Dalitz Plot Techniques

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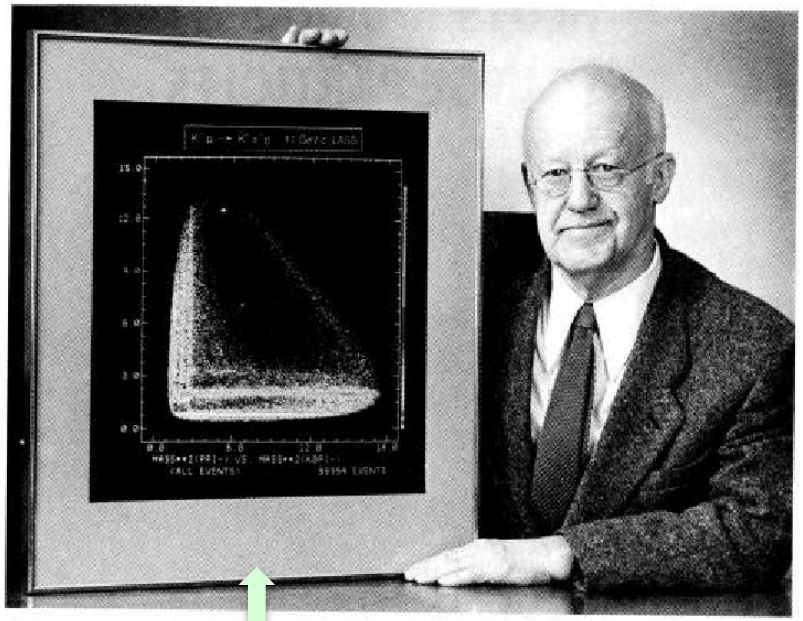
Outline

- Part I
  - A bit of history
  - Kinematic
  - Dynamic (Breit-Wigner, Barrier factors, angular distribution)
  - How to fit a DP
- Part II
  - Examples of analysis of DP

Dalitz…

Richard Dalitz (1925–2006)



$K^- p \rightarrow K^0_S \pi^- p$ at 11 GeV/c
~ 100k evts [SLAC-PUB-5151]

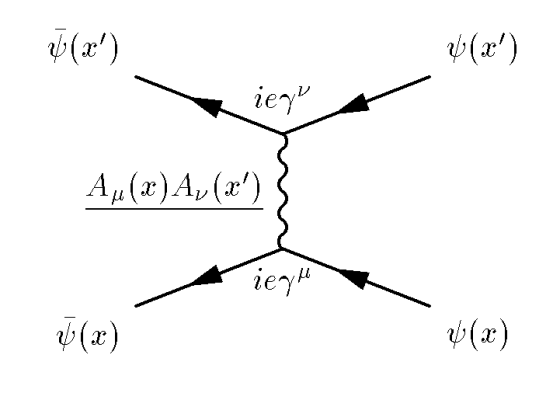
- Strange Baryon
- Baryon spectroscopy
- Hypernuclei
- Dalitz Pair:
  - $\pi^0 \rightarrow e^+ e^- \gamma$
  - $\pi^0 \rightarrow e^+ e^- e^+ e^-$
  - $D_s^{*+} \rightarrow D_s^+ e^+ e^-$
- **θ–τ puzzle**

...Plot

"A picture is worth a thousand words." Napoleon Bonaparte

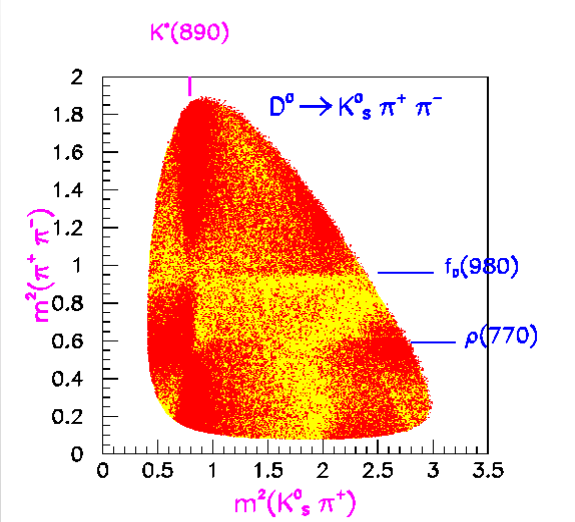
$$\bar{\psi}(x)\gamma^\mu\psi(x)\bar{\psi}(x')\gamma^\nu\psi(x')\underline{A_\mu(x)A_\nu(x')}$$

$$\underline{A_\mu(x)A_\nu(x')} = \int \frac{d^4k}{(2\pi)^4}\frac{-ig_{\mu\nu}}{k^2+i\epsilon}e^{-ik(x-x')}$$



"I visualize geometry better than numbers." Richard Dalitz

Dalitz plot by eye:
- Intermediated states
- Spin
- Interference patterns



The first "Dalitz Plot"

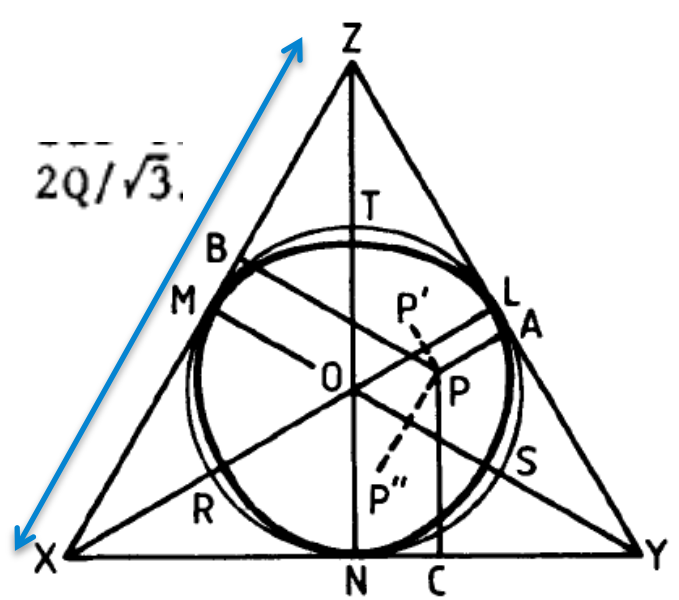


Fig. 1. Phase-space plot for $\tau^+ \to \pi^+\pi^+\pi^-$ events.

Viviani's theorem

$$\overline{PA} + \overline{PB} + \overline{PC} = K$$

$$T = \int \vec{v} \cdot d\vec{p}$$ Kinetic energy

Q = 3-body decay energy

If $T_1 = \overline{PA}, T_2 = \overline{PB}, T_3 = \overline{PC}$

Energy conservation

$$T_1 + T_2 + T_3 = Q$$

Points inside the inscribed circle preserve the momentum as well

Boundary distorted in relativistic regime

The first "Dalitz Plot"

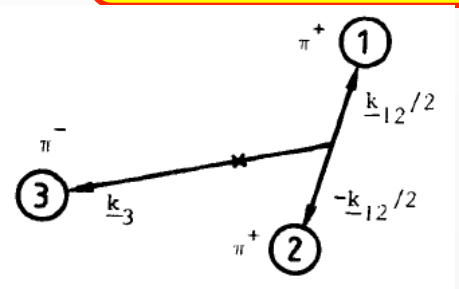
$$\theta^+ \longrightarrow \pi^+\pi^0$$

Assuming parity conservation:

$$P(\theta) = P(\pi^+)P(\pi^0)(-1)^L = (-1)^L$$

$$J^P = 0^+, 1^-, 2^+, etc...$$

$$\tau^+ \longrightarrow \pi^+\pi^-\pi^+$$



$$J = l_{12} + L$$

Assuming parity conservation:

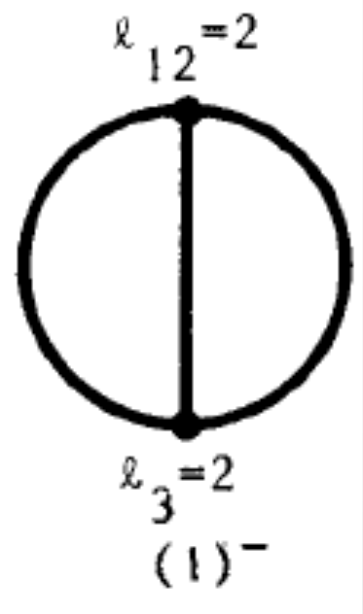
$$P(\tau^+) = P(\pi^+)P(\pi^-)P(\pi^+)(-1)^{l_{12}}(-1)^L = -1(-1)^{l_{12}}(-1)^L$$

By Bose symmetry, wavefunction symmetric under exchange of π's, so l_{12}=even

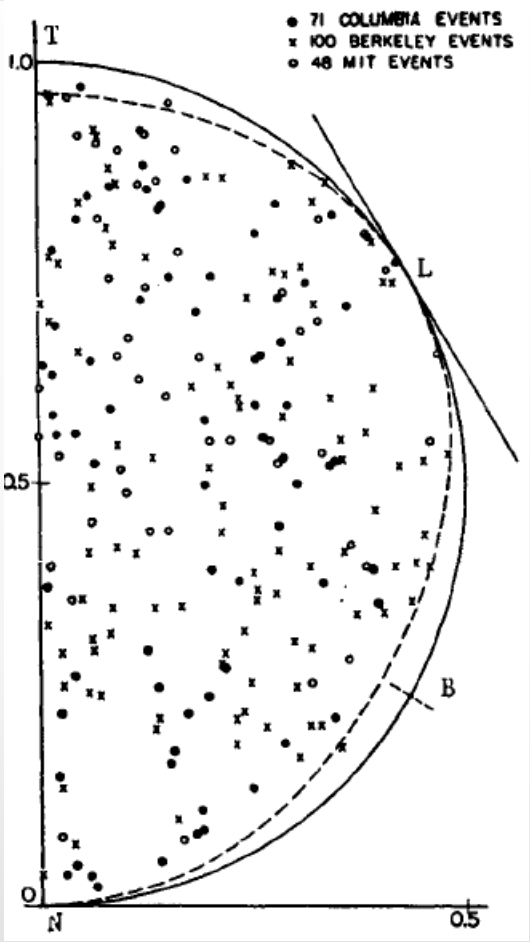
$$P(\theta^+) = -(-1)^L$$

$$J^P = 0^-, 1^\pm, 2^\pm, 3^\pm$$

The first "Dalitz Plot"



No indication that event distribution vanishes at the boundary



Θ and τ are different states?
***P* violation in weak interactions?**

Today we identify both states with K^+ and *P* symmetry is broken.

Dalitz Plot today

The scatter plot in m_{12}^2 e m_{23}^2 is called *Dalitz plot*

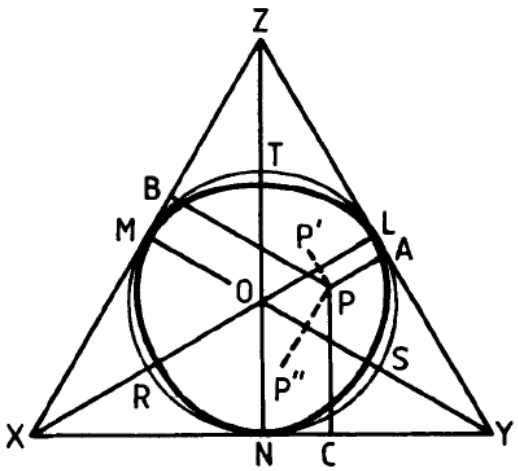
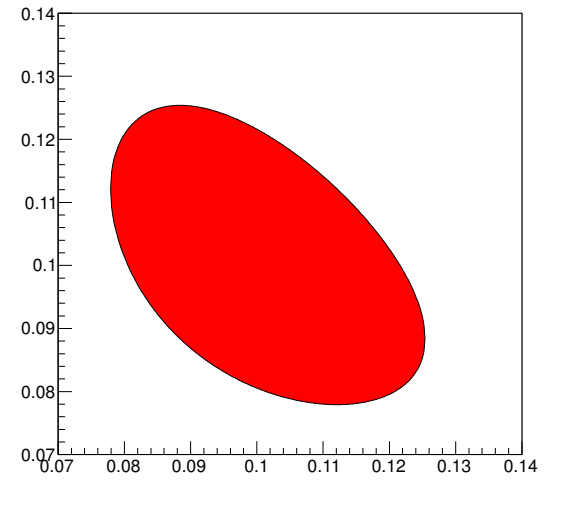


Fig. 1. Phase-space plot for $\tau^+ \to \pi^+\pi^+\pi^-$ events.

$$T_1 = \frac{(m_K - m_\pi)^2 - M_{23}^2}{2m_K}$$

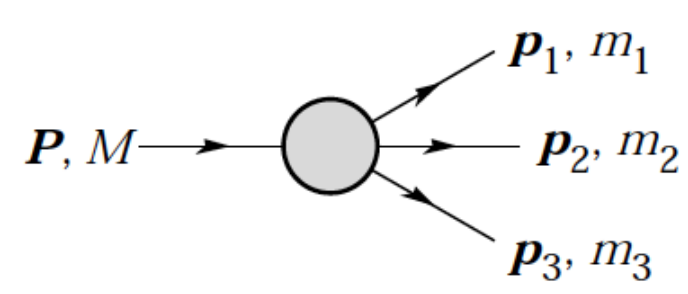


- M^2_{12} and M^2_{23} are covariant
- Complete description of the decay
  - 3-body decays proceed by intermediated resonances
  - Easy to identify resonances as bands on the DP

Dalitz Plot by eye

3-body decay. All particles are scalars ← **Default in this talk**

Constraints	Degree of freedom
3 four-vectors	12
4-momentum conservation	-4
3 masses	-3
3 Euler angles	-3
TOT	2



$$d\Gamma = \frac{1}{(2\pi)^3}\frac{1}{32M^3}\overline{|\mathcal{M}|^2}dm_{12}^2 dm_{23}^2$$

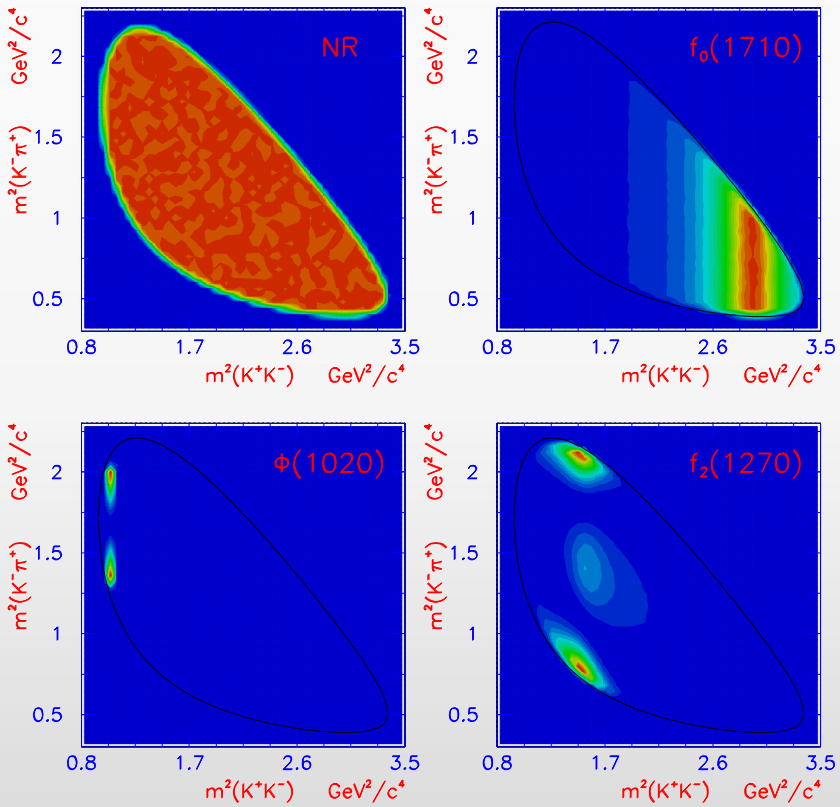
$\overline{|\mathcal{M}|^2} = Const \Rightarrow$ Dalitz uniformly populated

Nonuniformity $\Rightarrow$ Information on $\overline{|\mathcal{M}|^2}$

Dalitz Plot by eye

Simulation of $D_s^+ \rightarrow K^+K^-\pi^+$ decay through an intermediate resonance:

$$D_s^+ \rightarrow R\pi^+$$
$$\hookrightarrow K^+K^-$$



Number of zeros equals the spin of resonance

Dalitz Plot by eye

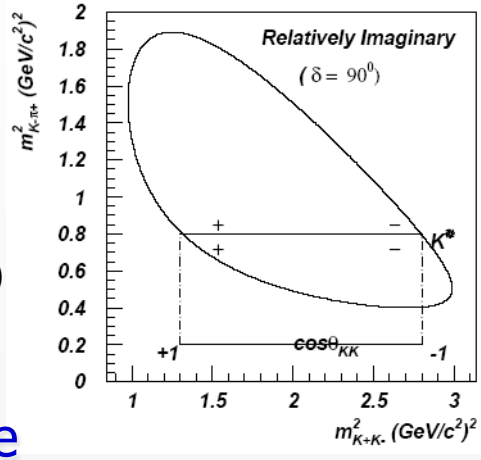
Interference patterns between vector and scalar:

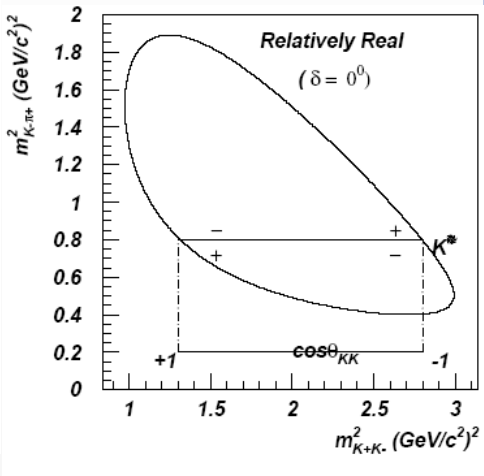
$$\mathcal{M} = \frac{\cos\theta_{KK}}{M_r^2 - M_{K\pi}^2 - i\Gamma M_r} + a(\cos\delta + i\sin\delta)$$

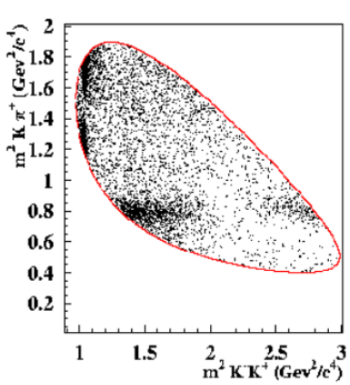
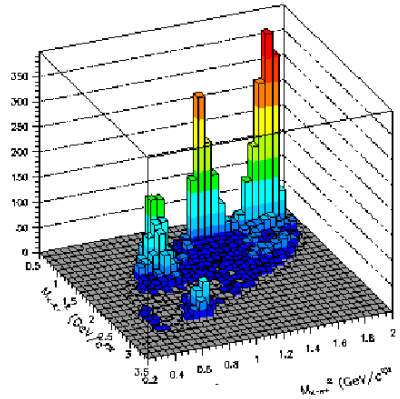
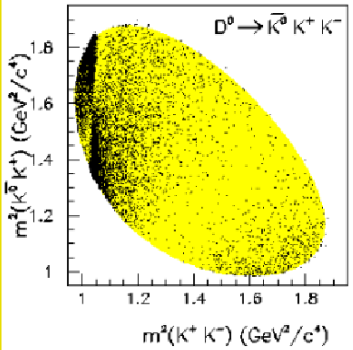
Vector

Constant

Relative phase

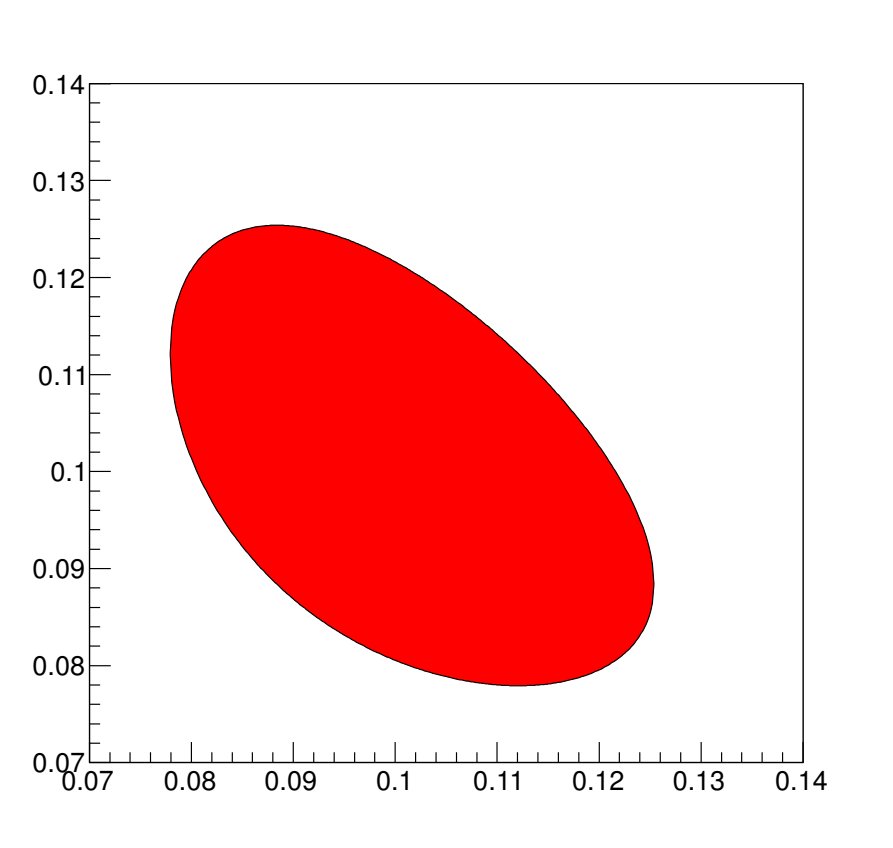




$D^+ \to K^+K^-\pi^+$	$D^0 \to \bar{K}^0K^+K^-$
 	

DP SHAPE

- Q→0 (Non-relativistic regime) ⟷ DP shape → "Egg"
- Q→∞ (Relativistic regime) ⟷ DP shape → Triangle



- $K^+ \rightarrow \pi^+\pi^-\pi^+$

DP SHAPE

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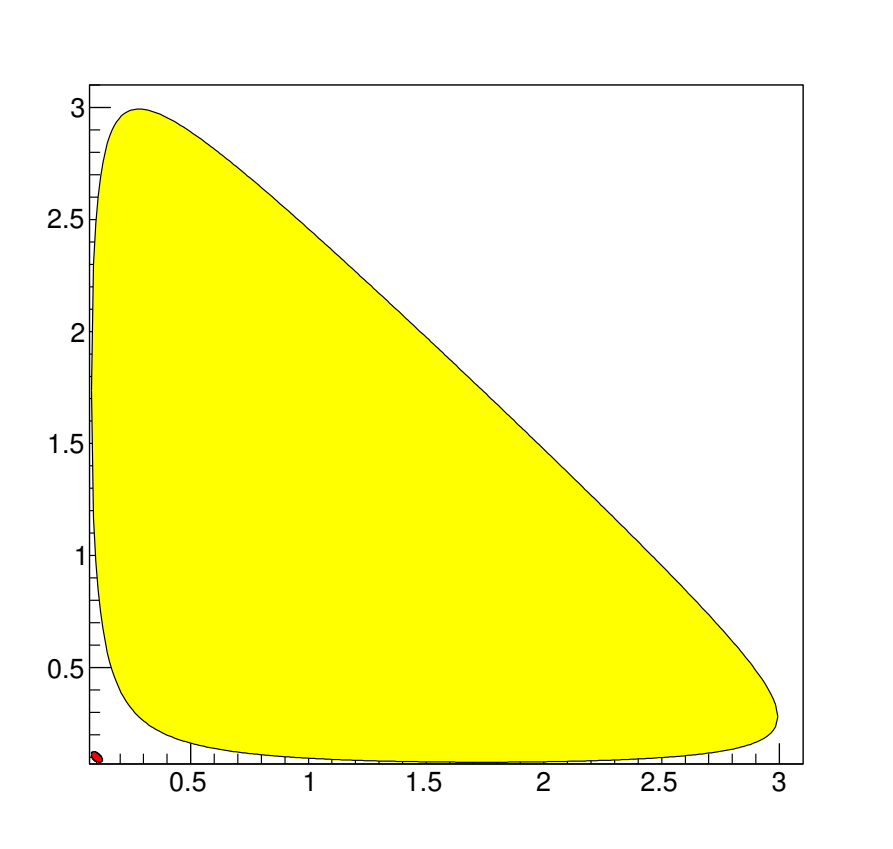

- $D^+ \rightarrow \pi^+\pi^-\pi^+$
- $K^+ \rightarrow \pi^+\pi^-\pi^+$

DP SHAPE

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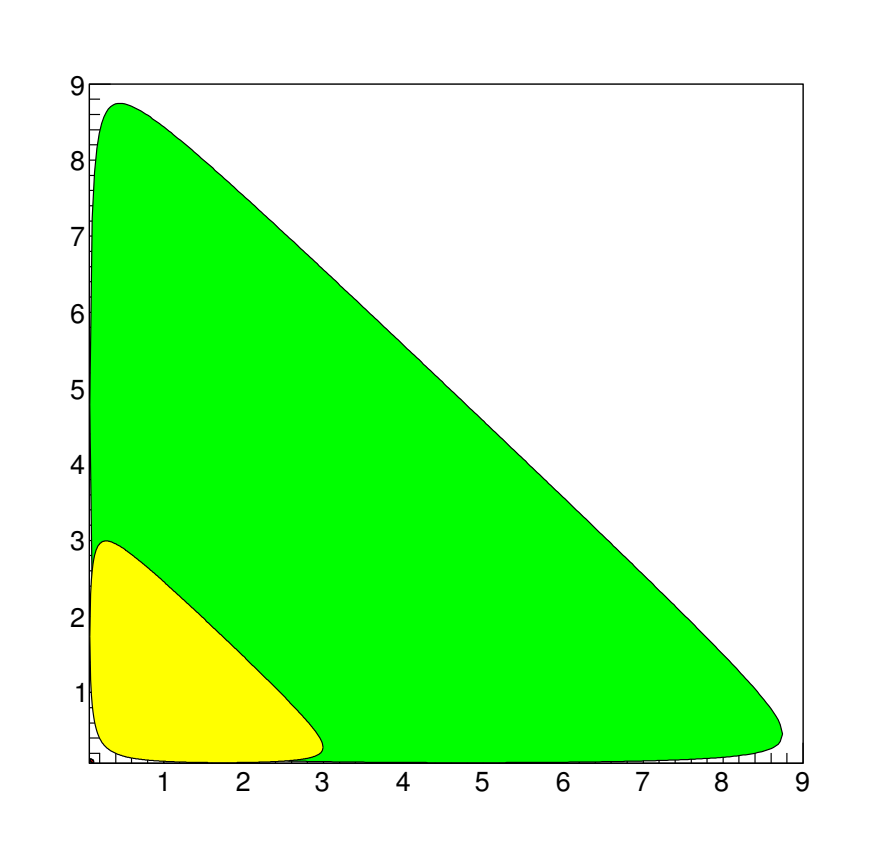
- Q→∞ (Relativistic regime) ⟷ DP shape → Triangle



- $J/\psi \rightarrow \pi^+\pi^-\pi^+$
- $D^+ \rightarrow \pi^+\pi^-\pi^+$
- $K^+ \rightarrow \pi^+\pi^-\pi^+$

DP SHAPE

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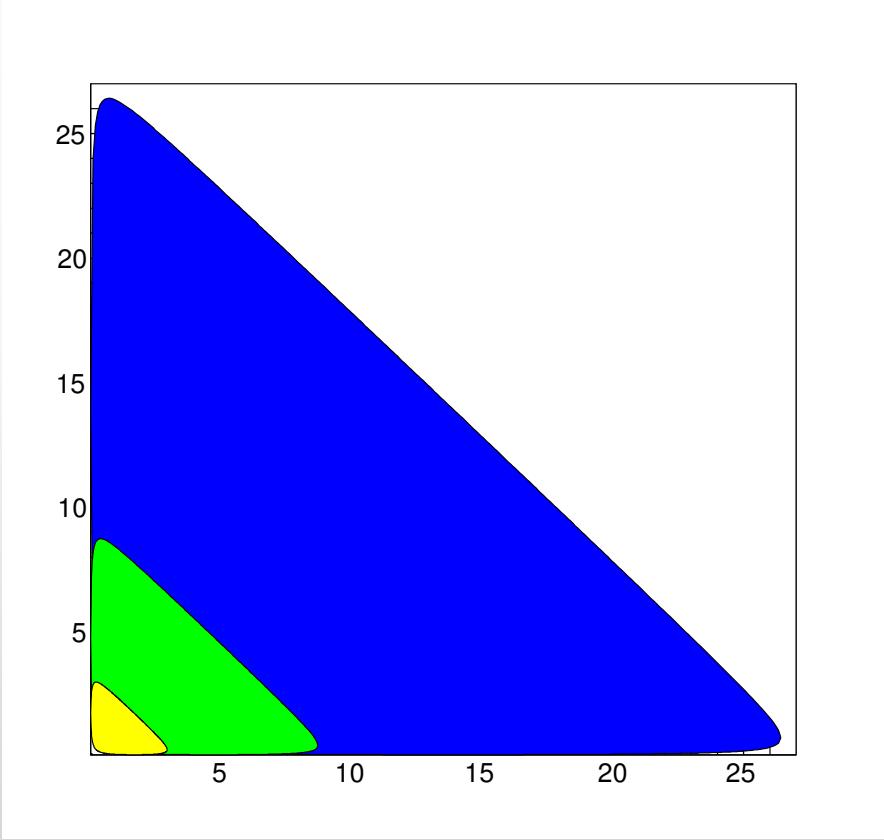


- $B^+ \rightarrow \pi^+\pi^-\pi^+$
- $J/\psi \rightarrow \pi^+\pi^-\pi^+$
- $D^+ \rightarrow \pi^+\pi^-\pi^+$
- $K^+ \rightarrow \pi^+\pi^-\pi^+$

Gottfried-Jackson Frame

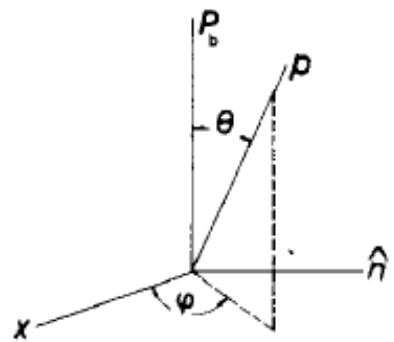
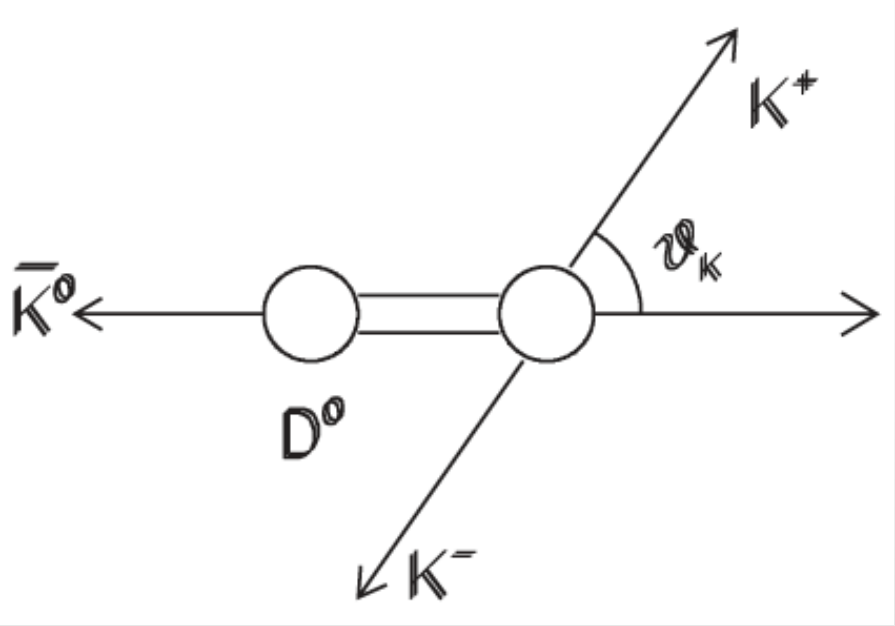


Fig. 2. – Definition of the decay angles (θ, φ) in the rest frame O_d of particle d. $\boldsymbol{p}$ is the momentum of one of the decay products of d, $\hat{\boldsymbol{n}}$ is perpendicular to the production plane, and $\boldsymbol{p}_b$ is the momentum of b as seen in O_d.

θ =Helicity Angle

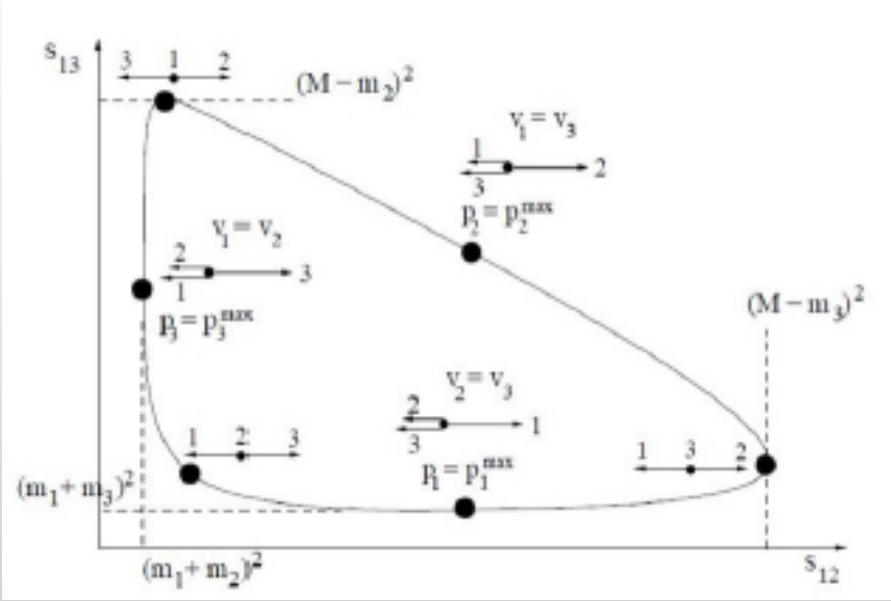


q = momentum of K^+ in resonance rest frame
p = momentum of D^0 in the resonance rest frame

Gottfried-Jackson Frame

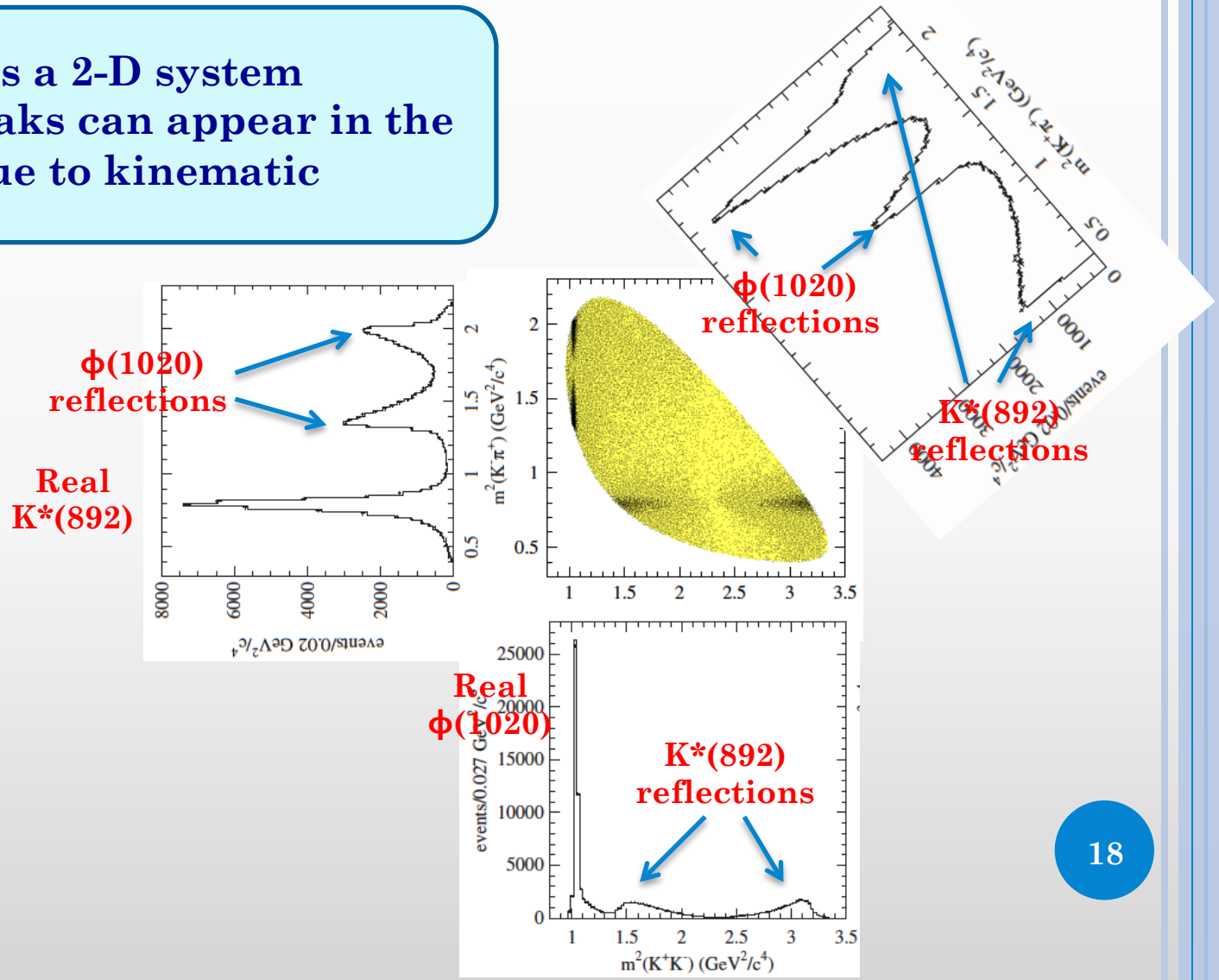
$$m_{13}^2 = (m_{13}^2)_{min}\frac{1+cos\theta}{2} + (m_{13}^2)_{max}\frac{1-cos\theta}{2}$$

$$cos\theta = -1 \Rightarrow (m_{13})^2_{min}$$
$$cos\theta = +1 \Rightarrow (m_{13})^2_{max}$$

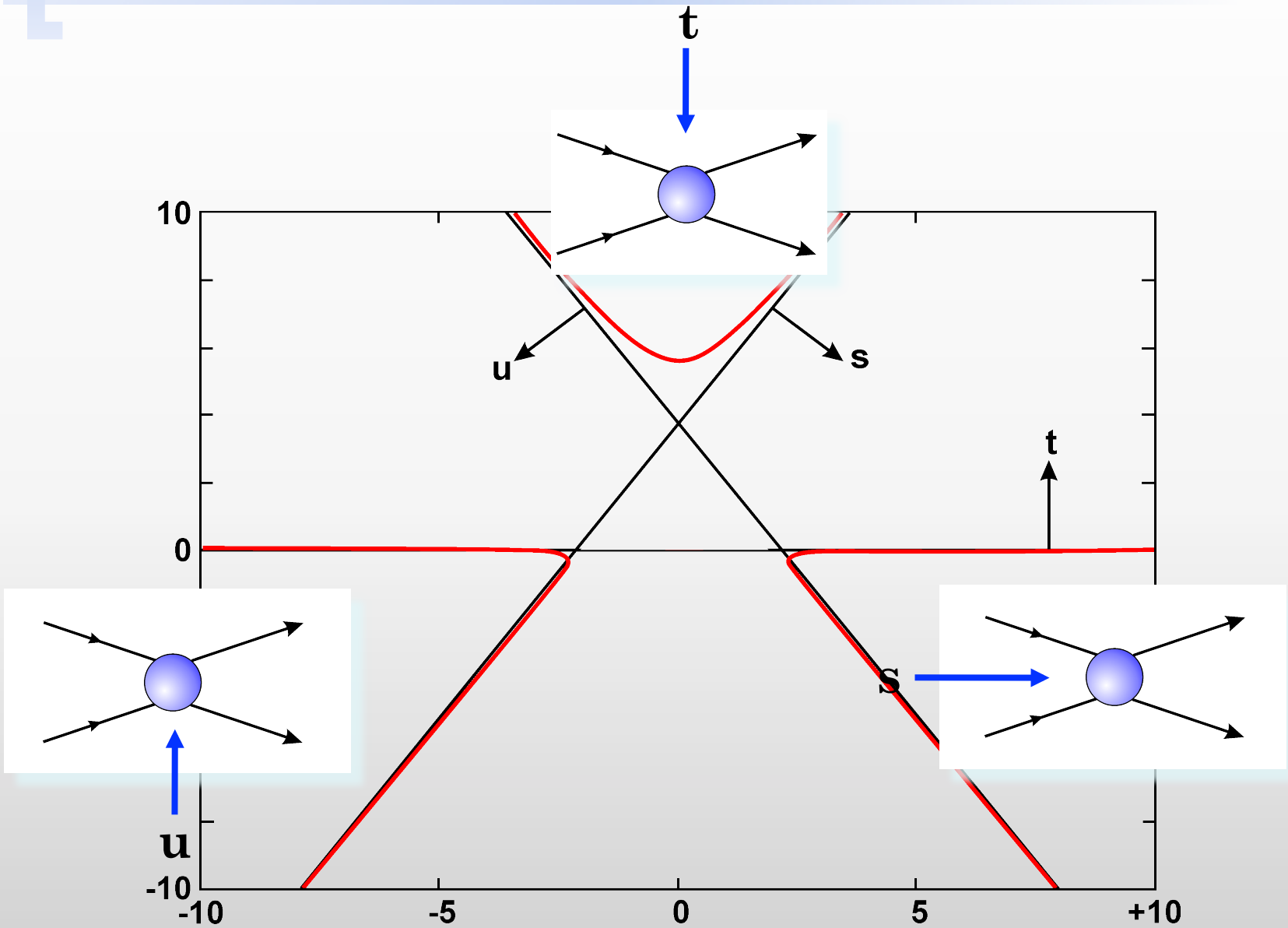


Kinematical Reflections/Shadows

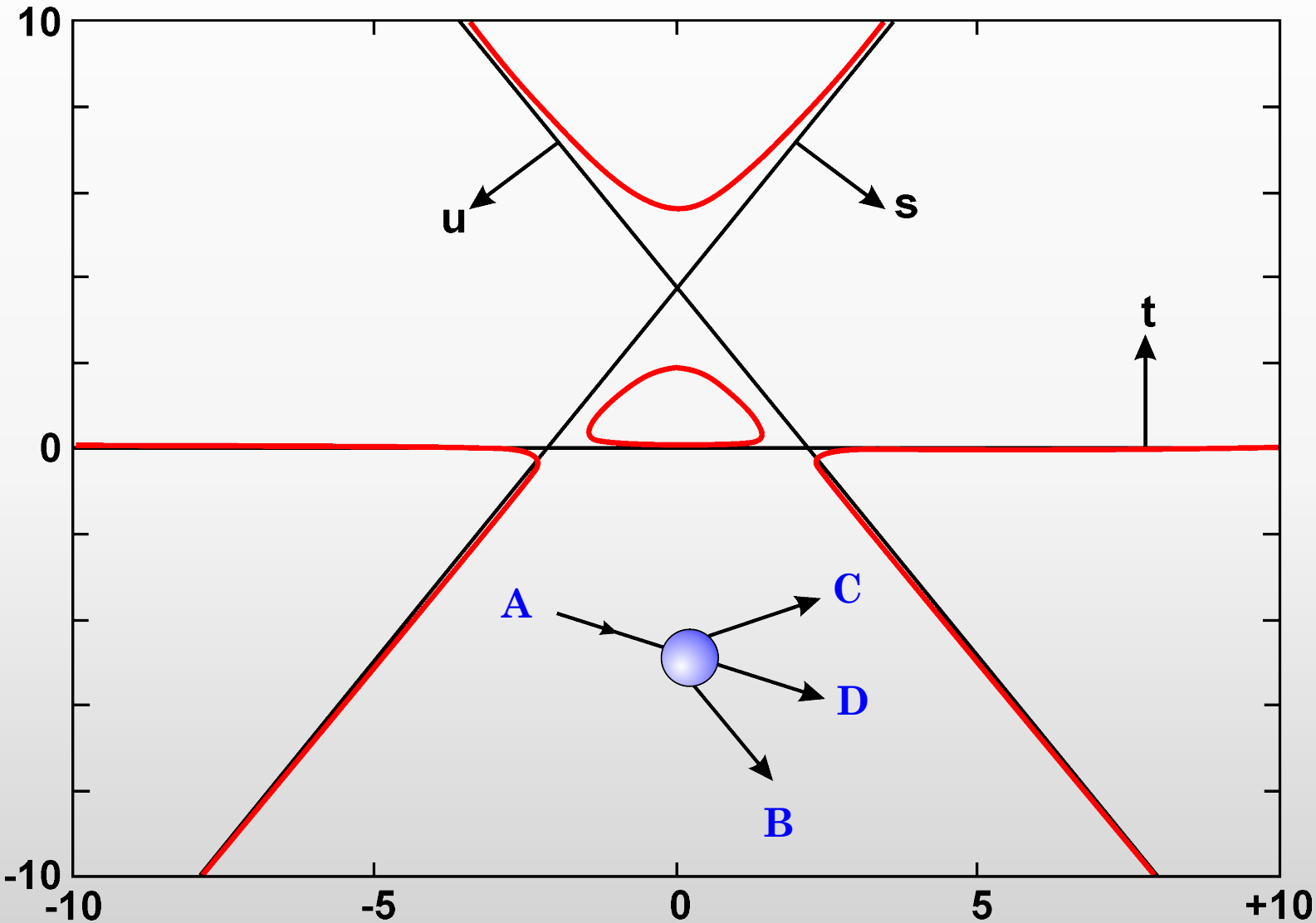
- A Dalitz plot is a 2-D system
- Fake mass peaks can appear in the projections due to kinematic



Scattering Region

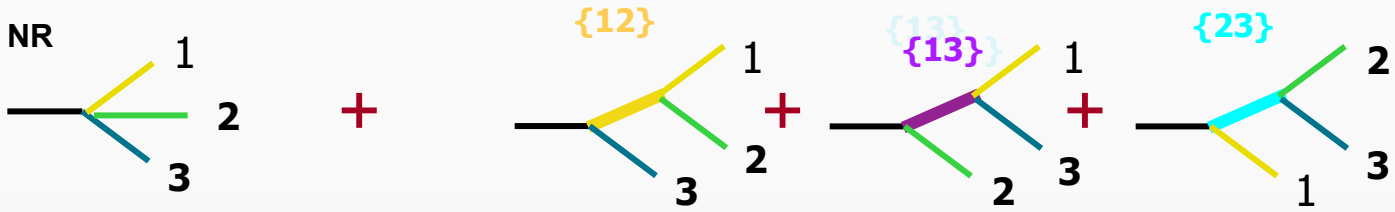


Decay Region



The Isobar Model

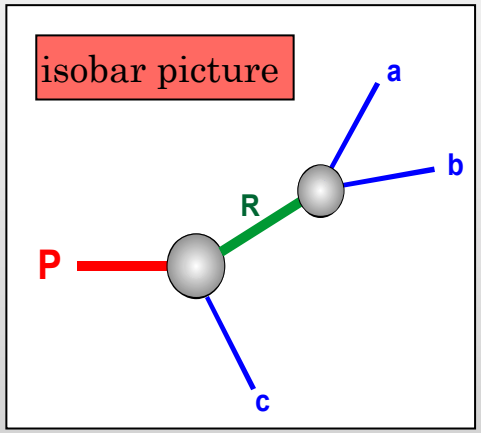
Isobar model: total decay amplitude as a coherent sum of processes where one daughter is spectator

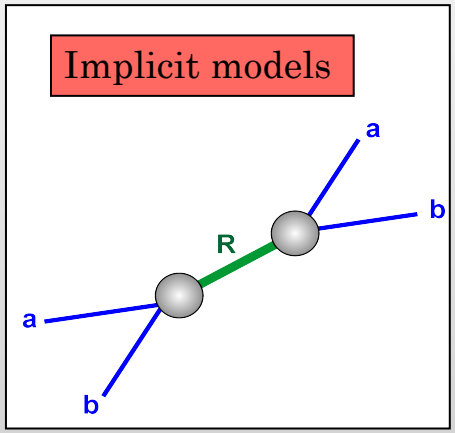


Three-body amplitude

$$\mathcal{A}_{\mathcal{D}}(s_{12}, s_{13}) = a_0 e^{i\delta_0} + \sum_r a_r e^{i\delta r} \mathcal{A}_r(s_{12}, s_{13})$$

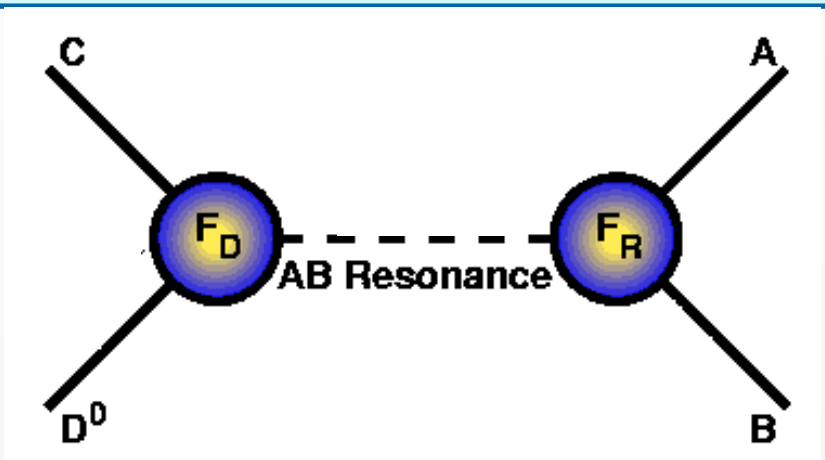
NR term(direct 3 body decay)





How to model a single term

D three-body decay D→ABC decaying through a resonance R→AB



$$\mathcal{A}_r(s_{12}, s_{13}) = F_D^J F_r^J \times M_r^J \times BW_r^J$$

Relativistic Breit-Wigner

$$BW_r^J(s) = \begin{cases} \dfrac{1}{M_r^2 - s - iM_r\Gamma_r(\sqrt{s})} \\ \dfrac{1}{M_r^2 - s - iM_r(\rho_1 g_1^2 + \rho_2 g_2^2)} \end{cases}$$

$a_0(980)/f_0(980)$

Angular distribution

D and *r* Blatt-Weisskopf form factors

Barrier factors

Near to the threshold (q→0), the conservation of angular momentum implies:

$$\mathcal{A} \propto \rho q^{2L}$$

Phase space

$$\rho = \frac{2q}{m}$$

Correction needed to prevent the amplitude from diverging for q→∞:

Blatt-Weisskopt form factor

Blatt-Weisskopt Barrier Factors

L	$B_L(q)$	$B'_L(q, q_0)$
0	1	1
1	$\sqrt{\frac{2z}{1+z}}$	$\sqrt{\frac{1+z_0}{1+z}}$
2	$\sqrt{\frac{13z^2}{(z-3)^2+9z}}$	$\sqrt{\frac{(z_0-3)^2+9z_0}{(z-3)^2+9z}}$
where $z = (\lvert q\rvert\, d)^2$		and $z_0 = (\lvert q_0\rvert\, d)^2$

$q \to 0 : B_L(q) \propto q^L$

$q \to 0 : B'_L(q, q_0) = K$

Mass-dependent width of a resonance:

$$\Gamma = \Gamma_0 \left(\frac{\rho}{\rho_0}\right) \frac{B_L^2(q)}{B_L^2(q_0)}$$

or

$$\Gamma = \Gamma_0 \left(\frac{q}{q_0}\right)^{2L+1} \left(\frac{m_0}{m}\right) B_L'^{\,2}(q, q_0)$$

S-Matrix

Consider a two-body scattering of the type ab→cd. The amplitude that an initial state |i> will be found in the final state |f>

$$S_{fi} = \langle f|S|i\rangle$$

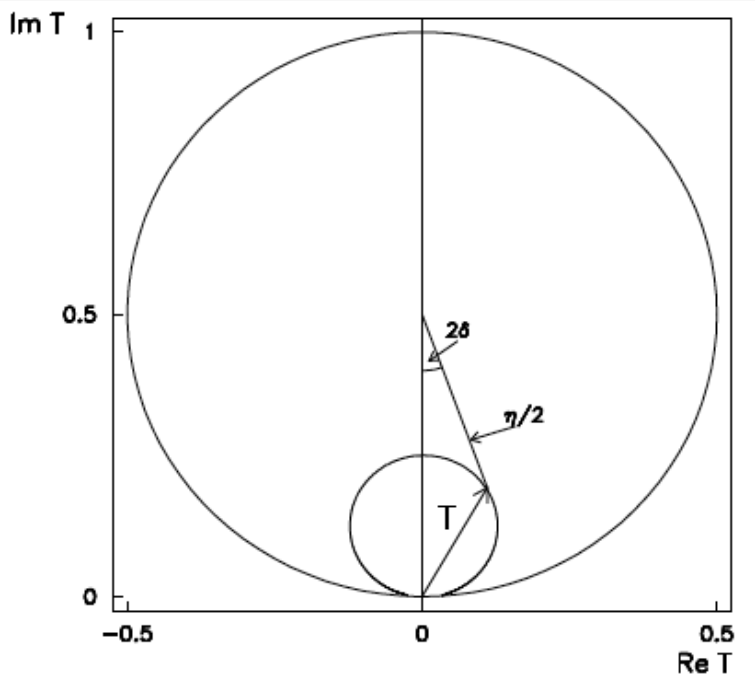
Probability Conservation

$$SS^\dagger = S^\dagger S = I$$

$$S = I + 2iT$$

T=Transition Operator

In elastic scattering, the trajectory of *T* in the complex plane (Argand diagram) is a circle of a unit diameter with its center at (0, i/2). In general the physically allowed T should remain at (elastic) or within this circle (inelastic)



K-Matrix

$$K^{-1} = T^{-1} + iI$$

→ ***K*** **results Hermitian and symmetric, i.e. real and symmetric**

↓

$$T = K(I - iK)^{-1} = (I - iK)^{-1}K$$

Resonances in the K-matrix formalism appear as a sum of poles in the K-matrix

$$K_{ij} = \sum_{\alpha} \frac{\sqrt{m_\alpha \Gamma_{\alpha i}}\sqrt{m_\alpha \Gamma_{\alpha j}}}{m_\alpha^2 - m^2}$$

Relativistic Breit-Wigner

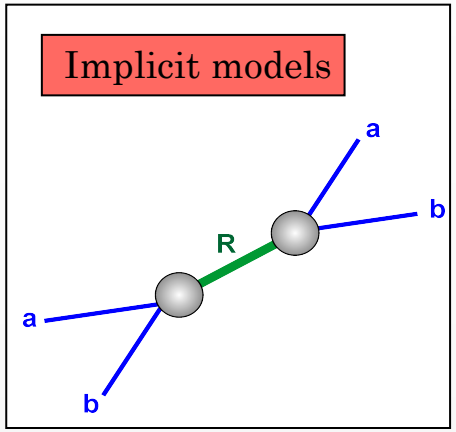
Consider a single pole in a single channel:

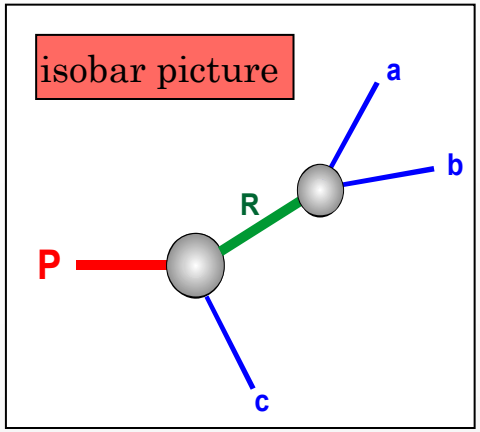
$$K = \frac{m_0 \Gamma(m)}{m_0^2 - m^2}$$

$$T = K(I - iK)^{-1} = \frac{m_0 \Gamma(m)}{m_0^2 - m^2 - i m_0 \Gamma(m)}$$

$$T = \left[\frac{m_0 \Gamma_0}{m_0^2 - m^2 - i m_0 \Gamma(m)} \right] \left[\frac{B_L(q)}{B_L(q_0)} \right]^2 \left(\frac{\rho}{\rho_0} \right)$$

Relativistic Breit-Wigner





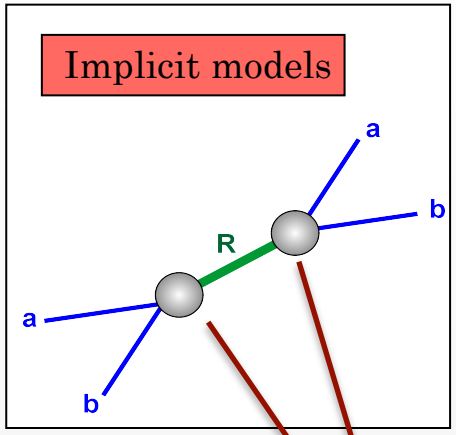
$$T = \left[\frac{m_0\Gamma_0}{m_0^2 - m^2 - im_0\Gamma(m)}\right]\left[\frac{B_L(q)}{B_L(q_0)}\right]^2\left(\frac{\rho}{\rho_0}\right)$$

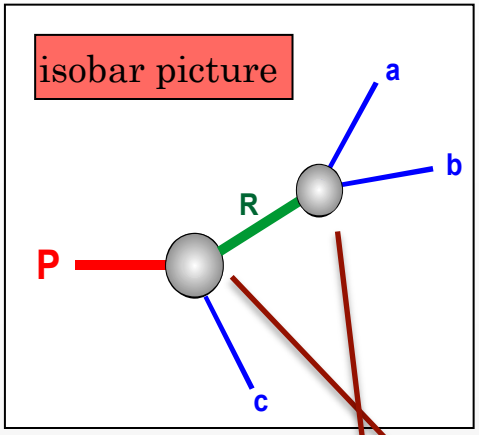
$$T = \left[\frac{m_0\Gamma_0}{m_0^2 - m^2 - im_0\Gamma(m)}\right]\left[\frac{B_L(q)}{B_L(q_0)}\right]^2\cancel{\left(\frac{\rho}{\rho_0}\right)}$$

Phase Space

$$d\Gamma = \frac{1}{(2\pi)^3}\frac{1}{32M^3}\overline{|\mathcal{M}|^2}\,dm_{12}^2\,dm_{23}^2$$

Relativistic Breit-Wigner





$$T = \left[\frac{m_0\Gamma_0}{m_0^2 - m^2 - im_0\Gamma(m)}\right]\left[\frac{B_L(q)}{B_L(q_0)}\right]^2\left(\frac{\rho}{\rho_0}\right)$$

$$T = \left[\frac{m_0\Gamma_0}{m_0^2 - m^2 - im_0\Gamma(m)}\right]\left[\frac{B_L(q)}{B_L(q_0)}\right]\left[\frac{B_L(p)}{B_L(p_0)}\right]$$

Relativistic Breit-Wigner

Which barrier factors?

$$T = \left[\frac{m_0\Gamma_0}{m_0^2 - m^2 - im_0\Gamma(m)}\right]\left[\frac{B_L(q)}{B_L(q_0)}\right]\left[\frac{B_L(p)}{B_L(p_0)}\right]$$

or

$$T = \left[\frac{m_0\Gamma_0}{m_0^2 - m^2 - im_0\Gamma(m)}\right] B'_L(q,q_0)B'_L(p,p_0)$$

?

It depends on what you use for the angular distribution!
The latter is the common choice

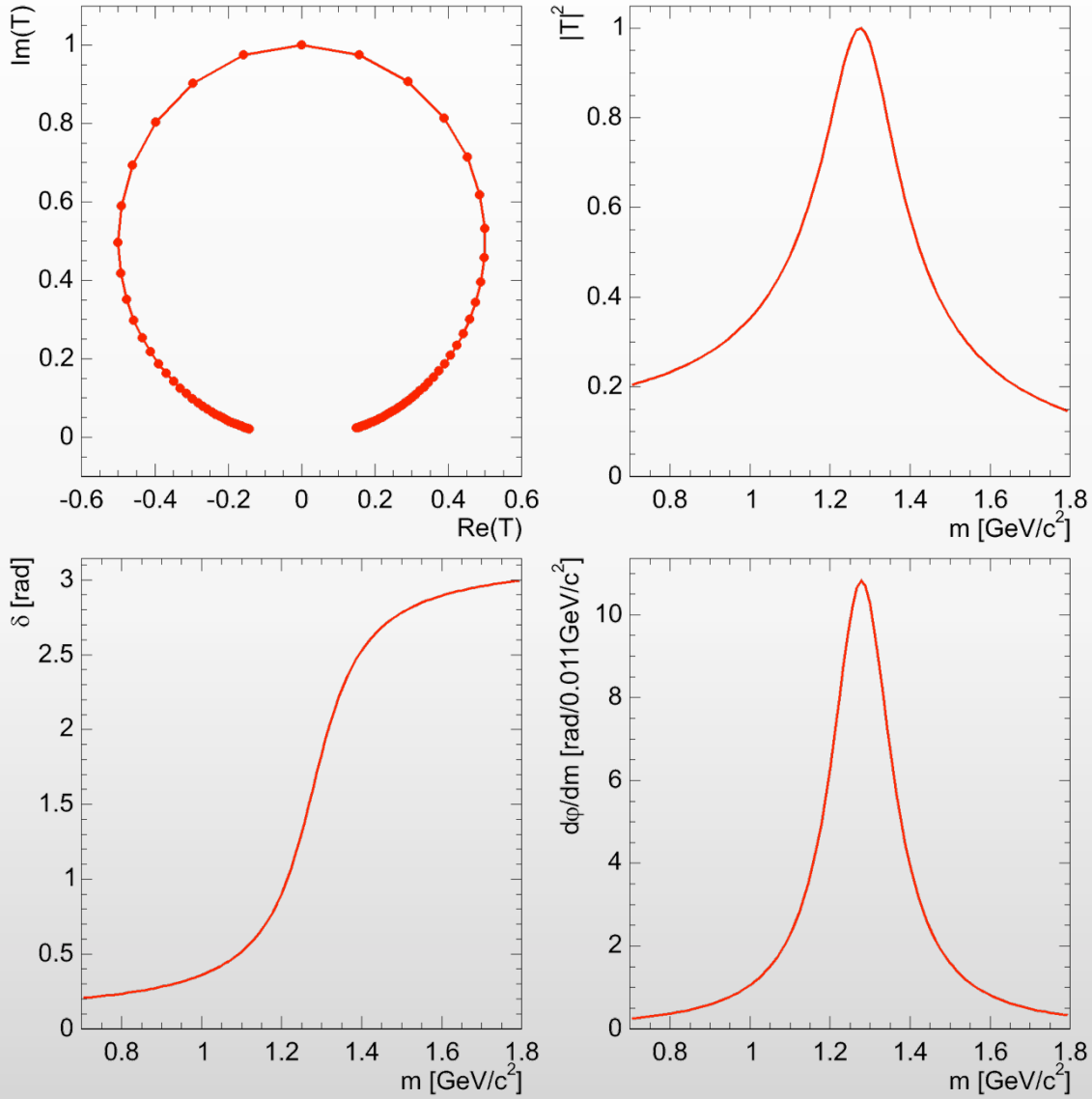
$$\mathcal{A}_r(s_{12},s_{13}) = F_D^J F_r^J \times M_r^J \times BW_r^J$$

$$\rightarrow T \leftarrow$$

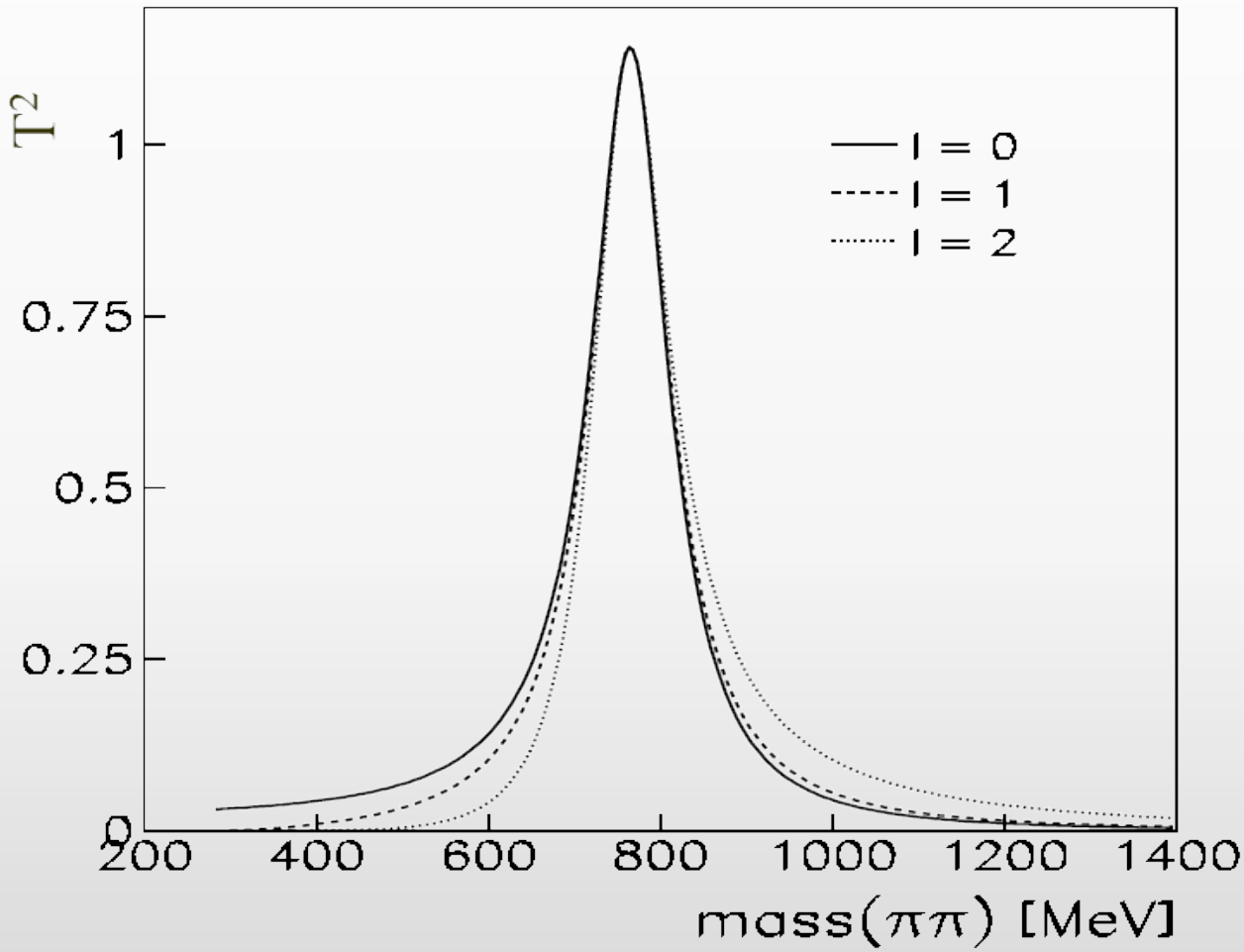
Reminder

$$p \rightarrow 0\; q \rightarrow 0 : \mathcal{A} \propto p^L q^L$$

Relativistic Breit-Wigner



Relativistic Breit-Wigner

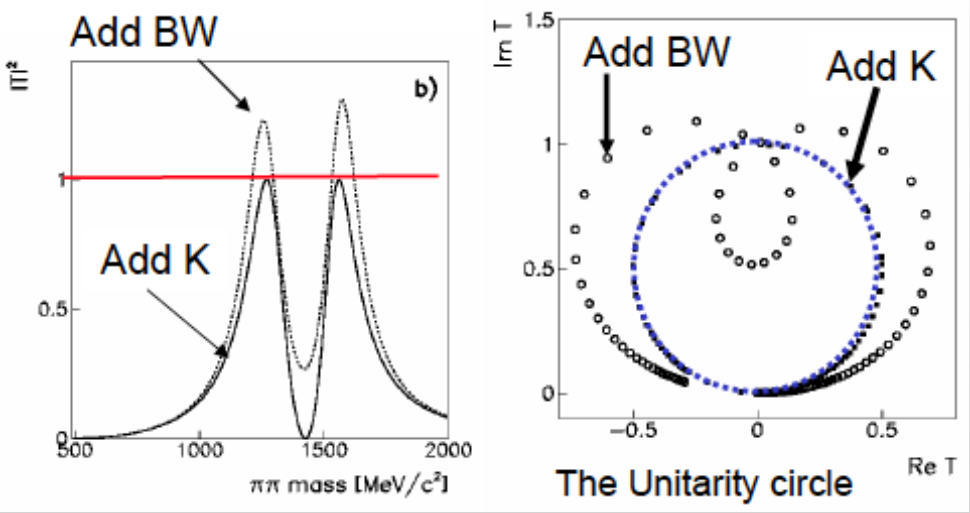


N Relativistic Breit-Wigner's

Consider two poles in a single channel:

$$K = \frac{m_a \Gamma_a(m)}{m_a^2 - m^2} + \frac{m_b \Gamma_b(m)}{m_b^2 - m^2}$$

$$T \simeq \frac{m_a \Gamma_a(m)}{m_a^2 - m^2 - i m_a \Gamma_a(m)} + \frac{m_b \Gamma_b(m)}{m_b^2 - m^2 - i m_b \Gamma(m)}$$



Flatte' Formula

Consider one single pole in two channels (e.g. S-wave scattering with the $a_0(980)$ coupling to pieta and KK final states)

$$\widehat{K}_{11} = \frac{\gamma_1^2 m_0 \Gamma_0}{m_0^2 - m^2} \qquad \widehat{K}_{22} = \frac{\gamma_2^2 m_0 \Gamma_0}{m_0^2 - m^2}$$

$$\widehat{K}_{12} = \widehat{K}_{21} = \frac{\gamma_1 \gamma_2 m_0 \Gamma_0}{m_0^2 - m^2}$$

$$\gamma_1^2 + \gamma_2^2 = 1$$

$$g_i = \gamma_i \sqrt{m_0 \Gamma_0}$$

$$\widehat{T} = \frac{\begin{pmatrix} g_1^2 & g_1 g_2 \\ g_1 g_2 & g_2^2 \end{pmatrix}}{m_0^2 - m^2 - i(\rho_1 g_1^2 + \rho_2 g_2^2)}$$

Angular Distributions

- **Zemach Tensors**
- **Helicity formalism**

The Zemach formalism is only valid for reactions where the daughters are spin-0. For final state particles with non-zero spin, the helicity formalism is required.

$$\text{Spin}\,0 : T(\Omega) = 1$$

$$\text{Spin}\,1 : T(\Omega) = M_{BC}^2 - M_{AC}^2 - \frac{(M_D^2 - M_C^2)(M_B^2 - M_A^2)}{M_{AB}^2} = -2\vec{p}\cdot\vec{q} \quad \propto pq$$

$$\text{Spin}\,2 : T(\Omega) = a_1^2 - \frac{1}{3}a_2 a_3 = \frac{16}{3}[3(\vec{p}\cdot\vec{q})^2 - (|\vec{p}||\vec{q}|)^2] \quad \propto p^2 q^2$$

$$a_1 = M_{BC}^2 - M_{AC}^2 + \frac{(M_{D_s}^2 - M_C^2)(M_A^2 - M_B^2)}{M_{AB}^2},$$

$$a_2 = M_{AB}^2 - 2M_{D_s}^2 - 2M_C^2 + \frac{(M_{D_s}^2 - M_C^2)^2}{M_{AB}^2},$$

$$a_3 = M_{AB}^2 - 2M_A^2 - 2M_B^2 + \frac{(M_A^2 - M_B^2)^2}{M_{AB}^2}.$$

$$B'_L(q, q_0) B'_L(p, p_0)$$

$$p \to 0\; q \to 0 : \mathcal{A} \propto p^L q^L$$

Maximum Likelihood Fit

$$\mathcal{L} = \prod_{i=i}^{N} P_i$$

Minuit

$$-2ln\mathcal{L} = -2\sum_{i=i}^{N} lnP_i$$

N is the total number of events

P_i is the probability of an event being in the infinitesimal region (dm^2_{12} dm^2_{23}) of the Dalitz Plot

$$\int_{DP} P_i = 1$$

$$P_i = P_{i,sig} + P_{i,bkg}$$

Maximum Likelihood Fit

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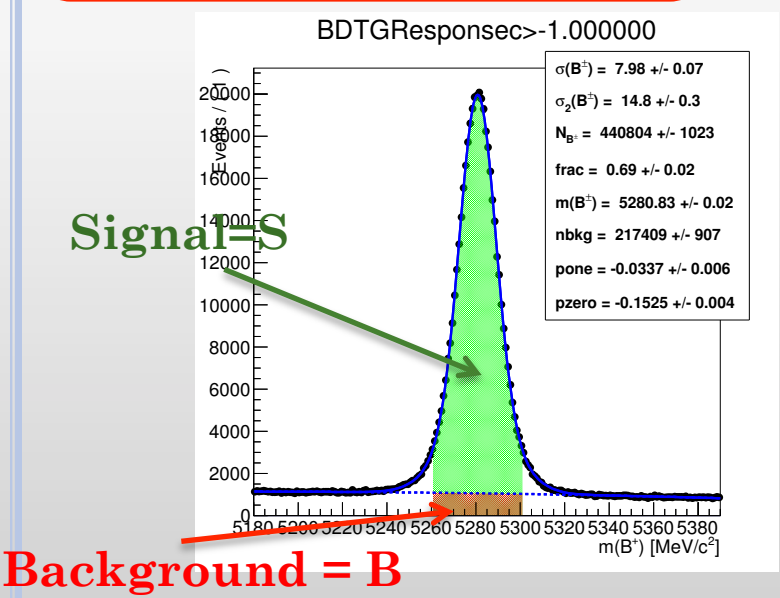

$$\int P_{i,sig} = f \qquad \int P_{i,bkf} = 1 - f$$

$$Purity = \frac{\mathcal{S}}{\mathcal{S} + \mathcal{B}}$$

Maximum Likelihood Fit

$$\mathcal{L} = \prod_{i=i}^{N} P_i$$

P_i is the probability of an event being in the infinitesimal region (dm^2_{12} dm^2_{23}) of the Dalitz Plot

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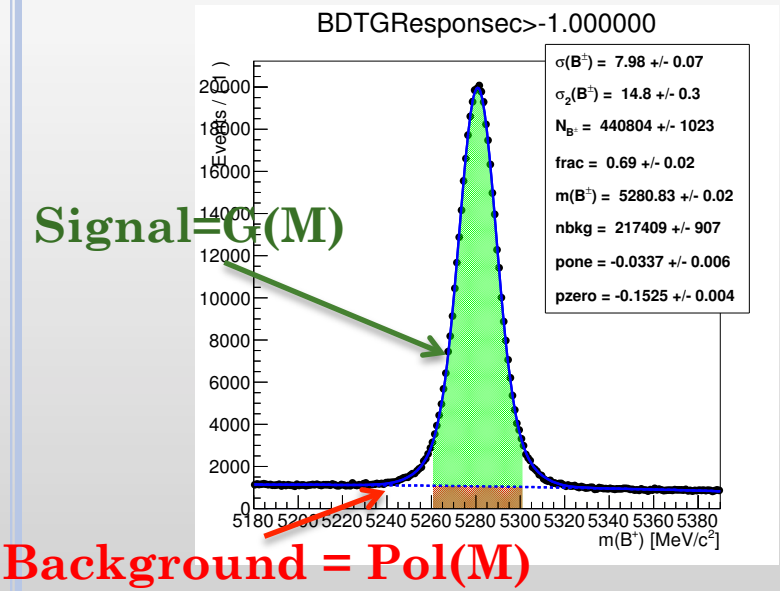


More sophisticated

$$\int_{DP} P_{i,sig} = f(M) \quad \int_{DP} P_{i,bkg} = 1 - f(M)$$

$$Purity = \frac{\mathcal{G}(M)}{\mathcal{G}(M) + Pol(M)}$$

Maximum Likelihood Fit

$$\mathcal{L} = \prod_{i=i}^{N} P_i$$

P_i is the probability of an event being in the infinitesimal region (dm^2_{12} dm^2_{23}) of the Dalitz Plot

$$\int_{DP} P_i = 1$$

Minuit

N is the total number of events

$$-2ln\mathcal{L} = -2\sum_{i=i}^{N} lnP_i$$

$$P_i = P_{i,sig} + P_{i,bkg}$$

$$d\Gamma = \frac{1}{(2\pi)^3}\frac{1}{32M^3}\overline{|\mathcal{M}|^2}dm^2_{12}dm^2_{23}$$

$$P_{i,sig} \propto |\mathcal{M}|^2$$

$$\xrightarrow{\int_{DP} P_i = 1}$$

$$P_{i,sig} = \frac{|\mathcal{M}|^2}{\int_{DP}\mathcal{M}^2 dDP}$$

Maximum Likelihood Fit

Case 1: Decay driven by kinematic only

$$\mathcal{M} = 1 \rightarrow P_{i,sig} = \frac{1}{Area_{DP}} \rightarrow$$

Dalitz plot uniformly populated

Case 2: Generic

$$\mathcal{M} = \sum_{j=1}^{N_R} c_j \mathcal{A}_j(m_{12}^2, m_{23}^2)$$

Isobar Model

$$P_{i,sig} = \frac{\left|\sum_{j=1}^{N_R} c_j \mathcal{A}_j(m_{i,12}^2, m_{i,23}^2)\right|^2}{\int_{DP} \left|\sum_{j=1}^{N_R} c_j \mathcal{A}_j(m_{12}^2, m_{23}^2)\right|^2 dDP} = \frac{\left|\sum_{j=1}^{N_R}\sum_{k=1}^{N_R} c_j c_k^* \mathcal{A}_j(m_{i,12}^2, m_{i,23}^2)\mathcal{A}_k^*(m_{i,12}^2, m_{i,23}^2)\right|^2}{\int_{DP} \left|\sum_{j=1}^{N_R}\sum_{k=1}^{N_R} c_j c_k^* \mathcal{A}_j(m_{12}^2, m_{23}^2)\mathcal{A}_k^*(m_{12}^2, m_{23}^2)\right|^2 dDP}$$

Maximum Likelihood Fit: Integration

- **Integration is time consuming**
- **Integration carried out by:**
  - **Numerical integration(e.g. Gaussian quadrature)**
  - **Events generated by phase space and then weighted by $P_{i,sig}$**

If all parameters don't depend on m^2_{12} and m^2_{23}, integration is not an issue

E.g. Only two amplitudes $\int_{DP} \left| \sum_{j=1}^{N_R} c_j \mathcal{A}_j(m^2_{12}, m^2_{23}) \right|^2 dDP = \int_{DP} |c_1 \mathcal{A}_1(m^2_{12}, m^2_{23}) + c_2 \mathcal{A}_2(m^2_{12}, m^2_{23})|^2 dDP$

$$\int_{DP} \{|c_1|^2 |\mathcal{A}_1(m^2_{12}, m^2_{23})|^2 + |c_2|^2 |\mathcal{A}_2(m^2_{12}, m^2_{23})|^2 + 2\Re[c_1 c_2^* A_1(m^2_{12}, m^2_{23}) A_2^*(m^2_{12}, m^2_{23})]\} dDP$$

$$|c_1|^2 \int_{DP} |\mathcal{A}_1(m^2_{12}, m^2_{23})|^2 dDP + |c_2|^2 \int_{DP} |\mathcal{A}_2(m^2_{12}, m^2_{23})|^2 dDP + 2\Re[c_1 c_2^*] \int_{DP} \Re[A_1(m^2_{12}, m^2_{23}) A_2^*(m^2_{12}, m^2_{23})] dDP - 2\Im[c_1 c_2^*] \int_{DP} \Im[A_1(m^2_{12}, m^2_{23}) A_2^*(m^2_{12}, m^2_{23})] dDP$$

Computed once at first loop

If one o more parameters are functions of the Dalitz variables, integration must be run at each iteration of minimization

Maximum Likelihood Fit

In a Dalitz plot analysis only relative phases and amplitudes matter. You are free to fix one phase and one amplitude

$$P_{i,sig} = \frac{|c_1\mathcal{A}_1(m^2_{i,12}, m^2_{i,23}) + c_2\mathcal{A}_2(m^2_{i,12}, m^2_{i,23})|^2}{\int_{DP}|c_1\mathcal{A}_1(m^2_{12}, m^2_{23}) + c_2\mathcal{A}_2(m^2_{12}, m^2_{23})|^2 dDP}$$

$$c = ke^{\delta}$$

k' δ'

$$P_{i,sig} = \frac{\cancel{k_1}|\mathcal{A}_1(m^2_{i,12}, m^2_{i,23}) + \frac{k_2}{k_1}e^{\delta_2-\delta_1}\mathcal{A}_2(m^2_{i,12}, m^2_{i,23})|^2}{\cancel{k_1}\int_{DP}|\mathcal{A}_1(m^2_{12}, m^2_{23}) + \frac{k_2}{k_1}e^{\delta_2-\delta_1}\mathcal{A}_2(m^2_{12}, m^2_{23})|^2 dDP}$$

Common choice: the amplitude with largest fit fraction(i.e. more visible) and/or the one spreading over a large region of DP

Maximum Likelihood Fit: Efficiency

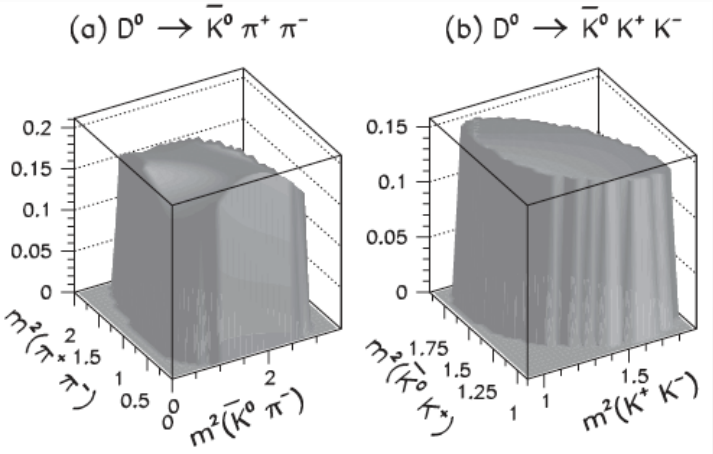
- **If the detector efficiency were uniform across the Dalitz plot, we can ignore it on the likelihood function (we are not sensitive to the absolute value of the efficiency but to its variation in a point respect to another one)**
- **It never happens in the real world and efficiency variation must be taken in account.**
- **Data selection must avoid steep variations or holes in the efficiency distribution**

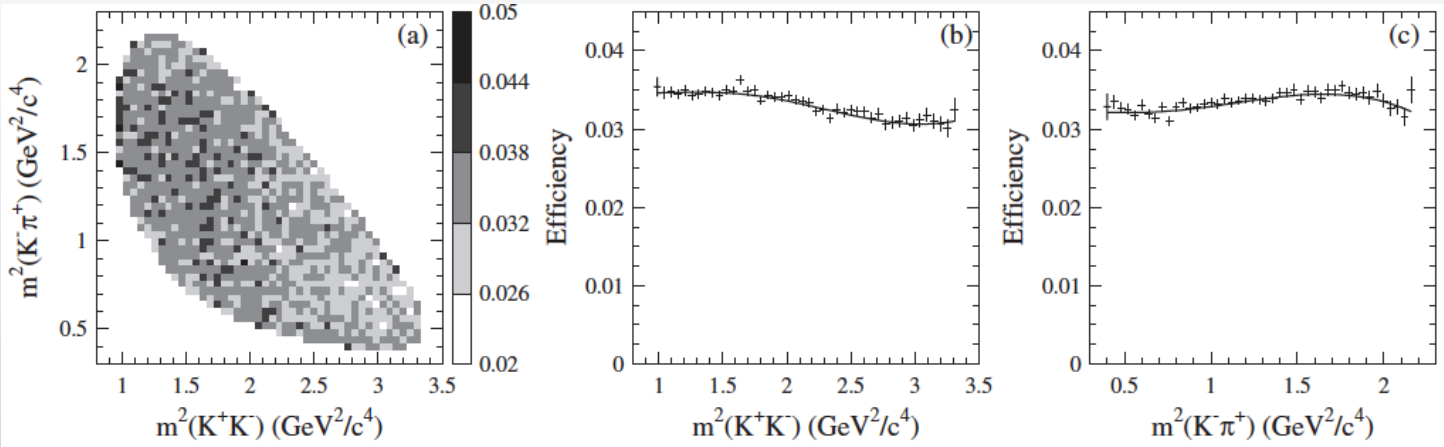
The selection efficiency is determined from a sample of Monte carlo events in which the decay is generated according to phase space (i.e. such that the Dalitz is uniformly populated). The generated events are passed through a detector simulation based on the GEANT4 toolkit, and subjected to the same reconstruction and event selection procedure as that applied to the data.

Maximum Likelihood Fit: Efficiency

The efficiency distribution insert directly into the likelihood or fitted by a 2-D polynomial function

$$\eta(x,y) = a_0 + a_1 x' + a_3 x'^2 + a_4 y'^2 + a_5 x' y' + a_6 x'^3 + a_7 y'^3,$$





Maximum Likelihood Fit: Background

$$\mathcal{L} = \prod_{i=i}^{N} P_i$$

Minuit

$$-2ln\mathcal{L} = -2\sum_{i=i}^{N} lnP_i$$

N is the total number of events

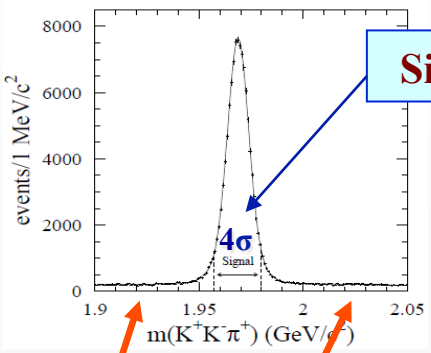
P_i is the probability of an event being in the infinitesimal region (dm^2_{12} dm^2_{23}) of the Dalitz Plot

$$\int_{DP} P_i = 1$$

$$P_i = P_{i,sig} + P_{i,bkg}$$

- **Sidebands regions used to parameterize the background shape**
- **Background assumed uniform. If not, one sideband**
- **Sidebands have different kinematical limit→boundary of the DP are different**
- **Mass constraint helps but events tend to shift**
- **Generally background is a issue. High purity signals are preferred**

Maximum Likelihood Fit: Background

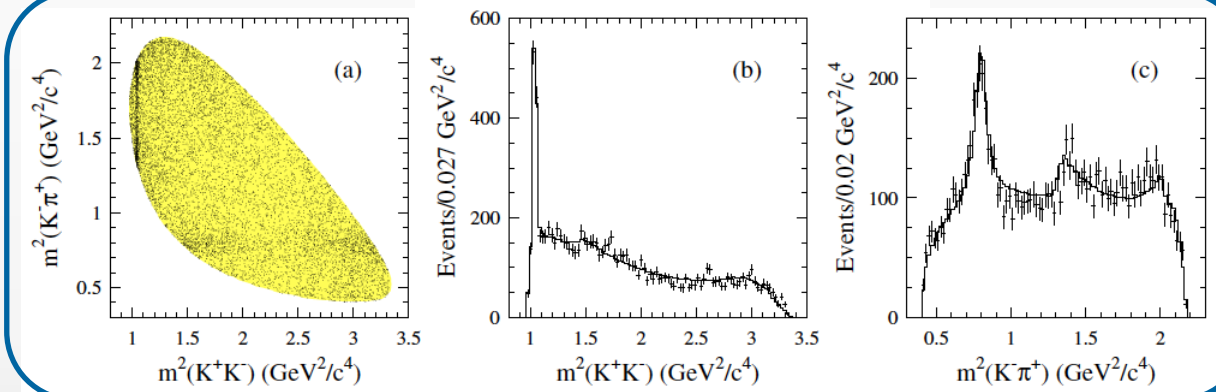


Signal Region

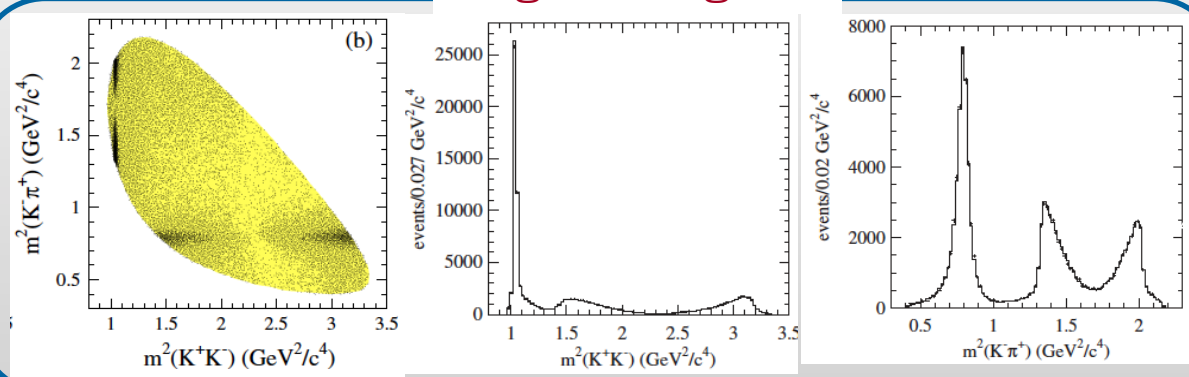
Events used to obtain Bkg shape: (-10σ, -6σ) and (6σ, 10σ)

$$\mathcal{L} = \prod_{n=1}^{N_B} \left[\frac{\sum_i k_i B_i}{\sum_i k_i I_{B_i}} \right]$$

Sideband Regions



Signal Region



Maximum Likelihood Fit: Ready to Go!

Signal shape

Efficiency

$$\mathcal{L} = \prod_{i=1}^{N_{evts}} \left[f(M) \frac{\epsilon(m_{i,12}^2, m_{i,23}^2) \left| \sum_{j=1}^{N_R} c_k \mathcal{A}_j(m_{i,12}^2, m_{i,23}^2) \right|^2}{\int_{DP} \epsilon(m_{12}^2, m_{23}^2) \left| \sum_{j=1}^{N_R} c_j \mathcal{A}_j(m_{12}^2, m_{23}^2) \right|^2 dDP} + (1 - f(M)) \frac{\sum_{j=1}^{N_B} |k_j|^2 |\mathcal{B}_j(m_{i,12}^2, m_{i,23}^2)|^2}{\int_{DP} \sum_{j=1}^{N_B} |k_j|^2 |B_j(m_{12}^2, m_{23}^2)|^2 dDP} \right]$$

Signal (Coherent)

Background (Incoherent)

- ✓ Complex parameters c (…and model parameters) ← **Minuit's job**
- ✓ Number of resonances
- ✓ Model
- ✓ Efficiency
- ✓ Background

} **Experimentalist's skill**

Fitting

Fitting a Dalitz plot is an iterative process

Step 0:

- **Choice of the reference resonance. Which?**
- **Add the main contributions . Which?**

Fitting

Fitting a Dalitz plot is an iterative process

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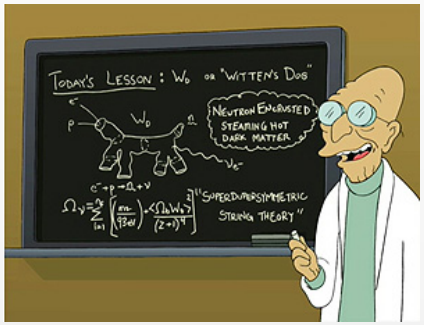

FITTING

Fitting a Dalitz plot is an iterative process

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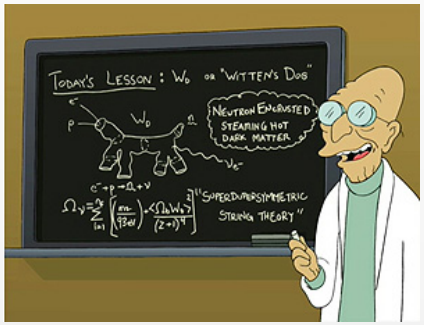
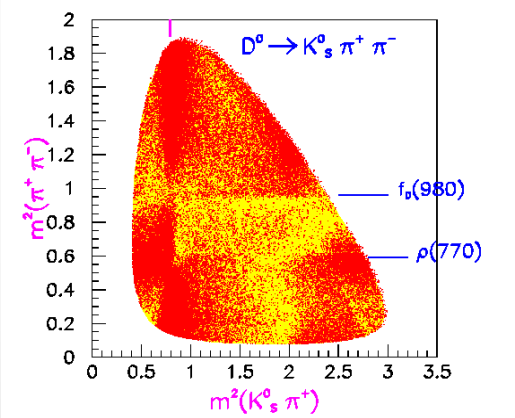


OR



LOOP

- **Add in sequence any other possible contribution**
- **Keep the one the largest change in likelihood**

LOOP

Loop until adding any other resonance makes the likelihood unchanged/No statistically significance

Goodness of fit

Is my fit good enough?

No rules, just guidelines. Check that no discrepancy in:

- 2-D χ^2 or pull distributions
- Every 1-D mass projections
- Moments

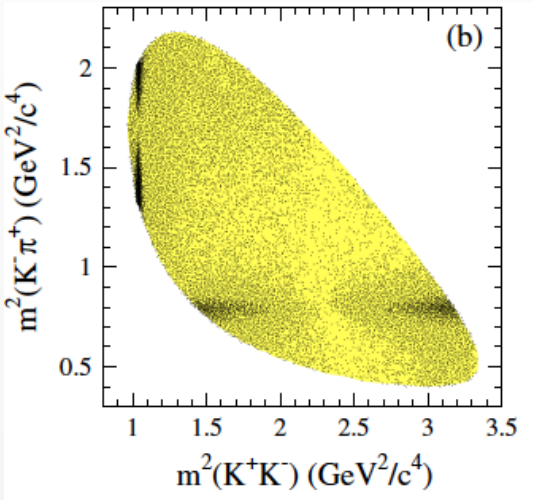
Goal is chi2/NDOF ~ 1 (Very difficult)
$1<\chi^2/NDOF<2$ still acceptable if no macroscopic discrepancy

Total fit fraction:

- **it could be > 100% because of interferences but >>100%(e.g. 500%) makes results suspicious**

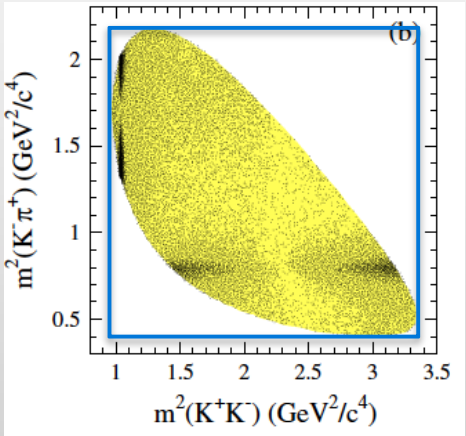
Adaptive binning

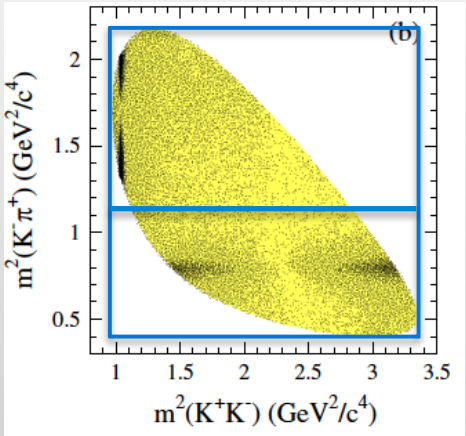
Adaptive binning is sometime mandatory

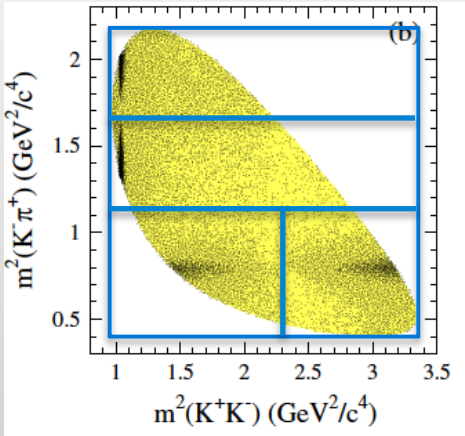


- ~90% of events gathered in a relative small region of the DP
- Large bin size → Not sensitive to the DP structures
- Small bin size → Many bins with no entries

Method 1







Adaptive binning

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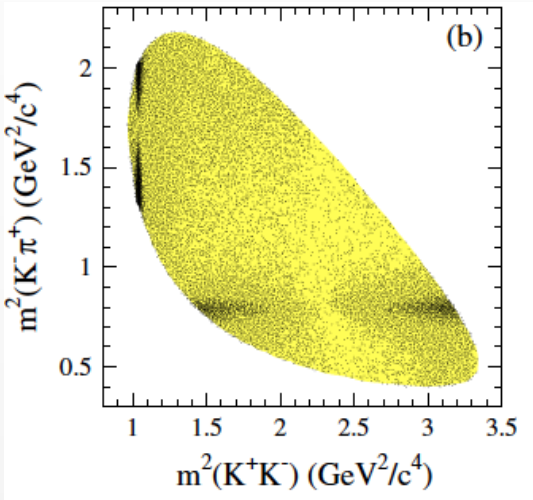

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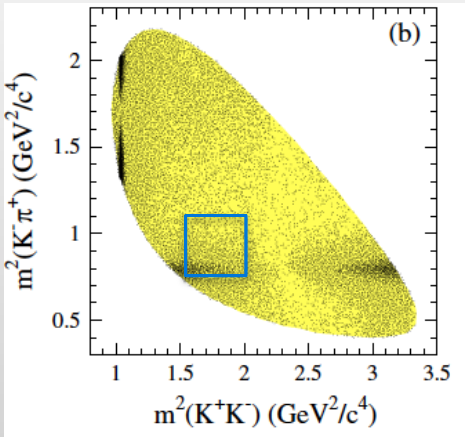
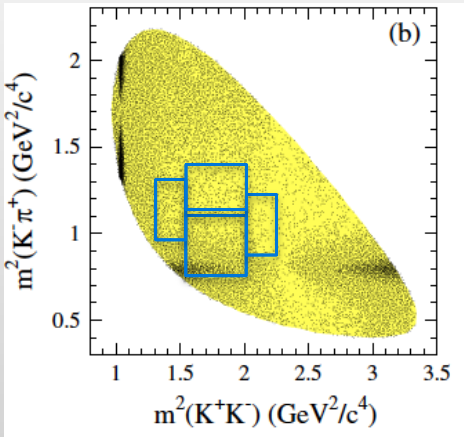
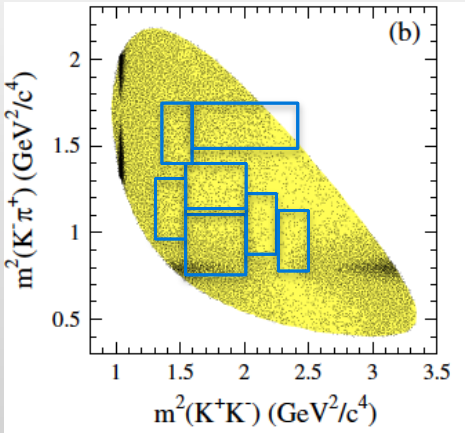
Method 2







Hints and Tricks

"I have tried any combination but fit results are not enough good. What to do?"

Global discrepancy:

- Better background description?
- Better efficiency distribution? A small change may end up in a huge improvement

Local discrepancy:

- Can you improve fit model? K-matrix parameterization, MIPWA?
- New states?

- Adding a resonance → - ln Likelihood must decrease or remain the same
- Stability of the solution?
- Different parameterization? Re and Im instead of magnitude and phase?
- Local minima? Different (random) starting values

Binned Fit

- Faster
- It might be not sensitive to tiny effects

- Generation of N event according phase space
- Event weighted by $P_{\{i,sig\}}$ (Background can be subtracted)
- Comparison to the data
- Minimize χ^2/NDF

Fit fractions

Fit results should be independent of convention, parameterization, reference decay mode, barrier factor

$$\sum \left\{ \begin{array}{c} \text{Fit fractions} \\ f_i = \dfrac{|c_i|^2 \int |A_i(x,y)|^2 dxdy}{\int |\sum_j c_j A_j(x,y)|^2 dxdy} \\ \\ \text{Interference fit fractions} \\ f_{kl} = \dfrac{2 \int \Re[c_k c_l^* A_k(x,y) A_l^*(x,y)] dxdy}{\int |\sum_j c_j A_j(x,y)|^2 dxdy} \end{array} \right\} = 1$$

Fit fractions

Fit fractions

$$\sum \left\{ f_i = \frac{|c_i|^2 \int |A_i(x,y)|^2 dxdy}{\int |\sum_j c_j A_j(x,y)|^2 dxdy} \right\} \neq 1$$

Interference fit fractions

$$f_{kl} = \frac{2 \int \Re[c_k c_l^* A_k(x,y) A_l^*(x,y)] dxdy}{\int |\sum_j c_j A_j(x,y)|^2 dxdy}$$

$$\sum \left\{ \quad \right\} = 1$$

Interference fit fraction between different waves on the same axis are null

TABLE VI. Fit fractions matrix of the best fit. T
diagonal elements give the fit fractions of the int
contribution integrates to zero. Quoted uncertainties are statistical only.

f_{kl} (%)	$\bar{K}^*(892)^0K^+$	$\phi(1020)\pi^+$	$f_0(980)\pi^+$	$\bar{K}_0^*(1430)^0K^+$	$f_0(1710)\pi^+$	$f_0(1370)\pi^+$
$\bar{K}^*(892)^0K^+$	47.9 ± 0.5	-4.36 ± 0.03	-2.4 ± 0.2	0.	-0.06 ± 0.03	0.08 ± 0.08
$\phi(1020)\pi^+$		41.4 ± 0.8	0.	-0.7 ± 0.2	0.	0.
$f_0(980)\pi^+$			16.4 ± 0.7	4.1 ± 0.6	-3.1 ± 0.2	-4.5 ± 0.3
$\bar{K}_0^*(1430)^0K^+$				2.4 ± 0.3	0.48 ± 0.08	-0.7 ± 0.1
$f_0(1710)\pi^+$					1.1 ± 0.1	0.86 ± 0.06
$f_0(1370)\pi^+$						1.1 ± 0.1

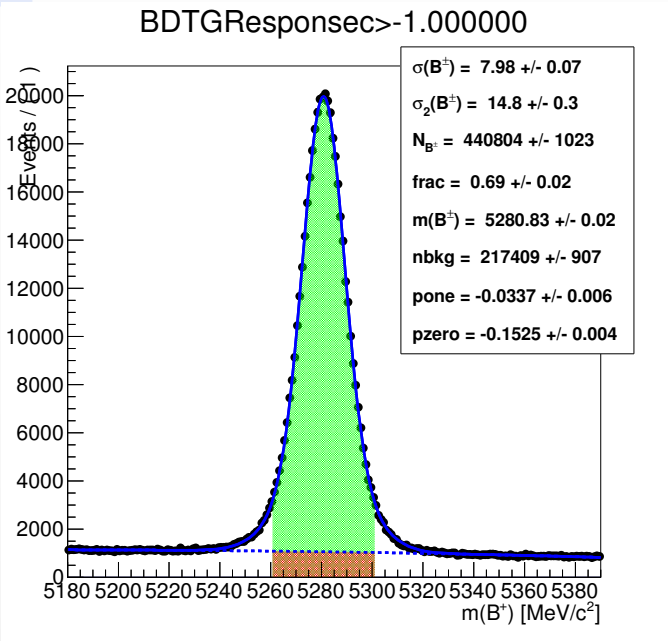
Fit fractions

$$\sum \left\{ \begin{array}{l} \textbf{Fit fractions} \\ \sum \left\{ f_i = \dfrac{|c_i|^2 \int |A_i(x,y)|^2 dxdy}{\int |\sum_j c_j A_j(x,y)|^2 dxdy} \right\} \neq 1 \\ \textbf{Interference fit fractions} \\ f_{kl} = \dfrac{2 \int \Re[c_k c_l^* A_k(x,y) A_l^*(x,y)] dxdy}{\int |\sum_j c_j A_j(x,y)|^2 dxdy} \end{array} \right\} = 1$$

Γ_{31}	$K^+K^-\pi^+$	[e]	(5.49±0.27) %
Γ_{32}	$\phi\pi^+$	[d,f]	(4.5 ±0.4) %
Γ_{33}	$\phi\pi^+$, $\phi \to K^+K^-$	[f]	(2.32±0.14) %
Γ_{34}	$K^+\overline{K}^*(892)^0$, $\overline{K}^{*0} \to K^-\pi^+$		(2.60±0.15) %
Γ_{35}	$f_0(980)\pi^+$, $f_0 \to K^+K^-$		(1.55±0.16) %
Γ_{36}	$f_0(1370)\pi^+$, $f_0 \to K^+K^-$		(2.4 ±0.4) $\times 10^{-3}$
Γ_{37}	$f_0(1710)\pi^+$, $f_0 \to K^+K^-$		(1.87±0.33) $\times 10^{-3}$
Γ_{38}	$K^+\overline{K}_0^*(1430)^0$, $\overline{K}_0^* \to K^-\pi^+$		(2.1 ±0.4) $\times 10^{-3}$

The use of the fit fractions from DP is not straightforward

Resolution



Detector Resolution displaces points beyond the Dalitz plot Border

3-body mass constrain fit: the momenta of daughters are adjusted till the mother mass is at its nominal value

What about the intermediated resonances?

- **If resolution << width, we don't care**
- **If resolution ~ width (e.g. φ, ω, etc)→Breit-Wigner convoluted with a Gaussian**

Detector resolution (MC)

Natural width Γ_0 fixed (PDG) or free

Squared Dalitz Plot

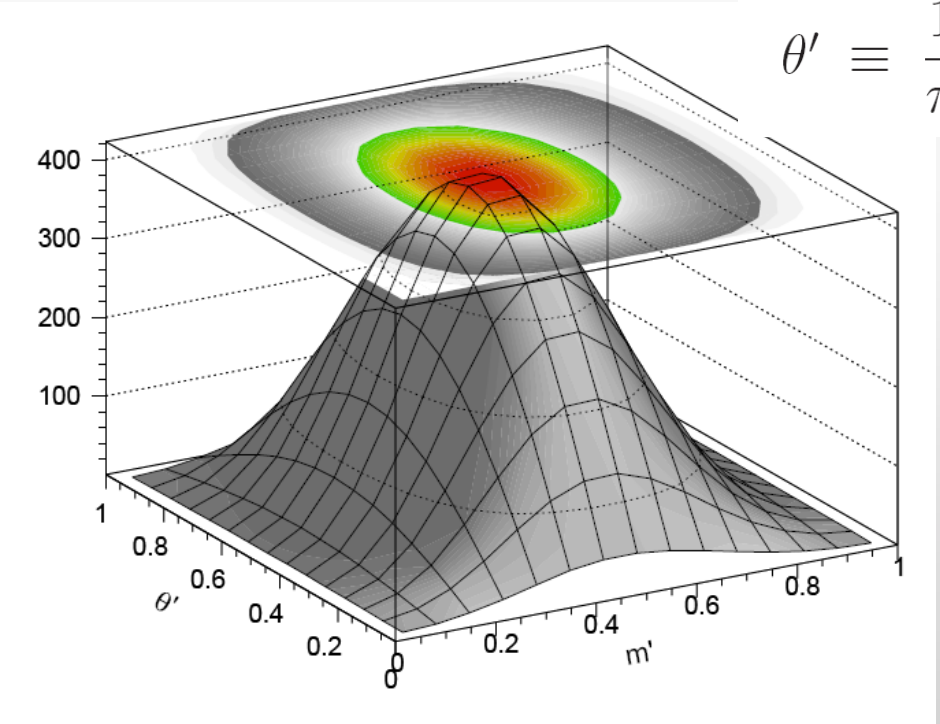
- Experimental effects such as efficiency variation often have to be included in a binned form, i.e. a histogram
- The conventional DP is a slightly awkward shape for binning – the kinematic boundary is not square
- In addition in B decay DPs most of the things we're interested in are clustered around edges
- So we need to have better resolution in those areas in order to describe the efficiency changes or background distribution better

Squared Dalitz Plot

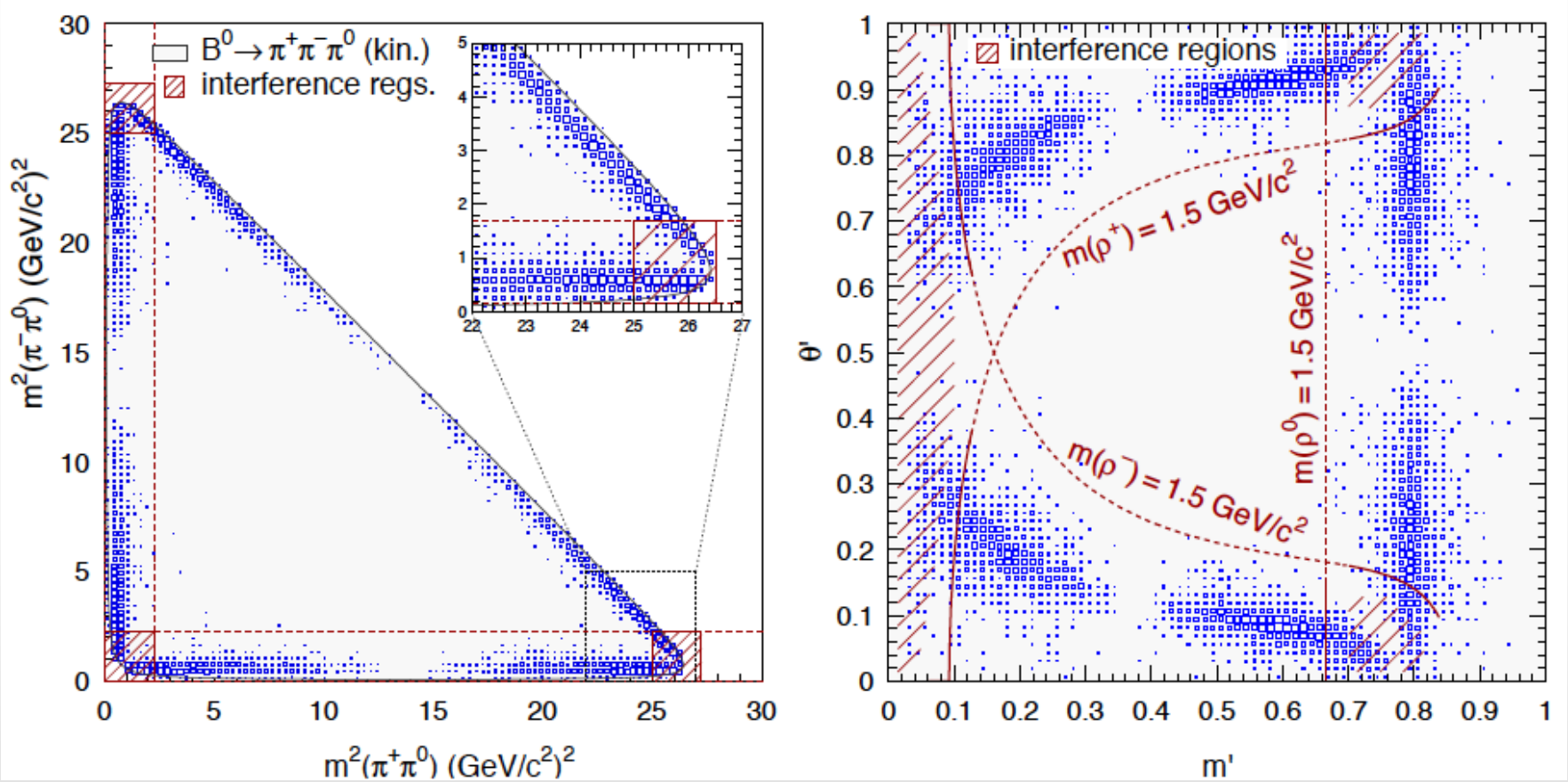
Both of these problems are solved by a transformation of variables into a square DP

$$m' \equiv \frac{1}{\pi}\arccos\left(2\frac{m_{K^+\pi^+} - m^{\min}_{K^+\pi^+}}{m^{\max}_{K^+\pi^+} - m^{\min}_{K^+\pi^+}} - 1\right),$$

$$\theta' \equiv \frac{1}{\pi}\theta_{K^+\pi^+},$$

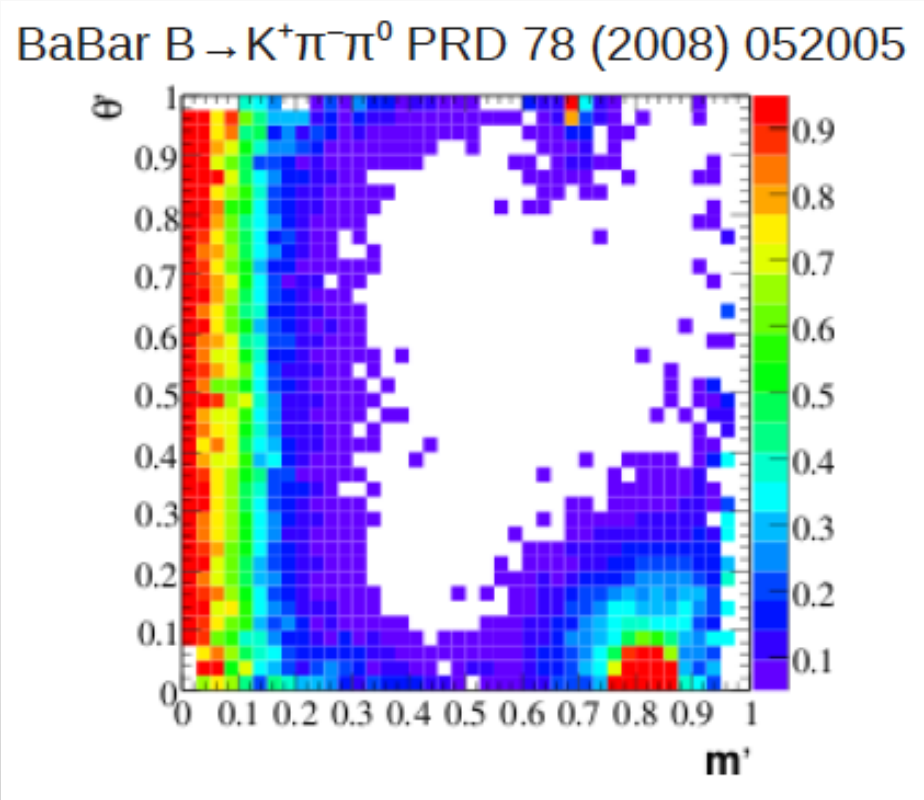


Squared Dalitz Plot



Misreconstruction (Self-Cross Feed)

Misreconstructed events: e.g. K→pi
Events are displaced by their original position
Very hard to model
MC simulation is used



Scalar → Vector Scalar Scalar (S→VSS)

Constraints	Degree of freedom
3 four-vectors	12
4-momentum conservation	-4
3 masses	-3
3 Euler angles	-3
Vector helicity	2
TOT	4

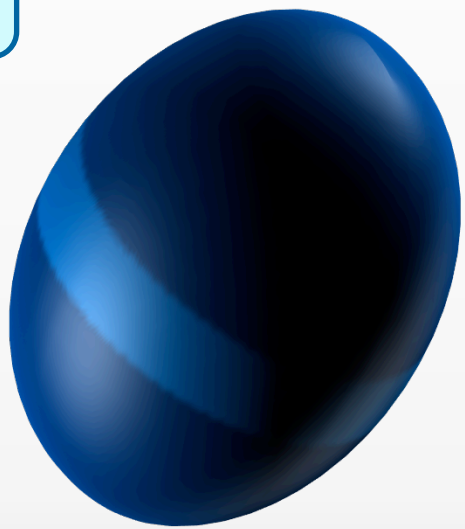
3-body decay with a vector state as a daughter

Helicity formalism is mandatory

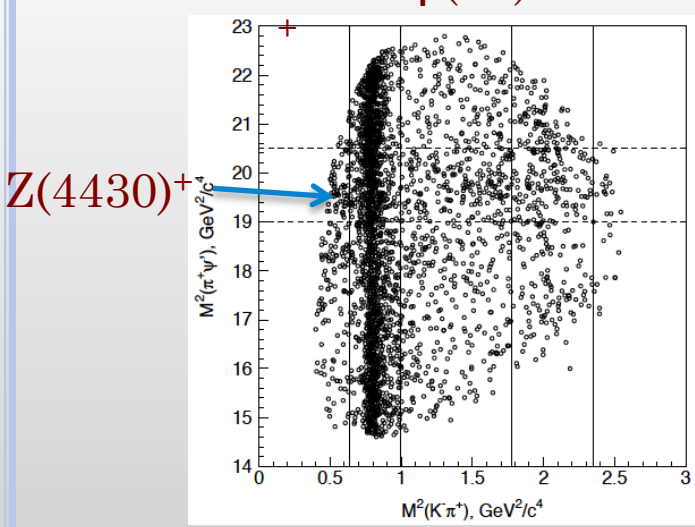
Scalar → Vector Scalar Scalar (S→VSS)

Recently the S→VSS decays got attention because of exotic states decaying to VS

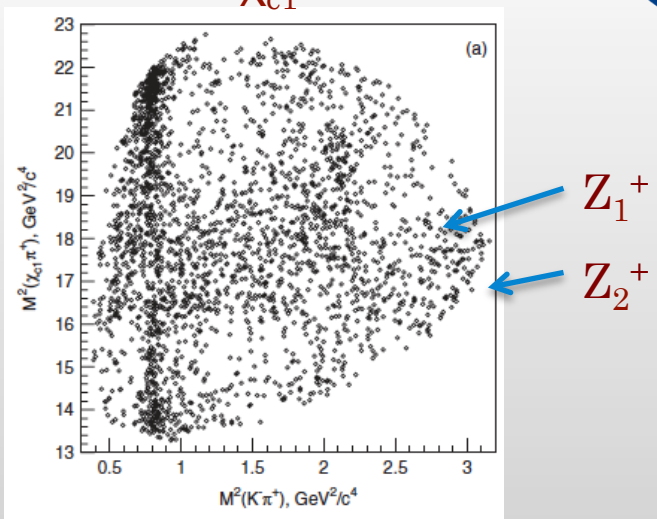
"4-D Dalitz plot"



$B^0 \to \psi(2S)\, K^- \pi$



$B^0 \to \chi_{c1}\, K^- \pi^+$

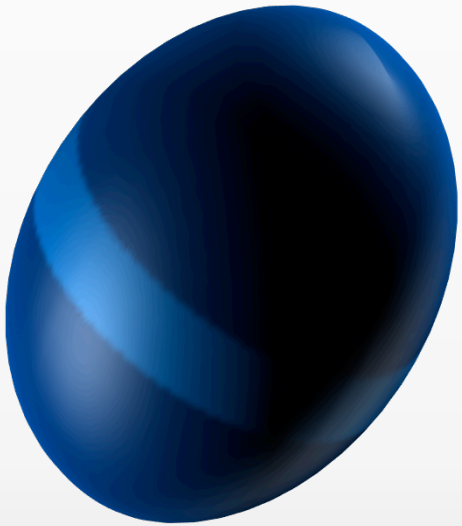


Shadows or new states?

4-Body Decay

Constraints	Degree of freedom
4 four-vectors	16
4-momentum conservation	-4
4 masses	-4
3 Euler angles	-3
TOT	5

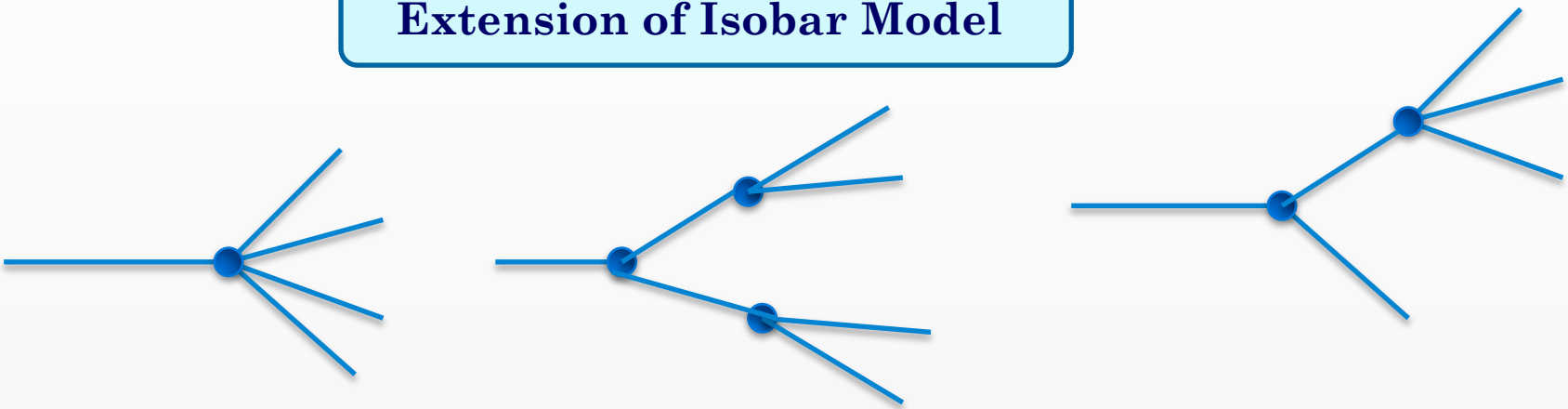
5-D Phase space or "5-D Dalitz plot"



The dynamic of decay is function of 5 independent variables
Common choice: m^2_{12}, m^2_{34}, $\cos\theta$, χ_1, χ_2

4-Body Decay

Extension of Isobar Model



Dalitz plot formalism in Ref. [19]). The amplitude for the decay chain $D^0 \to a_1(1260)^+\pi^-$, $a_1(1260)^+ \to \rho^0\pi^+$ (S-wave), $\rho^0 \to \pi^+\pi^-$, is

$$A = F_1^D \times \mathrm{BW}^{a_1} \times F_1^{a_1} \times \mathrm{BW}^{\rho} \times F_1^{\rho} \times \mathcal{M}. \quad (3)$$

The amplitude for the decay chain $D^0 \to \rho_1^0\rho_2^0$, $\rho_1^0 \to \pi^+\pi^-$, $\rho_2^0 \to \pi^+\pi^-$ is

$$A = F_l^D \times \mathrm{BW}^{\rho_1} \times F_1^{\rho_1} \times \mathrm{BW}^{\rho_2} \times F_1^{\rho_2} \times \mathcal{M}, \quad (4)$$

and for the chain $D^0 \to f_2(1270)\pi^+\pi^-$, $f_2(1270) \to \pi^+\pi^-$ the amplitude is

$$A = F_2^D \times \mathrm{BW}^{f_2} \times F_2^{f_2} \times \mathcal{M}. \quad (5)$$

Angular distribution: Zemach tensors applied in sequence

- Study of pseudo-scalar meson ($J^P = 0^-$) decaying to 3 scalars
- Search for CP violation:
  - Triple vector product (T violation)
  - Amplitude analysis