



NNLO $\tau^+\tau^-$ production cross section at threshold

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The threshold behaviour of the cross section $\sigma(e^+e^- \rightarrow \tau^+\tau^-)$ is analysed, taking into account the known higher-order corrections. At present, this observable can be determined to next-to-next-to-leading order (NNLO) in a combined expansion in powers of α_s and fermion velocities.

1. Introduction

The Tau-Charm Factory, a high-luminosity ($\sim 10^{33} \text{ cm}^{-2} \text{ s}^{-1}$) e^+e^- collider with a centre-of-mass energy near the $\tau^+\tau^-$ production threshold, has been proposed as a powerful tool to perform high-precision studies of the τ lepton, charm hadrons and the charmonium system [1]. In recent years, this energy region has been only partially explored by the Chinese BEBC machine ($\sim 10^{31} \text{ cm}^{-2} \text{ s}^{-1}$). The possibility to operate the Cornell CESR collider around the $\tau^+\tau^-$ threshold has revived again the interest on Tau-Charm Factory physics.

A precise understanding of the $e^+e^- \rightarrow \tau^+\tau^-$ production cross section near threshold is clearly required. The accurate experimental analysis of this observable could allow to improve the present measurement [2] of the τ lepton mass. The cross section $\sigma(e^+e^- \rightarrow \tau^+\tau^-)$ has already been analysed to $\mathcal{O}(\alpha^3)$ in refs. [3], including a resummation of the leading Coulomb corrections.

The recent development of non-relativistic effective field theories of QED (NRQED) and QCD (NRQCD) [4] has allowed an extensive investigation of the threshold production of heavy flavours at e^+e^- colliders. The threshold $b\bar{b}$ [5] and $t\bar{t}$ [6] production cross sections have been computed to the next-to-next-to-leading order (NNLO) in a combined expansion in powers of α_s and the fermion velocities. Making appropriate changes, those calculations can be easily applied to the study of $\tau^+\tau^-$ production [7]. One can then achieve a theoretical precision better than 0.1%.

2. Perturbative calculation to $\mathcal{O}(\alpha^4)$

At lowest order in QED, the τ leptons are produced by one-photon exchange in the s-channel, and the total cross section formula reads

$$\sigma_B(e^+e^- \rightarrow \tau^+\tau^-) = \frac{2\pi\alpha^2}{3s} v(3-v^2), \quad (1)$$

where $v = \sqrt{1 - 4m_\tau^2/s}$ is the velocity of the final τ leptons in the center-of-mass frame of the e^+e^- pair which makes σ_B vanish when $v \rightarrow 0$.

Electromagnetic corrections of $\mathcal{O}(\alpha)$ arise from the interference between the tree level result and 1-loop amplitudes. A factor α/v emerges in the 1-loop final state interaction between the tau leptons, making the cross section at threshold finite. Furry's theorem guarantees that contributions to $\sigma(e^+e^- \rightarrow \tau^+\tau^-)$ coming from initial, intermediate and final state corrections completely factorize at $\mathcal{O}(\alpha^3)$, including real photon emission.

Some undesirable features appear at $\mathcal{O}(\alpha^4)$: The two-loop $\tau^+\tau^-\gamma$ vertex develops an α^2/v^2 term which makes the cross section ill-defined when $v \rightarrow 0$, and multiple photon production of tau leptons by box-type diagrams and the non-zero interference of initial and final state radiation spoil exact factorization. However, as it has been recently shown [8], the squared amplitude of the $e^+e^- \rightarrow \tau^+\tau^-$ box diagram is proportional to $\alpha^4 v^2$, and so represents a N⁴LO correction in the combined expansion in powers of α and v , far beyond the scope of this analysis. In addition, contributions to the total cross section from diagrams with real photons emitted from the produced taus can be shown to begin at N³LO, and

factorization remains at NNLO. The total cross section can thus be written as an integration over the product of separate pieces including initial, intermediate and final state corrections:

$$\sigma(s) = \int^s F(s, w) \left| \frac{1}{1 + e^{2\Pi_{\text{em}}(w)}} \right|^2 \tilde{\sigma}(w) dw. \quad (2)$$

The radiation function $F(s, w)$ [9] describes initial state radiation, including virtual corrections. The integration accounts for the effective energy loss due to photon emission from the $e^+ e^-$ pair, and it includes the largest corrections coming from the emission of an arbitrary number of initial photons, which can sizeably suppress the total cross section. $\tilde{\sigma}$ collects only final-state interactions between the tau leptons, and it is usually written in terms of the tau spectral density R_τ ,

$$\tilde{\sigma}(e^+ e^- \rightarrow \gamma^* \rightarrow \tau^+ \tau^-) = R_\tau(s) \sigma_{pt}, \quad (3)$$

with $\sigma_{pt} = \frac{4\pi\alpha^2}{3s}$. The threshold behaviour of the total cross section will be ruled by the expansion of R_τ at low velocities.

3. Non-Relativistic Corrections: NRQED

A NNLO calculation of the cross section in the kinematic region where $v \sim \alpha$ has to account for all terms proportional to $v(\alpha/v)^n \times [1; \alpha; v; \alpha^2; \alpha v; v^2]$ with $n = 1, 2, \dots$. The leading divergences (i.e. $(\frac{\alpha}{v})^n$, $n > 1$) are resummed in the well-known Sommerfeld factor [10]

$$|\Psi_{c,E}(0)|^2 = \frac{\alpha\pi/v}{1 - \exp(-\alpha\pi/v)}, \quad (4)$$

multiplying the Born cross section (1). A systematic way to calculate higher-order corrections in this regime requires the use of a simplified theory which keeps the relevant physics at the scale $Mv \sim M\alpha$, characteristic of the Coulomb interaction. NRQED [4] was designed precisely for this purpose. It is an effective field theory of QED at low energies, applicable to fermions in non-relativistic regimes, i.e. with typical momenta $p/M \sim v \ll 1$. Interactions contained in the NRQED Lagrangian have a definite velocity counting but propagators and loop integrations can also generate powers of v . With appropriate

counting rules at hand, one can prove that all interactions between the non-relativistic pair $\tau^+ \tau^-$ can be described up to NNLO in terms of time-independent potentials [11], derived from the low-energy Lagrangian. Therefore, the low-energy expression of the τ spectral density is related with the non-relativistic Green's functions [12]:

$$R_\tau^{\text{NNLO}}(s) = \frac{6\pi}{M^2} \text{Im} \left(C_1 G(E) - \frac{4E}{3M} G_c(E) \right) \quad (5)$$

with C_1 a short distance coefficient to be determined by matching full and effective theory results and $E = m_\tau v^2$ the non-relativistic energy. The details of this derivation can be found in the Appendix B of [7]. The Green's function G obeys the Schrödinger equation corresponding to a two-body system interacting through potentials derived from $\mathcal{L}_{\text{NRQED}}$ at NNLO:

$$\left(-\frac{\nabla^2}{M} - \frac{\nabla^4}{4M^3} + V_c(\mathbf{r}) + V_{\text{BF}}(\mathbf{r}) + V_{\text{An}}(\mathbf{r}) - E \right) \times G(\mathbf{r}, \mathbf{r}', E) = \delta^{(3)}(\mathbf{r} - \mathbf{r}') \quad (6)$$

The term $-\frac{\nabla^4}{4M^3}$ is the first relativistic correction to the kinetic energy. V_c stands for the Coulomb potential with $\mathcal{O}(\alpha^2)$ corrections. At NNLO, the heavy leptons are only produced in triplet S-wave states, so we just need to consider the corresponding projection of the Breit-Fermi potential V_{BF} . Finally, V_{An} is a NNLO piece derived from a contact term in $\mathcal{L}_{\text{NRQED}}$, which reproduces the QED tree level s-channel diagram for the process $\tau^+ \tau^- \rightarrow \tau^+ \tau^-$.

A solution of eq. (6) must rely on numerical or perturbative techniques. In the QED case, a significant difference between both approaches is not expected, being α such a small parameter. Consequently we follow the perturbative approach, using recent results [12], where NLO and NNLO corrections to the Green's function are calculated analytically, via Rayleigh-Schrödinger time-independent perturbation theory around the known LO Coulomb Green's function G_c . We refer the reader to Appendix C of [7] for complete expressions of the Green's function corrections.

4. Vacuum Polarization

For a complete NNLO description of $\sigma(e^+e^- \rightarrow \tau^+\tau^-)$, two-loop corrections to the photon propagator should be included. The light lepton contributions to the vacuum polarization are the standard 1- and 2-loop perturbative expressions. For the τ contribution in the threshold vicinity $q^2 \gtrsim 4M^2$, resummation of singular terms in the limit $v \rightarrow 0$ is again mandatory. Under the assumption $\alpha \sim v$, we need to know NLO contributions to $\Pi_\tau(q^2)$, performing the direct matching for both real and imaginary parts.

In the hadronic sector, we can relate the hadronic vacuum polarization with the total cross section $\sigma(e^+e^- \rightarrow \gamma^* \rightarrow \text{had})$. Below 1 GeV, the electromagnetic production of hadrons is dominated by the ρ resonance and its decay to two charged pions. The photon mediated $\pi^+\pi^-$ production cross section is driven by the pion electromagnetic form factor $F(s)$. An analytic expression for $F(s)$ was obtained in Ref. [13], using Resonance Chiral Theory and the restrictions imposed by analyticity and unitarity. The obtained $F(s)$ provides an excellent description of experimental data up to energies of the order $s_\rho \sim 1 \text{ GeV}^2$.

For the integration region above s_ρ , we use $\text{Im}\Pi_{\text{had}}$ as calculated from pQCD. Our simple estimate has been proved [7] to deviate by less than 5% for the running of α at the scale $\sqrt{s} = M_Z$. Considering that Π_{had} modifies $\sigma(e^+e^- \rightarrow \tau^+\tau^-)$ near threshold by roughly 1%, our result has a global uncertainty smaller than 0.1% for the total cross section.

5. Bound states effect

Green's functions develop energy poles below threshold corresponding to spin triplet (n^3S_1) electromagnetic $\tau^+\tau^-$ bound states. The small width of these bound levels, dominated by their e^+e^- decay rate

$$\Gamma_{ee} = \frac{m_\tau \alpha^5}{6n^3} \approx \frac{6.1 \cdot 10^{-3} \text{ eV}}{n^3}, \quad (7)$$

make these states very difficult to be resolved experimentally. For the same reason, the bound states cannot affect the shape of the cross section

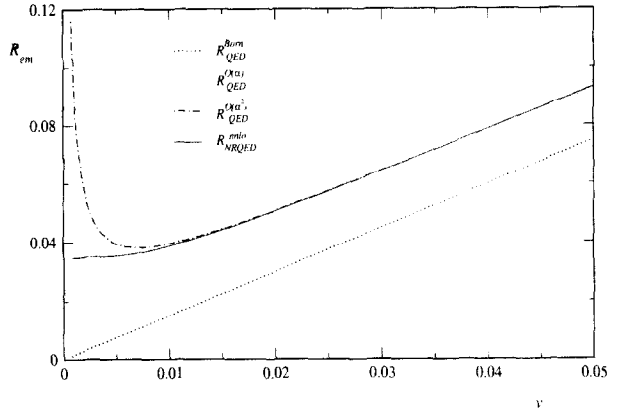


Figure 1. The spectral density R_τ at low velocities in both QED and NRQED.

at threshold, contrary to the case of heavy quark threshold production, where bound states play a crucial role.

6. Numerical analysis for $\sigma(e^+e^- \rightarrow \tau^+\tau^-)$

The need for performing resummations of the leading non-relativistic terms $(\alpha/v)^n [v, v\alpha, v^2, \dots]$ is evidenced in Fig. 1. The QED spectral density vanishes as $v \rightarrow 0$, due to the phase space velocity in formula (1), which is cancelled by the first v^{-1} term appearing in the $\mathcal{O}(\alpha)$ correction, making the cross section at threshold finite. More singular terms near threshold, v^{-2}, \dots arising in higher-order corrections completely spoil the expected good convergence of the QED perturbative series at low v . This is no longer the case for the effective theory perturbative series, whose convergence improves as we approach the threshold point, as shown in Fig. 2, and higher-order corrections reduce the perturbative uncertainty. In the whole energy range displayed in Fig. 2, the differences between the NNLO, NLO and LO results are below 0.8%, which indicates that the LO result, i.e. the Sommerfeld factor, contains the relevant physics to describe the threshold region. Adding the intermediate and initial state corrections we have a complete description of the total cross section of $\tau^+\tau^-$ production, as shown in Fig. 3. Coulomb interaction between the produced τ 's, becomes essential within few MeV's above the threshold,

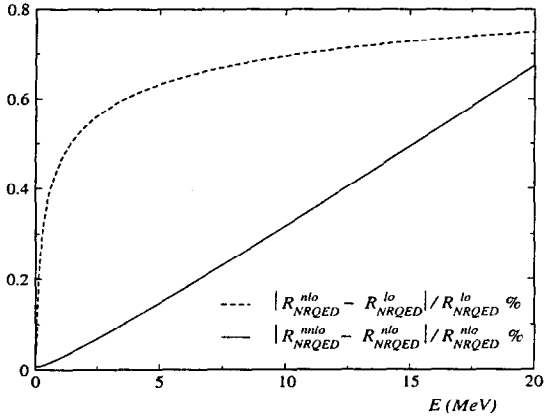


Figure 2. Relative sizes of corrections to $R_\tau(s)$ as calculated in NRQED.

and its effects have to be taken into account to all orders. Initial state radiation effectively reduces the available center-of-mass energy for τ production, lowering in this way the total cross section.

We should emphasize that NNLO corrections do not modify the predicted behaviour of the LO and NLO cross section, but are essential to guarantee that the truncated perturbative series at NLO gets small corrections from higher-order terms. Hence, we have shown that the theoretical uncertainty of our analysis of $\sigma(e^+e^- \rightarrow \tau^+\tau^-)$ is lower than 0.1%. Nevertheless, the statistical uncertainty of the most recent experiments is still much larger than the theoretical one due to low statistics, and we should wait for future machines to improve it.

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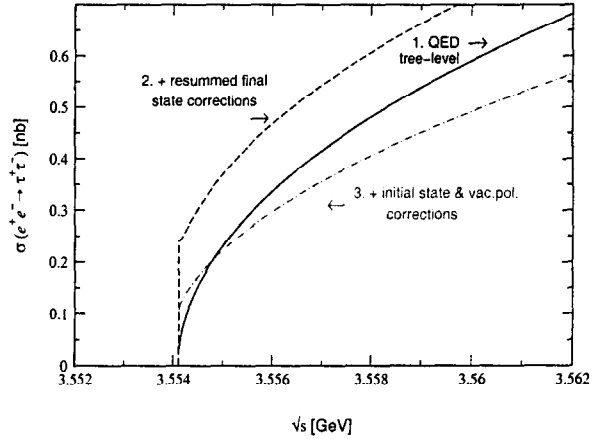


Figure 3. Total cross section $\sigma(e^+e^- \rightarrow \tau^+\tau^-)$ at threshold.

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