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EFFECTS OF THE TWO-PION EXCHANGE THREE-NUCLEON FORCE IN THE TRITON AND ^{3}He

W. GLÖCKLE

Institut für Theoretische Physik, Ruhr-Universität Bochum, 4630 Bochum, West-Germany

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Abstract: The two-pion-exchange three-nucleon force is incorporated into a Faddeev calculation in momentum space. Within the restricted space of states substantial additional binding energy shows up. A new expression for the charge form factor is derived which does not involve a partial-wave expansion of the total state. The form factor of ^{3}He shows sensitivity to the change in the wave-function induced by the three-nucleon force. The secondary maximum increases as a whole, and the radius comes closer to the experimental value; however the notorious discrepancy in the first minimum increases, too.

1. Introduction

The lightest nucleus, the deuteron, serves to parametrise the two-nucleon force. The next lightest nuclei, ^{3}H and ^{3}He, already withstand satisfactory theoretical explanations. The theoretical binding energies [1]) for various two-nucleon forces range between 7 and 7.5 MeV where the increase in binding energy goes along with a decrease in the d-state probability of the two-nucleon bound state. The experimental value is 8.48 MeV.

The charge form factor of ^{3}He [ref. [2])] is another observable which is still in an unsatisfactory state. Without mesonic exchange current contributions the theoretical momentum transfer at the first minimum is too high and the second maximum too low. Also the spatial extension, the radius, is theoretically too large, which is not surprising in view of the lack of binding energy.

Certainly one can expect that exchange current contributions [3]) play a role and appear to decrease the discrepancy between theory and experiment, but the calculations still lack a convincing consistency between the model used for the two-nucleon force, the choice of the dynamical equation or in other words the inclusion or exclusion of relativistic effects and the choice of diagrams for the interaction of the virtual photon with the currents in the system.

On the other hand it was always a general feeling that the three-nucleon forces might do most of the job. An interesting analysis [4]) of the experimental curve for the charge form factor appears to support strongly that point of view. Namely it is known [5]) that the two-pion-exchange three-nucleon force has the characteristic feature to be repulsive if one nucleon approaches the center of mass. This is not

only the case if all three nucleons are close together, which would be hindered already by the repulsive two-nucleon forces, but also if they are relatively far apart and lie on a line. Thus the inclusion of such a force could explain the dip in the "experimental" point density extracted in ref. [4]).

In that work we want to carry through our first study including the two-pion exchange three-nucleon force in a Faddeev calculation. In sects. 2 and 3 we present the three-nucleon force and describe how it is incorporated into a Faddeev calculation. In sect. 4 a modification of the Malfliet–Tjon procedure is introduced, which is necessary to eliminate unphysical states. The charge form factor is reconsidered in sect. 5 and it is shown how to avoid the tedious partial wave expansion of the total state. Our results are presented in the final sect. 6. The appendices collect some longer technical expressions.

2. The two-pion-exchange three-nucleon force

The two-pion-exchange three-nucleon force has been discussed in several papers [6]). Regarding fig. 1 the basic ingredient is obviously the off-shell (space-like pion momenta) πN scattering amplitude. On the one hand there have been trial

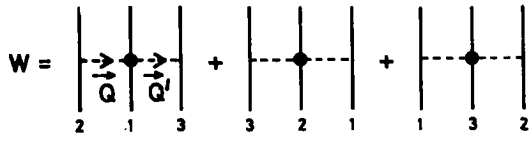

Fig. 1. The two-pion-exchange three-nucleon force.

models built for that amplitude, like an intermediate Δ-resonance, the coupling of the two pions to the nucleon through a σ-meson, the inclusion of the nucleon propagator to negative energy, etc. This is plagued by many uncertainties, like for instance the ambiguity [7]) in the off-shell Rarita–Schwinger propagator of the Δ-resonance. On the other hand one can connect the axial vector–nucleon amplitude through a Ward identity to the πN amplitude using current algebra and PCAC. A remaining axial vector background amplitude is assumed to be dominated by the $\Delta(1230)$ isobar. We shall use a three-nucleon force, which has been worked out with that second strategy [8]). In a recent paper [9]) that force was reconsidered. The part W_1, which singles out particle 1, shown in the first diagram of fig. 1, is

$$\langle p_1' p_2' p_3' | W_1 | p_1 p_2 p_3 \rangle$$

$$= \frac{1}{(2\pi)^6} \delta^3(p_1 + p_2 + p_3 - p_1' - p_2' - p_3') \frac{g^2}{4m^2} \frac{H(Q^2)}{Q^2 + \mu^2}$$

$$\times \frac{H(Q'^2)}{Q'^2 + \mu^2} \{ \tau_2 \cdot \tau_3 \sigma_2 \cdot Q \sigma_3 \cdot Q' [a + bQ \cdot Q' + c(Q^2 + Q'^2) + d p_1 \cdot p_1'] $$

$$+ i\tau_3 \times \tau_2 \cdot \tau_1 \sigma_2 \cdot Q \sigma_3 \cdot Q' \sigma_1 \cdot (Q \times Q')(d_3 + d_4) \} . \qquad (2.1)$$

One recognises the two-pion propagators together with $H(Q^2)$, the square of the pion form factor $K(Q^2)$ of the nucleon and the momentum and spin dependence of the off-shell πN amplitude in a p/m expansion as expressed in the curly bracket. (The nucleon propagator to positive energy is of course subtracted out.) Using just the Rarita–Schwinger propagator for the Δ-resonance as a model for the πN amplitude would lead only to the terms with $b \neq 0$ and $d_3 + d_4 = \frac{1}{4}b$, however with different magnitudes. The a and c terms are due to s-wave πN scattering. The $\boldsymbol{p}_1 \cdot \boldsymbol{p}_1'$ term shows up if one does not restrict the middle nucleon in the first diagram of fig. 1 to be at rest, an assumption which is seriously in conflict with the other two diagrams and with the need of the full momentum dependence of W for all three nucleons. Further we note that there are two parts which conserve or change the isospin in the 2–3 subsystem. Clearly they result from the isospin-even and isospin-odd parts of the πN amplitude in the crossed 2π channel. The strength parameters are fixed by the theory to be

$$a = 1.13\,\mu^{-1}, \quad b = -2.58\,\mu^{-3}, \quad c = 1.00\,\mu^{-3}, \quad d = -0.295\,\mu^{-3},$$
$$d_3 + d_4 = -0.753\,\mu^{-3}. \tag{2.2}$$

The weak piece in the expression (2.1) is the choice of the form factor, on which we shall comment in sect. 6.

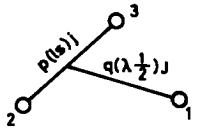

Fig. 2. Choice of Jacobi momenta and related angular momenta.

In ref. [9]) the problem is considered how to handle a force of the type (2.1) in a standard Faddeev calculation, which works in a truncated space of partial wave states of the triton. Related to fig. 2 these states are defined by

$$|pq\alpha\rangle \equiv |p(ls)jq(\lambda\tfrac{1}{2})J\mathcal{J}(t\tfrac{1}{2})T\rangle\,. \tag{2.3}$$

In obvious notation the angular momenta for the two-body subsystems and the spectator particle are $(ls)j$ and $(\lambda\frac{1}{2})J$ respectively. We use the Jacobi momenta

$$\begin{aligned} \boldsymbol{p} &= \tfrac{1}{2}(\boldsymbol{p}_2 - \boldsymbol{p}_3)\,, \\ \boldsymbol{q} &= \tfrac{2}{3}(\boldsymbol{p}_1 - \tfrac{1}{2}(\boldsymbol{p}_2 + \boldsymbol{p}_3))\,, \end{aligned} \tag{2.4}$$

expressed in terms of the individual nucleon momenta $\boldsymbol{p}_i$, $i = 1, 2, 3$.

Clearly for W_1 which singles out particle 1 the natural variables are the relative momenta for the two-body subsystems (21) and (31) and not the Jacobi momenta (2.4), which one would like to use in solving the Faddeev equation. Insisting on the Jacobi momenta therefore requires a nontrivial partial-wave expansion of W_1

in terms of the states (2.3). This has been carried out in ref. [9]) with the result

$$\langle p'q'\alpha'|W_1|pq\alpha\rangle = \sum_{\{\mu\}} H_{\{\mu\}}(p'q'pq) G^{\alpha'\alpha}_{\{\mu\}}(pq'pq)\,. \tag{2.5}$$

Here $\{\mu\}$ is a finite set of auxiliary discrete quantum numbers, $G^{\alpha'\alpha}_{\{\mu\}}$ are purely geometrical coefficients containing a known dependence on the momenta, and

$$H_{\{\mu\}}(p'q'pq) = \int\int\int P_{l_1}(x_1)P_{l_2}(x_2)P_{l_3}(x_3) \times A_1^{g_1}A_2^{g_2}\frac{H(Q^2)}{Q^2+\mu^2}\,\frac{H(Q'^2)}{Q'^2+\mu^2} \tag{2.6}$$

contains the dynamics.

The pion momenta are

$$\begin{aligned} Q^2 &= A_1^2+\tfrac{1}{4}A_2^2-A_1A_2x_1\,,\\ Q'^2 &= A_1^2+\tfrac{1}{4}A_2^2+A_1A_2x_1\,, \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} A_1 &= \sqrt{p^2+p'^2-2pp'x_2}\,,\\ A_2 &= \sqrt{q^2+q'^2-2qq'x_3}\,. \end{aligned} \tag{2.8}$$

We shall comment on the evaluation of $H_{\{\mu\}}$ in sect. 4.

3. The Faddeev equation including a three-nucleon force

In the framework of the Faddeev equation it appears natural to consider W as a sum over three parts:

$$W = \sum_{i=1}^{3} W_i\,. \tag{3.1}$$

In the model of fig. 1 this is obvious. Then the Schrödinger equation in integral form

$$\Psi = G_0 \sum_{i=1}^{3} (V_i + W_i)\bar{\Psi} \tag{3.2}$$

suggests to keep together $(V_i + W_i)$ and we get immediately the modified Faddeev equation

$$\psi = G_0 TP\psi \tag{3.3}$$

for the Faddeev component ψ. In (3.3) the two-body t-matrix t connected with V_1 is now replaced by the operator T defined through

$$T = (V_1 + W_1) + (V_1 + W_1)G_0T\,, \tag{3.4}$$

or

$$T = t + (1 + tG_0) W_1 (1 + G_0 T) . \tag{3.5}$$

The second form is convenient for a perturbative determination of T in powers of W_1. The permutation operator

$$P = P_{12} P_{23} + P_{13} P_{23} \tag{3.6}$$

brings in the other two Faddeev components which in the case of identical particles have the same functional form as ψ. Thus the total wave function is

$$\Psi = (1 + P)\psi . \tag{3.7}$$

The Faddeev equation (3.3) is solved in the representation spanned by the states (3.3). Setting

$$\varphi_\alpha(pq) \equiv pq\langle pq\alpha|\psi\rangle , \tag{3.8}$$

we get

$$\varphi_\alpha(pq) = \frac{1}{E - p^2 - \frac{3}{4}q^2} \sum_{\alpha'} \int_0^\infty \mathrm{d}q' q' \sum_{\alpha''} \int_0^\infty \mathrm{d}q'' q'' \times \int_{-1}^{1} \mathrm{d}x \frac{\langle pq\alpha|T|p_1 q''\alpha''\rangle}{p_1^{l''+1}} G^{\alpha''\alpha'}(q''q'x) \frac{\varphi_{\alpha'}(p_2 q')}{p_2^{l'+1}} , \tag{3.9}$$

with

$$\begin{aligned} p_1 &= \sqrt{q'^2 + \tfrac{1}{4}q''^2 + q'q''x} , \\ p_2 &= \sqrt{q''^2 + \tfrac{1}{4}q'^2 + q''q'x} . \end{aligned} \tag{3.10}$$

The function $G^{\alpha''\alpha'}(q''q'x)$ results from the permutation operator and is purely geometrical in nature. It is given in appendix A.

It is the inherent geometrical nature of the three-body problem, namely the presence of three two-body subsystems with corresponding third particles at the same time, which necessarily leads to the "skew" arguments p_1 and p_2 in (3.10). Now it is general experience for the triton that the q-dependence of $\varphi_\alpha(pq)$ drops within a few units of fm^{-1}, say $q_{\max}$, whereas the short-range nature of the repulsive part of the two-nucleon interaction induces very large p-components into φ_α. Using however the presentation of the permutation operator chosen in (3.9) the maximal value of p_2 as given in (3.10) is obviously $\frac{3}{2}q_{\max}$. Therefore the solution of (3.9) requires only the functions φ_α at $p \leq \frac{3}{2}q_{\max}$, $q \leq q_{\max}$, which obviously helps to keep the number of meshpoints small.

The occurrence of p_2 in φ_α makes an interpolation unavoidable. Since that technical problem is directly linked with the very geometrical nature of the three-body problem it may be justified that we describe our interpolation technique in this section.

Any interpolation algorithm like for instance spline interpolation needs function values at a certain number of meshpoints as the input to build up the polynomials. Using an iterative procedure to solve (3.9) the functions φ_α are available from the previous step and can therefore be interpolated. However that requires interpolation at every step of iteration, the number of which can be quite large. In order to avoid that we noticed that $\varphi_\alpha(p_2 q')$ is certainly a linear expression of the function values $\varphi_\alpha(p_i, q')$ at a conveniently chosen set of meshpoints $\{p_i\}$. Thus it should be possible to write

$$\varphi_\alpha(p_2 q) = \sum_i S_i(p_2)\varphi_\alpha(p_i q)\,, \tag{3.11}$$

where $S_i(p_2)$ are known functions depending only on the set of meshpoints $\{p_i\}$, the actual position p_2, and the type of interpolation chosen, but *not* on the function values $\varphi_\alpha(p_i, q)$. In the case of spline interpolation we worked [10]) out that modified representation (3.11) and showed that the smooth functions $S_i(p)$ can be determined easily through recurrence relations like the quantities which enter in the standard form of the spline algorithm.

Inserting (3.11) into (3.9) leads to the approximate set of integral equations based on a finite number of meshpoints:

$$\varphi_\alpha(p_i q) = \frac{1}{E - p_i^2 - \frac{3}{4}q^2} \sum_{\alpha'} \int \mathrm{d}q' q' \sum_{\alpha''} \int \mathrm{d}q'' q'' \times \sum_{i'} \left(\int_{-1}^{1} \mathrm{d}x \frac{\langle p_i q \alpha | T | p_1 q'' \alpha'' \rangle}{p_1^{i'+1} p_2^{i'+1}} G^{\alpha'' \alpha'}(q'' q' x) S_{i'}(p_2) \right) \varphi_{\alpha'}(p_{i'} q')\,. \tag{3.12}$$

Now one can carry through the x-integration without having to know the functions φ_α! Moreover this has to be done only *once*.

Finally using gaussian quadrature points in q we get the standard form of discretised integral equations in two variables:

$$\varphi_\alpha(p_i q_j) = \sum_{\alpha'} \sum_{i'j'} K_{\alpha\alpha'}(p_i q_j, p_{i'} q_{j'}) \varphi_{\alpha'}(p_{i'} q_{j'})\,. \tag{3.13}$$

That form would allow now also different algorithms for its solution than just the iteration procedures which are generally used.

Finally we present the operator T in the space of partial-wave states (2.3) in first order in W_1:

$$\begin{aligned} pq\langle pq\alpha | T | p'q'\alpha' \rangle p'q' &= \delta(q - q') \langle p\alpha | t(E - \tfrac{3}{4}q^2) | p'\alpha' \rangle \\ &\quad + \langle pq\alpha | W_1 | p'q'\alpha' \rangle + \sum_{\alpha''} \int \mathrm{d}p'' \langle p\alpha | t(E - \tfrac{3}{4}q^2) | p''\alpha'' \rangle \end{aligned}$$

$$\times \frac{1}{E - p''^2 - \frac{3}{4}q^2} \langle p''q\alpha'' | W_1 | p'q'\alpha' \rangle$$

$$+ \sum_{\alpha''} \int \mathrm{d}p'' \langle pq\alpha | W_1 | p''q''\alpha'' \rangle \frac{1}{E - p''^2 - \frac{3}{4}q''^2}$$

$$\times \langle p''\alpha'' | t(E - \tfrac{3}{4}q''^2) | p'\alpha' \rangle$$

$$+ \sum_{\alpha''} \int \mathrm{d}p'' \sum_{\alpha'''} \int \mathrm{d}p''' \langle p\alpha | t(E - \tfrac{3}{4}q^2) | p''\alpha'' \rangle$$

$$\times \frac{1}{E - p''^2 - \frac{3}{4}q^2} \langle p''q''\alpha'' | W_1 | p'''q'''\alpha''' \rangle \frac{1}{E - p'''^2 - \frac{3}{4}q'^2}$$

$$\times \langle p'''\alpha''' | t(E - \tfrac{3}{4}q'^2) | p'\alpha' \rangle . \qquad (3.14)$$

4. Numerical techniques

The first step taken to solve the infinite set of coupled integral equations (3.13) is to truncate it to a finite one in the number of partial-wave states α. The standard choice is 5 as given in table 1. This is of course based on the fact that the $^1S_0$ and $^3S_1$–$^3D_1$ two-nucleon forces are much stronger than the p-wave and higher l-wave forces. Also, it is a well-established experience that the two states nos. 4 and 5 with spectator angular momentum $\lambda = 2$ are important. They provide about 500 keV additional binding energy and are very important in the charge form factor. Including the three-nucleon force there is no obvious reason why the p-wave matrix elements of W are much smaller than the corresponding ones in table 1. So for the trivial reason of computer restrictions we were forced in the present work to stay within the 5-channel calculation.

TABLE 1

The five channels taken into account in the Faddeev component

No.	l	s	j	t	λ	J	L	S
1	0	0	0	1	0	$\frac{1}{2}$	0	$\frac{1}{2}$
2	0	1	1	0	0	$\frac{1}{2}$	0	$\frac{1}{2}$
3	2	1	1	0	0	$\frac{1}{2}$	2	$\frac{3}{2}$
4	0	1	1	0	2	$\frac{3}{2}$	2	$\frac{3}{2}$
5	2	1	1	0	2	$\frac{3}{2}$	0 1 2	$\frac{1}{2}$ $\frac{1}{2}, \frac{3}{2}$ $\frac{3}{2}$

The second step is the choice of meshpoints in the intervals $[0, q_{\max}]$ and $[0, \frac{3}{2}q_{\max}]$, for q and p, respectively. In q we choose gaussian quadrature points according to

$$q = \frac{1+x}{1/q_0 + (2/q_{\max} - 1/q_0)x}. \tag{4.1}$$

It also turned out that a corresponding mapping of the p-interval was a good set of grid points for the spline interpolation. Finally the x-integration was carried through by gaussian quadrature.

After the q-discretisation, the spline interpolation and truncation in α we face the homogeneous algebraic eigenvalue problem

$$\varphi_\mu = \sum_\nu K_{\mu\nu}(E)\varphi_\nu, \tag{4.2}$$

where μ, ν collect the indices for p, q, and α. In matrix notation this reads

$$\varphi = K(E)\varphi, \tag{4.3}$$

which is a special case of the eigenvalue problem

$$\eta_l \varphi_l = K(E)\varphi_l. \tag{4.4}$$

In the Malfliet–Tjon method [11]) an arbitrary starting function $\varphi^{(0)}$ is chosen and one considers the ratio

$$Q_n = \frac{K^{n+1}(E)\varphi^{(0)}}{K^n(E)\varphi^{(0)}}, \tag{4.5}$$

which obviously for $n \to \infty$ tends towards the eigenvalue η which is largest in magnitude. If that eigenvalue is positive it will be necessarily 1 at the triton binding energy, since the triton has only one bound state. Therefore one has simply to stabilise Q_n with respect to n and vary the energy until $Q_n \to 1$. This provides both, the energy and the Faddeev component φ. In the two-nucleon system the corresponding problem would involve the kernel $G_0(E)V$, where $G_0(E)$ is the free two-body propagator. Since there is a repulsive core clearly negative eigenvalues will exist, too, which correspond to bound states in the repulsive core region. In momentum space these unphysical bound states have very high momentum components. Corresponding types of unphysical bound states will show up in the three-nucleon problem with negative eigenvalues having high momentum components. It can turn out that the largest eigenvalue in magnitude is a negative one. Let us call it η'. In that case the Malfliet–Tjon method without modification will not work. We shall show below that this happens in the case of the triton and the Reid potential if the spectator momentum q is allowed for instance to extend up to $q_{\max} \approx 6\ \mathrm{fm}^{-1}$. Then Q_n tends towards η' at the (theoretical) triton binding energy and $K^n\varphi^{(0)}$ will be proportional to the corresponding unphysical eigenstate of K.

This does not happen for $q_{max} \approx 3\,\mathrm{fm}^{-1}$, since the higher momentum components in the kernel are missing and only a small η' in magnitude results.

As we shall see below the 3-body force requires $q_{max} \approx 6\,\mathrm{fm}^{-1}$ and we are confronted with the problem of $\eta' = -1.205$ besides the physical value $\eta = 1$.

Another technique for solving (4.2) is the Padé method [12]). It allows one to sum the Neumann series into the resolvent operator

$$(1-\lambda K(E))^{-1}\varphi^{(0)} = \varphi^{(0)} + \lambda K(E)\varphi^{(0)} + \lambda^2 K^2(E)\varphi^{(0)} + \cdots, \tag{4.6}$$

also outside the circle of convergence. The input to calculate the Padé ratio is again the various powers of K applied to an arbitrary starting function $\varphi^{(0)}$. The resolvent operator is a meromorphic function of λ with poles at $\lambda_l = \eta_l^{-1}$ and can be approximated by the Padé ratio. Near the three-body binding energy E_t the resolvent operator for $\lambda = 1$ tends towards

$$\frac{|\varphi\rangle\langle\bar{\varphi}|\varphi^{(0)}\rangle}{1-\eta(E)} \qquad \text{with } \eta(E_t) = 1\,.$$

By searching for that energy at which the inverse Padé ratio tends towards zero we can determine E_t. This is true for all values of p, q and α and one can choose an arbitrary point in the set p, q and α.

To determine not only E_t but also $\varphi_\alpha(pq)$ in this manner would require that we calculate the Padé ratio for *all* values p, q and α, which would be quite time consuming since about 20–30 iterations are necessary to stabilize the Padé ratio.

Once E_t is determined by the Padé method a more economic procedure is the following modified Malfliet–Tjon technique. At $E = E_t$ we know one eigenvalue $\eta = 1$; let us assume that the largest eigenvalue in magnitude, η', is negative and also known. This is sufficient to eliminate the undesired unphysical eigenstate φ' from the starting function $\varphi^{(0)}$. One can decompose

$$\varphi^{(0)} = c\varphi + c'\varphi' + \cdots, \tag{4.7}$$

and finds at $E = E_t$:

$$K\varphi^{(0)} = c\varphi + c'\eta'\varphi' + \cdots. \tag{4.8}$$

Multiplying (4.7) by η' and subtracting (4.8) we get

$$\eta'\varphi^{(0)} - K\varphi^{(0)} = c\varphi(\eta' - 1) + \cdots. \tag{4.9}$$

Thus the right-hand side has no longer a component in φ' and can be taken as the new starting function $\varphi^{(0)}_{new}$. The Malfliet–Tjon ratio will then tend towards 1 if we can assume that $\eta = 1$ is the next largest eigenvalue in magnitude.

On a computer the absence of φ' in (4.9) is of course at most correct within the number of digits available. The high-order application of K will finally build up φ' again in $K^n\varphi^{(0)}_{new}$. In practice it turned out that a renewed elimination of φ' was not necessary if φ' was eliminated within machine accuracy. To achieve that it is

not necessary to know η' to many digits. Let us assume that we know η' only approximately, $\eta'_{\rm appr}$. Then (4.9) will be replaced by

$$\eta'_{\rm appr}\varphi^{(0)} - K\varphi^{(0)} = c\varphi(\eta'_{\rm appr} - 1) + c'\varphi'(\eta'_{\rm appr} - \eta') + \cdots . \tag{4.10}$$

One reduces the coefficient of φ' now by repeating that elimination process several times. This yields after k-steps

$$c\varphi(\eta'_{\rm appr} - 1)^k + c'\varphi'(\eta'_{\rm appr} - \eta')^k + \cdots ,$$

which eliminates φ' to any desired accuracy.

How do we get $\eta'_{\rm appr}$? Since η' is assumed to be the largest eigenvalue in magnitude clearly Q_n will finally converge towards η'. This is however a very uneconomic procedure and the following procedure is better. Since E_t (or $\eta = 1$) has already been determined by the Padé technique the physical state φ can be eliminated in eactly the same manner as we described above for the elimination of φ'. Then Q_n will converge much faster towards η'. For instance for $q_{\max} = 6\ \mathrm{fm}^{-1}$ and using the Reid potential in a five-channel calculation, Q_n starting from $\varphi^{(0)} = 1$ needs about 80 iterations to stabilize at all points at the value $\eta' = -1.205$. Eliminating first the Faddeev component φ by a few subtractions reduces the number of iterations to about 15 to get η' within the same accuracy. Now knowing η' the φ' component is eliminated again by a few subtractions, and $K^n\varphi^{(0)}_{\rm new}$ tends towards the physical state within about 20–30 iterations.

We discussed that problem in some detail since a "strange unstable behaviour of Q_n" was known to occur [13]), and also that the modified Malfliet–Tjon method can be of more general interest.

The obvious breakdown of the Malfliet–Tjon method in the case of a strong negative eigenvalue for two nucleons has been discussed in ref. [14]). In that case however the kernel could be easily symmetrised and the starting function was explicitly orthogonalised with respect to the unphysical eigenstate. It was also noted that even the Padé technique did not work, since the iterates of the kernel built up very quickly, predominantly so the component in the direction of the unphysical eigenstate. That type of insight has been discussed again at length in ref. [15]), where in the case of the 3-nucleon system the pointwise Padé technique is recommended. We would like to comment that in eq. (2.9) of ref. [15]) one needs the states biorthogonal to $\phi_B(\omega)$ and $\phi_0(\omega)$ to make that relation valid (note that the Faddeev kernel is not symmetric). Because of that the orthogonalisation recommended below eq. (2.10) in ref. [15]) in the case of a very strong negative eigenvalue appears to be a quite involved procedure. Our proposal achieves exactly the desired starting function in eliminating the spurious component by easily accessible subtractions.

The calculation of the three-fold integral (3.6), the central piece of the partially wave-decomposed three-nucleon force, requires a comment. Improving on the treatment in ref. [9]) one can carry through the x_1 integration analytically with the

result

$$H_{\{\mu\}} = -(\Lambda^2-\mu^2)^4 \int\int P_{l_2}(x_2)P_{l_3}(x_3)A_1^{g_1-2}A_2^{g_2-2}$$
$$\times\Big\{A_1A_2\frac{3(\Lambda^2+B^2)^2-(\mu^2+B^2)^2}{(\Lambda^2+B^2)^3(\Lambda^4+2B^2(\Lambda^2-\mu^2)-\mu^4)}Q_{l_1}(\hat{x}_\Lambda)$$
$$-\frac{1}{(\Lambda^2+B^2)^2(\Lambda^4+2B^2(\Lambda^2-\mu^2)-\mu^4)}Q'_{l_1}(\hat{x}_\Lambda)$$
$$-A_1A_2\frac{2}{(\mu^2+B^2)(\Lambda^4+2B^2(\Lambda^2-\mu^2)-\mu^4)^2}Q_{l_1}(\hat{x}_\mu)\Big\}. \quad (4.11)$$

Thereby we used form-factors such that

$$H(Q^2)=\left(\frac{\Lambda^2-\mu^2}{\Lambda^2+Q^2}\right)^2, \quad (4.12)$$

and we have

$$B^2 = A_1^2+\tfrac{1}{4}A_2^2,$$
$$\hat{x}_\Lambda = (\Lambda^2+B^2)/A_1A_2, \quad (4.13)$$
$$\hat{x}_\mu = (\mu^2+B^2)/A_1A_2.$$

Regarding for instance the expression $1/(\mu^2+B^2)$ we see that it has a pole at $x_2=(1/2pp')(p^2+p'^2+\mu^2+\frac{1}{4}A_2^2)$ which can be very close to 1. Additional powers in those poles come from the associated Legendre functions Q_l. For $p\approx p'$ and $x_2\to 1$, $A_1\to 0$ and the argument $\hat{x}$ of Q_l will be very large. Then $Q_l(\hat{x})\sim 1/\hat{x}^{l+1}$ which brings in the additional poles. Since the integrand in x_2 can vary over many orders of magnitudes a straightforward gaussian quadrature integration is no longer possible. But one can use the first few terms in the asymptotic expansion of the Q_l and carry through the x_2 integration analytically. This leads to higher order derivatives of the Q_l which can be determined by recurrence relations but which is time consuming. A more elegant procedure [16]) is to apply a transformation

$$x_2 = \frac{x_2'+x_{20}}{1+x_{20}x_2'}, \quad (4.14)$$

which accumulates half the number of quadrature points for x_2' in x_2 to the right of x_{20}. A value of $x_{20}\sim 0.9$ turned out to be adequate to exhaust the integral in the case of a nearby pole, with sufficient accuracy. A corresponding transformation is necessary in x_3, where the same pole can show up for p and p' small but q and q' large. In all cases, with or without transformation, ten gaussian quadrature points were sufficient.

5. The charge form factor

It is well known that the standard calculation of the charge form factor requires a lot of partial waves namely about 40 [ref. [17])]. They are induced through the permutation operator P in (3.7) starting for instance from the standard 5 states in the Faddeev component ψ. This is an unpleasant and tedious numerical fact. Therefore we want to propose a formulation which avoids the partial wave decomposition of the total Ψ and includes therefore *all* partial waves starting from a given Faddeev component ψ.

Presumably a related evaluation of the trinucleon Coulomb energy including *all* partial waves however working in coordinate space has been performed in ref. [18]).

The charge form factor is defined by

$$F(\boldsymbol{Q}^2)=\frac{1}{4\pi}\int \mathrm{d}\hat{\boldsymbol{Q}}\int \mathrm{d}^3x_0\,\mathrm{e}^{-i\boldsymbol{Q}\cdot\boldsymbol{x}_0}$$

$$\times\langle\Psi|\sum_{i=1}^{3}\tfrac{1}{2}(1+\tau_3(i))f^{\mathrm{p}}(\boldsymbol{x}_0-\boldsymbol{x}_i)+\sum_{i=1}^{3}\tfrac{1}{2}(1-\tau_3(i)f^{\mathrm{n}}(\boldsymbol{x}_0-\boldsymbol{x}_i)|\Psi\rangle\,, \quad (5.1)$$

where

$$f^{\mathrm{p,n}}(\boldsymbol{x})=\frac{1}{(2\pi)^3}\int \mathrm{d}^3l\tilde{f}^{\mathrm{p,n}}(\boldsymbol{l})\,\mathrm{e}^{i\boldsymbol{l}\cdot\boldsymbol{x}} \quad (5.2)$$

is related to the nucleon form factors $\tilde{f}^{\mathrm{p,n}}(\boldsymbol{l})$. According to their properties,

$$\tilde{f}^{\mathrm{p}}(0)=1\,, \qquad \tilde{f}^{\mathrm{n}}(0)=0\,. \quad (5.3)$$

$F(\boldsymbol{Q}^2)$ is obviously normalized: $F(0)=Z$, where Z is the charge of the state $|\Psi\rangle$.

Clearly the three cyclic parts in $\langle\Psi(1+P)$ yield the same contribution and we can write (5.1) with a factor of 3, thereby replacing $\langle\Psi|$ with $\langle\psi|$. Let us indicate the straightforward evaluation of the resulting matrix element. We assume that the Faddeev component is given by a certain finite number of partial-wave states

$$|\psi\rangle=\sum_{\beta}\int p^2\,\mathrm{d}pq^2\,\mathrm{d}q\phi_\beta(pq)|pq\beta\rangle\,, \quad (5.4)$$

where we choose the LS coupling:

$$|pq\beta\rangle\equiv|pq(l\lambda)L(s\tfrac{1}{2})S\mathcal{J}(t\tfrac{1}{2})T\rangle\,. \quad (5.5)$$

Obviously the amplitudes ϕ_β are related to the amplitudes $\phi_\alpha\equiv\varphi_\alpha/pq$ of (3.8) in jJ coupling through

$$\phi_\beta(pq)=\sum_{jJ}\sqrt{\hat{j}\hat{J}\hat{L}\hat{S}}\begin{Bmatrix} l & s & j \\ \lambda & \frac{1}{2} & J \\ L & S & \mathcal{J}\end{Bmatrix}\phi_\alpha(pq) \quad (5.6)$$

($\hat{l} \equiv 2l+1$). We note that the 5 states of table 7 yield 8 amplitudes ϕ_β. The total state $|\Psi\rangle$ is then given as

$$|\Psi\rangle = \sum_\beta \int \phi_\beta(pq)(|pq\beta\rangle_1 + |pq\beta\rangle_2 + |pq\beta\rangle_3)\,, \tag{5.7}$$

where the indices indicate that the quantum numbers $pq\beta$ refer to the three different choices of two-body subsystems and related spectator particles.

In the matrix elements the isospin, spin and momentum parts factorize and we have

$$\begin{aligned}\langle\psi|\sum_{i=1}^{3} \tfrac{1}{2}(1\pm\tau_3(i))f(\boldsymbol{x}_0-\boldsymbol{x}_i)|\Psi\rangle \\ = \sum_{i=1}^{3}\int\int\int\sum_{j=1}^{3}\phi_\beta(pq)I_{ij}^{(\pm)}(tt'i)S_{ij}(ss') \\ \times {}_1\langle pq(l\lambda)L|f(\boldsymbol{x}_0-\boldsymbol{x}_i)|p'q'(l'\lambda')L\rangle_j\phi_{\beta'}(p'q')\,,\end{aligned} \tag{5.8}$$

with

$$I_{ij}^{(\pm)}(tt'i) \equiv {}_1\langle(t\tfrac{1}{2})TM_T|\tfrac{1}{2}(1\pm\tau_3(i))|(t'\tfrac{1}{2})TM_T\rangle_j \tag{5.9}$$

$$S_{ij}(ss') \equiv {}_1\langle(s\tfrac{1}{2})SM_S|(s'\tfrac{1}{2})SM_S\rangle_j\,. \tag{5.10}$$

It is straightforward exercise to evaluate the momentum space matrix elements and to carry through the integrals in (5.1) with the result

$$\begin{aligned}F(\boldsymbol{Q}^2) = \frac{3}{4\pi}\sum_{LS}\sum_{l\lambda st}\sum_{l'\lambda's't'}\{\delta_{ss'}[(I_{11}^{(\pm)}(tt'1)\tilde{f}^{\mathrm{p}}(\boldsymbol{Q}^2)+I_{11}^{(-)}(tt'1)\tilde{f}^{\mathrm{n}}(\boldsymbol{Q}^2))X_1 \\ +2(I_{11}^{(+)}(tt'2)\tilde{f}^{\mathrm{p}}(\boldsymbol{Q}^2)+I_{11}^{(-)}(tt'2)\tilde{f}^{\mathrm{n}}(\boldsymbol{Q}^2))X_2] \\ +2S_{12}(ss')[(I_{12}^{(+)}(tt'1)\tilde{f}^{\mathrm{p}}(\boldsymbol{Q}^2)+I_{12}^{(-)}(tt'1)\tilde{f}^{\mathrm{n}}(\boldsymbol{Q}^2))X_3 \\ +(I_{12}^{(+)}(tt'2)\tilde{f}^{\mathrm{p}}(\boldsymbol{Q}^2)+I_{12}^{(-)}(tt'2)\tilde{f}^{\mathrm{n}}(\boldsymbol{Q}^2))X_4 \\ +(I_{12}^{(+)}(tt'3)\tilde{f}^{\mathrm{p}}(\boldsymbol{Q}^2)+I_{12}^{(-)}(tt'3)\tilde{f}^{\mathrm{n}}(\boldsymbol{Q}^2))X_5]\}\,,\end{aligned} \tag{5.11}$$

where the functions X_2–X_5 are given in appendix B. If one evaluates (5.1) in the standard manner just the expression of type X_1,

$$\begin{aligned}X_1 = 2\pi\delta_{ll'}\delta_{\lambda\lambda'}\sum_{k=0}^{\lambda}\int p^2\,\mathrm{d}pq^2\,\mathrm{d}q\phi_\beta(pq) \\ \times\int \mathrm{d}x\,p_k(x)\frac{\phi_\beta(pq_1)}{q_1^\lambda}q^{\lambda-k}(\tfrac{2}{3}\boldsymbol{Q})^k\frac{\lambda!}{k!(\lambda-k)!}\,,\end{aligned} \tag{5.12}$$

with

$$q_1 = \sqrt{q^2+\tfrac{4}{3}\boldsymbol{Q}^2+\tfrac{4}{3}q\boldsymbol{Q}x} \tag{5.13}$$

results. Then however, the ϕ_β are components of the total wave function and the sum over the β quantum numbers includes many more terms than the ones kept in the Faddeev component. Though our form is more complicated we consider it to be more satisfactory, since for a given Faddeev component it is exact and involves only finite auxiliary sums. The only exceptions occur in $X_{4,5}$ in which unrestricted sums occur, but the convergence is extremely fast within a few terms.

6. Results

We carried through this investigation on the basis of the Reid two-nucleon interaction which also serves as a standard for checking the accuracy of the complex codes. Without a three-body force we reproduce [1]) binding energies of −6.48 MeV and −7.02 MeV for a three- and five-channel calculation, respectively. Typical

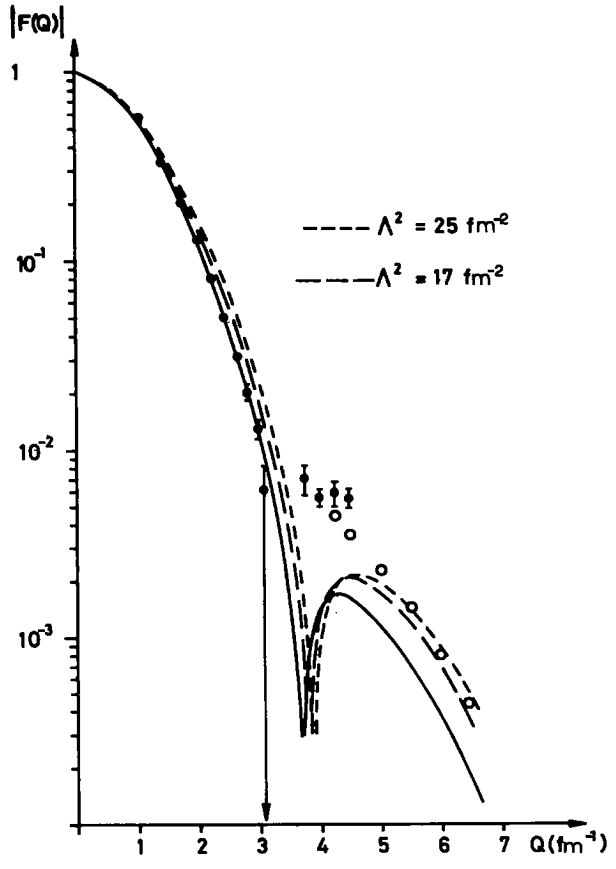

Fig. 3. The ^{3}He charge form factor. The experimental points are taken from ref. [23]) (closed circles) and from ref. [24]) (open circles). The unbroken and broken lines result from a 5-channel calculation without and with a three-body force, respectively.

parameters for the discretisations are $q_0 = 1\ \mathrm{fm}^{-1}$, $p_0 = 1\ \mathrm{fm}^{-1}$, number of q-points $N_q = 10$–12, number of p-points for the interval $0 \leq p \leq \frac{3}{2}q_{\max}$, $N_p = 20$–24, number of x-points $N_x = 6$. The dependence on the cut-off value $q_{\max}$ is negligible within 3 digits if $q_{\max} \geq 3.6$. The two-body Lippmann–Schwinger equation for the t-matrix was solved with a cut-off value of $p_{\max} = 20\ \mathrm{fm}^{-1}$. Between $p = \frac{3}{2}q_{\max}$ and $p_{\max}$ another 8 additional quadrature points were sufficient.

An up-to-date discussion of the trinucleon charge form factor without a three-nucleon force can be found in ref. [2]).

The charge form factor for a 5-channel calculation without a 3-body force is given in fig. 3. For the interesting secondary maximum we also show the result for a three-channel calculation without a 3-body force in fig. 4. As one can see, the inclusion of channels 4 and 5 give an important additional contribution. Also the height of the secondary maximum is sensitive to the choice of $q_{\max}$. It is only for $q_{\max} \geq 3.6$ that the height is independent of $q_{\max}$.

The calculation of the three-body force matrix elements are time consuming and computer restrictions forced us in this first investigation to restrict ourselves to the first three channels. In other words we assume that W_1 is active only in the first 3 channels, but we use that restricted force also in a five-channel calculation.

The function $H(Q^2)$ (4.12) arising from the πNN form factor depends on the cut-off mass Λ and regularises the short-range behaviour in the pair distances of particles (21) and (31) in the diagram of fig. 1. The form chosen for the form factor

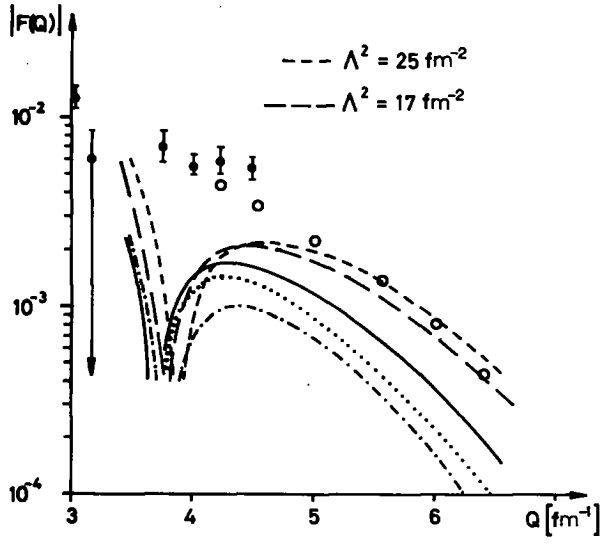

Fig. 4. The ^{3}He charge form factor. Experimental points as in fig. 3. The dashed dotted line: 3-channel calculation without W. The dotted and unbroken line: 5-channel calculations without W and choices of $q_{\max} = 3.0\ \mathrm{fm}^{-1}$ and $q_{\max} = 4$ or $6\ \mathrm{fm}^{-1}$, respectively. The broken lines 5-channel calculations with W and $q_{\max} = 6\ \mathrm{fm}^{-1}$.

is certainly dictated by simplicity and not physical insight which makes it absolutely necessary to check at least the dependence on the single parameter Λ.

The choice of $\Lambda \sim 5\ \mathrm{fm}^{-1}$ corresponds roughly to what is used in the OBEP [19]), whereas $\Lambda \sim 4\ \mathrm{fm}^{-1}$ is advocated [20]) by explaining the Goldberger–Treiman discrepancy. In that consideration the πNN form factor $K(q^2)$ should increase by about 3% from $Q^2 = 0$ to $Q^2 = \mu^2$. That requirement leads to $\Lambda = 4.12\ \mathrm{fm}^{-1}$ in the form $K(Q^2) = (\Lambda^2 - \mu^2)/(\Lambda^2 - Q^2)$, which we choose. Note however that this consideration is based on a very small neighbourhood of Q^2 around 0, whereas the short-range behaviour is governed by the behaviour of $H(\boldsymbol{Q}^2)$ at large $|\boldsymbol{Q}|$ values (moreover Q^2 is space-like).

A view on (4.12) tells us immediately that $H(\boldsymbol{Q}^2)$ decreases with decreasing Λ^2. Therefore a smaller Λ^2 should weaken the overall effect of W. This is also supported by a study [21]) which maps part of the three-body force in coordinate space for both Λ-values. Our results are shown in table 2.

Whereas for pure two-body forces the spectator particles feel only a kind of folding potential which is not violent at short distances like the two-nucleon force and therefore do not induce high momentum components in q, the three-nucleon force treats all three particles equally and the q-dependence of W is no longer so much different from the p-dependence. Thus the energy stabilises only around $q_{\max} \sim 6\ \mathrm{fm}^{-1}$.

As a numerical side remark we mention that the larger intervals in q and p required an increase in N_q and N_p to 12 and 24, respectively. It also leads to the occurrence of a negative eigenvalue of the Faddeev kernel with large magnitude and the necessity to eliminate the corresponding unphysical eigenstate as described in sect. 5.

Since we kept only the 3-channel × 3-channel coupling for W_1 also in the 5-channel calculation the similar gain in binding energy in both calculations (3 and 5 channels) is not too surprising. The two values $E = -8.96$ MeV for $\Lambda^2 = 25\ \mathrm{fm}^{-2}$ and $E = -8.30$ MeV for $\Lambda^2 = 17\ \mathrm{fm}^{-2}$ demonstrate the dependence on Λ but still stay within a reasonable neighbourhood of the experimental value of $E = -8.48$ MeV.

The calculated numbers may still be modified through the neglected partial-wave states of W_1. Further it is known [15]) that the neglected p-wave two-nucleon forces add another 100–200 keV to the binding energy.

As a warning we would like to point out that the three-body force (2.1) was derived through a p/m expansion. This expansion introduces powers of p in the numerator which in the original expression are not present and which may induce "spurious effects". Indeed virtually very high momenta occur in a nuclear wave function, for which p/m is not at all small with respect to 1.

In addition there are of course further diagrams which contribute to the real nuclear force. This has been recently pointed out in ref. [22]). Consistent with the OBEP three-nucleon forces with (π, ρ), (ρ, ρ), exchanges have to be considered,

too. In the estimate carried through in ref. [22]) they reduce the strong attraction of the 2π-exchange three-nucleon force.

From our first results one can say however that the 2π-exchange three-nucleon force is likely to yield an appreciable contribution to the binding energy of the three-nucleon system. One has also to expect corresponding effects for nuclei in general.

The numbers in table 2 we determined in keeping only the first-order contribution of W_1 in the T-operator, eq. (3.14). We carried through one three-channel calculation including the second-order term in W_1 for T, which increased the binding energy in the third digit by 3 units. This point however has to be checked more carefully with respect to the charge form factor in future work.

TABLE 2

The dependence on q_{max} of the theoretical triton binding energies for 5- and 3-channel calculations including a three-nucleon force

Λ^2	$25\ \mathrm{fm}^{-2}$		$17\ \mathrm{fm}^{-2}$
	E (MeV)		E (MeV)
q_{max}	5 channels	3 channels	5 channels
3.0	−8.05	−7.36	
4.0	−8.86	−7.82	
6.0	−8.96		−8.30

Let us now regard the charge form factor. It has been at the back of the minds of many people that a three-nucleon force may help to improve the serious discrepancy in that observable. We show in fig. 3 the charge form factor for ^{3}He for a five-channel calculation with the 3-body force also included. The two theoretical curves for $\Lambda^2 = 25\ \mathrm{fm}^{-2}$ and $\Lambda^2 = 17\ \mathrm{fm}^{-2}$ lie close together. We note that the theoretical curves are too high in the slope towards the first minimum. However for $Q \lesssim 1\ \mathrm{fm}^{-1}$ the agreement with the experimental values [23]) improves in comparison with the result without the three-nucleon forces. Thus the rms radius drops from $r = 2.05$ fm (without W) to $r = 1.85$ fm and $r = 1.90$ fm for $\Lambda^2 = 25\ \mathrm{fm}^{-2}$ and $\Lambda^2 = 17\ \mathrm{fm}^{-2}$ (with W) respectively. The experimental number is $r_{exp} = 1.88 \pm 0.05$ fm. As is shown in more detail in fig. 4 the position of the theoretical minimum is shifted a bit towards the wrong direction including W, but the situation improves above the secondary maximum. But clearly the sharp experimental rise towards the secondary maximum is not reproduced. The experimental numbers of ref. [24]) include magnetic contributions which for $Q \lesssim 6\ \mathrm{fm}^{-1}$ are completely negligible [25]).

In all calculations the underlying nucleon form factors are taken from ref. [26]).

Certainly the inclusion of a larger number of partial-wave matrix elements of W is needed in order to come to a conclusive result about the influence of the

2π-exchange three-nucleon force in the charge form factor. Judged from our preliminary results the effects can be appreciable. Of course perfect agreement should not be expected, since there should remain room for mesonic exchange-current contributions.

Clearly further numerical studies are necessary and existing codes are waiting to be run.

After finishing this work we received a preprint by the Grenoble group [27]). They solved the Faddeev equation in coordinate space, including a three-nucleon force. There the advantage of a coordinate space calculation based on local potentials and a nonrelativistic treatment shows up, which allows the inclusion of higher partial waves without running too fast into storage limits. So they could include up to 12 Faddeev components and moreover, solving the Faddeev equation in differential form meant that the three-nucleon force could be taken into account strictly nonperturbatively.. In the binding energy however little difference materialised between the expectation value of the force of −550 keV with respect to a 5-channel calculation and that more complete treatment which lead to −650 keV. Unfortunately only part of the force was kept and the effects are relatively small. They show however the same tendency.

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Appendix A

The geometrical coefficient in eq. (3.9):

$$G^{\alpha'\alpha}(q'qx)=\sum_k P_k(x)\sum_{l_1'+l_2'=l'}\sum_{l_1+l_2=l} q'^{l_2'+l_2}q^{l_1'+l_1}G^{\alpha'\alpha}, \tag{A.1}$$

$$G^{\alpha'\alpha}=-\sqrt{\hat{j}'\hat{J}'\hat{l}'\hat{\lambda}'\hat{s}'\hat{t}'}\,\sqrt{\hat{j}\hat{J}\hat{l}\hat{\lambda}\hat{s}\hat{t}}$$

$$\times\hat{k}\left(\tfrac{1}{2}\right)^{l_1+l_2}\sqrt{\frac{(2l+1)!}{(2l_1)!\,(2l_2)!}}\sqrt{\frac{(2l'+1)!}{(2l_1')!\,(2l_2')!}}$$

$$\times\begin{Bmatrix}\frac{1}{2} & \frac{1}{2} & t'\\ \frac{1}{2} & T & t\end{Bmatrix}\sum_{LS}\hat{L}\hat{S}\begin{Bmatrix}l' & s' & j'\\ \lambda' & \frac{1}{2} & J'\\ L & S & \mathcal{J}\end{Bmatrix}\begin{Bmatrix}l & s & j\\ \lambda & \frac{1}{2} & J\\ L & S & \mathcal{J}\end{Bmatrix}$$

$$\times \begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & s' \\ \frac{1}{2} & S & s \end{Bmatrix} \sum_{ff'} \begin{Bmatrix} \lambda & l_1 & f \\ l_2 & L & l \end{Bmatrix} (\lambda l_1 f, 00)$$

$$\times \begin{Bmatrix} l_1' & l_2' & l' \\ \lambda' & L & f' \end{Bmatrix} (l_2' \lambda' f', 00) \begin{Bmatrix} f' & l_1' & L \\ f & l_2 & k \end{Bmatrix}$$

$$\times (k l_1' f, 00)(k l_2 f', 00) \,. \tag{A.2}$$

Appendix B

The expressions X_2–X_5 in eq. (5.11) for the charge form factor are

$$X_2 = \pi \sqrt{\hat{l}\hat{\lambda}'} (-)^{l'+\lambda'+L} \sum_{k=0}^{l+l'} \sum_{k'=0}^{\lambda+\lambda'}$$

$$\times \int p^2 \, \mathrm{d}p q^2 \, \mathrm{d}q \int \mathrm{d}x P_k(x) \frac{\phi_\beta(p_1 q)}{p_1^l} \int \mathrm{d}y P_{k'}(y) \frac{\phi_{\beta'}(p q_1)}{q_1^{\lambda'}}$$

$$\times \sum_{l_1+l_2=l} p^{l_1} \sum_{\lambda_1'+\lambda_2'=\lambda'} q^{\lambda_1'} (\tfrac{1}{2}Q)^{l_2} (\tfrac{1}{3}Q)^{l\lambda_2'}$$

$$\times f(l l_1 l_2) f(\lambda' \lambda_1' \lambda_2') \hat{k}\hat{k}' (k l_1 l', 00)(k' \lambda_1' \lambda, 00)$$

$$\times \sum_f \begin{Bmatrix} f & l' & l \\ l_1 & l_2 & k \end{Bmatrix} (k l_2 f, 00) \begin{Bmatrix} f & \lambda & \lambda' \\ \lambda_1' & \lambda_2' & k' \end{Bmatrix} (k' \lambda_2' f, 00)$$

$$\times \begin{Bmatrix} l' & f & l \\ \lambda & L & \lambda' \end{Bmatrix}, \tag{B.1}$$

with

$$f(l l_1 l_2) = \sqrt{\frac{(2l+1)!}{(2l_1)!(2l_2)!}}, \tag{B.2}$$

$$\begin{aligned} p_1 &= \sqrt{p^2 + \tfrac{1}{4}Q^2 - pQx}\,, \\ q_1 &= \sqrt{q^2 + \tfrac{1}{9}Q^2 - \tfrac{2}{3}qQx}\,; \end{aligned} \tag{B.3}$$

$$X_3 = \pi \sqrt{\hat{l}'\hat{\lambda}'} (-)^{l'} \sum_{k=0}^{\lambda} \sum_{k'=0}^{l'+\lambda'+\min(l\lambda)}$$

$$\times \int p^2 \, \mathrm{d}p q^2 \, \mathrm{d}q \int \mathrm{d}x P_k(x) \frac{\phi_\beta(p q_1)}{q_1^{\lambda}} \int \mathrm{d}y P_{k'}(y) \frac{\phi_{\beta'}(p_2 q_2)}{p_2^{l'} q_2^{\lambda'}}$$

$$\times q^{\lambda-k} \sum_{l_1'+l_2'=l'} (\tfrac{1}{2}p)^{l_1'} (\tfrac{3}{4}q)^{l_2'} \sum_{\lambda_1'+\lambda_2'=\lambda'} p^{\lambda_1'} (\tfrac{1}{2}q)^{\lambda_2'}$$

$$\times (\tfrac{2}{3}Q)^k \frac{f(\lambda\lambda - kk)(k\lambda - k\lambda, 00)}{\sqrt{\hat{\lambda}}} (-)^k \hat{k}' (-)^{k'}$$

$$\times f(l'l_1'l_2')f(\lambda'\lambda_1'\lambda_2')(-)^{\lambda_2'}\sum_{f_1'f_2'}\begin{Bmatrix} l_1' & l_2' & l' \\ \lambda_1' & \lambda_2' & \lambda' \\ f_1' & f_2' & L \end{Bmatrix}$$

$$\times(l_1'\lambda_1'f_1',00)(l_2'\lambda_2'f_2',00)\sqrt{\hat{f}_1'\hat{f}_2'}$$

$$\times\begin{Bmatrix} \lambda & l & L \\ f_1' & f_2' & k' \end{Bmatrix}(k'f_1'l,00)(k'f_2'\lambda,00)(-)^{f_1'+f_2'+L}, \tag{B.4}$$

with

$$\begin{aligned} q_1&=\sqrt{q^2+\tfrac{4}{9}Q^2-\tfrac{4}{3}qQx}\,,\\ p_2&=\sqrt{\tfrac{1}{4}p^2+\tfrac{9}{16}q^2+\tfrac{3}{4}pqy}\,,\\ q_2&=\sqrt{p^2+\tfrac{1}{4}q^2-pqy}\,; \end{aligned} \tag{B.5}$$

$$X_4=\tfrac{1}{2}\pi\sqrt{\hat{l}\hat{\lambda}\hat{l}'\hat{\lambda}'}\sum_{k_1k_2k'=0}^{\infty}\int p^2\,\mathrm{d}p q^2\,\mathrm{d}q\int \mathrm{d}x P_{k_1}(x)$$

$$\times\int \mathrm{d}y P_{k_2}(y)\frac{\phi_\beta(p_1q_1)}{p_1^{l}q_1^{\lambda}}\int \mathrm{d}z P_{k'}(z)\frac{\phi_{\beta'}(p_2q_2)}{p_2^{l'}q_2^{\lambda'}}$$

$$\times\sum_{l_1+l_2=l}p^{l_1}\sum_{\lambda_1+\lambda_2=\lambda}q^{\lambda_1}\sum_{l_1'+l_2'=l'}(-\tfrac{1}{2}p)^{l_1'}(-\tfrac{3}{4}q)^{l_2'}$$

$$\times\sum_{\lambda_1'+\lambda_2'=\lambda'}p^{\lambda_1'}(-\tfrac{1}{2}q)^{\lambda_2'}(-\tfrac{1}{2}Q)^{l_2}(\tfrac{1}{3}Q)^{\lambda_2}\hat{k}_1\hat{k}_2\hat{k}'$$

$$\times(-)^{k'}f(ll_1l_2)f(\lambda\lambda_1\lambda_2)f(l'l_1'l_2')f(\lambda'\lambda_1'\lambda_2')$$

$$\times\sum_{f_1f_2}\hat{f}_1\begin{Bmatrix} l_1 & l_2 & l \\ \lambda_1 & \lambda_2 & \lambda \\ f_1 & f_2 & L \end{Bmatrix}(l_2\lambda_2f_2,00)(k_1k_2f_2,00)$$

$$\times\sum_{k_6k_7}\begin{Bmatrix} l_1 & \lambda_1 & f_1 \\ k_1 & k_2 & f_2 \\ k_6 & k_7 & L \end{Bmatrix}(l_1k_1k_6,00)(\lambda_1k_2k_7,00)$$

$$\times\sum_{f_1'f_2'}\begin{Bmatrix} l_1' & l_2' & l' \\ \lambda_1' & \lambda_2' & \lambda' \\ f_1' & f_2' & L \end{Bmatrix}(l_1'\lambda_1'f_1',00)(l_2'\lambda_2'f_2',00)$$

$$\times\sqrt{\hat{f}_1'\hat{f}_2'}\begin{Bmatrix} k_7 & k_6 & L \\ f_1' & f_2' & k' \end{Bmatrix}(k'f_1'k_6,00)(k'f_2'k_7,00)\,, \tag{B.6}$$

with

$$
\begin{aligned}
p_1 &= \sqrt{p^2 + \tfrac{1}{4}Q^2 - pQx}\,, \\
q_1 &= \sqrt{q^2 + \tfrac{1}{9}Q^2 + \tfrac{2}{3}qQy}\,, \\
p_2 &= \sqrt{\tfrac{1}{4}p^2 + \tfrac{9}{16}q^2 + \tfrac{3}{4}pqz}\,, \\
q_2 &= \sqrt{p^2 + \tfrac{1}{4}q^2 - pqz}\,.
\end{aligned}
\tag{B.7}
$$

In the expression for X_5 an additional phase $(-)^{k_1+l_2}$ occurs.

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