EE 341 — 11 Apr '16

Review: Time Invariance

Intuition: system behavior doesn't change with time

Formal proof: $y[n-n_0] = T\left[x[n-n_0]\right]$

$$TD = DT$$

You can also show a system is not TI with a single counter example.

Eyeball indicators:

- n outside $x[\ ]$
- operation on n inside $x[\ ]$ other than addition

Note: eyeball indicators do <u>not</u> constitute a formal proof.

n in summation limits is tricky

Consider the following three systems:

$$y[n] = \sum_{k=0}^{n} x[k] \qquad y[n] = \sum_{k=-\infty}^{n} x[k] \qquad y[n] = \sum_{k=-\infty}^{n} \alpha_k x[k]$$

From previ page

not TI

TI

not TI

$$y[n-n_0] = \sum_{k=-\infty}^{n-n_0} \alpha_k x[k]$$

$\neq$

$$T[x[n-n_0]] = \sum_{k=-\infty}^{n} \alpha_k x[k-n_0] = \sum_{k'=-\infty}^{n-n_0} \alpha_{k'+n_0} x[k']$$

$k' = k - n_0$

Consider the following two systems, where $M > 0$ is a constant:

$$y[n] = \sum_{k=0}^{M} \alpha_k x[k] \qquad y[n] = \sum_{k=0}^{M} \alpha_k x[n-k]$$

fixed window of $x[\cdot]$

moving window of $x[\cdot]$

$$y[n-n_0] = \sum_{k=0}^{M} \alpha_k x[k]$$

$\neq$

$$T[x[n-n_0]] = \sum_{k=0}^{M} \alpha_k x[k-n_0]$$

not TI

$$y[n-n_0] = \sum_{k=0}^{M} \alpha_k x[n-n_0-k]$$

$$T[x[n-n_0]] = \sum_{k=0}^{M} \alpha_k x[n-k-n_0]$$

TI

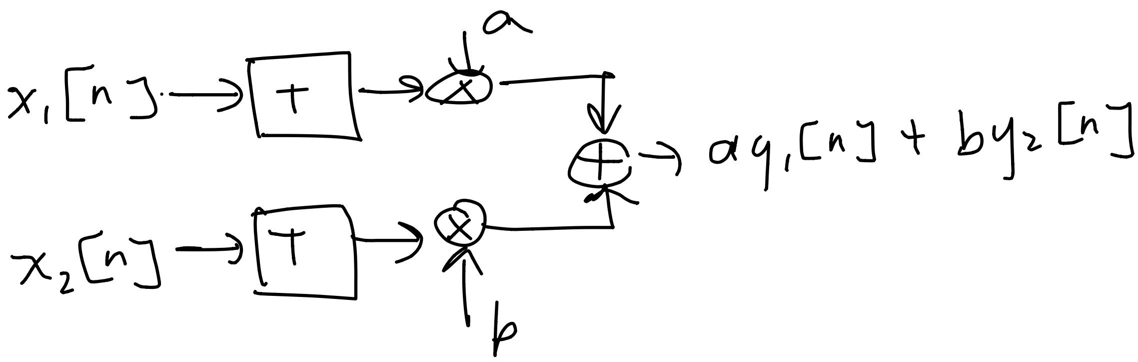
Linearity

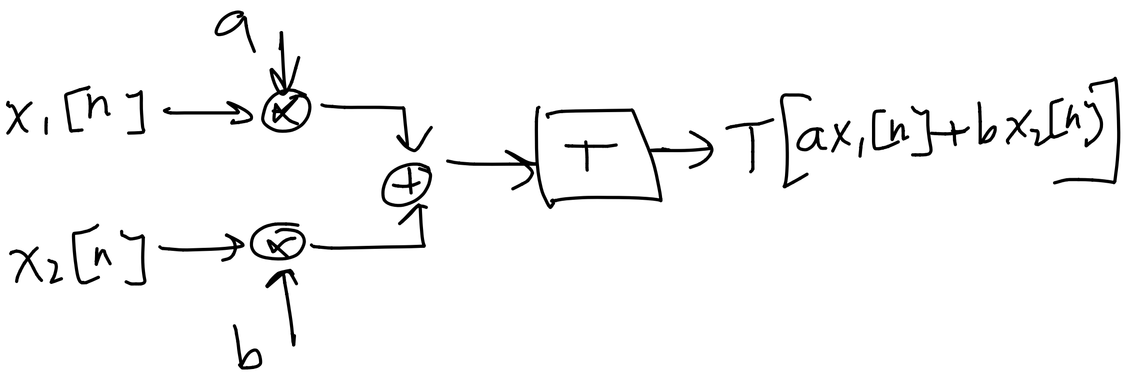
<u>Formal definition:</u> A system is linear if both additivity and scaling hold:

$$T[x_1[n]] = y_1[n] \text{ and } T[x_2[n]] = y_2[n] \Rightarrow$$
$$T[ax_1[n] + bx_2[n]] = ay_1[n] + by_2[n]$$

<u>Intuition:</u> Weighted combinations in give weighted combinations out

General (formal) test for linearity:





Note: You can show something is not linear by showing either Scaling or additivity fail

$ay[n] \neq T[ax[n]]$

$y_1[n] + y_2[n] \neq T[x_1[n] + x_2[n]]$

a single example works for the counter example proof

Special case test for non-linear systems: Zero input must produce zero output due to scaling property:

$$T[ax_1[n]] = ay_1[n]$$

Let $a = 0$, then $T[0] = 0$.

consider the system

$$y[n] = 2x[n] + 1$$

$$x[n] = 0 \Longrightarrow y[n] = 1$$

not linear

This constitutes a formal proof!

Test Linearity for the following systems:

$$y[n] = \sum_{k=0}^{M} \alpha_k x[n-k]$$

$$ay_1[n] + by_2[n] = a\sum_{k=0}^{M} \alpha_k x_1[n-k] + b\sum_{k=0}^{M} \alpha_k x_2[n-k]$$

$$= \sum_{k=0}^{M} \alpha_k (ax_1[n-k] + bx_2[n-k])$$

$$= T[ax_1[n] + bx_2[n]]$$

Linear!

$$y[n] = x[n]x[n-1]$$

$$ay_1[n] + by_2[n] = ax_1[n]x_1[n-1] + bx_2[n]x_2[n-1]$$

$$T[ax_1[n] + bx_2[n]] = (ax_1[n] + bx_2[n])(ax_1[n-1] + bx_2[n-1])$$

$$= a^2 x_1[n]x_1[n-1] + b^2 x_2[n]x_2[n-1] + \text{cross terms}$$

$\neq$

Not Linear

Alternatively: show scaling doesn't hold

$$ay[n] = ax[n]x[n-1]$$

$$T[ax[n]] = a^2 x[n]x[n-1]$$

$\neq$

When a system is both linear and time-invariant (LTI), it has some special properties that we'll learn about in the next chapter.

Examples: Test for linearity and time-invariance

$$y[n] = \sin(x[n]) \qquad\qquad y[n] = \cos(\omega_0 n + x[n])$$

time-invariant
not linear

not time invariant
not linear

TI: $y[n-n_0] = \sin(x[n-n_0])$ $=$
$T[x[n-n_0]] = \sin(x[n-n_0])$

$y[n-n_0] = \cos(\omega_0(n-n_0) + x[n-n_0])$ $\neq$
$T[x[n-n_0]] = \cos(\omega_0 n + x[n-n_0])$

Linearity: scaling counter examples

$x[n] = \pi/2 \rightarrow y[n] = \sin(\pi/2) = 1$ $\neq$
$a = 2$ $\quad 2x[n] = \pi \rightarrow T[2x[n]] = \sin(\pi) = 0$

$x[n] = \pi/4 \rightarrow y[n] = \cos(\omega_0 n + \pi/4)$
$a = 2 \Rightarrow a y[n] = 2\cos(\omega_0 n + \pi/4)$
$\Rightarrow T[a x[n]] = \cos(\omega_0 n + \pi/2)$

$$y[n] = \begin{cases} x[n-1] & x[n] \geq 0 \\ 0 & x[n] < 0 \end{cases} \qquad\qquad y[n] = \begin{cases} x[n] & n \geq 0 \\ 0 & n < 0 \end{cases}$$

time-invariant
not linear

not TI
linear

$y[n-n_0] = \begin{cases} x[n-n_0-1] & x[n-n_0] \geq 0 \\ 0 & x[n-n_0] < 0 \end{cases}$
$=$
$T[x[n-n_0]] = \begin{cases} x[n-1-n_0] & x[n-n_0] \geq 0 \\ 0 & x[n-n_0] < 0 \end{cases}$

$y[n-n_0] = \begin{cases} x[n-n_0] & n-n_0 \geq 0 \\ 0 & n-n_0 < 0 \end{cases}$ $\neq$
$T[x[n-n_0]] = \begin{cases} x[n-n_0] & n \geq 0 \\ 0 & n < 0 \end{cases}$

$a y[n] = \begin{cases} a x[n-1] & x[n] \geq 0 \\ 0 & x[n] < 0 \end{cases} = \begin{cases} -x[n-1] & x[n] \geq 0 \\ 0 & x[n] < 0 \end{cases}$
$\neq$
$T[a x[n]] = \begin{cases} a x[n-1] & a x[n] \geq 0 \\ 0 & a x[n] < 0 \end{cases} = \begin{cases} -x[n-1] & x[n] \leq 0 \\ 0 & x[n] > 0 \end{cases}$
if $a = -1 \Rightarrow$

$a y_1[n] + b y_2[n]$
$= \begin{cases} a x_1[n] + b x_2[n] & n \geq 0 \\ 0 & n < 0 \end{cases}$
$= T[a x_1[n] + b x_2[n]]$

From Friday class assignments: causal but not TI systems

Some examples

a) $y[n] = 5n\,x[n]$

b) $y[n] = (n+1)\,x[n]$

c) $y[n] = (n+3)\,x[n-1] - 1$

d) $y[n] = x[n]^n$

e) $y[n] = n\,x[n] + 5$

f) $y[n] = n^5\left(x[n] - x[n-2]\right)$

g) $y[n] = n\,x[n-1]\,x[n-2]$

h) $y[n] = (\sin(n))^2\,x[n]$

i) $y[n] = n^2\cos(n)\,x[n]$

j) $y[n] = n\,x[n-1]\,u[n]$

k) $y[n] = x[n] + 5n^2$

l) $y[n] = e^n x[n] + 1$

m) $y[n] = (5-n)\,x[n]$

n) $y[n] = x[n]\,\delta[n]$

o) $y[n] = \dfrac{x[n]}{n}$

○ NOT memoryless

□ NOT stable

☆ NOT linear

There are 4 more nonlinear systems

There are only 2 stable systems

can you find them?