NLA PROJECT REPORT

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Here you can check our project out!!!

1 Introduction

Our project is based on the article "Enhanced image approximation using shifted rank-1 reconstruction". Low rank approximation is of great interest in many applications like object detection, video denoising, seismic data processing. Variety of methods exists, we are going to introduce approximation using shifting rank-1 matrices. These matrices are of the form $S_\lambda(uv^*)$ where $u \in C^M$,$v \in C^N$ and $\lambda \in Z^N$. These kind of shifts naturally appear in applications, where an object u is observed in N measurements at different positions indicated by the shift λ. The vector v gives the observation intensity. Furthermore, efficient algorithm to calculate shifted rank-1 approximation in $O(NM \log M)$ will be mentioned.

2 Team contribution

In the course of the research our team has solved some tasks. Firstly, all team members have analysed different articles according to the chosen theme. Based on it, was chosen the main vector of the work on the project. The roles of the team were divided in the following way;
- Vitaly was responsible for programming part, as we had only article without any existing code with it. He was managed to implement all algorithms mentioned in the article, apply it to different data types, obtain results, estimate errors and plot some valid graphics, test it on our own data (our selfie);
- Kristina parsed an article from scientific point, theory advising for team members, report and presentation making;
- Veronika - presentation and report corrections.

3 Task description and data construction

As we know from the course, rectangular data type presented as matrices $A \in C^{n\times m}$ can be approximated by low rank via SVD as it gives minimum to Frobenius norm

$$\|A - \sum_{k=1}^{L} \sigma_k (u^k (v^k)^*)\|_F \tag{1}$$

We seek to find an approximation of A using L rank-1 matrices $u^k(v^k)^*$ where the columns of each matrix can be shifted arbitrarily. $A \in \mathbb{C}^{M\times N}$, approximation matrices are of the form $S_\lambda(uv^*)$, where $u \in \mathbb{C}^M, v \in \mathbb{C}^N$ and $\lambda \in \mathbb{Z}^N$. The operator S_{λ^k} circularly shifts the k-th column of uv^* by λ_k. In our work we consider the following optimisation problem:

$$\min_{\lambda^1,\dots,\lambda^L u^1,\dots,u^L v^1,\dots,v^L} \|A - \sum_{k=1}^{L} S_{\lambda^k}(u^k (v^k)^*)\|_F \tag{2}$$

L denotes the number of shifted rank-1 matrices. The motivation behind this new approach is given by applications such as seismic data processing, where the given data A consists of several time signals recorded at different locations.

Assume each time signal records the same events (such as earthquakes or explosions) where each event is shifted according to the distance between source and sensor. Then an event can be written as such a shifted rank-1 matrix where u_k is the seismic wave, v_k is the intensity and λ_k gives the time of arrival at the different sensors for each event. This model also holds for other applications.

Shift operator. The shift operator S_λ. Let $\mathbb{I} \in \mathbb{C}^{(M-1)\times(M-1)}$ be the identity matrix and define

$$\tilde{S} := \begin{bmatrix} 0 & 1 \\ \mathbb{I} & 0 \end{bmatrix} \in \mathbb{C}^{M\times M}$$

,

$$\tilde{S}^k := \prod_{j=1}^{k} S$$

Denote the columns of the matrix A as $A = [a^1, ..., a^N]$.
For $\lambda \in \mathbb{Z}^N$ define the shift operator $S_\lambda : \mathbb{C}^{M\times N} \to \mathbb{C}^{M\times N}$ as

$$S_\lambda A := [\tilde{S}^{\lambda_1} a^1, ..., \tilde{S}^{\lambda_N} a^N]$$

S_λ is linear operator.

3.1 Algorithms.

It is hard to solve problem (2) for all variables at once $\to$ an iterative method is developed, where the number of unknowns on each step is drastically reduced. In each iteration only one of the L unknown shifted rank-1 matrices is recovered. Afterwards the data A is updated by subtracting the shifted rank-1 matrix. This process can be iterated L times to reconstruct all unknowns.
In each step we seek the solution of

$$\min_{\lambda,u,v} \|A - S_\lambda(uv^*)\|_F \tag{3}$$

Apply an inverse shift operator $S_{-\lambda}$ and using its linearity, we get:

$$\min_{\lambda,u,v} \|S_{-\lambda}(A) - uv^*\|_F$$

$$\|S_{-\lambda}(A) - uv^*\|_F^2 = \|A\|_F^2 - \|S_{-\lambda}(A)\|_2^2$$

Thus, we can find λ by solving:

$$\max_{\lambda\in\mathbb{Z}^N} \|S_{-\lambda}(A)\|_2^2 \tag{4}$$

Unfortunately, maximizing the spectral norm over a discrete set is still a hard problem. The shifted rank-1 approximation algorithm can be summarized as follows:

Algorithm 1:

Input: A, L
for $k = 1, \ldots, L$ **do**
  Solve (4) for λ^k
  Calculate u^k, v^k using SVD of $S_{-\lambda}(A)$
  Update $A \leftarrow A - S_\lambda(u^k(v^k)^*)$
end for
return λ^k, u^k, v^k for $k = 1, \ldots, L$

Given A the algorithm can extract L unique features that can be described as shifted rank-1 matrices.

One of possible applications. Seismic exploration seeks for subsurface deposits of e.g., gas or oil. Therefore, an artificial seismic wave is generated and multiple sensors are placed along the testing area. In this setup earth layers play an important role as their boundaries reflect the seismic wave. The reflected wave varies depending on the material of the layer. Moreover, the time of arrival of the reflection at different sensors depends on the distance to the layer. A decomposition of the seismic image as given by our algorithm can be useful. The shift vectors λ_k give information about the position of each layer while the seismic wave u_k (and also v_k) contain information about the material.

3.2 Reconstruction of λ.

A naive approach to solve this problem leads to exponential run time.
In Fourier domain some useful properties can be obtained. First, problem (4) can be relaxed in Fourier domain. The relaxed problem has no adequate formulation in time domain but its maximum can be described easily. Second, performing the algorithm in Fourier domain decreases the run time to $O(NM\log M)$ since it basically evaluates cross-correlations that transfer to element-wise multiplications in Fourier domain. Another helpful inequality: rank inequality of the Hadamart product.

$$rank(A) = rank(\hat{A}) = rank(\hat{A} \circ P_{-\lambda}P_{\lambda}) \leq rank(\hat{A} \circ P_{-\lambda})rank(P_{\lambda}) \leq rank(S_{-\lambda}A)rank(P_{\lambda}) \tag{5}$$

Let u be the first left singular vector of A. Suppose we have given a shift vector λ such that

$$\|uA\|_2 \leq \|uS_{-\lambda(A)}\|_2 \tag{6}$$

holds. Then it follows that:

$$\|uS_{-\lambda(A)}\|_2 - \|uA\|_2 = \sum_{k=1}^{N} |u^* \tilde{S}^{-\lambda_k} a^k|^2 - |u^* a^k|^2 \tag{7}$$

Hence, the spectral norm increases.
Updating all entries of λ at once leads to an unstable method. Hence, we solve the following problem.

$$\max_{k,\lambda_k} |u^* \tilde{S}^{-\lambda_k} a^k|^2 - |u^* a^k|^2 \tag{8}$$

If the maximum equal to zero $\rightarrow$ that no choice of λ_k for any column k can increase the spectral norm and thus we converged to a local maximum. If the maximum is > 0. Let k and λ_k be the position where the maximum is achieved. Then shifting the k-th column by -λ_k will increase the spectral norm. We update A accordingly, calculate the new first left singular vector u and repeat this procedure until ending in a local maximum. This is guaranteed to happen after a finite number of steps since there are only finitely many possibilities for λ_k.
Define $B = |F^{-1}(\hat{u} \circ \hat{A})|2$ and let b_1 be its first row. Then problem can be written as finding the maximum of $B - Ib_1$.

Algorithm 2:

Input: A
Set $s = 1$
while $s \neq 0$ **do**
  Calculate $\hat{u}$ the first left singular vector of $\hat{A}$
  Set $B = |F^{-1}(\bar{\hat{u}} \odot \hat{A})|^2$
  Subtract the first row of B from every row
  Get the position (k, s) of the maximum in B
  Shift the k-th column of A by $-s$
  Update $\lambda_k \leftarrow \lambda_k + s$
end while
return λ

Remark Algorithm 2 requires the computation of element-wise multiplications, (inverse) Fourier transforms and the calculation of the first left singular vector of A. The multiplications scale linear in M and N, the inverse Fourier

transform can be calculated in $O(MN \log M)$. The first left singular vector: we can use the previous singular vector as a starting guess for any iterative method, e.g., power iteration. Since the matrix was only shifted in one column, this will be a very good guess. Altogether, the local optimization has a complexity of $O(MNlogM)$ per iteration.The Fourier transforms only need to be calculated once. Shifting the matrix A can also be implemented in Fourier domain by multiplying with a phase shift.
The choice of the starting guess in Algorithm2 is crucial.
Problem (4) in Fourier domain

$$\max_{\lambda \in \mathbb{Z}^N} \|\hat{A} \circ P_{-\lambda}\|_2^2 \tag{9}$$

, where $P_{-\lambda}$ is a phase matrix.

Theorem. It holds that

$$\||\hat{A}|\|_2^2 = max_{P,|p_{jk}|=1} \|\hat{A} \circ P\|_2^2 \tag{10}$$

Let u^{opt} be the first left singular vector of $|\hat{A}|$ and $\hat{u}^p$ the first left singular vector of $\hat{A} \circ P$. Then $\|\hat{A} \circ P\|_2 = \||\hat{A}|\|_2 \Rightarrow |\hat{u}^p| = |\hat{u}^opt|$ Furthermore, if $A = S_\lambda(uv^*)$ is a shifted rank-1 matrix, then

$$|\hat{u}^p| = |\hat{u}| = |\hat{u}^opt| \tag{11}$$

Theorem gives us the maximum value of the relaxed problem. Taking a starting guess close to the necessary condition of the relaxed problem and with a large first singular value, we may be close to a global optimum of relaxed problem. To force the largest singular value towards the necessary conditions we add an additional step to integrate the optimal amplitude. Update for u:

$$\hat{u} \leftarrow |\hat{u}^{opt}| \circ phase(\hat{u}) \tag{12}$$

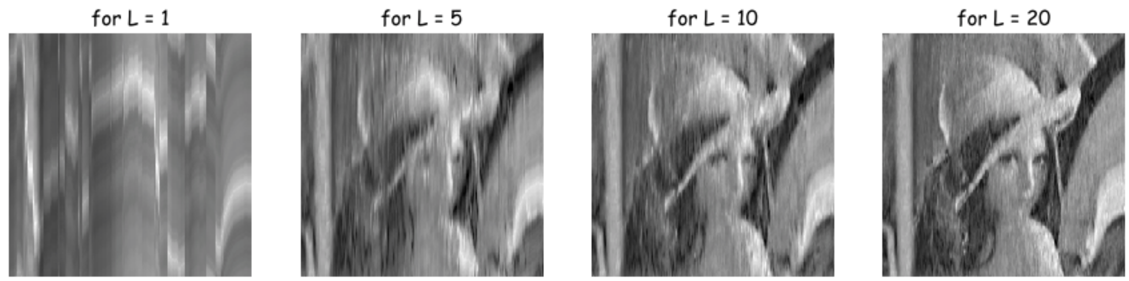
Algorithm 3:
1: Input: A
2: Calculate $B = |F^{-1}(|\hat{A}| \odot \hat{A})|^2$
3: Set λ to the indices of the maximum values of B in each column and update $A \leftarrow S_{-\lambda}A$
4: Update λ and A using Algorithm 2 with (12)
5: Update λ and A using Algorithm 2
6: **return** λ

Algorithm 3 derives the shift λ in three steps. First, a starting guess is calculated (Line 3). Afterwards, it tries to approximate a global solution in Line 4. Last, local optimization is applied.

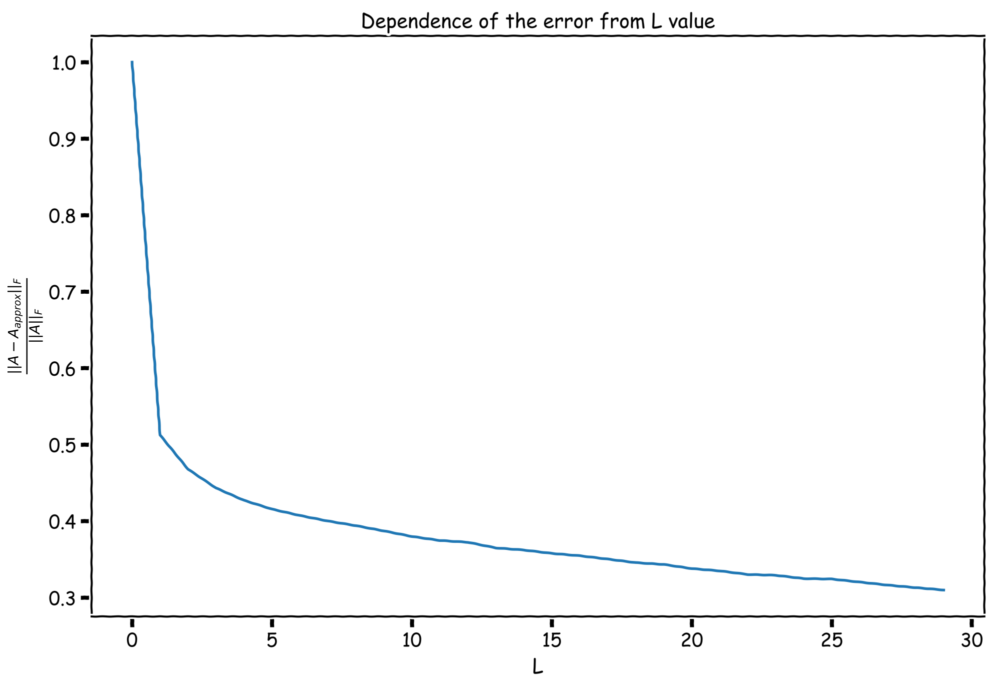
4 Obtained results.

We take a Lenna picture:

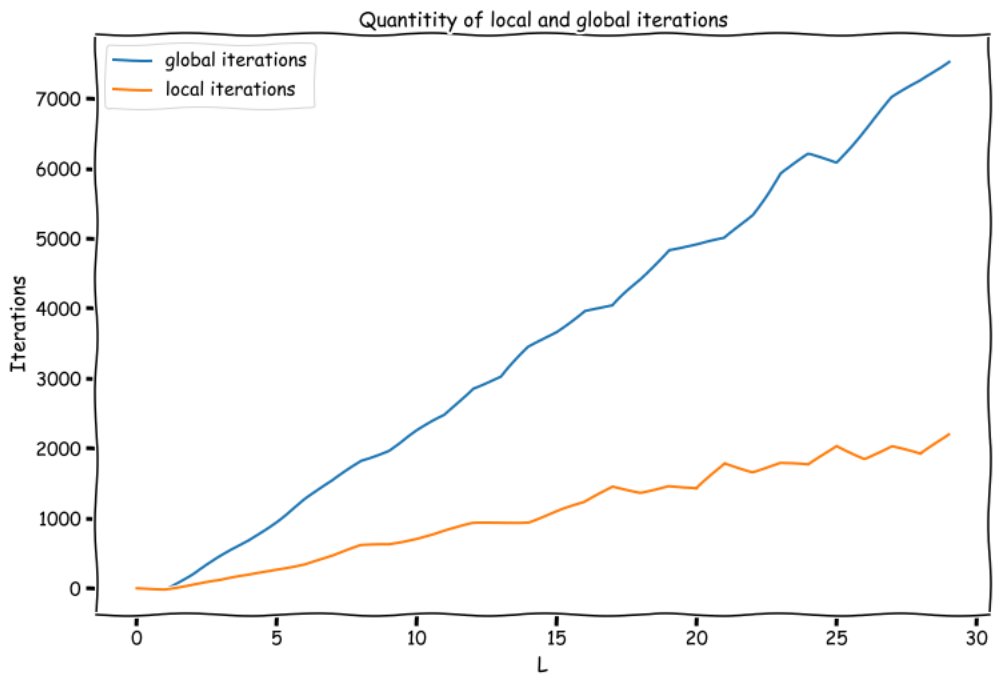
Applied described above algorithm to it we get as a result for different values of L (number of matrices in approximation) several pictures. One can notice that quality of approximation pictures raise with larger value of L.



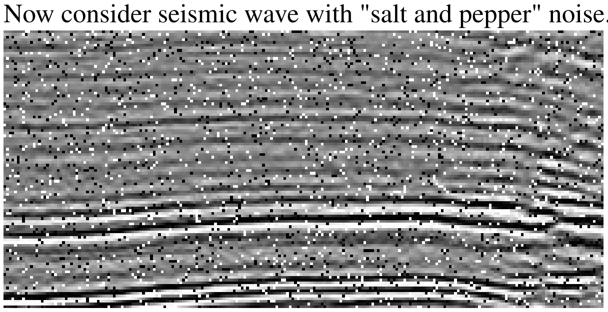
Estimations. Average approximation error over "lena.png" data.



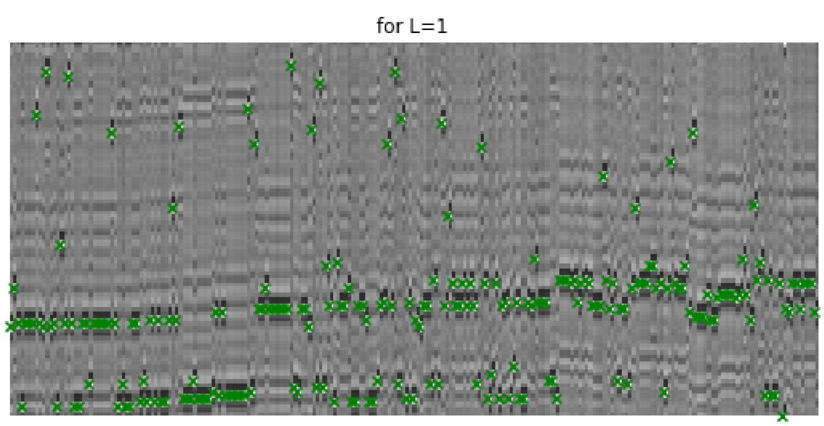
Global approximation greatly improve the results. Local optimization only causes a minor improvement.The calculated shift is already quite optimal such that local optimization terminates after comparatively less iterations.



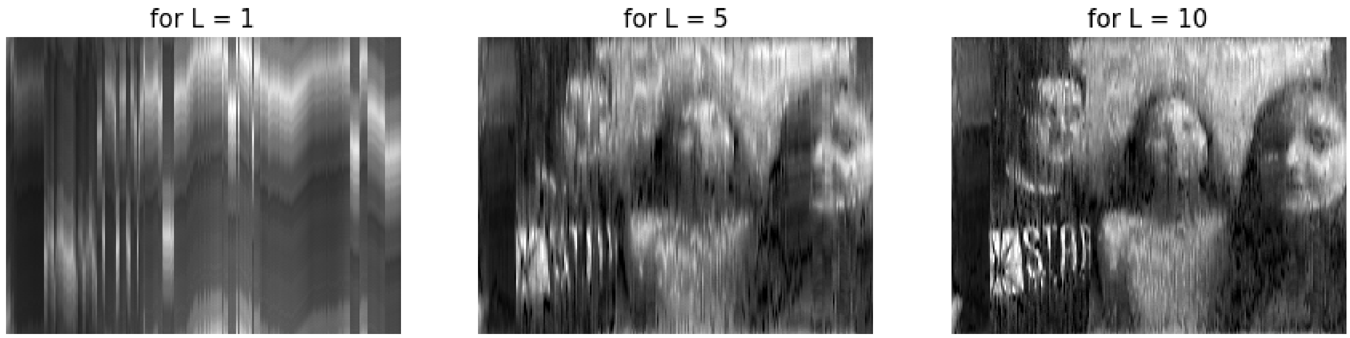
Now consider seismic wave with "salt and pepper" noise.



We use our algorithm with L = 1 to identify earth layer reflection at the bottom of the image. The shift vector λ_1 is plotted on bottom of the images to indicate the identified earth layer. So, this approach helps separate parts on the picture.



Last but not least test was of restoring our selfie. As we can see from that and previous examples, by shifting the columns we are able to follow horizontal structures in the image. Vertical structures such as the hair cannot be reconstructed in such detail using column shifts.



5 Runtime and Storage.

Storing one shifted rank-1 matrix requires M + N doubles and N integers. In Remark we stated that SR1 ("shift + rank-1") has a runtime of $O(LMN \log M)$.

5.1 Conclusion and future work

- On the base of rank-1 approximation with shifts we managed to recover the image with different quality depending on the L - number of matrices of rank - 1.
- We note that SR1 seismic images is perfect performing method. For these type of images the shifted rank-1 model fully applies.
- We were able to make 'object detection'. Applied our approach to image of seismic wave we could separate the main object and the background.
- We have validated this method on our own data - the photo of our team. According to the label on the Vitaly's T-shirt we can see how the quality of recovering increases from L = 1 to L = 10.
- We only need N + M + N entries to store one shifted rank-1 matrix

We presented a generalization of the low-rank approximation, which allows to individually shift the column of rank-1 matrices. This model was designed to represent objects that move through the data. Basic properties of the occurring shift operators and shifted rank-1 matrices were stated. The new approach comes with some disadvantages, that in original paper were mentioned. The sparse approximation is slightly worse than comparable methods. Big advantages of this model are the extracted parameters. It has been shown that the obtained parameters can directly be used to extract crucial information. In the future method can be improved as it is not better than others for many types of problems. For example, smoothness of the shift vector will be extremely useful. For more applications object detection on video can be mentioned, the authors of the article were managed to reach some results, but not perfect.

References

[1] Bofsman and Ma. Enhanced image approximation using shifted rank-1 reconstruction Harbin Institute of Technology, China. IEEE, 2019. Link: https://arxiv.org/pdf/1810.01681.pdf