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Vertex reconstruction by means of the method of Kalman filtering

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The concept of Kalman filtering is described as a tool to find and reconstruct the location of interaction points in high-energy physics events. We discuss the method for using charged particle tracks, and we propose for the first time to apply it also for neutral particles, and to combine the information from both charged and neutral particles originating from a common point in space. The formalism is derived in detail for a particular choice of parametrisation, which we consider representative for a typical particle physics experiment.

1. Introduction

In the past, the most common method to find and reconstruct the origin and precise location of interactions, called the *vertex*, was the *method of least squares fitting* [1]. In this paper we promote another linear mathematical technique, which has only recently become more popular among the HEP community: the method of *Kalman filtering*. Despite the fact that it was "invented" for other purposes [2], it has already been successfully applied as a tool for reconstructing charged particle tracks, see e.g. ref. [3].

The general description of the concept of track and vertex reconstruction with a Kalman filter technique can be found in refs. [4] * and [5]. In the application of vertex reconstruction one can derive the underlying formulae also by minimization of a chisquare of a special form, which we call a local chisquare. Since the technique of minimizing a chisquare is common practice, this approach leads to an easier understanding of this particular application of the Kalman filter formalism. Although within this framework the vertex reconstruction can be completely described (appendix A), the dynamical structure of the Kalman filter formalism, which is important for many other applications, will not become transparent. Therefore the more interested reader is referred to the textbooks. In this paper we describe the formalism of vertex reconstruction by the method of Kalman filtering in the sense of minimization of a local chisquare. We derive in detail the formulae for a particular choice of parametrization, which is representative of applications in a typical high-energy physics detector. The particular experiment in mind here is the H1-experiment [6] at the electron–proton collider HERA in Hamburg (Germany). We describe the case of charged particles and, for the first time, propose a method to consistently include also neutral particles into the formalism in combination with charged particles.

The paper is organized as follows: In section 2 the general formalism is described. In section 3 we introduce our particular choices of parametrisation for the different vectors used in the formalism. The parametrisation is most critical and has to be chosen very carefully. By fixing the parametrisation, the

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* We believe that the variable $\mathbf{E}_k$ in eq. (28) on p. 449 should read: $\mathbf{E}_k = -\mathbf{C}_k \mathbf{A}_k^{\mathrm{T}} \mathbf{G}_k \mathbf{B}_k \mathbf{W}_k$.

mapping between the measured particle track parameters and the parameters of the fit, the "measurement mapping", is defined, which is listed explicitly. The case of charged particles is considered and a similar formalism is introduced for neutral particles. In section 4 we describe the combined treatment of charged and neutral objects, which can be applied for example in the full reconstruction of the successive steps in exclusive particle decay chains. In the following, *objects* can be interpreted as a single charged or neutral particle, as well as a combination of multiple particles, like for instance a jet. Our findings are summarized in section 5, and in the appendices a complete derivation of the filter equations based on the minimization of a local chisquare is given, and the Jacobian matrices of the measurement mapping are written out explicitly.

2. Kalman filter and vertex reconstruction

In the standard least squares fit formalism for vertex reconstruction all the tracks within an event are fitted to a single vertex in one single step. It is in this sense a *global* method. Therefore, the dimensions of matrices and vectors in this formalism are proportional to the number N of measured tracks of the event. Because the required processing time for the inversion of a matrix is proportional to N^3, this behaviour is unsatisfactory, particularly in the view of today's existing and proposed high-energy physics collider experiments, which feature quite a high multiplicity of charged particles. In addition, due to its global structure, this formalism is less flexible in handling different vertex hypotheses within a single event, such as removing spurious tracks or taking only a few tracks of the event together to search for secondary vertices.

On the other hand, due to the original task of the Kalman filter to optimize dynamical systems, its basic idea is to use the information of different particle trajectories about the vertex consecutively one after the other. In this sense it can be considered as a *local* technique.

Beginning with a start value for the vertex position, one compares this position with the information about the vertex from one track. By a correct weighting of the start value and the information of the track, which is in fact the minimization of a local chisquare, one calculates a new estimate for the vertex position. This position is then compared with the information of the next particle track of the event yielding another new estimate. This procedure is then repeated for each track one wants to fit to a single vertex, until one obtains the final vertex position. This technique is called *filter*. Apart from the vertex position one needs to know also the momenta of the particles at this vertex. The calculation of these momenta at the final vertex position is done afterwards and is called *smoothing*. Although the described structure differs strongly from the standard technique of the least squares fit method, the estimates of the vertex are the same, being the best attainable estimates in the mathematical sense.

Before describing the formalism in more detail, we define our notation:

$\boldsymbol{x}_k$ = estimate of the vertexposition after using the information of k tracks,

$\boldsymbol{x}^{\mathrm{t}}$ = true vertex position,

$\mathbf{C}_k = \mathrm{cov}(\boldsymbol{x}_k) := \mathrm{cov}(\boldsymbol{x}_k - \boldsymbol{x}^t)$,

$\boldsymbol{q}_k$ = estimate of the momentum of particle k at $\boldsymbol{x}_k$,

$\boldsymbol{q}_k^{\mathrm{t}}$ = true momentum of particle k at $\boldsymbol{x}^{\mathrm{t}}$,

$$\mathbf{D}_k = \mathrm{cov}(\boldsymbol{q}_k) := \mathrm{cov}(\boldsymbol{q}_k - \boldsymbol{q}_k^{\mathrm{t}}), \tag{1}$$

$\mathbf{E}_k = \mathrm{cov}(\boldsymbol{x}_k, \boldsymbol{q}_k) := \mathrm{cov}\big((\boldsymbol{x}_k - \boldsymbol{x}^{\mathrm{t}}), (\boldsymbol{q}_k - \boldsymbol{q}_k^{\mathrm{t}})\big)$,

$\boldsymbol{m}_k$ = measurement vector, five measured parameters of track k (see below),

$\boldsymbol{v}_k$ = measurement noise (disturbance),

$\mathbf{V}_k = \mathrm{cov}(\boldsymbol{v}_k)$ = covariance matrix,

$\mathbf{G}_k = \mathbf{V}_k^{-1}$ = weightmatrix of particle k.

The measurement equation describes a mapping function $\boldsymbol{h}$ of the true vertex position $\boldsymbol{x}^{\mathrm{t}}$ and the true momentum $\boldsymbol{q}_k^{\mathrm{t}}$ of track k at this position to the measured parameters $\boldsymbol{m}_k$ of this track k distorted by the measurement noise $\boldsymbol{v}_k$,

$$\boldsymbol{m}_k = \boldsymbol{h}_k(\boldsymbol{x}^{\mathrm{t}}, \boldsymbol{q}_k^{\mathrm{t}}) + \boldsymbol{v}_k. \tag{2}$$

All $\boldsymbol{v}_k$ are assumed to be independent, unbiased and of finite variance. In the application of Kalman filtering, the measurement equation must be linear. Therefore we assume the following form:

$$\boldsymbol{h}_k(\boldsymbol{x}^{\mathrm{t}}, \boldsymbol{q}_k^{\mathrm{t}}) \approx \boldsymbol{h}_k(\boldsymbol{x}_k^{(0)}, \boldsymbol{q}_k^{(0)}) + \mathbf{A}_k(\boldsymbol{x}^{\mathrm{t}} - \boldsymbol{x}_k^{(0)}) + \mathbf{B}_k(\boldsymbol{q}_k^{\mathrm{t}} - \boldsymbol{q}_k^{(0)}) = \boldsymbol{c}_k^{(0)} + \mathbf{A}_k\boldsymbol{x}^{\mathrm{t}} + \mathbf{B}_k\boldsymbol{q}_k^{\mathrm{t}}, \tag{3}$$

with

$$\mathbf{A}_k = \left.\frac{\partial \mathbf{h}_k}{\partial \boldsymbol{x}^{\mathrm{t}}}\right|_{(\boldsymbol{x}_k^{(0)}, \boldsymbol{q}_k^{(0)})}, \qquad \mathbf{B}_k = \left.\frac{\partial \mathbf{h}_k}{\partial \boldsymbol{q}_k^{\mathrm{t}}}\right|_{(\boldsymbol{x}_k^{(0)}, \boldsymbol{q}_k^{(0)})}, \qquad \boldsymbol{c}_k^{(0)} = \boldsymbol{h}_k(\boldsymbol{x}_k^{(0)}, \boldsymbol{q}_k^{(0)}) - \mathbf{A}_k\boldsymbol{x}_k^{(0)} - \mathbf{B}_k\boldsymbol{q}_k^{(0)}.$$

A reasonable choice for the point of linearisation $(\boldsymbol{x}_k^{(0)}, \boldsymbol{q}_k^{(0)})$ is for $\boldsymbol{x}_k^{(0)}$ the estimate after $k-1$ tracks $\boldsymbol{x}_{k-1}$, and for $\boldsymbol{q}_k^{(0)}$ the momentum at the point on the track k, which is closest to $\boldsymbol{x}_k^{(0)}$.

The vertex reconstruction with the Kalman filter then proceeds as follows: First, a start value for the vertex position $\boldsymbol{x}_0$ and for its covariance matrix $\mathbf{C}_0$ is chosen. If there is no a priori information about the vertex position available, a good choice for $\boldsymbol{x}_0$ is the origin of the coordinate system, and for the covariance matrix $\mathbf{C}_0$ a diagonal matrix with infinitely large diagonal elements. If an a priori estimate for the vertex position with a correct covariance matrix exists (e.g. from the beam position), one has to take these values. Next, this $\boldsymbol{x}_0$ is compared with the information about the vertex position obtained from the measured parameters $\boldsymbol{m}_1$ of one track of the event. This is accomplished through a weighted addition of the vertex information we have from one measured track $\boldsymbol{m}_1$ to the chisquare of our previous guess for the vertex position $\boldsymbol{x}_0$. This chisquare obtains the form

$$\chi^2_{\mathrm{KF}}(\boldsymbol{x}, \boldsymbol{q}) = (\boldsymbol{x} - \boldsymbol{x}_0)^{\mathrm{T}}(\mathbf{C}_0)^{-1}(\boldsymbol{x} - \boldsymbol{x}_0) + (\boldsymbol{m}_1 - \boldsymbol{c}_1^{(0)} - \mathbf{A}_1\boldsymbol{x} - \mathbf{B}_1\boldsymbol{q})^{\mathrm{T}}\mathbf{G}_1(\boldsymbol{m}_1 - \boldsymbol{c}_1^{(0)} - \mathbf{A}_1\boldsymbol{x} - \mathbf{B}_1\boldsymbol{q}). \tag{4}$$

The position $\boldsymbol{x}$ and the momentum $\boldsymbol{q}$ that minimize this χ^2_{KF} are the first estimates for $\boldsymbol{x}_1$ and for $\boldsymbol{q}_1$ at this vertex. A correct error propagation (see appendix A) yields the covariance matrix $\mathbf{C}_1$ of $\boldsymbol{x}_1$. Note, that the χ^2_{KF} depends only on the measured parameters of one track, whereas the chisquare of a global least squares fit contains the measured parameters of all tracks of the event which are included. It is for this reason, that we refer to χ^2_{KF} as a local chisquare.

Next, this first estimate of the vertex position $\boldsymbol{x}_1$ is compared with the next track $\boldsymbol{m}_2$ by minimization of

$$\chi^2_{\mathrm{KF}}(\boldsymbol{x}, \boldsymbol{q}) = (\boldsymbol{x} - \boldsymbol{x}_1)^{\mathrm{T}}(\mathbf{C}_1)^{-1}(\boldsymbol{x} - \boldsymbol{x}_1) + (\boldsymbol{m}_2 - \boldsymbol{c}_2^{(0)} - \mathbf{A}_2\boldsymbol{x} - \mathbf{B}_2\boldsymbol{q})^{\mathrm{T}}\mathbf{G}_2(\boldsymbol{m}_2 - \boldsymbol{c}_2^{(0)} - \mathbf{A}_2\boldsymbol{x} - \mathbf{B}_2\boldsymbol{q}).$$

This procedure is then repeated for every single track one wishes to include in the fit, resulting in the final estimate for the vertex position $\boldsymbol{x}_N$ and its covariance matrix $\mathbf{C}_N$, where N is the total number of used tracks. This technique is called *filtering*.

The momentum $\boldsymbol{q}_k$ of track k is calculated at the vertex $\boldsymbol{x}_k$ including only the information of k tracks. After the filtering one has to recalculate the momenta of all tracks at this final vertex position $\boldsymbol{x}_N$. This is accomplished in the *smoothing* part.

Since especially for a small number of tracks the final vertex position $\boldsymbol{x}_N$ may still depend on the start position $\boldsymbol{x}_0$, this $\boldsymbol{x}_N$ can be chosen as the new start-position and the entire procedure restarts until it converges according to some predefined criteria. We use for the convergence criteria the change in the total chisquare χ^2_N defined below.

Thus the following equations hold (a complete derivation for the filter equations based on the minimization of the local chisquare is given in appendix A):

Filtering:

$$\begin{aligned} \boldsymbol{x}_k &= \mathbf{C}_k\left[(\mathbf{C}_{k-1})^{-1}\boldsymbol{x}_{k-1} + \mathbf{A}_k^{\mathrm{T}}\mathbf{G}_k^{\mathbf{B}}\left(\boldsymbol{m}_k - \boldsymbol{c}_k^{(0)}\right)\right], \\ \boldsymbol{q}_k &= \mathbf{W}_k\mathbf{B}_k^{\mathrm{T}}\mathbf{G}_k\left(\boldsymbol{m}_k - \boldsymbol{c}_k^{(0)} - \mathbf{A}_k\boldsymbol{x}_k\right); \end{aligned} \tag{5}$$

$$\begin{aligned} \mathbf{C}_k &= \left((\mathbf{C}_{k-1})^{-1} + \mathbf{A}_k^{\mathrm{T}}\mathbf{G}_k^{\mathbf{B}}\mathbf{A}_k\right)^{-1}, \\ \mathbf{D}_k &= \mathbf{W}_k + \mathbf{W}_k\mathbf{B}_k^{\mathrm{T}}\mathbf{G}_k\mathbf{A}_k\mathbf{C}_k\mathbf{A}_k^{\mathrm{T}}\mathbf{G}_k\mathbf{B}_k\mathbf{W}_k, \\ \mathbf{E}_k &= -\mathbf{C}_k\mathbf{A}_k^{\mathrm{T}}\mathbf{G}_k\mathbf{B}_k\mathbf{W}_k, \end{aligned} \tag{6}$$

with

$$\begin{aligned} &\mathbf{W}_k = \left(\mathbf{B}_k^{\mathrm{T}}\mathbf{G}_k\mathbf{B}_k\right)^{-1}, \qquad \mathbf{G}_k^{\mathbf{B}} = \mathbf{G}_k - \mathbf{G}_k\mathbf{B}_k\mathbf{W}_k\mathbf{B}_k^{\mathrm{T}}\mathbf{G}_k, \\ &\mathrm{cov}(\boldsymbol{x}_k) = \mathbf{C}_k, \qquad \mathrm{cov}(\boldsymbol{q}_k) = \mathbf{D}_k, \quad \mathrm{cov}(\boldsymbol{x}_k, \boldsymbol{q}_k) = \mathbf{E}_k, \end{aligned}$$

and for the chisquare

$$\chi_k^2 = \chi_{k-1}^2 + \chi_{\mathrm{KF}}^2,$$

with

$$\begin{aligned} \chi_{\mathrm{KF}}^2 &= \boldsymbol{r}_k^{\mathrm{T}}\mathbf{G}_k\boldsymbol{r}_k + (\boldsymbol{x}_k - \boldsymbol{x}_{k-1})^{\mathrm{T}}(\mathbf{C}_{k-1})^{-1}(\boldsymbol{x}_k - \boldsymbol{x}_{k-1}), \\ \boldsymbol{r}_k &= \left(\boldsymbol{m}_k - \boldsymbol{c}_k^{(0)} - \mathbf{A}_k\boldsymbol{x}_k - \mathbf{B}_k\boldsymbol{q}_k\right). \end{aligned} \tag{7}$$

Smoothing:

$$\begin{aligned} \boldsymbol{x}_k^N &= \boldsymbol{x}_N, \\ \boldsymbol{q}_k^N &= \mathbf{W}_k\mathbf{B}_k^{\mathrm{T}}\mathbf{G}_k\left(\boldsymbol{m}_k - \boldsymbol{c}_k^{(0)} - \mathbf{A}_k\boldsymbol{x}_N\right); \end{aligned} \tag{8}$$

$$\begin{aligned} \mathbf{C}_k^N &= \mathbf{C}_N, \\ \mathbf{D}_k^N &= \mathbf{W}_k + \mathbf{W}_k\mathbf{B}_k^{\mathrm{T}}\mathbf{G}_k\mathbf{A}_k\mathbf{C}_N\mathbf{A}_k^{\mathrm{T}}\mathbf{G}_k\mathbf{B}_k\mathbf{W}_k, \\ \mathbf{E}_k^N &= -\mathbf{C}_N\mathbf{A}_k^{\mathrm{T}}\mathbf{G}_k\mathbf{B}_k\mathbf{W}_k. \end{aligned} \tag{9}$$

The covariance matrix for the correlation between the momenta $\mathbf{Q}_{kj}^N$ resulting from the smoothing is given by

$$\begin{aligned} \mathbf{Q}_{kj}^N &= \operatorname{cov}\left(\boldsymbol{q}_k^N, \boldsymbol{q}_j^N\right) := \operatorname{cov}\left(\boldsymbol{q}_k^N - \boldsymbol{q}_k^{\mathrm{t}}, \boldsymbol{q}_j^N - \boldsymbol{q}_j^{\mathrm{t}}\right), \\ \mathbf{Q}_{kj}^N &= \mathbf{W}_k \mathbf{B}_k^{\mathrm{T}} \mathbf{G}_k \mathbf{A}_k \mathbf{C}_N \mathbf{A}_j^{\mathrm{T}} \mathbf{G}_j \mathbf{B}_j \mathbf{W}_j. \end{aligned} \tag{10}$$

From this it follows, that by incorporating the information of each track separately, this formalism is very flexible in changing the tracks that one wants to include in the fit. This is important for searching for outliers and secondary vertices. In addition, the dimension of the matrices never exceeds five, the number of parameters to describe the track of a charged particle (see below).

A useful criterium to test a given vertex hypothesis, i.e. to decide whether the chosen tracks really belong to one vertex, is the value of the total chisquare of the fit, χ_N^2. In the optimal case the variable χ_N^2 behaves like a χ^2-distribution with a uniform probability distribution between 0 and 1 (see ref. [7] for explicit formulae). The number of degrees of freedom, ν, depends on the number of tracks N included in the vertex fit. If no a priori information of the vertex position is known, ν amounts to $\nu = 2N - 3$, in the other case $\nu = 2N$.

The formulae for the filtering and the smoothing given above are independent of the special choice of the parametrisation describing the tracks, the momenta and the position of the vertex. Apart from fitting only charged particles, we have also applied this formalism to fitting neutral objects as well as any combination of charged and neutral ones. Although the parametrisations and mappings for charged and neutral particles are different (as will be shown in the next section), this does not cause any principle problems within the framework of this formalism. In every iteration step in the filter, one basically only has to apply the corresponding mapping function. This simple treatment of different track types is again a direct consequence of the fact, that the information about the vertex is taken into account independently for each track.

3. Choice of parametrisation

The general formalism developed above for the vertex reconstruction contains three different vectors:

$\boldsymbol{x}_k$ = estimate for the vertex position by using the information of k tracks,
$\boldsymbol{q}_k$ = estimate for the momentum of particle k at $\boldsymbol{x}_k$,
$\boldsymbol{m}_k$ = the measured parameters of the track k.

In the practical application, the choice of parametrisation for these vectors is of great importance, and in general the two following requirements should be met:
(a) The errors of all these vectors should be unbiased, independent and normally distributed.
(b) The measurement mapping, which is a mapping from $\boldsymbol{x}_k$ and $\boldsymbol{q}_k$ to the measured track parameters $\boldsymbol{m}_k$, should be linear.

The choice of the parametrisation strongly influences both these conditions and they cannot be fully satisfied in our task of vertex reconstruction.

3.1. Parametrisation for charged particles

The trajectory of a charged particle in a uniform magnetic field is a helix. Five parameters are needed to define such a helix. We choose the origin of our coordinate system to be at the centre of a cylindrical detector with the z-axis being the axis of symmetry of the detector. A homogenuos magnetic field is assumed also in the $+z$-direction. Then the parameters for the measurement vector defining the helix are (with DCA being the point on the helix which has minimal distance to the z-axis, and with the allowed range of values included in brackets)

$$\begin{aligned}
&\boldsymbol{m}_{c_k}(1) = \kappa && (-\infty \mid +\infty) \text{ signed curvature,} \\
&\boldsymbol{m}_{c_k}(2) = \phi && (-\pi \mid +\pi) \text{ azimuth angle at DCA,} \\
&\boldsymbol{m}_{c_k}(3) = \theta && (0 \mid +\pi) \text{ polar angle,} \\
&\boldsymbol{m}_{c_k}(4) = \mathrm{dca} && (-\infty \mid +\infty) \text{ distance of closest approach,} \\
&\boldsymbol{m}_{c_k}(5) = z && (-\infty \mid +\infty)\ z \text{ at DCA.}
\end{aligned} \tag{11}$$

The subscript c stands for charged tracks and k denotes the track index. The projection of the helix onto the transverse x–y-plane is a circle and its inverse radius is termed the curvature κ (in units of cm^{-1}),

$$\kappa = -cz_{\mathrm{e}}B\frac{1}{p_{\mathrm{t}}}, \tag{12}$$

where $z_{\mathrm{e}}e$ is the charge of the particle, B the value of the magnetic field in tesla (chosen in the $+z$-direction), c the speed of light in nm/s and p_{t} the transverse momentum of the particle in GeV/c. The error of the inverse of the transverse momentum is normally distributed and therefore the curvature as well [7]. A positive (negative) sign of κ means a counter-clockwise (clockwise) rotation of the particle trajectory in the x–y-plane as viewed from the $+z$-direction. These definitions are shown explicitely in fig. 1.

ϕ is the angle between the transverse momentum p_{t} at DCA and the x-axis.

θ is the slope of the helix and defined by $\theta = \arccos(\mathrm{d}z/\mathrm{d}s)$, where s is the path length along the helix, increasing when moving in the particle direction. Note, that other parametrisations for $\boldsymbol{m}_{c_k}(3)$ have been used in the past, such as $\tan(\theta)$ or $\sin(\theta)$. The particular choice depends primarily on the detector geometry (material and multiple scattering) and on the particle momentum ranges considered.

dca is the minimal distance from the helix to the z-axis. The sign of dca is the sign of the vectorproduct ($\mathbf{DCA} \times \boldsymbol{p}_{\mathrm{t}}$) with **DCA** being the vector form the origin of the coordinate system to DCA.

z is the z-coordinate of DCA.

The signs of κ and dca are by no means trivial. They are defined such that there is no discontinuity in the parameters by changing one of the parameters (see fig. 1). Because this work was performed with the practical application within the H1-experiment in mind, our choice of parametrisation and the definition of eq. (12) conform with the standard H1-conventions for particle tracks, in which a positive particle possesses a negative κ-value (see refs. [6,8]).

One practical advantage of this parametrisation is that because the values are given at the DCA being close to the real vertex position, there is no additional material between the DCA and the vertex, and therefore there is in general no multiple scattering.

For the vertex position we choose Cartesian coordinates. When employing for example spherical coordinates and putting the true vertex position at the origin of the coordinate system, it is obvious that the error in the radius cannot be unbiased, as it is always positive. Similar problems arise for ϕ and θ. Therefore we find that in spherical or cylindrical coordinates the convergence of the fit strongly depends on the start values. One possibility to overcome these problems would be to set the origin of the coordinate system far away from the centre of the detector. We do not consider this to be a satisfactory solution of the problem. However, none of these problems arise when adopting Cartesian coordinates.

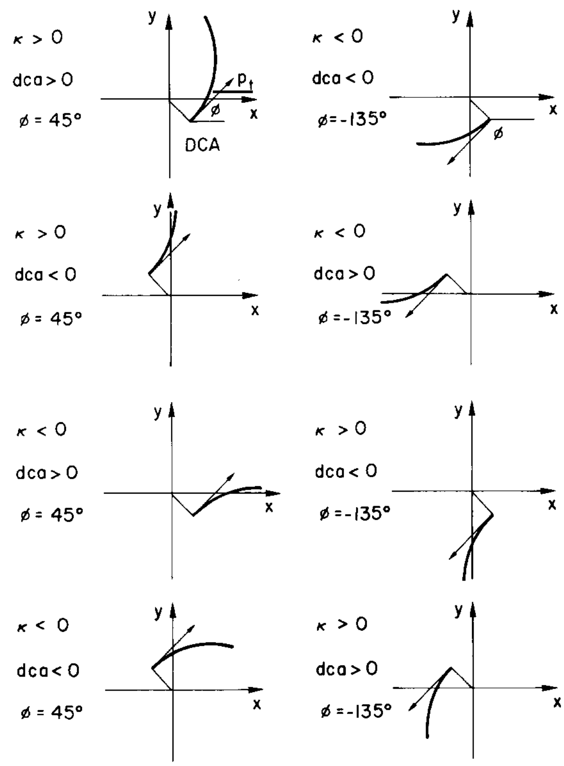

Fig. 1. Definition of the conventions used for the parameters in the equations. The different cases of projecting a track helix onto the $x-y$-plane illustrate the signs of κ and dca in eq. (11).

For the particle momentum, only geometrical parameters are used, hence it is referred to as the geometrical momentum,

$$
\begin{aligned}
&\boldsymbol{q}_{c_k}(1) = \kappa \quad (-\infty \,|\, +\infty) \text{ signed curvature,}\\
&\boldsymbol{q}_{c_k}(2) = \phi \quad (-\pi \,|\, +\pi) \text{ azimuth angle,}\\
&\boldsymbol{q}_{c_k}(3) = \theta \quad (0 \,|\, +\pi) \text{ polar angle.}
\end{aligned} \tag{13}
$$

The kinematical momentum can be calculated by means of eq. (12). This is the best choice by looking at requirement (b), since the mapping of the two parameters κ and θ to the measured parameters $\boldsymbol{m}_{c_k}(1)$ and $\boldsymbol{m}_{c_k}(2)$ is linear. In the experimental range of their values, they are also in good agreement with requirement (a).

The choice of the parameters also fixes the measurement equation, which is now a mapping from a point $P_0 = (x_0, y_0, z_0)$ on the helix and the geometrical momentum $\boldsymbol{q}_0 = (\kappa_0, \phi_0, \theta_0)$ at this point to the five parameters of the measurement vector at DCA, as illustrated in detail in fig. 2. There the rotation angle γ in the x–y-plane is defined by

$$\gamma = -\arctan(b/c) \quad \text{with} \quad b = x_0 \cos\phi_0 + y_0 \sin\phi_0,$$

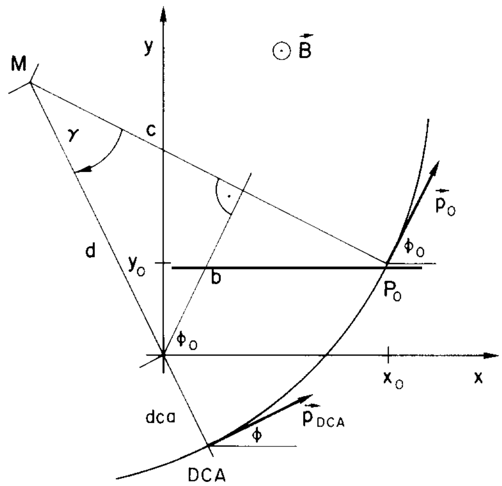

Fig. 2. Illustration of the geometrical meaning of the abbreviations used in eq. (14) for charged particles.

where b is the sum of the projections of x_0 and y_0 on the tangent in P_0, and

$$c = \frac{1}{\kappa_0} + y_0 \cos\phi_0 - x_0 \sin\phi_0$$

is the radius minus the sum of the projections of x_0 and y_0 on $\boldsymbol{MP}_0$, and therefore

$$\gamma = \arctan\left(\frac{x_0 \cos\phi_0 + y_0 \sin\phi_0}{x_0 \sin\phi_0 - y_0 \cos\phi_0 - 1/\kappa_0}\right), \qquad \mathrm{dca} = \frac{1}{\kappa_0} - \frac{c}{\cos\gamma}.$$

Hence the measurement equation assumes the form

$$\begin{aligned}
&\boldsymbol{m}_{c_k} = \boldsymbol{h}_c(x_0, y_0, z_0; \kappa_0, \phi_0, \theta_0),\\
&\kappa = \kappa_0, \qquad \phi = \phi_0 + \gamma, \qquad \theta = \theta_0,\\
&\mathrm{dca} = \frac{1}{\kappa_0} - \frac{(1/\kappa_0 + y_0 \cos\phi_0 - x_0 \sin\phi_0)}{\cos\gamma}, \qquad z = z_0 + \gamma\frac{\cos\theta_0}{\sin\theta_0}\frac{1}{\kappa_0}.
\end{aligned} \tag{14}$$

This particular choice of parametrisation gives a closed representation of the measurement equation. The only singularities occur at $\kappa_0 = 0$ and at $\theta_0 = 0$. The value $\kappa_0 = 0$ means that the radius is infinite. Tracks of charged particles have always finite radii, and tracks with $\theta = 0$ cannot be measured in practice due to the cylindrical form of the detector. By combining tracks of decay products of a mother particle (see below), the value $\theta = 0$ can occur for the mother particle. In practice, values in the vicinity of zero have to be treated separately to ensure proper numerical handling (derivatives, matrix inversions etc.). To complete the description of the formalism, the first derivatives of this function, $\mathbf{A}_c = \partial \boldsymbol{h}c/\partial \boldsymbol{x}$ and $\mathbf{B}_c = \partial \boldsymbol{h}c/\partial \boldsymbol{q}c$, applied for the linear approximation in the filter equations, can be found in appendix B.

3.2. Parametrisation for neutral particles

So far we have dealt only with tracks of charged particles. However, when fitting objects in the more general sense, we suggest that neutral particles should also be taken into consideration. This can be useful for instance within the context of a complete kinematical reconstruction of exclusive particle decay channels, where a neutral *mother* particle may decay into two charged *daughter* particles, or in the case of treating particle jets (conelike sprays of particles) as reconstructed objects. To achieve this, we have to choose a slightly different parametrisation for trajectories of neutral particles, being straight lines. The problem arises with the variable κ, the curvature, which is not defined for neutral particles because eq. (12) is only valid for charged particles.

To evade this difficulty, the following parameters are chosen for the measurement vector:

$$\begin{aligned}
&\boldsymbol{m}_{n_k}(1) = p && (0 \mid +\infty) \text{ absolute value of the total momentum,}\\
&\boldsymbol{m}_{n_k}(2) = \phi && (-\pi \mid +\pi) \text{ azimuth angle at DCA,}\\
&\boldsymbol{m}_{n_k}(3) = \theta && (0 \mid +\pi) \text{ polar angle,}\\
&\boldsymbol{m}_{n_k}(4) = \text{dca} && (-\infty \mid +\infty) \text{ distance of closest approach,}\\
&\boldsymbol{m}_{n_k}(5) = z && (-\infty \mid +\infty)\ z \text{ at DCA,}
\end{aligned} \tag{15}$$

and for the momentum:

$$\begin{aligned}
&\boldsymbol{q}_{n_k}(1) = p && (0 \mid +\infty) \text{ absolute value of the total momentum,}\\
&\boldsymbol{q}_{n_k}(2) = \phi && (-\pi \mid +\pi) \text{ azimuth angle at DCA,}\\
&\boldsymbol{q}_{n_k}(3) = \theta && (0 \mid +\pi) \text{ polar angle.}
\end{aligned} \tag{16}$$

For the vertex position we employ Cartesian coordinates as well. For a neutral object it is not possible to express the momentum with physically meaningful geometrical quantities only. Therefore we select the absolute value of the kinematical momentum for $\boldsymbol{q}_{n_k}(1)$ and $\boldsymbol{m}_{n_k}(1)$. In accordance with the definitions for charged particles, we still call $\boldsymbol{q}_{n_k}$ a geometrical momentum and reserve the term kinematical momentum for the parametrisation $\boldsymbol{p} = (p_x, p_y, p_z)$.

Again the vector $\boldsymbol{m}_{n_k}$ contains five parameters. In fact, the four parameters ϕ, θ, dca and z fully define a straight line. But in some applications, neutral particles only arise as composed objects, e.g. mother particles, where its momentum is obtained by adding the momenta of the daughters. In this case, the absolute value of the mother momentum is correlated to the other parameters of its measurement vector. In order not to lose information, we have to take all five parameters into the mapping (see also the next section where the relations between mother and daughters will be given).

With the definitions as outlined in fig. 3 the measurement equation reads

$$\begin{aligned}
&\boldsymbol{m}_{n_k} = \boldsymbol{h}_n(x_0, y_0, z_0; p_0, \phi_0, \theta_0),\\
&p = p_0, \qquad \phi = \phi_0 \qquad \theta = \theta_0, \qquad \text{dca} = r_\perp \sin(\xi),\\
&z = z_0 - r_\perp \cos(\xi) \frac{1}{\tan(\theta_0)},
\end{aligned} \tag{17}$$

with the following abbreviations:

$$r_\perp = \sqrt{x_0^2 + y_0^2}, \qquad \phi_0' = \arccos\left(\frac{x_0}{r_\perp}\right), \qquad \xi = \phi_0 - \phi_0'.$$

The only singularity lies at $\theta_0 = 0$ and its treatment proceeds along the same lines as in the case for charged particles. Again the first derivatives of this function, $\mathbf{A}_n = \partial \boldsymbol{h}_n / \partial \boldsymbol{x}$ and $\mathbf{B}_n = \partial \boldsymbol{h}_n / \partial \boldsymbol{q}_n$, applied for the linear approximation in the filter equations can be found in appendix B.

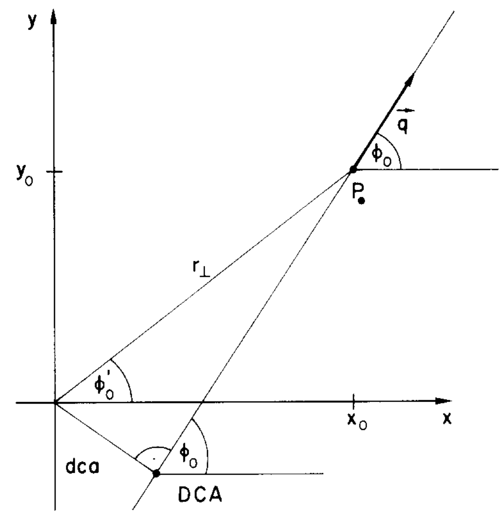

Fig. 3. Illustration of the geometrical meaning of the abbreviations used in eq. (17) for neutral particles.

4. Combined fit of charged and neutral particles

Consider for example the determination of the lifetime of a neutral particle (e.g. B^0 or D^0 meson) by means of the impact parameter measurement method. This exploits the fact that the particle (the mother) travels a reasonable distance through the detector and then decays at a secondary vertex position into other stable particles (its daughters). In such a case, one needs to know the position of the decay point and the momentum of the mother as well as a good estimate for the location of the primary vertex, which is defined as the common point of origin of the mother and other stable particles (not its own daughters). To include such a mother in a fit of primary vertex, we have to calculate its measurement vector and covariance matrix from the fitted momenta and covariance matrices of the decay products at the secondary vertex. This is in principle a straightforward calculation, but one has to take care of the differences between charged and neutral tracks, as described below.

The calculation of the measurement parameters of a combined object, which we shall generically refer to as mother in the following, is done in four steps:

(a) Determine the kinematical momenta from the fitted geometrical momenta of the particles at the secondary vertex.
(b) Add the kinematical momenta of the daughters to yield the kinematical momentum of the mother.
(c) Obtain the geometrical momentum of the mother from its kinematical momentum.
(d) Calculate the proper measurement parameters of the mother at its DCA.

In the following we use $\boldsymbol{q}$ for geometrical momenta, $\boldsymbol{p}$ for kinematical momenta, the subscript n for neutral tracks, the subscript c for charged tracks and the subscripts k, i, j for the track numbers.

In step (a) we have to convert the fitted and smoothed geometrical momenta of the tracks to kinematical momenta, while distinguishing between charged and neutral tracks, as the parametrisation for the geometrical momenta is different.

(i) Charged particle tracks

The parameters of the geometrical momentum are listed in eq. (13). The curvature is defined such that the sign of the charge $z_e e$ is given by $sign(z_e) = -sign(\kappa B)$ (see eq. (12)), where B is the magnetic field in the z-direction. Because the transverse momentum p_t has to be positive, we find $p_t = c\,|z_e B/\kappa|$. With this p_t and $\boldsymbol{p}_c = (p_x, p_y, p_z)$, the mapping $\boldsymbol{p}_c = \boldsymbol{f}_c(\boldsymbol{q}_c)$ then becomes

$$p_x = p_t \cos(\phi), \qquad p_y = p_t \sin(\phi), \qquad p_z = p_t \cot(\theta). \tag{18}$$

The Jacobian matrix for the error propagation, $\mathbf{W}_c = (\partial \boldsymbol{f}_c/\partial \boldsymbol{q})|_{\boldsymbol{q}_c^{(0)}}$, reads then

$$\mathbf{W}_c|_{\boldsymbol{q}_c^{(0)}} = \begin{pmatrix} -p_t(1/\kappa_0)\cos(\phi_0) & -p_t \sin(\phi_0) & 0 \\ -p_t(1/\kappa_0)\sin(\phi_0) & p_t \cos(\phi_0) & 0 \\ -p_t(1/\kappa_0)\cot(\theta_0) & 0 & -p_t[1/\sin^2(\theta_0)] \end{pmatrix}. \tag{19}$$

(ii) Neutral particles

The parameters of the geometrical momentum are defined in eq. (16). Using $\boldsymbol{p}_n = (p_x, p_y, p_z)$, the following mapping $\boldsymbol{p}_n = \boldsymbol{f}_n(\boldsymbol{q}_n)$ holds:

$$p_x = p \cos(\phi)\sin(\theta), \qquad p_y = p \sin(\phi)\sin(\theta), \qquad p_z = p\cos(\theta). \tag{20}$$

The Jacobian matrix for the error propagation, $\mathbf{W}_n = (\partial \boldsymbol{f}_n/\partial \boldsymbol{q})|_{\boldsymbol{q}_n^{(0)}}$, then becomes:

$$\mathbf{W}_n|_{\boldsymbol{q}_n^{(0)}} = \begin{pmatrix} \sin(\theta_0)\cos(\phi_0) & -p_0 \sin(\theta_0)\sin(\phi_0) & p_0\cos(\theta_0)\cos(\phi_0) \\ \sin(\theta_0)\sin(\phi_0) & p_0\sin(\theta_0)\cos(\phi_0) & p_0\cos(\theta_0)\sin(\phi_0) \\ \cos(\theta_0) & 0 & -p_0\sin(\theta_0) \end{pmatrix}. \tag{21}$$

In step (b) the kinematical momenta of all tracks are added to obtain the total momentum

$$\boldsymbol{p}_{tot} = \sum_{k=1}^{L} \boldsymbol{p}_{n_k} + \sum_{k=1}^{N} \boldsymbol{p}_{c_k}. \tag{22}$$

where L is the number of neutral tracks and N the number of charged tracks to be fitted to the vertex, with $M = N + L$ the sum of the two.

In step (c) the kinematical momentum of the mother is converted to the geometrical, which is needed in the subsequent step. Again the charge of the mother has to be taken into account.

(i) Charged mother particle with charge $z_{tot}e$

The parameters of the geometrical momentum are defined in eq. (13). Using $\boldsymbol{p}_{tot} = (p_x, p_y, p_z)$, the mapping $\boldsymbol{q}_{tot} = \boldsymbol{g}_c(\boldsymbol{p}_{tot})$ then corresponds to

$$\kappa = -\frac{c z_{tot} B}{\sqrt{p_x^2 + p_y^2}}, \qquad \phi = \arccos\left(\frac{p_x}{\sqrt{p_x^2 + p_y^2}}\right), \qquad \theta = \arccos\left(\frac{p_z}{\sqrt{p_x^2 + p_y^2 + p_z^2}}\right). \tag{23}$$

The Jacobian matrix for the error propagation, $\mathbf{W}_c^- = (\partial \boldsymbol{g}_c/\partial \boldsymbol{p})|_{\boldsymbol{p}_{tot}^{(0)}}$, results in

$$\mathbf{W}_c^-|_{\boldsymbol{p}_{tot}^{(0)}} = \begin{pmatrix} \dfrac{c z_{tot} B p_{x0}}{\left(p_{x0}^2 + p_{y0}^2\right)^{3/2}} & \dfrac{c z_{tot} B p_{y0}}{\left(p_{x0}^2 + p_{y0}^2\right)^{3/2}} & 0 \\ -\dfrac{p_{y0}}{p_{x0}^2 + p_{y0}^2} & \dfrac{p_{x0}}{p_{x0}^2 + p_{y0}^2} & 0 \\ \dfrac{p_{x0} p_{z0}}{p_0^2\sqrt{p_{x0}^2 + p_{y0}^2}} & \dfrac{p_{y0} p_{z0}}{p_0^2\sqrt{p_{x0}^2 + p_{y0}^2}} & -\dfrac{\sqrt{p_{x0}^2 + p_{y0}^2}}{p_0^2} \end{pmatrix} = (\mathbf{W}_c)^{-1}, \tag{24}$$

with $p_0 = \sqrt{p_{x0}^2 + p_{y0}^2 + p_{z0}^2}$.

(ii) neutral mother particle

The parameters of the geometrical momentum are listed in eq. (16). Using $\boldsymbol{p}_{\text{tot}} = (p_x, p_y, p_z)$, the mapping $\boldsymbol{q}_{\text{tot}} = \boldsymbol{g}_{\text{n}}(\boldsymbol{p}_{\text{tot}})$ is described by

$$p = \sqrt{p_x^2 + p_y^2 + p_z^2}, \qquad \phi = \arccos\left(\frac{p_x}{\sqrt{p_x^2 + p_y^2}}\right), \qquad \theta = \arccos\left(\frac{p_z}{\sqrt{p_x^2 + p_y^2 + p_z^2}}\right). \tag{25}$$

The Jacobian matrix for the error propagation, $\mathbf{W}_{\text{n}}^- = (\partial \boldsymbol{g}_{\text{n}}/\partial \boldsymbol{p})|_{\boldsymbol{p}_{\text{tot}}^{(0)}}$, is the following:

$$\mathbf{W}_n^- |_{\boldsymbol{p}_{\text{tot}}^{(0)}} = \begin{pmatrix} \dfrac{p_{x0}}{p_0} & \dfrac{p_{y0}}{p_0} & \dfrac{p_{z0}}{p_0} \\ -\dfrac{p_{y0}}{p_{x0}^2 + p_{y0}^2} & \dfrac{p_{x0}}{p_{x0}^2 + p_{y0}^2} & 0 \\ \dfrac{p_{x0} p_{z0}}{p_0^2 \sqrt{p_{x0}^2 + p_{y0}^2}} & \dfrac{p_{y0} p_{z0}}{p_0^2 \sqrt{p_{x0}^2 + p_{y0}^2}} & -\dfrac{\sqrt{p_{x0}^2 + p_{y0}^2}}{p_0^2} \end{pmatrix} = (\mathbf{W}_{\text{n}})^{-1}, \tag{26}$$

with $p_0 = \sqrt{p_{x0}^2 + p_{y0}^2 + p_{x0}^2}$.

In step (d), the measurement parameters of the mother are calculated for a further vertex fit. Therefore we have to map the momentum of the mother particle given at the position of the secondary vertex to its DCA. This is exactly the same mapping as we used in the measurement equation. Therefore eq. (14) is valid for the mapping of charged mother particles and eq. (17) for neutral particles. The Jacobian matrices $\mathbf{A}_{\text{c}}$ and $\mathbf{B}_{\text{c}}$ of the mapping of charged particles are given in the appendix, eqs. (B.3) and (B.4) and $\mathbf{A}_{\text{n}}$ and $\mathbf{B}_{\text{n}}$ for neutral particles in eqs. (B.5) and (B.6).

For use in subsequent fits, the covariance matrix of the mother has to be calculated by a proper error propagation through all the four steps described above.

We use the following generalized abbreviations, where the covariance matrices are the smoothed covariance matrices after the vertex fit (see eqs. (1), (8), (9) and (10):

$\mathbf{C} = \text{cov}(\boldsymbol{x}_M)$, covariance matrix of the smoothed vertex position,

$\mathbf{E}_{\text{n}_i} = \text{cov}\left(\boldsymbol{x}_M, \boldsymbol{q}_{\text{n}_i}^M\right)$, for neutral tracks,

$\mathbf{E}_{\text{c}_i} = \text{cov}\left(\boldsymbol{x}_M, \boldsymbol{q}_{\text{c}_i}^M\right)$, for charged tracks,

$\mathbf{Q}_{\text{n}_{i,j}} = \text{cov}\left(\boldsymbol{q}_{\text{n}_i}^M, \boldsymbol{q}_{\text{n}_j}^M\right)$, correlation between neutral tracks,

$\mathbf{Q}_{\text{n}_{i,i}} = \mathbf{D}_{\text{n}_i}^M$,

$\mathbf{Q}_{\text{c}_{i,j}} = \text{cov}\left(\boldsymbol{q}_{\text{c}_i}^M, \boldsymbol{q}_{\text{c}_j}^M\right)$, correlation between charged tracks,

$\mathbf{Q}_{\text{c}_{i,i}} = \mathbf{D}_{\text{c}_i}^M$,

$\mathbf{Q}_{\text{nc}_{i,j}} = \text{cov}\left(\boldsymbol{q}_{\text{n}_i}^M, \boldsymbol{q}_{\text{c}_j}^M\right)$, correlation between neutral and charged tracks.

With the above defined Jacobian matrices we obtain the following calculation scheme for the error propagation:

Step (a) + (b):

$$\begin{array}{ll} \text{neutral tracks} & \text{charged tracks} \\ \mathbf{E}_{\mathrm{n}} = \sum_i \mathbf{E}_{\mathrm{n}_i} \mathbf{W}_{\mathrm{n}_i}^{\mathrm{T}}, & \mathbf{E}_{\mathrm{c}} = \sum_i \mathbf{E}_{\mathrm{c}_i} \mathbf{W}_{\mathrm{c}_i}^{\mathrm{T}}, \\ \mathbf{Q}_{\mathrm{n}} = \sum_{i,j} \mathbf{W}_{\mathrm{n}_i} \mathbf{Q}_{\mathrm{n}_{i,j}} \mathbf{W}_{\mathrm{n}_j}^{\mathrm{T}}, & \mathbf{Q}_{\mathrm{c}} = \sum_{i,j} \mathbf{W}_{\mathrm{c}_i} \mathbf{Q}_{\mathrm{c}_{i,j}} \mathbf{W}_{\mathrm{c}_j}^{\mathrm{T}}, \end{array} \tag{27}$$

and for the combination of charged and neutral tracks:

$$\mathbf{Q}_{\mathrm{nc}} = \sum_{i,j} \mathbf{W}_{\mathrm{n}_i} \mathbf{Q}_{\mathrm{nc}_{i,j}} \mathbf{W}_{\mathrm{c}_j}^{\mathrm{T}} + \sum_{i,j} \mathbf{W}_{\mathrm{c}_j} \mathbf{Q}_{\mathrm{nc}_{i,j}}^{\mathrm{T}} \mathbf{W}_{\mathrm{n}_i}^{\mathrm{T}}, \tag{28}$$

$$\mathbf{C}_{\mathrm{tot}} = \mathbf{C}, \qquad \mathbf{E}_{\mathrm{tot}} = \mathbf{E}_{\mathrm{n}} + \mathbf{E}_{\mathrm{c}}, \qquad \mathbf{Q}_{\mathrm{tot}} = \mathbf{Q}_{\mathrm{n}} + \mathbf{Q}_{\mathrm{c}} + \mathbf{Q}_{\mathrm{nc}}. \tag{29}$$

After steps (c) and (d), the final covariance matrix of the measurement vector of the mother **G** is given for charged mother particles by

$$\mathbf{G}_{\mathrm{c}} = \mathbf{A}_{\mathrm{c}} \mathbf{C} \mathbf{A}_{\mathrm{c}}^{\mathrm{T}} + \mathbf{A}_{\mathrm{c}} \mathbf{E}_{\mathrm{tot}} \left(\mathbf{W}_{\mathrm{c}}^{-1}\right)^{\mathrm{T}} \mathbf{B}_{\mathrm{c}}^{\mathrm{T}} + \mathbf{B}_{\mathrm{c}} \mathbf{W}_{\mathrm{c}}^{-1} \mathbf{E}_{\mathrm{tot}}^{\mathrm{T}} \mathbf{A}_{\mathrm{c}}^{\mathrm{T}} + \mathbf{B}_{\mathrm{c}} \mathbf{W}_{\mathrm{c}}^{-1} \mathbf{Q}_{\mathrm{tot}} \left(\mathbf{W}_{\mathrm{c}}^{-1}\right)^{\mathrm{T}} \mathbf{B}_{\mathrm{c}}^{\mathrm{T}}, \tag{30}$$

and for a neutral object by

$$\mathbf{G}_{\mathrm{n}} = \mathbf{A}_{\mathrm{n}} \mathbf{C} \mathbf{A}_{\mathrm{n}}^{\mathrm{T}} + \mathbf{A}_{\mathrm{n}} \mathbf{E}_{\mathrm{tot}} \left(\mathbf{W}_{\mathrm{n}}^{-1}\right)^{\mathrm{T}} \mathbf{B}_{\mathrm{n}}^{\mathrm{T}} + \mathbf{B}_{\mathrm{n}} \mathbf{W}_{\mathrm{n}}^{-1} \mathbf{E}_{\mathrm{tot}}^{\mathrm{T}} \mathbf{A}_{\mathrm{n}}^{\mathrm{T}} + \mathbf{B}_{\mathrm{n}} \mathbf{W}_{\mathrm{n}}^{-1} \mathbf{Q}_{\mathrm{tot}} \left(\mathbf{W}_{\mathrm{n}}^{-1}\right)^{\mathrm{T}} \mathbf{B}_{\mathrm{n}}^{\mathrm{T}}. \tag{31}$$

From these formulae, one can directly see that **G** is symmetric as it is supposed to be.

With this formalism for charged and neutral particles, and any combination thereof, we have a very flexible tool for the estimation of primary and secondary vertices and for the momenta of the particles at these vertices. Any combination of tracks in the event can be fitted to a common vertex, and then the parameters of the hypothetical mother calculated to be used in further fits.

The afore described concept of vertex finding and reconstructing has been successfully applied within the context of the H1-experiment at HERA. Because the actual demonstration of the fit capabilities is closely related to the detector performance (e.g. position and momentum resolutions etc. of a detector that just started running), they will be the topic of a separate, forthcoming experimental paper.

5. Summary

In conclusion, we have described in detail a technique to find and reconstruct primary and secondary vertices within a high-energy physics experiment, which is based on linear Kalman filtering. We have for the first time proposed a combined treatment of charged particles and neutral objects. The method has been successfully applied with the H1-experiment.

This Kalman filter technique has advantages over the conventional least squares fit method, because it incorporates the tracking information sequentially (locally), as opposed to globally. Hence it is more flexible in adding and removing tracks, and at the same time always only employs matrices and vectors of dimension five, independent of the number of tracks. This requires less processing time, particularly in cases where the total number of objects is becoming quite large, as is predicted to be the case in most future collider experiments. To fully exploit the potential of combining objects, a very high resolution detector is most advantageous, such as the ones being proposed e.g. for the SSC or the LHC collider projects.

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Appendix A. Derivation of the filter equations based on the minimization of a local chisquare

Here we describe a complete derivation of the filter equations. First we derive some general properties for the minimization of a chisquare. The measurement equation assumes the form

$$\boldsymbol{m} = \mathbf{H}\boldsymbol{a} + \boldsymbol{v}, \tag{A.1}$$

where $\boldsymbol{m}$ are the measured values, $\boldsymbol{v}$ is the measurement noise and $\boldsymbol{a}$ are the values to be estimated. With the covariance matrix $\operatorname{cov}(\boldsymbol{v}) = \mathbf{V}_m = \mathbf{V}_m^{\mathrm{T}}$ and the weightmatrix $\mathbf{W}_m = \mathbf{V}_m^{-1}$, the chisquare has the form

$$\chi^2(\boldsymbol{a}) = (\boldsymbol{m} - \mathbf{H}\boldsymbol{a})^{\mathrm{T}}\mathbf{W}_m(\boldsymbol{m} - \mathbf{H}\boldsymbol{a}). \tag{A.2}$$

The value $\boldsymbol{a}_0$, which minimizes the chisquare is the estimate for our vector $\boldsymbol{a}$,

$$\chi^2(\boldsymbol{a}_0) = \min \Leftrightarrow \frac{\partial\chi^2(\boldsymbol{a})}{\partial\boldsymbol{a}}\Big|_{\boldsymbol{a}_0} = \mathbf{0}. \tag{A.3}$$

Using eqs. (A.2) and (A.3),

$$\frac{\partial\chi^2(\boldsymbol{a})}{\partial\boldsymbol{a}}\Big|_{\boldsymbol{a}_0} = 2\mathbf{H}^{\mathrm{T}}\mathbf{W}_m\mathbf{H}\boldsymbol{a}_0 - 2\mathbf{H}^{\mathrm{T}}\mathbf{W}_m\boldsymbol{m} = \mathbf{0},$$

we find

$$\boldsymbol{a}_0 = \mathbf{F}\boldsymbol{m}, \tag{A.4a}$$

with

$$\mathbf{F} = \left(\mathbf{H}^{\mathrm{T}}\mathbf{W}_m\mathbf{H}\right)^{-1}\mathbf{H}^{\mathrm{T}}\mathbf{W}_m. \tag{A.4b}$$

The covariance matrix of $\boldsymbol{a}_0$ is given by the error propagation

$$\begin{aligned}
\mathbf{V}_{a_0} &= \mathbf{F}\mathbf{V}_m\mathbf{F}^{\mathrm{T}} \\
&= \left(\mathbf{H}^{\mathrm{T}}\mathbf{W}_m\mathbf{H}\right)^{-1}\mathbf{H}^{\mathrm{T}}\underbrace{\mathbf{W}_m\mathbf{V}_m}_{=1}\left[\left(\mathbf{H}^{\mathrm{T}}\mathbf{W}_m\mathbf{H}\right)^{-1}\mathbf{H}^{\mathrm{T}}\mathbf{W}_m\right]^{\mathrm{T}} \\
&= \left[\left(\mathbf{H}^{\mathrm{T}}\mathbf{W}_m\mathbf{H}\right)^{\mathrm{T}}\right]^{-1} = \left(\mathbf{H}^{\mathrm{T}}\mathbf{W}_m\mathbf{H}\right)^{-1}.
\end{aligned} \tag{A.5}$$

The application of vertex fitting with the general formalism of Kalman can in some sense be reduced to the minimization of a local chisquare of the following form (see also eq. (4)):

$$\begin{aligned}
\chi^2_{\mathrm{kf}}(\boldsymbol{x}, \boldsymbol{q}) = {} & (\boldsymbol{x} - \boldsymbol{x}_{k-1})^{\mathrm{T}}(\mathbf{C}_{k-1})^{-1}(\boldsymbol{x} - \boldsymbol{x}_{k-1}) \\
& + \left(\boldsymbol{m}_k - \boldsymbol{c}_k^{(0)} - \mathbf{A}_k\boldsymbol{x} - \mathbf{B}_k\boldsymbol{q}\right)^{\mathrm{T}}\mathbf{G}_k\left(\boldsymbol{m}_k - \boldsymbol{c}_k^{(0)} - \mathbf{A}_k\boldsymbol{x} - \mathbf{B}_k\boldsymbol{q}\right).
\end{aligned} \tag{A.6}$$

We can rewrite eq. (A.6) in the more general form (A.2) with

$$\boldsymbol{m}=\begin{pmatrix} x_{k-1} \\ m_k-c_k^{(0)} \end{pmatrix},\quad \boldsymbol{a}=\begin{pmatrix} x \\ q \end{pmatrix},\quad \mathbf{H}=\begin{pmatrix} 1 & 0 \\ \mathbf{A}_k & \mathbf{B}_k \end{pmatrix},\quad \mathbf{W}_m=\begin{pmatrix} (\mathbf{C}_{k-1})^{-1} & 0 \\ 0 & \mathbf{G}_k \end{pmatrix}, \tag{A.7}$$

where $\mathbf{G}_k$ denotes the weightmatrix of the measurement noise of track k.

Using eq. (A.4b) we obtain for

$$\begin{aligned}\mathbf{F}&=\left(\begin{pmatrix} 1 & \mathbf{A}_k^{\mathrm{T}} \\ 0 & \mathbf{B}_k^{\mathrm{T}} \end{pmatrix}\begin{pmatrix} (\mathbf{C}_{k-1})^{-1} & 0 \\ 0 & \mathbf{G}_k \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \mathbf{A}_k & \mathbf{B}_k \end{pmatrix}\right)^{-1}\begin{pmatrix} 1 & \mathbf{A}_k^{\mathrm{T}} \\ 0 & \mathbf{B}_k^{\mathrm{T}} \end{pmatrix}\begin{pmatrix} (\mathbf{C}_{k-1})^{-1} & 0 \\ 0 & \mathbf{G}_k \end{pmatrix} \\ &=\begin{pmatrix} (\mathbf{C}_{k-1})^{-1}+\mathbf{A}_k^{\mathrm{T}}\mathbf{G}_k\mathbf{A}_k & \mathbf{A}_k^{\mathrm{T}}\mathbf{G}_k\mathbf{B}_k \\ \mathbf{B}_k^{\mathrm{T}}\mathbf{G}_k\mathbf{A}_k & \mathbf{B}_k^{\mathrm{T}}\mathbf{G}_k\mathbf{B}_k \end{pmatrix}^{-1}\begin{pmatrix} (\mathbf{C}_{k-1})^{-1} & \mathbf{A}_k^{\mathrm{T}}\mathbf{G}_k \\ 0 & \mathbf{B}_k^{\mathrm{T}}\mathbf{G}_k \end{pmatrix}.\end{aligned} \tag{A.8}$$

Using eq. (A.5) the covariance matrix of $\boldsymbol{a}_0=\binom{x_k}{q_k}$ then reads

$$\mathbf{V}_{a_0}=\begin{pmatrix} (\mathbf{C}_{k-1})^{-1}+\mathbf{A}_k^{\mathrm{T}}\mathbf{G}_k\mathbf{A}_k & \mathbf{A}_k^{\mathrm{T}}\mathbf{G}_k\mathbf{B}_k \\ \mathbf{B}_k^{\mathrm{T}}\mathbf{G}_k\mathbf{A}_k & \mathbf{B}_k^{\mathrm{T}}\mathbf{G}_k\mathbf{B}_k \end{pmatrix}^{-1}, \tag{A.9}$$

which is conveniently abbreviated by

$$\mathbf{V}_{a_0}=\begin{pmatrix} \mathbf{C}_k & \mathbf{E}_k \\ \mathbf{E}_k^{\mathrm{T}} & \mathbf{D}_k \end{pmatrix}, \tag{A.10}$$

with the matrices defined by $\mathbf{C}_k=\operatorname{cov}(x_k)$, $\mathbf{D}_k=\operatorname{cov}(q_k)$ and $\mathbf{E}_k=\operatorname{cov}(x_k,\, q_k)$.

For the inversion of the matrix on the right side of eq. (A.9), we use the general Ansatz

$$\begin{pmatrix} \mathbf{R} & \mathbf{S} \\ \mathbf{S}^{\mathrm{T}} & \mathbf{T} \end{pmatrix}\begin{pmatrix} \mathbf{C} & \mathbf{E} \\ \mathbf{E}^{\mathrm{T}} & \mathbf{D} \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{A.11}$$

with $\mathbf{R}=\mathbf{R}^{\mathrm{T}}$, $\mathbf{T}=\mathbf{T}^{\mathrm{T}}$, $\mathbf{C}=\mathbf{C}^{\mathrm{T}}$ and $\mathbf{D}=\mathbf{D}^{\mathrm{T}}$, we find a solution

$$\mathbf{C}=(\mathbf{R}-\mathbf{S}\mathbf{T}^{-1}\mathbf{S}^{\mathrm{T}})^{-1},\qquad \mathbf{E}=-\mathbf{C}\mathbf{S}\mathbf{T}^{-1},\qquad \mathbf{D}=\mathbf{T}^{-1}(1-\mathbf{S}^{\mathrm{T}}\mathbf{E}). \tag{A.12}$$

With this Ansatz, $\mathbf{V}_{a_0}$ can be calculated. Inserting

$$\mathbf{R}=(\mathbf{C}_{k-1})^{-1}+\mathbf{A}_k^{\mathrm{T}}\mathbf{G}_k\mathbf{A}_k,\qquad \mathbf{S}=\mathbf{A}_k^{\mathrm{T}}\mathbf{G}_k\mathbf{B}_k,\qquad \mathbf{T}=\mathbf{B}_k^{\mathrm{T}}\mathbf{G}_k\mathbf{B}_k, \tag{A.13}$$

and using the abbreviations

$$\mathbf{W}_k=(\mathbf{B}_k^{\mathrm{T}}\mathbf{G}_k\mathbf{B}_k)^{-1}\qquad \mathbf{G}_k^{\mathbf{B}}=\mathbf{G}_k-\mathbf{G}_k\mathbf{B}_k\mathbf{W}_k\mathbf{B}_k^{\mathrm{T}}\mathbf{G}_k, \tag{A.14}$$

and eqs. (A.10)–(A.14), we obtain the filtered covariance matrices

$$\begin{aligned}\mathbf{C}_k&=\left[(\mathbf{C}_{k-1})^{-1}+\mathbf{A}_k^{\mathrm{T}}\mathbf{G}_k\mathbf{A}_k-\mathbf{A}_k^{\mathrm{T}}\mathbf{G}_k\mathbf{B}_k\mathbf{W}_k\mathbf{B}_k^{\mathrm{T}}\mathbf{G}_k\mathbf{A}_k\right]^{-1} \\ &=\left[(\mathbf{C}_{k-1})^{-1}+\mathbf{A}_k^{\mathrm{T}}\mathbf{G}_k^{\mathbf{B}}\mathbf{A}_k\right]^{-1}, \\ \mathbf{E}_k&=-\mathbf{C}_k\mathbf{A}_k^{\mathrm{T}}\mathbf{G}_k\mathbf{B}_k\mathbf{W}_k,\qquad \mathbf{D}_k=\mathbf{W}_k+\mathbf{W}_k\mathbf{B}_k^{\mathrm{T}}\mathbf{G}_k\mathbf{A}_k\mathbf{C}_k\mathbf{A}_k^{\mathrm{T}}\mathbf{G}_k\mathbf{B}_k\mathbf{W}_k.\end{aligned} \tag{A.15}$$

To calculate the filtered vertex position x_k and momentum q_k at this position, we make use of eq. (A.4a), (A.4b), (A.8) and (A.10):

$$\begin{pmatrix} x_k \\ q_k \end{pmatrix} = \begin{pmatrix} \mathbf{C}_k & \mathbf{E}_k \\ \mathbf{E}_k^{\mathrm{T}} & \mathbf{D}_k \end{pmatrix} \begin{pmatrix} (\mathbf{C}_{k-1})^{-1} & \mathbf{A}_k^{\mathrm{T}}\mathbf{G}_k \\ 0 & \mathbf{B}_k^{\mathrm{T}}\mathbf{G}_k \end{pmatrix} \begin{pmatrix} x_{k-1} \\ m_k - c_k^{(0)} \end{pmatrix},$$

$$\begin{pmatrix} x_k \\ q_k \end{pmatrix} = \begin{pmatrix} \mathbf{C}_k & \mathbf{E}_k \\ \mathbf{E}_k^{\mathrm{T}} & \mathbf{D}_k \end{pmatrix} \begin{pmatrix} (\mathbf{C}_{k-1})^{-1} x_{k-1} + \mathbf{A}_k^{\mathrm{T}}\mathbf{G}_k\left(m_k - c_k^{(0)}\right) \\ \mathbf{B}_k^{\mathrm{T}}\mathbf{G}_k\left(m_k - c_k^{(0)}\right) \end{pmatrix}.$$

By employing eq. (A.15), a straightforward calculation yields the final results:

$$x_k = \mathbf{C}_k\left[(\mathbf{C}_{k-1})^{-1} x_{k-1} + \mathbf{A}_k^{\mathrm{T}}\mathbf{G}_k^{\mathbf{B}}\left(m_k - c_k^{(0)}\right)\right], \tag{A.16}$$

$$q_k = \mathbf{W}_k\mathbf{D}_k^{\mathrm{T}}\mathbf{G}_k\left(m_k - c_k^{(0)} - \mathbf{A}_k x_k\right). \tag{A.17}$$

Appendix B. First derivatives of the measurement mapping

For the linear approximation of the measurement mapping we need the two matrices $\mathbf{A}$ and $\mathbf{B}$, given by the first derivatives of the measurement mapping $\boldsymbol{h}$. For completeness, these matrices are listed here explicitly.

(a) Charged particles

With the coordinates of the centre of the circle of the projected helix

$$m_x = x_0 - \frac{1}{\kappa_0}\sin\phi_0, \qquad m_y = y_0 + \frac{1}{\kappa_0}\cos\phi_0, \tag{B.1}$$

the derivatives of the rotation angle γ are then

$$\begin{aligned} &\frac{\partial\gamma}{\partial x_0} = -\frac{m_y}{m_x^2 + m_y^2}, \qquad \frac{\partial\gamma}{\partial y_0} = \frac{m_x}{m_x^2 + m_y^2}, \qquad \frac{\partial\gamma}{\partial z_0} = 0, \\ &\frac{\partial\gamma}{\partial\kappa_0} = -\frac{1}{\kappa_0^2}\frac{(x_0\cos\phi_0 + y_0\sin\phi_0)}{\left(m_x^2 + m_y^2\right)}, \qquad \frac{\partial\gamma}{\partial\phi_0} = y_0\frac{\partial\gamma}{\partial x_0} - x_0\frac{\partial\gamma}{\partial y_0}, \qquad \frac{\partial\gamma}{\partial\theta_0} = 0. \end{aligned} \tag{B.2}$$

The matrices are expressed as follows:

$$\mathbf{A}_{\mathrm{c}} = \frac{\partial \boldsymbol{h}_{\mathrm{c}}}{\partial \boldsymbol{x}}, \tag{B.3}$$

$$\frac{\partial h_1}{\partial x_0} = 0, \qquad \frac{\partial h_1}{\partial y_0} = 0, \qquad \frac{\partial h_1}{\partial z_0} = 0,$$

$$\frac{\partial h_2}{\partial x_0} = \frac{\partial\gamma}{\partial x_0}, \qquad \frac{\partial h_2}{\partial y_0} = \frac{\partial\gamma}{\partial y_0}, \qquad \frac{\partial h_2}{\partial z_0} = 0,$$

$$\frac{\partial h_3}{\partial x_0} = 0, \qquad \frac{\partial h_3}{\partial y_0} = 0, \qquad \frac{\partial h_3}{\partial z_0} = 0,$$

$$\frac{\partial h_4}{\partial x_0} = \left[\cos\gamma\sin\phi_0 - \sin\gamma\frac{\partial\gamma}{\partial x_0}\left(\frac{1}{\kappa_0} - x_0\sin\phi_0 + y_0\cos\phi_0\right)\right]\frac{1}{\cos^2\gamma},$$

$$\frac{\partial h_4}{\partial y_0} = -\left[\cos\gamma\cos\phi_0 + \sin\gamma\frac{\partial\gamma}{\partial y_0}\left(\frac{1}{\kappa_0} - x_0\sin\phi_0 + y_0\cos\phi_0\right)\right]\frac{1}{\cos^2\gamma},$$

$$\frac{\partial h_4}{\partial z_0} = 0,$$

$$\frac{\partial h_5}{\partial x_0} = \frac{\partial\gamma}{\partial x_0}\frac{\cos\theta_0}{\sin\theta_0}\frac{1}{\kappa_0}, \qquad \frac{\partial h_5}{\partial y_0} = \frac{\partial\gamma}{\partial y_0}\frac{\cos\theta_0}{\sin\theta_0}\frac{1}{\kappa_0}, \qquad \frac{\partial h_5}{\partial z_0} = 1.$$

$$\mathbf{B}_{\mathrm{c}} = \frac{\partial \boldsymbol{h}_{\mathrm{c}}}{\partial \boldsymbol{q}}, \tag{B.4}$$

$$\frac{\partial h_1}{\partial \kappa_0} = 1, \qquad \frac{\partial h_1}{\partial \phi_0} = 0, \qquad \frac{\partial h_1}{\partial \theta_0} = 0,$$

$$\frac{\partial h_2}{\partial \kappa_0} = \frac{\partial\gamma}{\partial \kappa_0}, \qquad \frac{\partial h_2}{\partial \phi_0} = 1 + \frac{\partial\gamma}{\partial \phi_0}, \qquad \frac{\partial h_2}{\partial \theta_0} = 0,$$

$$\frac{\partial h_3}{\partial \kappa_0} = 0, \qquad \frac{\partial h_3}{\partial \phi_0} = 0, \qquad \frac{\partial h_3}{\partial \theta_0} = 1,$$

$$\frac{\partial h_4}{\partial \kappa_0} = -\frac{1}{\kappa_0^2} + \left[\frac{1}{\kappa_0^2}\cos\gamma - \sin\gamma\frac{\partial\gamma}{\partial\kappa_0}\left(\frac{1}{\kappa_0} - x_0\sin\phi_0 + y_0\cos\phi_0\right)\right]\frac{1}{\cos^2\gamma},$$

$$\frac{\partial h_4}{\partial \phi_0} = \left[(x_0\cos\phi_0 + y_0\sin\phi_0)\cos\gamma - \sin\gamma\frac{\partial\gamma}{\partial\phi_0}\left(\frac{1}{\kappa_0} - x_0\sin\phi_0 + y_0\cos\phi_0\right)\right]\frac{1}{\cos^2\gamma},$$

$$\frac{\partial h_4}{\partial \theta_0} = 0,$$

$$\frac{\partial h_5}{\partial \kappa_0} = \frac{1}{\kappa_0}\frac{\cos\phi_0}{\sin\phi_0}\left(\frac{\partial\gamma}{\partial\kappa_0} - \frac{1}{\kappa_0}\gamma\right), \qquad \frac{\partial h_5}{\partial \phi_0} = \frac{\partial\gamma}{\partial\phi_0}\frac{\cos\theta_0}{\sin\theta_0}\frac{1}{\kappa_0}, \qquad \frac{\partial h_5}{\partial \theta_0} = -\gamma\frac{1}{\sin^2\theta_0}\frac{1}{\kappa_0}.$$

) Neutral particles

$$\mathbf{A}_{\mathrm{n}} = \frac{\partial \boldsymbol{h}_{\mathrm{n}}}{\partial \boldsymbol{x}}, \tag{B.5}$$

$$\frac{\partial h_1}{\partial x_0} = 0, \qquad \frac{\partial h_1}{\partial y_0} = 0, \qquad \frac{\partial h_1}{\partial z_0} = 0,$$

$$\frac{\partial h_2}{\partial x_0} = 0, \qquad \frac{\partial h_2}{\partial y_0} = 0, \qquad \frac{\partial h_2}{\partial z_0} = 0,$$

$$\frac{\partial h_3}{\partial x_0} = 0, \qquad \frac{\partial h_3}{\partial y_0} = 0, \qquad \frac{\partial h_3}{\partial z_0} = 0,$$

$$\frac{\partial h_4}{\partial x_0} = \sin(\phi_0), \qquad \frac{\partial h_4}{\partial y_0} = -\cos(\phi_0), \qquad \frac{\partial h_4}{\partial z_0} = 0,$$

$$\frac{\partial h_5}{\partial x_0} = -\cos(\phi_0)\frac{1}{\tan(\theta_0)}, \qquad \frac{\partial h_5}{\partial y_0} = -\sin(\phi_0)\frac{1}{\tan(\theta_0)}, \qquad \frac{\partial h_5}{\partial z_0} = 1.$$

$$\mathbf{B}_n = \frac{\partial \boldsymbol{h}_n}{\partial \boldsymbol{q}}, \tag{B.6}$$

$$\frac{\partial h_1}{\partial p_0} = 1, \quad \frac{\partial h_1}{\partial \phi_0} = 0, \quad \frac{\partial h_1}{\partial \theta_0} = 0,$$

$$\frac{\partial h_2}{\partial p_0} = 0, \quad \frac{\partial h_2}{\partial \phi_0} = 1, \quad \frac{\partial h_2}{\partial \theta_0} = 0,$$

$$\frac{\partial h_3}{\partial p_0} = 0, \quad \frac{\partial h_3}{\partial \phi_0} = 0, \quad \frac{\partial h_3}{\partial \theta_0} = 1,$$

$$\frac{\partial h_4}{\partial p_0} = 0, \quad \frac{\partial h_4}{\phi_0} = r_\perp \cos(\xi), \quad \frac{\partial h_4}{\partial \theta_0} = 0,$$

$$\frac{\partial h_5}{\partial p_0} = 0, \quad \frac{\partial h_5}{\partial \phi_0} = r_\perp \sin(\xi) \frac{1}{\tan(\theta_0)}, \quad \frac{\partial h_5}{\partial \theta_0} = r_\perp \cos(\phi_0) \frac{1}{\sin^2(\theta_0)},$$

with

$$r_\perp = \sqrt{x_0^2 + y_0^2}, \qquad \phi_0' = \arccos\left(\frac{x_0}{r_\perp}\right), \qquad \xi = \phi_0 - \phi_0'.$$

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