HYERS–ULAM–RASSIAS STABILITY OF GENERALIZED DERIVATIONS

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Abstract. The generalized Hyers–Ulam–Rassias stability of generalized derivations on unital Banach algebras into Banach bimodules is established.

1. Introduction

One of interesting questions in the theory of functional equations is the problem of the stability of functional equations as follows: "When is it true that a mapping satisfying a functional equation approximately must be close to an exact solution of the given functional equation?"

The first stability problem was raised by S. M. Ulam during his talk at University of Wisconsin in 1940 [18]:

Given a group G_1, a metric group (G_2, d) and a positive number ε, does there exist a $\delta > 0$ such that if a mapping $f : G_1 \to G_2$ satisfies the inequality $d(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G_1$ then there exists a homomorphism $T : G_1 \to G_2$ such that $d(f(x), T(x)) < \varepsilon$ for all $x \in G_1$.

Ulam's problem was partially solved by D. H. Hyers in 1941 in the context of Banach spaces with $\delta = \varepsilon$ as the following [7]:

Suppose that E_1, E_2 are Banach spaces and $f : E_1 \to E_2$ is a mapping for which there exists $\varepsilon > 0$ such that $\|f(x + y) - f(x) - f(y)\| < \varepsilon$ for all $x, y \in E_1$. Then there is a unique additive mapping $T : E_1 \to E_2$ defined by $Tx = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$ such that $\|f(x) - T(x)\| < \varepsilon$ for all $x \in E_1$.

Now assume that E_1 and E_2 are real normed spaces with E_2 complete, $f : E_1 \to E_2$ is a mapping such that for each fixed $x \in E_1$ the mapping $t \mapsto f(tx)$ is continuous on $\mathbb{R}$, and let there exist $\varepsilon \geq 0$ and $p \neq 1$ such that

$$\|f(x + y) - f(x) - f(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p)$$

for all $x, y \in E_1$.

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It was shown by Th. M. Rassias [15] for $p \in [0,1)$ (and indeed $p < 1$) and Z. Gajda [4] following the same approach as in [15] for $p > 1$ that there exists a unique linear map $T : E_1 \to E_2$ such that

$$\|f(x) - T(x)\| \leq \frac{2\varepsilon}{|2^p - 2|}\|x\|^p$$

for all $x \in E_1$. This phenomenon is called *Hyers–Ulam–Rassias stability*. It is shown that there is no analogue of Th. M. Rassias result for $p = 1$ (see [4, 17]).

In 1992, a generalization of the Rassias theorem was obtained by Găvruta as follows [5]:

Suppose $(G, +)$ is an abelian group, E is a Banach space and the so-called admissible control function $\varphi : G \times G \to [0, \infty)$ satisfies

$$\widetilde{\varphi}(x,y) := \frac{1}{2}\sum_{n=0}^{\infty} 2^{-n}\varphi(2^n x, 2^n y) < \infty$$

for all $x, y \in G$. If $f : G \to E$ is a mapping with

$$\|f(x+y) - f(x) - f(y)\| \leq \varphi(x,y)$$

for all $x, y \in G$, then there exists a unique mapping $T : G \to E$ such that $T(x+y) = T(x) + T(y)$ and $\|f(x) - T(x)\| \leq \widetilde{\varphi}(x,x)$ for all $x, y \in G$.

Since then several stability problems of various functional equations have been investigated by many mathematicians. The reader is referred to [3, 16] for a comprehensive account of the subject.

Generalized derivations were first appeared in the context of operator algebras [8]. Later, these were introduced in the framework of pure algebra [6]. There is also other generalization of the notion of derivation which is called (σ, τ)-derivation; cf. [9].

Let $\mathcal{A}$ be an algebra and let $\mathcal{X}$ be an $\mathcal{A}$-bimodule. A linear mapping $\mu : \mathcal{A} \to \mathcal{X}$ is called a *generalized derivation* if there exists a derivation (in the usual sense) $\delta : \mathcal{A} \to \mathcal{X}$ such that $\mu(ab) = a\mu(b) + \delta(a)b$ for all $a, b \in \mathcal{A}$. Familiar examples are the derivations from $\mathcal{A}$ to $\mathcal{X}$ and all so-called inner generalized derivations i.e. those are defined by $\mu_{x,y}(a) = xa - ay$ for fixed arbitrary elements $x, y \in \mathcal{X}$. Moreover, every right multiplier (i.e., an additive map h of $\mathcal{A}$ satisfying $h(xy) = h(x)y$ for all $x, y \in \mathcal{A}$) is a generalized derivation.

The stability of derivations was studied by C.-G. Park in [13, 14]. A discussion of stability of the so-called $(\sigma - \tau)$-derivations and a study of the so-called generalized (θ, ϕ)-derivations are given in [10] and [1], respectively. The present paper is devoted to study of stability of generalized derivations. The results of this paper are a generalization of those of Park's papers [13, 14].

Throughout the paper A denotes a unital normed algebra with unit 1 and $\mathcal{X}$ is a unit linked Banach $\mathcal{A}$-bimodule in the sense that $1x = x1 = x$ for all $x \in \mathcal{X}$.

2. Main Results.

Our aim is to establish the generalized Hyers–Ulam–Rassias stability of generalized derivations. We extend main results of C.-G. Park [13] to generalized derivations from a unital normed algebra to a unit linked Banach $\mathcal{A}$-bimodule. We apply the direct method which was first devised by D. H. Hyers [7] to construct an additive function from an approximate one and use some ideas of [10] and [14].

Theorem 2.1. *Suppose $f : \mathcal{A} \to \mathcal{X}$ is a mapping with $f(0) = 0$ for which there exist a map $g : \mathcal{A} \to \mathcal{X}$ and a function $\varphi : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \times \mathcal{A} \to [0, \infty)$ such that*

$$\widetilde{\varphi}(a, b, c, d) := \frac{1}{2}\sum_{n=0}^{\infty} 2^{-n}\varphi(2^n a, 2^n b, 2^n c, 2^n d) < \infty \tag{2.1}$$

$$\|f(\lambda a + \lambda b + cd) - \lambda f(a) - \lambda f(b) - cf(d) - g(c)d\| \leq \varphi(a, b, c, d) \tag{2.2}$$

for all $\lambda \in \mathbb{T} = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ and all $a, b, c, d \in \mathcal{A}$. Then there exists a unique generalized derivation $\mu : \mathcal{A} \to \mathcal{X}$ such that

$$\|f(a) - \mu(a)\| \leq \widetilde{\varphi}(a, a, 0, 0) \tag{2.3}$$

for all $a \in \mathcal{A}$.

Proof. Setting $c = d = 0$ and $\lambda = 1$ in (2), we have

$$\|f(a + b) - f(a) - f(b)\| \leq \varphi(a, b, 0, 0), \tag{2.4}$$

for all $a, b \in \mathcal{A}$. Now we use the Th. M. Rassias method on inequality (2.4) (see [5] and [11]). One can use induction on n to show that

$$\|\frac{f(2^n a)}{2^n} - f(a)\| \leq \frac{1}{2}\sum_{k=0}^{n-1} 2^{-k}\varphi(2^k a, 2^k a, 0, 0) \tag{2.5}$$

for all $n \in \mathbb{N}$ and all $a \in \mathcal{A}$, and that

$$\|\frac{f(2^n a)}{2^n} - \frac{f(2^m a)}{2^m}\| \leq \frac{1}{2}\sum_{k=m}^{n-1} 2^{-k}\varphi(2^k a, 2^k a, 0, 0)$$

for all $n > m$ and all $a \in \mathcal{A}$. It follows from the convergence (2.1) that the sequence $\{\frac{f(2^n a)}{2^n}\}$ is Cauchy. Due to the completeness of $\mathcal{X}$, this sequence is convergent. Set

$$\mu(a) := \lim_{n\to\infty} \frac{f(2^n a)}{2^n} \tag{2.6}$$

Putting $c=d=0$ and replacing a,b by $2^na, 2^nb$, respectively, in (2.2), we get

$$\|2^{-n}f(2^n(\lambda a+\lambda b))-2^{-n}\lambda f(2^na)-2^{-n}\lambda f(2^nb)\|\leq 2^{-n}\varphi(2^na,2^nb,0,0).$$

Taking the limit as $n\to\infty$ we obtain

$$\mu(\lambda a+\lambda b)=\lambda\mu(a)+\lambda\mu(b), \tag{2.7}$$

for all $a,b\in\mathcal{A}$ and all $\lambda\in\mathbb{T}$.

Next, let $\gamma=\theta_1+\mathbf{i}\theta_2\in\mathbb{C}$ where $\theta_1,\theta_2\in\mathbb{R}$. Let $\gamma_1=\theta_1-[\theta_1], \gamma_2=\theta_2-[\theta_2]$. Then $0\leq\gamma_i<1,\ (1\leq i\leq 2)$ and by using Remark 2.2.2 of [12] one can represent γ_i as $\gamma_i=\frac{\lambda_{i,1}+\lambda_{i,2}}{2}$ in which $\lambda_{i,j}\in\mathbb{T},\ (1\leq i,j\leq 2)$. Since μ satisfies (2.7) we infer that

$$\begin{aligned}
\mu(\gamma x) &= \mu(\theta_1x)+\mathbf{i}\mu(\theta_2x)\\
&= [\theta_1]\mu(x)+\mu(\gamma_1x)+\mathbf{i}\big([\theta_2]\mu(x)+\mu(\gamma_2x)\big)\\
&= \big([\theta_1]\mu(x)+\frac{1}{2}\mu(\lambda_{1,1}x+\lambda_{1,2}x)\big)+\mathbf{i}\big([\theta_2]\mu(x)+\frac{1}{2}\mu(\lambda_{2,1}x+\lambda_{2,2}x)\big)\\
&= \big([\theta_1]\mu(x)+\frac{1}{2}\lambda_{1,1}\mu(x)+\frac{1}{2}\lambda_{1,2}\mu(x)\big)+\mathbf{i}\big([\theta_2]\mu(x)+\frac{1}{2}\lambda_{2,1}\mu(x)+\frac{1}{2}\lambda_{2,2}\mu(x)\big)\\
&= \theta_1\mu(x)+\mathbf{i}\theta_2\mu(x)\\
&= \gamma\mu(x).
\end{aligned}$$

for all $x\in\mathcal{A}$. So μ is $\mathbb{C}$-linear.

Moreover, it follows from (2.5) and (2.6) that $\|f(a)-\mu(a)\|\leq\widetilde{\varphi}(a,a,0,0)$ for all $a\in\mathcal{A}$. It is known that additive mapping μ satisfying (2.3) is unique [2].

Putting $\lambda=0$, $x=y=0$ and replacing c,d by $2^nc, 2^nd$, respectively, in (2.2) we obtain

$$\|f(2^{2n}cd)-2^ncf(2^nd)-2^ng(2^nc)d\|\leq\varphi(0,0,2^nc,2^nd),$$

whence

$$\|2^{-2n}f(2^{2n}cd)-2^{-n}cf(2^nd)-2^{-n}g(2^nc)d\|\leq 2^{-2n}\varphi(0,0,2^nc,2^nd) \tag{2.8}$$

Put $d=1$ in (2.8). By (2.6), $\lim\limits_{n\to\infty}2^{-2n}f(2^{2n}a)=\mu(a)$ and by the convergence of series (2.1), $\lim\limits_{n\to\infty}2^{-2n}\varphi(0,0,2^nc,2^nd)=0$. Hence the sequence $\{2^{-n}g(2^nc)\}$ is convergent. Set $\delta(c):=\lim\limits_{n\to\infty}2^{-n}g(2^nc),\ c\in\mathcal{A}$. Let n tend to ∞ in (2.8),. Then

$$\mu(cd)=c\mu(d)+\delta(c)d \tag{2.9}$$

Next we claim that δ is a derivation. Put $d = 1$ in (2.9). Then $\delta(c) = \mu(c) - c\mu(1)$. Hence δ is linear. Further,

$$\begin{aligned}
\delta(c_1c_2) &= \mu(c_1c_2) - c_1c_2\mu(1) \\
&= (c_1\mu(c_2) + \delta(c_1)c_2) - c_1c_2\mu(1) \\
&= c_1\mu(c_2) + (\mu(c_1) - c_1\mu(1))c_2 - c_1c_2\mu(1) \\
&= c_1(\mu(c_2) - c_2\mu(1)) + (\mu(c_1) - c_1\mu(1))c_2 \\
&= c_1\delta(c_2) + \delta(c_1)c_2.
\end{aligned}$$

Thus δ satisfies Leibnitz' rule. It then follows from (2.9) that μ is a generalized derivation. □

Remark 2.2. The significance of functional equation (2.2) is that the required derivation δ is naturally constructed. In other words, we do not need any additional functional inequality for existence of δ.

Remark 2.3. Due to $\mathcal{A}$ is unital, the mapping δ appeared in the definition of generalized derivation is unique. In fact, $\delta(a) = \mu(a) - a\mu(1)$.

Corollary 2.4. *Suppose that $f : \mathcal{A} \to \mathcal{X}$ is a mapping with $f(0) = 0$ for which there exist constants $\beta \geq 0$ and $p < 1$ such that*

$$\|f(\lambda a + \lambda b + cd) - \lambda f(a) - \lambda f(b) - cf(d) - g(c)d\| \leq \beta(\|a\|^p + \|b\|^p + \|c\|^p + \|d\|^p)$$

for all $\lambda \in \mathbb{T}$ and all $a, b, c, d \in \mathcal{A}$.
Then there is a unique generalized derivation $\mu : \mathcal{A} \to \mathcal{X}$ such that

$$\|f(a) - \mu(a)\| \leq \frac{\beta\|a\|^p}{1 - 2^{p-1}}$$

for all $a \in \mathcal{A}$.

Proof. Put $\varphi(a, b, c, d) = \beta(\|a\|^p + \|b\|^p + \|c\|^p + \|d\|^p)$ in Theorem 2.1. □

Proposition 2.5. *Suppose that $f : \mathcal{A} \to \mathcal{X}$ is a mapping with $f(0) = 0$ for which there exists a function $\varphi : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \times \mathcal{A} \to [0, \infty)$ such that*

$$\widetilde{\varphi}(a, b, c, d) := \frac{1}{2}\sum_{n=0}^{\infty} 2^{-n}\varphi(2^n a, 2^n b, 2^n c, 2^n d) < \infty$$

$$\|f(\lambda a + \lambda b + cd) - \lambda f(a) - \lambda f(b) - cf(d) - g(c)d\| \leq \varphi(a, b, c, d)$$

for $\lambda = 1, \mathbf{i}$ and for all $a, b, c, d \in \mathcal{A}$. If for each fixed $a \in \mathcal{A}$ the function $t \mapsto f(ta)$ is continuous on $\mathbb{R}$ then there exists a unique generalized derivation $\mu : \mathcal{A} \to \mathcal{X}$ such that $\|f(a) - \mu(a)\| \leq \widetilde{\varphi}(a, a, 0, 0)$ for all $a \in \mathcal{A}$.

Proof. Put $c = d = 0$ and $\lambda = 1$ in (2.2). It follows from the proof of Theorem 2.1 that there exists a unique additive mapping $\mu : \mathcal{A} \to \mathcal{X}$ given by $\mu(a) = \lim_{n\to\infty} \frac{f(2^n a)}{2^n}, a \in \mathcal{A}$. By the same reasoning as in the proof of the main theorem of [15], the mapping μ is $\mathbb{R}$-linear.

Assuming $b = c = d = 0$ and $\lambda = \mathbf{i}$, it follows from (2.2) that $\|f(\mathbf{i}a) - \mathbf{i}f(a)\| \leq \varphi(a, 0, 0, 0), a \in \mathcal{A}$. Hence $\frac{1}{2^n}\|f(2^n\mathbf{i}a) - \mathbf{i}f(2^n a)\| \leq \varphi(2^n a, 0, 0, 0)$ for all $n \in N$ and $a \in \mathcal{A}$. The right hand side tends to zero as $n \to \infty$, so that

$$\begin{aligned} \mu(\mathbf{i}a) &= \lim_{n\to\infty} \frac{f(2^n\mathbf{i}a)}{2^n} \\ &= \lim_{n\to\infty} \frac{if(2^n a)}{2^n} \\ &= \mathbf{i}\mu(a) \end{aligned}$$

for all $a \in \mathcal{A}$. For each $\lambda \in \mathbb{C}, \lambda = r_1 + \mathbf{i}r_2$ $(r_1, r_2 \in \mathbb{R})$. Hence

$$\begin{aligned} \mu(\lambda a) &= \mu(r_1 a + \mathbf{i}r_2 a) = r_1\mu(a) + r_2\mu(\mathbf{i}a) \\ &= r_1\mu(a) + \mathbf{i}r_2\mu(a) = (r_1 + \mathbf{i}r_2)\mu(a) \\ &= \lambda\mu(a). \end{aligned}$$

Thus μ is $\mathbb{C}$-linear. That μ is a generalized derivation can be deduced in the same fashion as in the proof of Theorem 2.1. □

Proposition 2.6. *Let $\mathcal{A}$ be a unital C^*-algebra. Suppose that $f : \mathcal{A} \to \mathcal{X}$ is a mapping with $f(0) = 0$ for which there exists a function $\varphi : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \times \mathcal{A} \to [0, \infty)$ such that*

$$\widetilde{\varphi}(a, b, c, d) := \frac{1}{2}\sum_{n=0}^{\infty} 2^{-n}\varphi(2^n a, 2^n b, 2^n c, 2^n d) < \infty$$

$$\|f(\lambda a + \lambda b + cd) - \lambda f(a) - \lambda f(b) - cf(d) - g(c)d\| \leq \varphi(a, b, c, d)$$

$$\|f(2^n u^*) - f(2^n u)^*\| \leq \varphi(2^n u, 2^n u, 0, 0) \tag{2.10}$$

for all $\lambda \in \mathbb{T}$, all $a, b, c, d \in \mathcal{A}$, all nonnegative integers n and all unitaries u in $\mathcal{A}$. Then there exists a unique generalized derivation $\mu : \mathcal{A} \to \mathcal{X}$ such that $\|f(a) - \mu(a)\| \leq \widetilde{\varphi}(a, a, 0, 0)$ for all $a \in \mathcal{A}$.

Proof. It follows from the proof of Theorem 2.1 that there exists a unique generalized derivation $\mu : \mathcal{A} \to \mathcal{X}$ given by $\mu(a) = \lim_{n\to\infty} \frac{f(2^n a)}{2^n}, a \in \mathcal{A}$ satisfying (2.3).
Using (2.10), we have

$$\|2^{-n}f(2^n u^*) - 2^{-n}f(2^n u)^*\| \leq 2^{-n}\varphi(2^n u, 2^n u, 0, 0).$$

Letting $n \to \infty$ we conclude that $\mu(u^*) = \mu(u)^*$. Since μ is linear and every element of a C^*-algebra can be represented as a linear combination of unitaries [12], we deduce that $\mu(a^*) = \mu(a)^*$. $\square$

Now let $\mathcal{A}$ be a unital Banach algebra. The mapping $f : \mathcal{A} \to \mathcal{A}$ is called an *approximately generalized derivation* if $f(0) = 0$ and there exist a positive number ε and a mapping $g : \mathcal{A} \to \mathcal{A}$ such that

$$\|f(\lambda a + \lambda b + cd) - \lambda f(a) - \lambda f(b) - cf(d) - g(c)d\| \leq \varepsilon$$

for all $\lambda \in \mathbb{T}$ and all $a, b, c, d \in \mathcal{A}$.

Theorem 2.7. *Let $\mathcal{A}$ be a unital Banach algebra and $f : \mathcal{A} \to \mathcal{A}$ be an approximately generalized derivation with the corresponding mapping g. Then f is a generalized derivation and g is a derivation.*

Proof. Put $\varphi(a, b) = \varepsilon$ in Theorem 2.1. Then we get a generalized derivation μ defined by $\mu(a) := \lim_{n\to\infty} \frac{f(2^n a)}{2^n}$ such that

$$\|\mu(a) - f(a)\| \leq \varepsilon$$

for all $a \in \mathcal{A}$. We have

$$\begin{aligned}
\|2^n(f(2^m a) - 2^m f(a))\| &\leq \|2^n 1 f(2^m a) - g(2^n 1)2^m a - f((2^n 1)(2^m a))\| \\
&\quad + \|f((2^n 1)(2^m a)) - g(2^n 1)2^m a - 2^{n+m} 1 f(a)\| \\
&\leq \varepsilon + \|f((2^n 1)(2^m a)) - g(2^n 1)2^m a - 2^{n+m} 1 f(a)\| \\
&\leq \varepsilon + \|f((2^n 1)(2^m a)) - \mu((2^n 1)(2^m a))\| \\
&\quad + \|\mu((2^n 1)(2^m a)) - 2^{n+m} 1 f(a) - g(2^n 1)2^m a\| \\
&\leq 2\varepsilon + \|\mu((2^n 1)(2^m a)) - 2^{n+m} 1 f(a) - g(2^n 1)2^m a\| \\
&\leq 2\varepsilon + 2^m \|\mu(2^n 1 a) - f(2^n 1 a)\| \\
&\quad + 2^m \|f(2^n 1 a) - 2^n 1 f(a) - g(2^n 1)a\| \\
&\leq (2 + 2^{m+1})\varepsilon,
\end{aligned}$$

for all nonnegative integers m, n and all $a \in \mathcal{A}$. Fix m and let n tend to ∞ in the following inequality

$$\|f(2^m a) - 2^m f(a)\| \leq \frac{2 + 2^{m+1}}{2^n} \varepsilon.$$

Then $f(2^m a) = 2^m f(a)$ for all m and all $a \in \mathcal{A}$. Therefore $\mu(a) = \lim_{m\to\infty} \frac{f(2^m a)}{2^m} = f(a)$ for all $a \in \mathcal{A}$. $\square$

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