well as to suppress the contribution of the large s domain in the dispersion integral we use the Borel transformation. Put $s = s_0 e^{i\phi}$ ($\phi = 0$ on the upper edge of the cut) and make the Borel transformation in s_0. As a result, we get the following sum rules for the real and imaginary parts of (81):

$$\int_0^\infty \exp\Big(\frac{s}{M^2}\cos\phi\Big)\cos\Big(\frac{s}{M^2}\sin\phi\Big)\,(v_1 - a_1)(s)\,\frac{\mathrm{d}s}{2\pi^2}$$
$$= f_\pi^2 + \sum_{k=1}^\infty (-1)^k \frac{\cos(k\phi)\,O_{2k+2}}{k!\,M^{2k}} \tag{82}$$

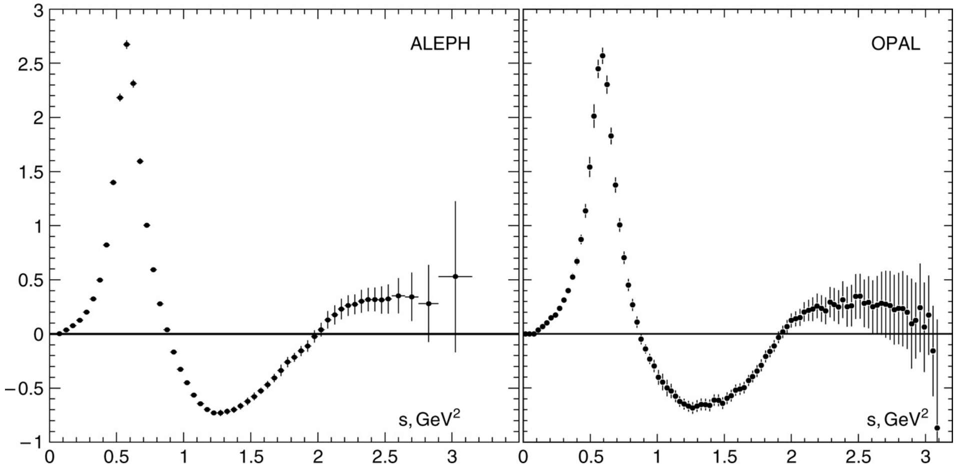

Fig. 5. The measured difference $v_1(s) - a_1(s)$. This figure is from [48] and [49], reproduced in [51].

$$\int_0^\infty \exp\left(\frac{s}{M^2}\cos\phi\right)\,\sin\left(\frac{s}{M^2}\sin\phi\right)(v_1 - a_1)(s)\,\frac{\mathrm{d}s}{2\pi^2 M^2}$$
$$= \sum_{k=1}^{\infty}(-1)^k \frac{\sin(k\phi)\,O_{2k+2}}{k!\,M^{2k+2}}. \tag{83}$$

The use of the Borel transformation along the rays in the complex plane has a number of advantages. The exponent index is negative at $\pi/2 < \phi < 3\pi/2$. Choose ϕ in the region $\pi/2 < \phi < \pi$. In this region, on one hand, the shadowed area in Fig. 1 in the integrals (82) and (83) is touched to a lesser degree and, on the other hand, the contribution of large s, particularly $s > m_\tau^2$, where experimental data are absent, is exponentially suppressed. At definite values of ϕ the contribution of some condensates vanishes, which may also be used. In particular, the condensate O_8 does not contribute to (82) at $\phi = 5\pi/6$ or to (83) at $\phi = 2\pi/3$, while the contribution of O_6 to (82) vanishes at $\phi = 3\pi/4$. Finally, a well known advantage of the Borel sum rules is factorial suppression of higher dimension terms of the OPE. Figs. 6 and 7 present the results of the calculations of the left-hand parts of Eqs. (82) and (83) on the basis of the ALEPH [48] experimental data compared with OPE predictions—the right-hand part of these equations.

When comparing the theoretical curves with experimental data it must be kept in mind that the value of f_π, which in the figures was taken to be equal to the experimental one $f_\pi = 130.7$ MeV, in fact has a theoretical uncertainty of the order of $(\Delta f_\pi^2/f_\pi^2)_{theor} \sim m_\pi^2/m_\rho^2$, where m_ρ is the characteristic hadronic scale (say, the ρ-meson mass). This uncertainty is caused by chiral symmetry violation in QCD. In particular, taking account of this uncertainty may lead to a better agreement of the theoretical curve with the data in Fig. 6(b). The calculation of instanton contributions (Eq. (65)) reveals that in all cases considered above, they are less than 0.5×10^{-3} at $M^2 > 0.8\ \mathrm{GeV}^2$, i.e. are well below the errors. (In some cases they improve the agreement with the data.) The best fit of the data

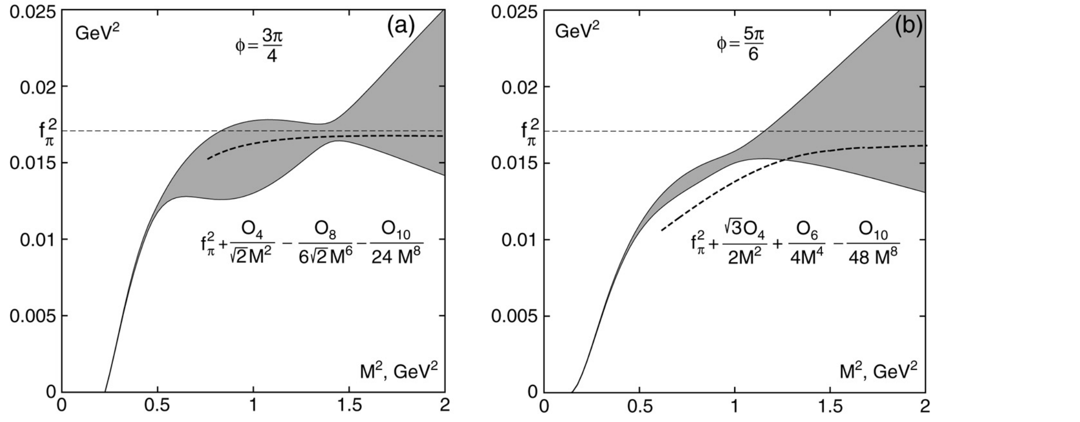

Fig. 6. Eq. (82): the left-hand part is obtained on the basis of the experimental data; the shaded region corresponds to experimental errors; the right-hand part—the theoretical one—is represented by the dotted curve; numerical values of the condensates O_4, O_8, O_{10}, O_6 are taken according to (78)–(80) and (84); (a) $\phi = 3\pi/4$, (b) $\phi = 5\pi/6$.

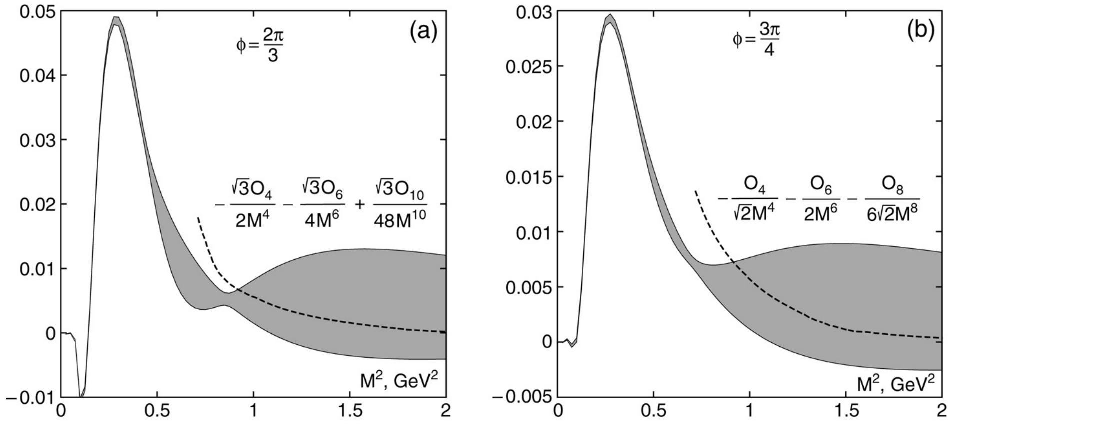

Fig. 7. The same as Fig. 6, but for Eq. (83); (a) $\phi = 2\pi/3$, (b) $\phi = 3\pi/4$.

(the dashed curves at Figs. 6 and 7) was achieved at the value

$$O_6 = -4.4 \times 10^{-3}\ \mathrm{GeV}^6. \tag{84}$$

It follows from (84), after separating out the α_s correction $[1 + (89/48)\alpha_s/\pi) = 1.33]$, that

$$\alpha_s \langle 0|\bar{q}q|0\rangle^2 = 1.5 \times 10^{-4}\ \mathrm{GeV}^6. \tag{85}$$

The error may be estimated as 30%. The value of (85) in the limit of errors agrees with the previous estimation [51]. The contribution of dimension 10 is negligible in all cases at $M^2 \geq 1\ \mathrm{GeV}^2$. It is worth mentioning that the theory, i.e. the OPE, agrees with the data at $M^2 > 0.8\ \text{GeV}^2$. The good agreement of the theoretical curves with the data confirms the chosen value of O_8 (78) and, therefore, the use of the factorization hypothesis. From (84), with the use of $\alpha_s(1\ \text{GeV}^2) = 0.55$ (see Fig. 3), the value of the quark condensate at 1 GeV can be found:

$$\langle 0|qq|0\rangle_{1\ \text{GeV}} = -1.65 \times 10^{-2}\ \text{GeV}^3 = -(254\ \text{MeV})^3 \tag{86}$$

and a convenient parameter is

$$a_{\bar{q}q}(1\ \text{GeV}^2) \equiv -(2\pi)^2\langle 0|\bar{q}q|0\rangle_{1\ \text{GeV}} = 0.65\ \text{GeV}^3. \tag{87}$$

The magnitude of the quark condensate (86) is close to that which follows from the Gell-Mann–Oakes–Renner relation (Eq. (22)).

In the last few years there have been many attempts [53,92–98] to determine quark condensates using $V - A$ spectral functions measured in τ-decay. Unlike the approach presented above, where the polarization operator analytical properties were exploited in the whole complex q^2-plane, which allowed one to separate out the contribution of operators of different dimensions, the authors of [53,92–98] considered the finite energy sum rules—FESR (or integrals over contours)—with chosen weight functions. In [92,94] the $N_c \to \infty$ limit was used. In [93–95,98] an attempt was made to find higher dimension condensates (up to 18 in [87], up to 16 in [93,94] and up to 12 in [95]). Determination of higher dimension condensates requires fine-tuning of the upper limit of integration in FESR. If the upper limit of integration s_0 in FESR is below 2 GeV^2 (e.g., such an upper limit, $s_0 = 1.47\ \text{GeV}^2$, was chosen in [98]), then instanton-like corrections, not given by the OPE, are of importance. (See Section 4.2.) The same remark applies to the case of weight factors singular at $s = 0$, like s^{-l}, $l > 0$ [53], when there is an enhancement of the contribution from low s, where the OPE breaks down. Keeping in mind these remarks, we have a satisfactory agreement of the values of the condensate (84) presented above with those found in [53,94,98].

5.2. *Determination of condensates from $V + A$ and V structure functions of the τ-decay*

Let us turn now to studying the $V + A$ correlator in the domain of low Q^2, where the OPE terms play a much more essential role than in the determination of R_τ. A general remark is in order here. As was discussed in [28] and stressed recently by Shifman [29], the condensates cannot be defined in a rigorous way, because there is some arbitrariness in the separation of their contributions from the perturbative part. Usually [28,29], they are defined by the introduction of some normalization point μ^2 with a magnitude of a few Λ^2_{QCD}. The integration over momenta in the domain below μ^2 is related to condensates, above μ^2, to perturbation theory. In such a formulation the condensates are μ dependent, $\langle O_D\rangle = \langle O_D\rangle_\mu$, and, strictly speaking, they also depend on the way in which the infrared cut-off μ^2 is introduced. The problem becomes more severe when the perturbative expansion is performed up to higher order terms and the calculation pretends to high precision. We mention that this remark does not refer to chirality violating condensates, because perturbative terms do not contribute to chirality violating structures in the limit of massless quarks. For this reason, in principle, chirality violating condensates, e.g. $\langle 0|\bar{q}q|0\rangle$, can be determined with higher precision than chirality conserving ones. Here I use the definition of condensates which can be called n-loop condensates. As was formulated in Section 4, we treat the renormalization group equation, (54), and the equation for the polarization operator, (55), in the n-loop approximation as exact ones; the expansion in inverse logarithms is not performed. Specific values of condensates relate to such a procedure. Of course, their numerical values depend on the number of loops accounted for; that is why the condensates defined in this way are called n-loop condensates.

Consider the polarization operator $\Pi = \Pi^{(1)}_{V+A} + \Pi^{(0)}_A$ defined in (50) and its imaginary part:

$$\omega(s) = v_1(s) + a_1(s) + a_0(s) = 2\pi \operatorname{Im} \Pi(s + i0). \tag{88}$$

In the parton model, $\omega(s) \to 1$ at $s \to \infty$. Any sum rule can be written in the following form:

$$\int_0^{s_0} f(s)\, \omega_{\text{exp}}(s)\, \mathrm{d}s = i\pi \oint f(s)\, \Pi_{\text{theor}}(s)\, \mathrm{d}s \tag{89}$$

where $f(s)$ is some function analytical in the integration region. In what follows we use $\omega_{\text{exp}}(s)$, obtained from τ-decay invariant mass spectra published in Ref. [48] for $0 < s < m_\tau^2$. The experimental error of the integral (89) is computed as a double integral with the covariance matrix $\overline{\omega(s)\omega(s')} - \overline{\omega}(s)\overline{\omega}(s')$, which can also be obtained from the data available in [48]. In the theoretical integral in (89) the contour goes from $s_0 + i0$ to $s_0 - i0$ anticlockwise around all the poles and cuts of the theoretical correlator $\Pi(s)$. Because of the Cauchy theorem the unphysical cut must be inside the integration contour.

The choice of the function $f(s)$ in Eq. (89) is actually a matter of taste. First let us consider the usual Borel transformation:

$$B_{\text{exp}}(M^2) = \int_0^{m_\tau^2} \mathrm{e}^{-s/M^2} \omega_{\text{exp}}(s)\, \frac{\mathrm{d}s}{M^2} = B_{\text{pt}}(M^2) + 2\pi^2 \sum_n \frac{\langle O_{2n}\rangle}{(n-1)!\, M^{2n}}. \tag{90}$$

We separated out the purely perturbative contribution B_{pt}, which is computed numerically according to (89) and Eqs. (52)–(55). Recall that the Borel transformation improves the convergence of the OPE series because of the factors $1/(n-1)!$ in front of the operators and suppresses the contribution of the high energy tail, where the experimental error is large. But it does not suppress the unphysical perturbative cut, the main source of error in this approach; it may even increase it, since $\mathrm{e}^{-s/M^2} > 1$ for $s < 0$. So the perturbative part $B_{\text{pt}}(M^2)$ can be reliably calculated only for $M^2 \gtrsim 0.8$–$1\ \text{GeV}^2$ and higher; below this value the influence of the unphysical cut is out of control.

Both B_{exp} and B_{pt} in the four-loop approximation for $\alpha_s(m_\tau^2) = 0.355$ and 0.330 are shown in Fig. 8. The shaded areas display the theoretical error. They are taken equal to the contribution of the last term in the perturbative Adler function expansion $K_4 a^4$ (52).

The contribution of the O_8 operator is of order O_8^{V-A}/N_c^2 and negligible [51]. (In fact, it depends on the factorization procedure and is uncertain for this reason.) The contributions of $D = 4$ and $D = 6$ operators are positive [see (59)]. So, the theoretical perturbative curve must go below the experimental points. The result shown in Fig. 8 favours the lower value of the QCD coupling constant $\alpha_s(m_\tau^2) = 0.330$ (or, perhaps, $\alpha_s(m_\tau^2) = 0.340$). As is seen from Fig. 8, the theoretical curve (perturbative at $\alpha_s(m_\tau^2) = 0.330$ plus OPE terms) is in agreement with experiment at $M^2 \geq 0.9\ \text{GeV}^2$.

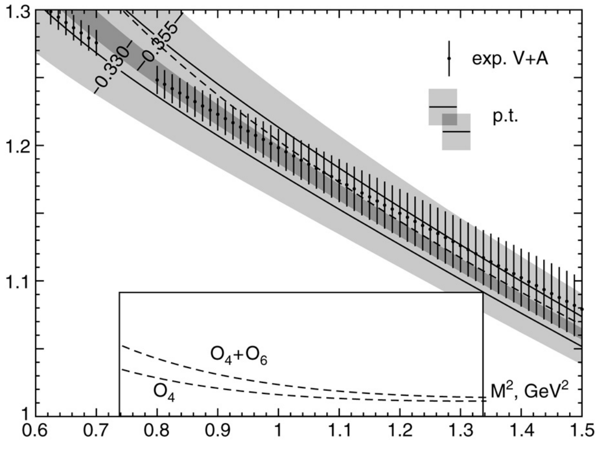

Fig. 8. The results of the Borel transformation of the $V + A$ correlator for two values $\alpha_s(m_\tau^2) = 0.355$ and $\alpha_s(m_\tau^2) = 0.330$. The widths of the bands correspond to PT errors, dots with dashed errors, to experimental data. The dashed curve is the sum of the perturbative contribution at $\alpha_s(m_\tau^2) = 0.330$, and O_4 (Eqs. (59) and (92)) and O_6 (Eqs. (59) and (85)) condensate contributions.

In order to separate the contribution of the gluon condensate let us perform the Borel transformation along the rays in the complex s-plane in the same way as was done in Section 5.1; the real part of the Borel transform at $\phi = 5\pi/6$ does not contain the $d = 6$ operator:

$$\mathrm{Re}\, B_{\exp}(M^2 e^{i5\pi/6}) = \mathrm{Re}\, B_{pt}(M^2 e^{i5\pi/6}) + \pi^2 \frac{\langle O_4 \rangle}{M^4}. \tag{91}$$

The results are shown in Fig. 9. If we accept the lower value of $\alpha_s(m_\tau^2)$, we get the following restriction on the value of the gluon condensate:

$$\left\langle \frac{\alpha_s}{\pi} G^a_{\mu\nu} G^a_{\mu\nu} \right\rangle = 0.006 \pm 0.012\,\mathrm{GeV}^4, \qquad \alpha_s(m_\tau^2) = 0.330 \quad \text{and} \quad M^2 > 0.8\,\mathrm{GeV}^2. \tag{92}$$

The theoretical and experimental errors are added in quadrature in Eq. (92).

Turn now to analysis of the vector correlator (the vector spectral function was published by ALEPH in [99]). In principle this cannot give any new information in comparison with $V - A$ and $V + A$ cases. However the analysis of the vector current correlator is important since it can also be performed using the experimental data on e^+e^- annihilation. The imaginary part of the electromagnetic current correlator, measured there, is related to the charged current correlator (45) by the isotopic symmetry. The statistical error in e^+e^- experiments is less than in τ-decays because of the significantly larger number of events. So it would be interesting to perform a similar analysis with e^+e^- data, which is a matter for separate research.

First we consider usual the Borel transformation for the vector current correlator, since it was originally applied in [100] for sum rule analysis. It is defined as (89) with the experimental spectral function $\omega_{\exp} = 2v_1$ instead of $v_1 + a_1 + a_0$ (the normalization

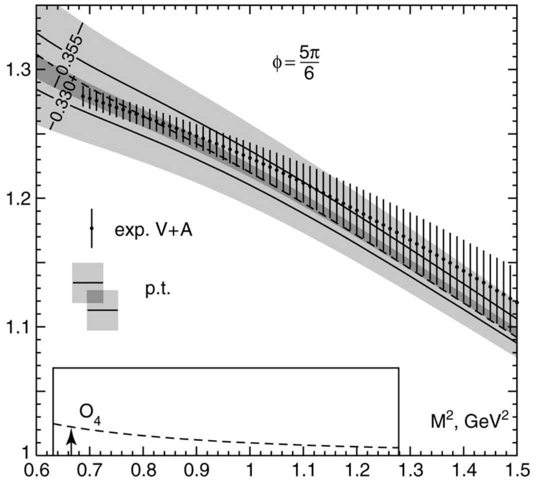

Fig. 9. Real part of the Borel transform (91) along the ray at the angle $\phi = 5\pi/6$ to the real axes. The dashed line corresponds to the gluonic condensate given by the central value of (92).

is $v_1(s) \to 1/2$ at $s \to \infty$ in the parton model). Correspondingly, in the r.h.s. one should take the vector operators $2O^V = O^{V+A} + O^{V-A}$. The numerical results are shown in Fig. 10. The perturbative theoretical curves are the same as in Fig. 8 with the $V + A$ correlator. The dashed lines display the contributions of the gluonic condensate given by Eq. (92), $2O_6^V = -3.5 \times 10^{-3}$ GeV^6 and $2O_8^V = O_8^{V-A} = -2.8 \times 10^{-3}$ GeV^8, added to the $\alpha_s(m_\tau^2) = 0.330$ perturbative curve. The contribution of each condensate is shown in the box below. Notice that for such condensate values the total OPE contribution is small, since positive O_4 compensates negative O_6 and O_8. The agreement is observed for $M^2 > 0.8$ GeV^2.

The Borel transformations along the rays in the complex plane lead to the same conclusion; at $M^2 > 0.8$–0.9 GeV^2 agreement with experiment at the 2% level is achieved at $\alpha_s(m_\tau^2) = 0.33$–$0.34$ and at the values of the quark and gluon condensates given by (84) and (92). There is some discrepancy in the vector spectral functions obtained in τ-decay and in e^+e^- annihilation (see [54], [101] and references herein): the e^+e^- data are below the τ-decay ones by 5–10% in the interval $s = 0.6$–0.8 GeV^2. The substitution of e^+e^- data instead of τ-decay data in the sum rule presented in Fig. 10 does not spoil the agreement of the theory with experiment in the limit of errors.

A few words about instanton contributions are in order. They can be calculated in the same way as in the case of $V - A$ correlators. At the chosen values of the instanton gas parameters instanton contributions are small, less than 0.5×10^{-3} at $M^2 > 0.8$ GeV^2, and do not spoil the agreement of the theory with experiment.

6. Determination of the quark condensate from QCD sum rules for baryon masses

Since in QCD with massless quarks the baryon masses arise due to spontaneous violation of the chiral symmetry and, in a good approximation, the proton mass (as well as

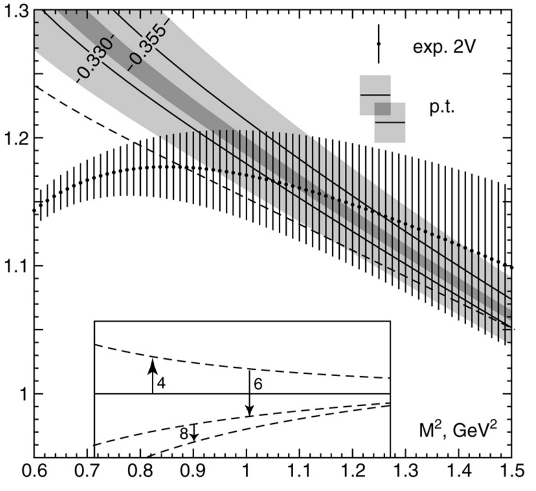

Fig. 10. Borel transformation for vector currents.

the Δ-isobar) can be expressed through the quark condensate [32], the QCD sum rules for baryon masses are a suitable tool for determination of the quark condensate assuming that baryon masses are known. The sum rules can be derived by considering the polarization operator

$$\Pi(p) = i \int d^4x e^{ipx} \langle 0|T\{\eta(x), \bar{\eta}(0)\}|0\rangle \tag{93}$$

where $\eta(x)$ is the quark current with baryon quantum numbers. In the case of the proton the most suitable current is [32,102]

$$\eta(x) = \varepsilon^{abc}(u^a C\gamma_\mu u^b)\gamma_5\gamma_\mu d^c \tag{94}$$

where u^a, d^c are u and d quark fields, a, b, c are colour indices, C is the charge conjugation matrix. After Borel transformation the sum rules for the proton mass have the form [32,37,44]

$$\begin{aligned} &M^6 E_2(s_0/M^2)L^{-4/9}\left[1+\left(\frac{53}{12}+\gamma_E\right)\frac{\alpha_s(M^2)}{\pi}\right]+\frac{1}{4}M^2 b E_0(s_0/M^2)L^{-4/9} \\ &\quad +\frac{4}{3}a_{\bar{q}q}^2\left[1+f(M^2)\frac{\alpha_s(M^2)}{\pi}\right]-\frac{1}{3}a_{\bar{q}q}^2\frac{m_0^2}{M^2}=\bar{\lambda}_p^2 e^{-m^2/M^2} \end{aligned} \tag{95}$$

$$\begin{aligned} &2a_{\bar{q}q}M^4 E_1(s_0/M^2)\left[1+\frac{3}{2}\,\frac{\alpha_s(M^2)}{\pi}\right]+\frac{272}{81}\,\frac{\alpha_s(M^2)}{\pi}\,\frac{a_{\bar{q}q}^3}{M^2}-\frac{1}{12}a_{\bar{q}q}b \\ &\quad = m\bar{\lambda}_p^2 e^{-m^2/M^2}. \end{aligned} \tag{96}$$

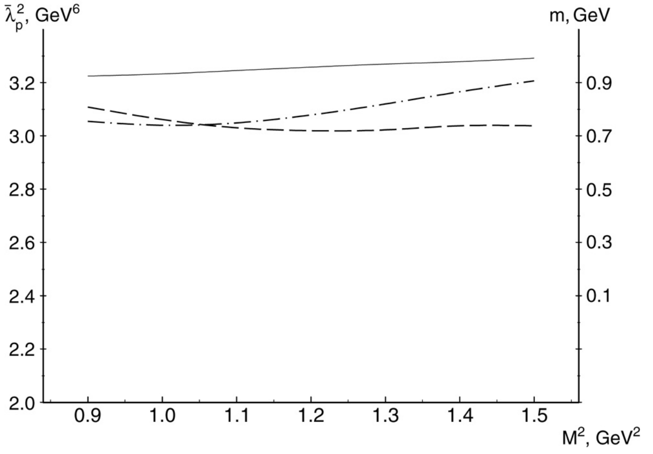

Fig. 11. The sum rules for proton mass Eqs. (95) and (96). The dashed and dash–dotted curves give $\bar{\lambda}_p^2$, determined correspondingly from (95) and (96); the experimental value of m was substituted in (left scale). The solid line gives m as the ratio of (96) to (95).

Here M is the Borel parameter, m is the nucleon mass, $a_{\bar{q}q}$ is given by (87), $\gamma_E = 0.577$,

$$b = (2\pi)^2 \langle 0|\frac{\alpha_s}{\pi} G^2|0\rangle \tag{97}$$

$$E_n(x) = \frac{1}{n!}\int_0^x \mathrm{d}z z^n \mathrm{e}^{-z} \tag{98}$$

$$L = \frac{\alpha_s(\mu^2)}{\alpha_s(M^2)}, \tag{99}$$

L corresponds to anomalous dimensions, s_0 is the continuum threshold and μ^2 is the normalization point, chosen as $\mu^2 = 1$ GeV^2. The constant $\bar{\lambda}_p$ is defined as $\bar{\lambda}_p^2 = 2(2\pi)^4\lambda_p^2$:

$$\langle 0|\eta|p\rangle = \lambda_p v_p, \tag{100}$$

where v_p is the proton spinor. The α_s corrections to the proton sum rules were found in [103]. The function $f(s)$ is small, $|f| < 0.2$ at $0.9 < M^2 < 1.5$ GeV^2, and the α_s correction to the term proportional to $a_{\bar{q}q}^2$ can be neglected. The sum rules (95) and (96) were calculated at the following values of the parameters: $\langle 0|(\alpha_s/\pi)G^2|0\rangle = 0.005$ GeV^4 ($b = 0.20$ GeV^4), $s_0 = 2.5$ GeV^2. The numerical value of the quark condensate was not fixed by the value given in (87), but considered as a free parameter. For the best fit of the sum rules it was chosen to be $a_{\bar{q}q} = 0.60$ (cf. (87)). First, the value of λ_p^2 was found from (95) and (96), where the experimental value of the proton mass was substituted; Fig. 11, left scale. Then Eq. (96) was divided by (95) and the theoretical value of the proton mass was found; Fig. 11, right scale.

As is seen from Fig. 11, the $\bar{\lambda}_p^2$ determined from (95) and (96) are almost independent of M^2 and coincide with one another, as they should. The proton mass value coincides with the experimental one with a precision better than 3%. The conclusion is that the value

$$a_{\bar{q}q} = 0.60 \text{ GeV}^3 \tag{101}$$

describes the proton mass sum rule well. The main source of the error is the large α_s correction (about 0.8) to the first term in (95). If we suppose that its uncertainty is 20%, then the corresponding error in $a_{\bar{q}q}$ is ± 0.1 GeV^3. Therefore, we get from proton mass sum rules

$$a_{\bar{q}q} = (0.60 \pm 0.10) \text{ GeV}^3. \tag{102}$$

A remark about a possible role of instantons in the sum rules for proton mass is in order. As was found in [104,105], if the quark current with proton quantum numbers is given by (95), then instantons do not change the sum rule (95). Their contribution to (96) is moderate in the instanton gas model if the model parameters are chosen as in (71) [104, 105] and may shift the value of the quark condensate (102) by 10–20%, i.e. in the limit of the quoted error.

7. The gluon condensate and determination of the charmed quark mass from the charmonium spectrum

7.1. The method of moments; The results

The existence of a gluon condensate was first demonstrated by Shifman, Vainshtein and Zakharov [1]. They considered the polarization operator $\Pi_c(q^2)$ of the vector charmed current

$$\Pi_c(q^2)(q_\mu q_\nu - \delta_{\mu\nu} q^2) = i \int \mathrm{d}^4x \mathrm{e}^{iqx} \langle 0|T J_\mu(x), J_\nu(0)|0\rangle \tag{103}$$

$$J_\mu(x) = \bar{c}\gamma_\mu c \tag{104}$$

and calculated the moments of $\Pi_c(q^2)$:

$$M_n(Q^2) = \frac{4\pi^2}{n!} \left(-\frac{\mathrm{d}}{\mathrm{d}Q^2}\right)^n \Pi_c(Q^2), \tag{105}$$

with ($Q^2 = -q^2$) at $Q^2 = 0$. The OPE for $\Pi(Q^2)$ was used and only one term in the OPE series was accounted for—the gluonic condensate. In the perturbative part of $\Pi(Q^2)$ only the first-order term in α_s was accounted for and a small value of α_s was chosen, $\alpha_s(m_c) \approx 0.2$. The moments were saturated by the contribution of charmonium states and in this way the value of the gluon condensate (27) was found. The SVZ approach [1] was criticized in [106], where it was shown that the higher order terms of the OPE, namely, the contributions of G^3 and G^4 operators, are of importance at $Q^2 = 0$. Reinders, Rubinstein and Yazaki [107] demonstrated, however, that SVZ results may be restored if one considers nonsmall values $Q^2 > 0$ instead of $Q^2 = 0$. Later there were many attempts to determine the gluon condensate by considering various processes within various approaches. In some of them the value of (27) (or one higher by a factor of 1.5) was confirmed [100,108,109]; in others it was claimed that the actual value of the gluon condensate is higher than that of (27) by a factor 2–5 [110].

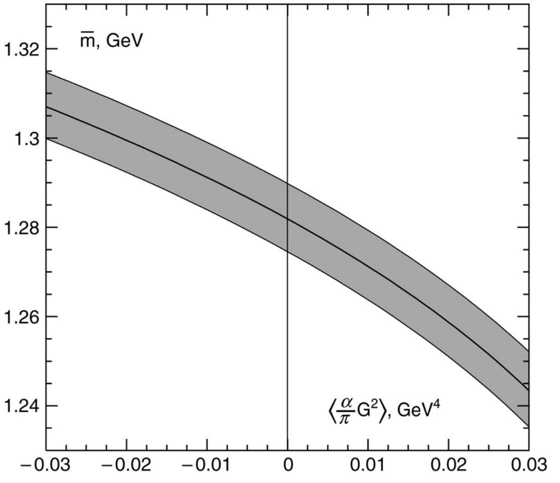

Fig. 12. The dependence of $\overline{m}(\overline{m})$ on $\langle 0|\alpha_s/\pi)G^2|0\rangle$ obtained at $n = 10$, $Q^2 = 0.98 \cdot 4m^2$ and $\alpha_s(Q^2 + \overline{m}^2)$.

From today's point of view, the calculations performed in [1] have a serious drawback. Only the first-order (NLO) perturbative correction was accounted for in [1] and it was taken at a rather low value of α_s, later not confirmed by the experimental data. The contribution of the next, dimension 6, operator G^3 was neglected, so the convergence of the operator product expansion was not tested.

There are recent publications [111] where the charmonium as well as bottomonium sum rules are analysed at $Q^2 = 0$ with account taken of α_s^2 perturbative corrections in order to extract the charm and bottom quark masses in various schemes. The condensate is usually taken to be 0 or some another fixed value. However, the charm mass and the condensate values are entangled in the sum rules. This can be easily understood for large Q^2, where the mass and condensate corrections to the polarization operator behave as some series in negative powers of Q^2, and one may eliminate the condensate contribution to a great extent by slightly changing the quark mass. Vice versa, different condensate values may vary the charm quark mass within a few per cent (see Fig. 12).

Therefore, in order to perform reliable calculations of the gluon condensate by studying the moments of the charmed current polarization operator it is necessary to account for α_s^2 perturbative corrections to the moments, α_s corrections to the gluon condensate contribution, the $\langle G^3\rangle$ term in the OPE and to find the region in (n, Q^2) space where all these corrections are small. This programme was realized in [112]. The basic points of this consideration are presented below.

The dispersion representation for $\Pi(q^2)$ has the form

$$R(s) = 4\pi \,\mathrm{Im}\, \Pi_c(s + i0), \qquad \Pi_c(q^2) = \frac{q^2}{4\pi^2} \int_{4m_c^2}^{\infty} \frac{R(s)\,\mathrm{d}s}{s(s - q^2)}, \tag{106}$$

where $R(\infty) = 1$ in the partonic model. In the approximation of infinitely narrow widths of resonances, $R(s)$ can be written as a sum of contributions from resonances and the continuum:

$$R(s) = \frac{3\pi}{Q_c^2\, \alpha_{\rm em}^2(s)} \sum_{\psi} m_\psi \Gamma_{\psi\to ee}\, \delta(s - m_\psi^2) + \theta(s - s_0) \tag{107}$$

where $Q_c = 2/3$ is the charge of the charmed quarks, s_0 is the continuum threshold (in what follows, $\sqrt{s_0} = 4.6$ GeV), $\alpha(s)$ is the running electromagnetic constant, $\alpha(m_{J/\psi}^2) = 1/133.6$. The polarization operator moments are expressed through R as

$$M_n(Q^2) = \int_{4m_c^2}^{\infty} \frac{R(s)\, {\rm d}s}{(s + Q^2)^{n+1}}. \tag{108}$$

According to (108), the experimental values of the moments are determined by the equality

$$M_n(Q^2) = \frac{27\pi}{4\alpha_{\rm em}^2} \sum_{\psi=1}^{6} \frac{m_\psi \Gamma_{\psi\to ee}}{(m_\psi^2 + Q^2)^{n+1}} + \frac{1}{n(s_0 + Q^2)^n}. \tag{109}$$

In the sum in (109) the following resonances were accounted for: $J/\psi(1S)$, $\psi(2S)$, $\psi(3770)$, $\psi(4040)$, $\psi(4160)$, $\psi(4415)$; their $\Gamma_{\psi\to ee}$ widths were taken from PDG data [13]. It is reasonable to consider the ratios of moments $M_{n1}(Q^2)/M_{n2}(Q^2)$ from which the uncertainty due to error in $\Gamma_{J/\psi\to ee}$ markedly falls out. The theoretical value for $\Pi(q^2)$ is represented as a sum of perturbative and nonperturbative contributions. It is convenient to express the perturbative contribution through $R(s)$, making use of (106) and (108):

$$R(s) = \sum_{n\ge 0} R^{(n)}(s, \mu^2)\, a^n(\mu^2) \tag{110}$$

where $a(\mu^2) = \alpha_s(\mu^2)/\pi$. Nowadays, three terms of expansion in (110) are known: $R^{(0)}$ [113] $R^{(1)}$ [114], $R^{(2)}$ [115]. They are represented as functions of the quark velocity $v = \sqrt{1 - 4m_c^2/s}$, where m_c is the pole mass of quark. Since they are cumbersome, I will not present them here (see [112] for details).

Nonperturbative contributions to the polarization operator have the form (restricted by $D = 6$ operators)

$$\Pi_{nonpert}(Q^2) = \frac{1}{(4m_c^2)^2} \left\langle 0 \left| \frac{\alpha_s}{\pi} G^2 \right| 0 \right\rangle [f^{(0)}(z) + a f^{(1)}(z)]$$
$$+ \frac{1}{(4m_c^2)^3} g^3 f^{abc} \langle 0| G^a_{\mu\nu} G^b_{\nu\lambda} G^c_{\lambda\mu} |0\rangle F(z), \qquad z = -\frac{Q^2}{4m_c^2}. \tag{111}$$

Functions $f^{(0)}(z)$, $f^{(1)}(z)$ and $F(z)$ were calculated in [1], [116] and [117], respectively. The use of the quark pole mass is, however, unacceptable. The problem is that in this case the PT corrections to moments are very large in the region of interest and the perturbative series seems to diverge.

So, it is reasonable to use the $\overline{MS}$ mass $\overline{m}(\mu^2)$, taken at the point $\mu^2 = \overline{m}^2$. The calculations performed in Ref. [100] show that in the region near the diagonal in the (Q^2, n)-plane, $Q^2/4m^2 = n/5 - 1$, all above-mentioned corrections are small. For example,

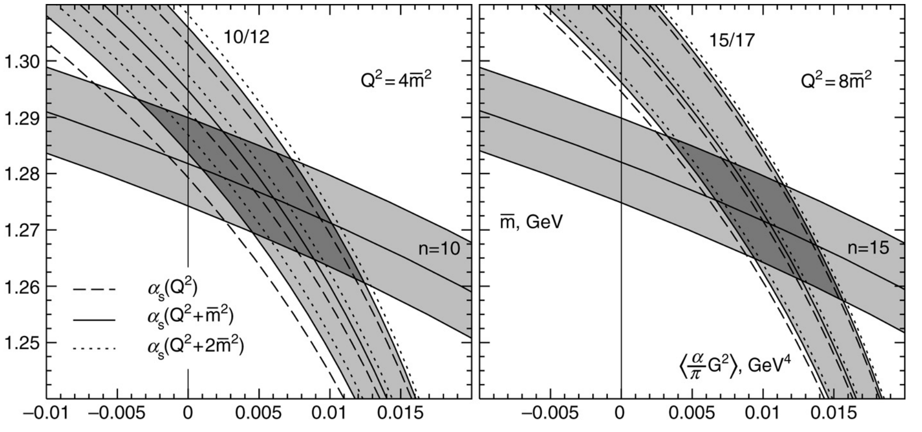

Fig. 13. The dependence of $\overline{m}(\overline{m})$ on $\langle 0|(\alpha_s/\pi)G^2|0\rangle$ obtained from the moments (horizontal bands) and their ratios (vertical bands) at different α_s. Left-hand figure: $Q^2 = 4\overline{m}^2$, $n = 10$, M_{10}/M_{12}; right-hand figure: $Q^2 = 8\overline{m}^2$, $n = 15$, M_{15}/M_{17}.

$$n = 10,\, Q^2 = 4\bar{m}_c^2: \qquad \frac{\bar{M}^{(1)}}{\bar{M}^{(0)}} = 0.045, \qquad \frac{\bar{M}^{(2)}}{\bar{M}^{(0)}} = 1.136, \qquad \frac{\bar{M}^{(G,1)}}{\bar{M}^{(G,0)}} = -1.673 \tag{112}$$

(here $M^{(k)}$ stands for the coefficients of the contributions of the terms $\sim a^k$ to the moments; $M^{(G,k)}$ are similar coefficients for the gluonic condensate contribution).

At $a \sim 0.1$ and at the ratios of moments given by (112), there is a good reason to believe that the PT series converges well. Such a good convergence holds (at $n > 5$) only in the case of large enough Q^2; at $Q^2 = 0$ one does not succeed in finding such n that perturbative corrections to the moments, α_s corrections to the gluonic condensates and the term $\sim\langle G^3\rangle$ contribution would be simultaneously small.

It is also necessary to choose the scale normalization point μ^2 where $\alpha_s(\mu^2)$ is taken. In (110) $R(s)$ is a physical value and cannot depend on μ^2. Since, however, we take into account in (110) only three terms, at unsuitable choices of μ^2 such μ^2 dependence may arise due to neglected terms. At large Q^2 the natural choice is $\mu^2 = Q^2$. It can be thought that at $Q^2 = 0$ a reasonable scale is $\mu^2 = \overline{m}^2$, though a numerical factor is not excluded in this equality. That is why it is reasonable to take interpolation form

$$\mu^2 = Q^2 + \overline{m}^2, \tag{113}$$

but to check the dependence of the final results on a possible factor at $\overline{m}^2$. Setting the theoretical value of some moment at fixed Q^2 (in the region where $M_n^{(1)}$ and $M_n^{(2)}$ are small) equal to its experimental value, one can find the dependence of $\overline{m}$ on $\langle(\alpha_s/\pi)G^2\rangle$ (neglecting the terms $\sim\langle G^3\rangle$). Such a dependence for $n = 10$ and $Q^2/4m^2 = 0.98$ is presented in Fig. 12.

To fix both $\overline{m}$ and $\langle(\alpha_s/\pi)G^2\rangle$ one should, except for moments, take their ratios. Fig. 13 shows the value of $\overline{m}$ obtained from the moment M_{10} and the ratio M_{10}/M_{12} at $Q^2 = 4m^2$

and from the moment M_{15} and the ratio M_{15}/M_{17} at $Q^2 = 8m^2$. The best values of the masses of the charmed quark and gluonic condensate are obtained from Fig. 13:

$$\bar{m}(\bar{m}^2) = 1.275 \pm 0.015\,\mathrm{GeV}, \qquad \left\langle \frac{\alpha_s}{\pi} G^2 \right\rangle = 0.009 \pm 0.007\,\mathrm{GeV}^4. \tag{114}$$

The calculation shows that the influence of the continuum—the last term in Eq. (107)—is completely negligible. Up to now the corrections $\sim\langle G^3\rangle$ were not taken into account. It appears that in the region of n and Q^2 used to find $\overline{m}$ and the gluonic condensate they are comparatively small and, almost, do not change $\overline{m}$; however, there is an increase in $\langle(\alpha_s/\pi)G^2\rangle$ by 10–20% if the term $\sim\langle G^3\rangle$ is estimated according to (29) at $\rho_c = 0.5$ fm.

It should be noted that improvement of the accuracy of $\Gamma_{J/\psi\to ee}$ would make it possible to obtain the value of gluonic condensate precisely: the widths of the horizontal bands in Fig. 13 are determined mainly just by this error. In particular, this would perhaps allow one to exclude the zero value of the gluonic condensate, which would be extremely important. Unfortunately, Eq. (114) does not allow one to do this for certain. The diminution of the theoretical errors which determine the width of the vertical bands seems to be less substantial.

In order to check the results (114) for the gluon condensate the pseudoscalar and axial vector channels in charmonia were considered. The same method of moments was used and the regions in the space (n, Q^2) where higher order perturbative and OPE terms are small were found. In the pseudoscalar case it was obtained [118] that, if for $\overline{m}$ the value of (114) is accepted and the contribution of the $\langle 0|G^4|0\rangle$ condensate may be neglected, then there follows the upper limit for the gluon condensate

$$\left\langle 0 \left| \frac{\alpha_s}{\pi}\, G^2 \right| 0 \right\rangle < 0.008\ \mathrm{GeV}^4. \tag{115}$$

The contribution of the $D = 6$ condensate $\langle 0|G^3|0\rangle$ is shown to be small. If the $\langle G^4\rangle$ condensate is accounted for and its value is estimated from the factorization hypothesis, then the upper limit for the gluon condensate increases to

$$\left\langle 0 \left| \frac{\alpha_s}{\pi}\, G^2 \right| 0 \right\rangle < 0.015\ \mathrm{GeV}^4. \tag{116}$$

In [119] the case of the axial vector channel in charmonia was investigated and very strong limitations on the gluon condensate were found:

$$\left\langle 0 \left| \frac{\alpha_s}{\pi}\, G^2 \right| 0 \right\rangle = 0.005^{+0.001}_{-0.004}\ \mathrm{GeV}^4. \tag{117}$$

Unfortunately, (117) does not allow one to exclude the zero value of the gluon condensate. It should be mentioned that the allowed region in (n, Q^2) space, where all corrections are small, is very narrow in this case, which prevents us, in [119], from checking the result of (117) by studying some other regions in (n, Q^2), as was done in the two previous cases—vector and pseudoscalar.

Let us now turn the problem around and try to predict the width $\Gamma_{J/\psi\to ee}$ theoretically. In order to avoid incorrect circle argumentation we do not use the condensate value just obtained, but take the limit $\langle\frac{\alpha_s}{\pi}G^2\rangle = 0.006 \pm 0.012\,\mathrm{GeV}^4$ found from τ-decay data. Then, the mass limits $\bar{m} = 1.28$–1.33 GeV can be found from the moment ratios exhibited above, which do not depend on $\Gamma_{J/\psi\to ee}$ if the contributions of higher resonances are approximated by a continuum (the accuracy of such an approximation is about 3%). The substitution of these values of $\bar{m}$ into the moments gives

$$\Gamma^{\text{theor}}_{J/\psi\to ee} = 4.9 \pm 0.8\,\text{keV} \tag{118}$$

in comparison with the experimental value $\Gamma_{J/\psi\to ee} = 5.26 \pm 0.37\,\text{keV}$. Such a good coincidence of the theoretical prediction with experimental data is a very impressive demonstration of the QCD sum rule effectiveness. It must be stressed that while obtaining (118) no additional inputs were used besides the condensate restriction taken from Eq. (92) and the value of $\alpha_s(m_\tau^2)$.

7.2. The attempts to sum the Coulomb-like corrections; Recent publications

Sometimes when considering the heavy quarkonia sum rules the Coulomb-like corrections are summed [120–124]. The basic argumentation for such summation is that at $Q^2 = 0$ and high n only small quark velocities $v \lesssim 1/\sqrt{n}$ are essential and the problem becomes nonrelativistic. So it is possible to perform the summation with the help of well known formulae of nonrelativistic quantum mechanics for $|\psi(0)|^2$ in the case of Coulomb interaction (see [125]).

This method was not used here for the following reasons:

1. The basic idea of our approach is to calculate the moments of the polarization operator in QCD by applying the perturbation theory and OPE (l.h.s. of the sum rules) and to compare them with the r.h.s. of the sum rules, represented by the contribution of charmonium states (mainly by J/ψ in vector channel). Therefore it is assumed that the theoretical side of the sum rule is dual to the experimental one, i.e. the same domains of coordinate and momentum spaces are of importance on both sides. But the charmonium states (particularly, J/ψ) are by no means Coulomb systems. A particular argument in favour of this statement is provided by the ratio $\Gamma_{J/\psi\to ee}/\Gamma_{\psi'\to ee} = 2.4$. If charmonia were a nonrelativistic Coulomb system, $\Gamma_{\psi\to ee}$ would be proportional to $|\psi(0)|^2 \sim 1/(n_r+1)^3$, and since ψ' is the first radial excitation with $n_r = 1$, this ratio would be equal to 8 (see also [125]).

2. The heavy quark–antiquark Coulomb interaction at large distances $r > r_{\text{conf}} \sim 1\,\text{GeV}^{-1}$ is screened by gluon and light quark–antiquark clouds, resulting in string formation. Therefore the summation of Coulombic series makes sense only when the Coulomb radius r_{Coul} is below r_{conf}. (It must be borne in mind that higher order terms in Coulombic series represent the contributions of large distances, $r \gg r_{\text{Coul}}$.) For charmonia we have

$$r_{\text{Coul}} \approx \frac{2}{m_c C_F \alpha_s} \approx 4\,\text{GeV}^{-1}. \tag{119}$$

It is clear that the necessary condition $R_{\text{Coul}} < R_{\text{conf}}$ is badly violated for charmonia. This means that the summation of the Coulomb series in the case of charmonium would be an erroneous step.

3. The analysis is performed at $Q^2/4\bar{m}^2 \geq 1$. At large Q^2 the Coulomb corrections are suppressed in comparison with the $Q^2 = 0$ case. It is easy to estimate the characteristic values of the quark velocities. At large n they are $v \approx \sqrt{(1+Q^2/4m^2)/n}$. In the region (n, Q^2) the quark velocity $v \sim 1/\sqrt{5} \approx 0.45$ exploited above is not small and not in the nonrelativistic domain, where the Coulomb corrections are large and legitimate.

Nevertheless let us look at the expression for R_c, obtained after summation of the Coulomb corrections in the nonrelativistic theory [126]. It reads (to go from QED to QCD one has to replace $\alpha \to C_F\alpha_s$, $C_F = 4/3$)

$$R_{c,\,\mathrm{Coul}} = \frac{3}{2}\,\frac{\pi C_F \alpha_s}{1-\mathrm{e}^{-x}} = \frac{3}{2}\,v\left(1+\frac{x}{2}+\frac{x^2}{12}-\frac{x^4}{720}+\cdots\right) \tag{120}$$

where $x = \pi C_F\alpha_s/v$. At $v = 0.45$ and $\alpha_s \approx 0.26$ the first three terms in the expansion (120), accounted for in our calculations, reproduce the exact value of $R_{c,\mathrm{Coul}}$ with the accuracy 1.6%. Such a deviation leads to an error of the mass $\bar{m}$ of order $(1\text{–}2)\times 10^{-3}$ GeV, which is completely negligible. In order to avoid misunderstanding, it must be mentioned that the value of $R_{c,\mathrm{Coul}}$, computed by summing the Coulomb corrections in nonrelativistic theory, does not have too much in common with the real physical situation. Numerically, at chosen values of the parameters, $R_{c,\mathrm{Coul}} \approx 1.8$, while the real value (both experimental and in the perturbative QCD) is about 1.1. The goal of the arguments presented above was to demonstrate that even in the case of a Coulombic system our approach would have a good accuracy of calculation.

At $v = 0.45$ the momentum transfer from quark to antiquark is $\Delta p \sim 1$ GeV. (This is a typical domain for QCD sum rule validity.) In coordinate space it corresponds to $\Delta r_{q\bar{q}} \sim 1\ \mathrm{GeV}^{-1}$. Comparison with potential models [126] demonstrates that in this region the effective potential strongly differs from the Coulombic one.

4. Substantial compensation of various terms in the expression for the moments in the $\overline{\mathrm{MS}}$ scheme is not achieved if only the Coulomb terms are taken into account. This means that the terms of non-Coulombic origin are more important here than Coulombic ones.

For all these reasons the summation of the nonrelativistic Coulomb corrections is inadequate for the problem considered: it will not improve the accuracy of calculations, and would be misleading.

In the recent publication [127] it is claimed that the gluon condensate is much larger than the values presented above; $\langle 0|(\alpha_s/\pi)\ G^2|0\rangle = 0.062 \pm 0.019\ \mathrm{GeV}^4$ was found. The author of [127] considered the model where the hadronic spectrum is represented by an infinite number of vector mesons. The polarization operator calculated in this model was set equal to those in QCD given by perturbative and OPE terms. The value of the gluon condensate was found from this equality. The zero-width approximation was used for vector mesons. It is clear, however, that the account taken of nonzero widths results in terms of the same type, proportional to $1/Q^4$, as those from the contribution of the gluon condensate. The sign of these terms is such that they lead to diminishing of the gluon condensate. That is, after accounting for the ρ-meson width, the value of the gluon condensate decreases by a factor of 2. For this reason the results of [127] are not reliable.

8. Valence quark distributions in the nucleon at low Q^2 and the condensates

Quark and gluon distributions in hadrons are not fully understood in QCD. QCD predicts the evolution of these distributions with Q^2 in accord with the Dokshitzer–Gribov–Lipatov–Altarelli–Parisi (DGLAP) [128–130] equations, but not the initial values from which this evolution starts. The standard way of determining quark and gluon distributions in nucleons is the following [131–135] (for a recent review see [136]). At some $Q^2 = Q_0^2$ (usually, at low or intermediate $Q^2 \sim 2$–5 GeV^2) the form of the quark (valence and sea) and gluon distributions is assumed and characterized by the number of free parameters. Then, by using DGLAP equations, quark and gluon distributions are calculated at all Q^2 and x and compared with the whole set of the data on deep inelastic lepton–nucleon scattering (sometimes also with prompt photon production, jets at high $p_\perp$ etc). The best fit for the parameters is found and, therefore, quark and gluon distributions are determined at all Q^2, including their initial values $q(Q_0^2, x)$, $g(Q_0^2, x)$. Evidently, such an approach is not completely satisfactory from the theoretical point of view—it would be desirable to determine the initial distribution directly from QCD. In QCD calculations, valence quark distributions in nucleons essentially depend on the vacuum condensate, particularly on the gluon condensate. Therefore, comparing valence quark distributions calculated in QCD with those found by fitting to the data allows one to check the values of condensates obtained by consideration of quite different physical phenomena. For all these reasons it is desirable to find quark and gluon distributions in nucleons at low $Q^2 \sim 2$–5 GeV^2 directly on the basis of QCD.

The method of calculation of valence quark distributions at low Q^2 ($Q^2 = 2$–5 GeV^2) was suggested in [137] and developed in [138–140]. Recently, the method was improved and valence quark distributions in pions [141] and transversely and longitudinally polarized ρ-mesons [142] were calculated, which was impossible in the initial version of the method. The idea of the approach (in the improved version) is to consider the imaginary part (in the s-channel) of a four-point correlator $\Pi(p_1, p_2, q, q')$ corresponding to the non-forward scattering of two quark currents, of which one has the quantum numbers of the hadron of interest (in our case, of the proton) and the other is electromagnetic (or weak). It is supposed that the virtualities of the photon q^2, q'^2 and hadron currents p_1^2, p_2^2 are large and negative: $|q^2| = |q'^2| \gg |p_1^2|, |p_2^2| \gg R_c^{-2}$, where R_c is the confinement radius. It was shown in [137] that in this case the imaginary part in the s-channel [$s = (p_1 + q)^2$] of $\Pi(p_1, p_2; q, q')$ is dominated by a small distance contribution at intermediate x. (The standard notation is used: x is the Bjorken scaling variable, $x = -q^2/2\nu$, $\nu = p_1 q$). The proof of this statement is given in [138]. So, in the above-mentioned domain of q^2, q'^2, p_1^2, p_2^2 and intermediate x, Im $\Pi(p, p_2; q, q')$ can be calculated using the perturbation theory and the operator product expansion in both sets of variables $q^2 = q'^2$ and p_1^2, p_2^2. Only the lowest twist terms, corresponding to the conditions $|p_1^2/q^2| \ll 1$, $|p_2^2/q^2| \ll 1$, are considered.

The approach is inapplicable at small x and x close to 1. This can be easily understood for physical reasons. In deep inelastic scattering at large $|q^2|$ the main interaction region in space–time is the light cone domain and longitudinal distances along the light cone are proportional to $1/x$ and become large at small x [143,144]. For OPE validity it is necessary for these longitudinal distances along the light cone to also be small, which is not the case at small x. At $1 - x \ll 1$ another condition of applicability of the method is violated. The total energy squared $s = Q^2(1/x - 1) + p_1^2$, $Q^2 = -q^2$ is not large at $1 - x \ll 1$. Numerically, the typical values to be used below are $Q^2 \sim 5$ GeV^2, $p_1^2 \sim -1$ GeV^2. Then, even at $x \approx 0.7$, $s \approx 1$ GeV^2; i.e., at such x we are in the resonance region, but not in the scaling region. So, one may predict beforehand that our method could work only up to $x \approx 0.7$. The inapplicability of the method at small and large x manifests itself in the blow-up of higher order terms of the OPE. More precise limits on the applicability domain in x will be found from the magnitudes of these terms.

A further procedure is common for QCD sum rules. On one hand, the four-point correlator $\Pi(p_1, p_2; q, q')$ is calculated using perturbation theory and the OPE. On the other hand, the double-dispersion representation in p_1^2, p_2^2 in terms of physical state contributions is written for the same correlator and the contribution of the lowest state is extracted using the Borel transformation. By setting these two expressions equal the desired quark distribution is found. Valence quark distributions in protons according to this method were calculated in [145]. The basic results of [145] are presented below.

Consider the four-current correlator which corresponds to the virtual photon scattering on the quark current with the quantum number of the proton:

$$T^{\mu\nu}(p_1, p_2, q, q') = -i \int \mathrm{d}^4x \mathrm{d}^4y \mathrm{d}^4z \cdot \mathrm{e}^{i(p_1x+qy-p_2z)} \cdot \langle 0|T\{\eta(x), j_\mu^{u,d}(y), j_\nu^{u,d}(0), \bar{\eta}(z)\}|0\rangle, \tag{121}$$

where $\eta(x)$ is the three-quark current (94). Choose the currents in the form $j_\mu^u = \bar{u}\gamma_\mu u$, $j_\mu^d = \bar{d}\gamma_\mu d$, i.e. as an electromagnetic current which interacts only with a u (d) quark (with unit charges). Such a choice allows us to get sum rules separately for distribution functions of u and d quarks. Let us take the hadronic current momenta to be nonequal, perform the independent Borel transformation over p_1^2 and p_2^2 and only at very end set the Borel parameters M_1^2 and M_2^2 equal. The procedure described allows one to kill nondiagonal transition matrix elements of the type

$$\langle 0|j^h|h^*\rangle\langle h^*|j_\mu^{el}(y)j_\nu^{el}(0)|h\rangle\langle h|j^h|0\rangle \tag{122}$$

and thus makes it possible to separate the diagonal transition of interest:

$$\langle 0|j^h|h\rangle\langle h|j_\mu^{el}(y)j_\nu^{el}(0)|h\rangle\langle h|j^h|0\rangle. \tag{123}$$

As was shown in Ref. [145], the sum rules for nucleons have the form

$$\frac{2\pi}{4M^4}\frac{\bar{\lambda}_p^2}{32\pi^4}\, xq^{u,d}(x)\mathrm{e}^{-m^2/M^2} = \mathrm{Im}\, T_{u,d}^0 + \text{Power corrections}. \tag{124}$$

Here the l.h.s. is the phenomenological side of the sum rule—the proton state contribution; $\bar{\lambda}_p$ is defined in (100). In numerical calculations they will be set equal: $\bar{\lambda}_p^2 = 3.0\ \mathrm{GeV}^6$. The right-hand side is calculated in QCD. The excited state contribution—the continuum—is identified with the contribution of the bare loop diagram, starting from continuum threshold value s_0, and is transferred to the l.h.s. of the sum rule. The bare loop contribution to the sum rules is represented in Fig. 14.

The results after the double Borel transformation are

$$\mathrm{Im}\, T_{u(d)}^0 = \varphi_0^{u(d)}(x)\frac{M^2}{32\pi^3}\, E_2(s_0/M^2) \tag{125}$$

where

$$\varphi_0^u(x) = x(1-x)^2(1+8x), \qquad \varphi_0^d(x) = x(1-x)^2(1+2x), \tag{126}$$

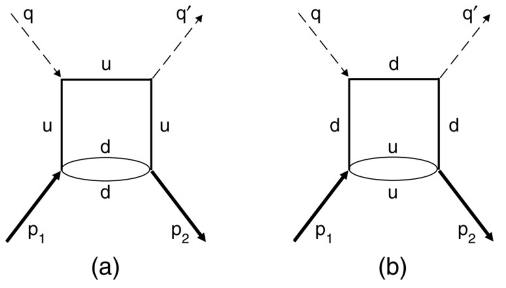

Fig. 14. Bare loop diagrams, corresponding to the unit operator contribution for u and d quarks (respectively, (a) and (b)).

and $E_2(z)$ is given by (98). The substitution of Eq. (125) into the sum rules (124) results in

$$xq(x)_0^{u(d)} = \frac{2M^6 e^{m^2/M^2}}{\bar{\lambda}_p^2}\,\varphi_0^{u(d)}(x)\cdot E_2\left(\frac{s_0}{M^2}\right). \tag{127}$$

Making use of the relation $\bar{\lambda}_p^2 e^{-m^2/M^2} = M^6 E_2$ which follows from the sum rule for the nucleon mass (see (95) in the same approximation), we get

$$\int_0^1 d_0(x)\mathrm{d}x = 1, \qquad \int_0^1 u_0(x)\mathrm{d}x = 2. \tag{128}$$

In the bare loop approximation there also appears the sum rule for the second moment:

$$\int_0^1 x(q_0^u(x) + q_0^d(x))\mathrm{d}x = 1. \tag{129}$$

Analogously to [138] one can show that relations (128) and (129) also hold when taking into account power corrections proportional to the quark condensate square in the sum rules for the four-point correlator and in the sum rules for the nucleon mass. Relations (128) reflect the fact that the proton has two u quarks and one d quark. Relation (129) expresses the momentum conservation law—in the bare loop approximation all momentum is carried by valence quarks. Therefore, the sum rules (128) and (129) demonstrate that the zero-order approximation is reasonable.

Let us calculate the perturbative corrections to the bare loop and restrict ourselves to the leading order (LO) corrections proportional to $\ln Q_0^2/\mu^2$, where Q_0^2 is the point where the quark distribution $q(x, Q_0^2)$ is calculated and μ^2 is the normalization point. In our case it is reasonable to choose μ^2 to be equal to the Borel parameter $\mu^2 = M^2$. The results take the form

$$d^{LO}(x) = d_0(x)\left\{1 + \frac{4}{3}\ln(Q_0^2/M^2)\cdot\frac{\alpha_s(Q_0^2)}{2\pi}\cdot\left[1/2 + x + \ln((1-x)^2/x) + \frac{-5-17x+16x^2+12x^3}{6(1-x)(1+2x)} - \frac{(3-2x)x^2\ln(1/x)}{(1-x)^2(1+2x)}\right]\right\} \tag{130}$$

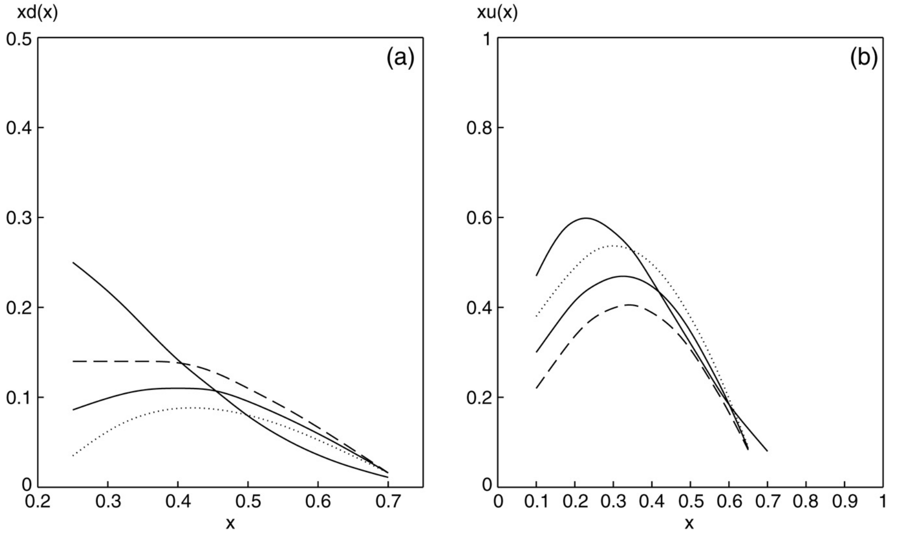

Fig. 15. d and u quark distributions at various values of the gluon condensate ($\langle(\alpha_s/\pi)G^2\rangle = 0.012$, 0.06 and 0 GeV^4; respectively dotted, solid and dashed lines). The thick solid line corresponds to the results of [131].

$$u^{LO}(x) = u_0(x)\cdot\left\{1+\frac{4}{3}\,\frac{\alpha_s(Q_0^2)}{2\pi}\ln(Q_0^2/M^2)\left[1/2+x+\ln(1-x)^2/x\right.\right.$$
$$\left.\left. +\ \frac{7-59x+46x^2+48x^3}{6(1-x)(1+8x)}-\frac{(15-8x)x^2\ln(1/x)}{(1-x)^2(1+8x)}\right]\right\} \tag{131}$$

where $u_0(x)$ and $d_0(x)$ are bare loop contributions, given by (127).

The contributions of the gluon condensate to the u and d quark distributions were found to be (the ratios to bare loop contributions are presented)

$$\frac{u(x)_{\langle G^2\rangle}}{u_0(x)} = \frac{\langle(\alpha_s/\pi)G^2\rangle}{M^4}\cdot\frac{\pi^2}{12}\frac{(11+4x-31x^2)}{x(1-x)^2(1+8x)}E_0(s_0/M^2)/E_2\left(\frac{s_0}{M^2}\right) \tag{132}$$
$$\frac{\mathrm{d}(x)_{\langle G^2\rangle}}{d_0(x)} = -\frac{\langle(\alpha_s/\pi)G^2\rangle}{M^4}\frac{\pi^2}{6}\frac{(1-2x^2)}{x^2(1-x)^2(1+2x)}E_0(s_0/M^2)/E_2\left(\frac{s_0}{M^2}\right). \tag{133}$$

The contributions of the quark condensate—the terms proportional to $\alpha_s(M^2)\langle 0|\bar{q}q|0\rangle^2$—are a few times smaller than the contributions of the gluon condensate and are not presented here. (They can be found in Ref. [145].)

The final results for the valence quark distribution in the proton are of the form

$$xu(x) = \frac{M^6 e^{m^2/M^2}}{\bar{\lambda}_N^2}2x(1-x)^2(1+8x)E_2\left(\frac{s_0}{M^2}\right)\left\{\left[1+\frac{u^{LO}(x,Q_0^2)}{u_0(x)}\right]\right.$$
$$\left. +\ \frac{1}{u_0(x)}[u(x)_{\langle G^2\rangle}+u(x)_{\alpha_s\langle\bar{q}q\rangle^2}]\right\} \tag{134}$$

$$xd(x) = \frac{M^6 e^{m^2/M^2}}{\bar{\lambda}_N^2} 2x(1-x)^2(1+2x)E_2\left(\frac{s_0}{M^2}\right)\left\{\left[1+\frac{d^{LO}(x,Q_0^2)}{d_0(x)}\right] + \frac{1}{d_0(x)}[d(x)_{\langle G^2\rangle} + d(x)_{\alpha_s\langle\bar{q}q\rangle^2}]\right\}. \tag{135}$$

The valence u and d quark distributions calculated according to (134) and (135) for various $\langle(\alpha_s/\pi)G^2\rangle = 0.00, 0.006, 0.012\ \mathrm{GeV}^4$ are shown in Fig. 15.

The following values of parameters were used: $\alpha_s\langle 0|\bar{q}q|0\rangle^2$ given by (85), $\bar{\lambda}_p^2 = 3.0\ \mathrm{GeV}^6$, $\lambda_{\mathrm{QCD}} = 250$ GeV, $s_0 = 2.5\ \mathrm{GeV}^2$, $Q_0^2 = 5\ \mathrm{GeV}^2$. The contribution of the $\langle G^3\rangle$ condensate was estimated on the basis of the instanton model—Eq. (29). This contribution may influence u and d quark distributions at $x \lesssim 0.2$ and increase both of them by 10–20%. The limits of applicability of QCD calculations are: for the u quark—$0.2 < x < 0.65$; for the d quark—$0.3 < x < 0.65$. The lower limit arises from the gluon condensate contribution—it was required that this contribution does not exceed 30%; the upper limit was determined by increasing perturbative corrections and $\alpha_s\langle\bar{q}q\rangle^2$ terms. For comparison, Fig. 15 presents the results of the fit to data on the basis of the solution of the DGLAP equation. The LO order fit [131] is chosen, but not the more precise NLO fits [132–136], because QCD calculations of quark distributions were performed in LO. As is seen from Fig. 15, the initial (at $Q_0^2 = 5\ \mathrm{GeV}^2$) valence quark distributions at $0.3 < x < 0.7$ calculated in QCD are in satisfactory agreement with those found from the data. The preferred value of the gluonic condensate is $\langle 0|(\alpha_s/\pi)G^2|0\rangle = 0.006\ \mathrm{GeV}^4$; values higher than $0.012\ \mathrm{GeV}^2$ and lower than 0 may probably be excluded. These statements are in full accord with those obtained in the previous sections.

9. Conclusion

The basic parameters of QCD—$\alpha_s(Q^2)$—and the values of the vacuum condensates, determining the hadron physics at low momentum transfers ($Q^2 \sim 1$–$5\ \mathrm{GeV}^2$), are reliably determined by the theory. It was demonstrated that the values of these parameters found by consideration of various processes are in good agreement with one another. The values of u, d, s quark masses and their ratios are known now with a good precision—about 10–15% in the ratios and about 20% in the mass absolute values. The precision of the charmed quark mass value $\overline{m}_c(\overline{m}_c)$ in the $\overline{MS}$ renormalization scheme is extremely high—about 1%. The knowledge of $\alpha_s(Q^2)$ and the condensates makes it possible to find the polarization operators of vector and axial currents at $Q^2 \geq 1\ \mathrm{GeV}^2$ with high precision. In such calculations high order perturbative terms—$\sim\alpha_s^2$ and, in some cases, $\sim\alpha_s^3$—must be accounted for. Therefore, we have now a good basis for the theoretical description of many physical phenomena in low energy QCD—hadron masses, their static properties, quark distributions in hadrons etc. Of course, at even lower momentum transfer, $Q^2 \lesssim 1\ \mathrm{GeV}^2$, the approach exploited in this review, and based on perturbation theory and the OPE, does not work: the confinement mechanism and the mechanism of spontaneous symmetry breaking are acting at full strength. The construction of various models is unavoidable here. But for such models the knowledge of basic QCD parameters is also quite important—they may play the role of cornerstones for the models.

I summarize here the final values of α_s and the condensates:

$$\alpha_s(m_\tau^2) = 0.340 \pm 0.015 \qquad \overline{MS} - \text{scheme} \tag{136}$$

$$\langle 0|\frac{\alpha_s}{\pi} G_{\mu\nu}^2|0\rangle = 0.005 \pm 0.004 \text{ GeV}^2 \tag{137}$$

$$\langle 0|\bar{q}q|0\rangle_{1 \text{ GeV}} = -(1.65 \pm 0.15) \times 10^{-2} \text{ GeV}^3, \qquad q = u, d \tag{138}$$

$$\alpha_s \langle 0|\bar{q}q|0\rangle^2 = (1.5 \pm 0.2) \times 10^{-4} \text{ GeV}^6. \tag{139}$$

(In the determination of (139), the factorization hypothesis is assumed.) The values of the errors, given by (137)–(139), are a bit uncertain, since the procedures for averaging of errors in different processes are subjective in an essential way.

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