

General Physics 2 Summary

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1 矢量场

向量运算

$$\nabla(\varphi\psi) = \psi\nabla\varphi + \varphi\nabla\psi$$

$$\nabla \cdot (\varphi\vec{g}) = \nabla\varphi \cdot \vec{g} + \varphi\nabla \cdot \vec{g}$$

$$\nabla \times (\varphi\vec{g}) = \nabla\varphi \times \vec{g} + \varphi\nabla \times \vec{g}$$

$$\nabla \cdot (\vec{g} \times \vec{f}) = (\nabla \times \vec{g}) \cdot \vec{f} - \vec{g} \cdot (\nabla \times \vec{f})$$

$$\nabla \times (\vec{g} \times \vec{f}) = (\vec{f} \cdot \nabla)\vec{g} - (\nabla \cdot \vec{g})\vec{f} + (\nabla \cdot \vec{f})\vec{g} - (\vec{g} \cdot \nabla)\vec{f}$$

$$\nabla \times (\nabla \times \vec{h}) = \nabla(\nabla \cdot \vec{h}) - \nabla^2\vec{h}$$

$$\vec{c} \times (\vec{a} \times \vec{b}) = (\vec{c} \cdot \vec{b})\vec{a} - (\vec{c} \cdot \vec{a})\vec{b}$$

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{b} \cdot (\vec{c} \times \vec{a}) = \vec{c} \cdot (\vec{a} \times \vec{b})$$

无旋/无源场

$$\nabla \times \vec{F} = 0 \implies \exists \varphi \text{ s.t. } \vec{F} = \nabla\varphi \quad (1.0.1)$$

$$\nabla \cdot \vec{F} = 0 \implies \exists \vec{C} \text{ s.t. } \vec{F} = \nabla \times \vec{C} \quad (1.0.2)$$

积分

$$flux = \int_{\partial\Omega} \vec{F} \cdot d\vec{S} = \int_{\Omega} \nabla \cdot \vec{F} dV \quad (1.0.3)$$

$$\begin{aligned} circulation &= \int_{\partial\Sigma} \vec{F} \cdot d\vec{l} = \int_{\Sigma} (\nabla \times \vec{F}) \cdot d\vec{S} \\ &= \int_S \sum \left(\frac{\partial f_y}{\partial z} - \frac{\partial f_z}{\partial y} \right) dydz \end{aligned}$$

1.1 Maxwell equations

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \quad (1.1.1)$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad (1.1.2)$$

$$\nabla \times \vec{B} = \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} + \frac{\vec{j}}{c^2\epsilon_0} \quad (1.1.3)$$

$$\nabla \cdot \vec{B} = 0 \quad (1.1.4)$$

2 静电学

2.1 basics

库仑定律

$$\vec{F}_1 = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r_{12}^2} \hat{r}_{12} \quad (2.1.1)$$

电场

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{r} \quad (2.1.2)$$

叠加原理

$$\begin{aligned} \vec{E} &= \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho dV_2}{r^2} \hat{r} \\ E_x &= \frac{1}{4\pi\epsilon_0} \int_V \frac{(\Delta x)\rho dV_2}{r^3} \end{aligned}$$

电势

$$\vec{E} = -\nabla\varphi \quad (2.1.3)$$

$$\Delta\phi = -\int_{\gamma} \vec{E} \cdot d\vec{s} \quad \text{path free}$$

Gauss 定理: 闭合面 S , 有

$$\int_S \vec{E} \cdot d\vec{S} = \frac{Q_{\text{内部}}}{\epsilon_0} \quad (2.1.4)$$

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \quad (2.1.5)$$

examples. 均匀带电球体的场 (径向)

$$E(r) = \frac{1}{4\pi\epsilon_0} \begin{cases} \frac{Q}{r^2}, & r > R_0 \\ \frac{Qr}{R_0^3}, & r < R_0 \end{cases} \quad (2.1.6)$$

均匀带电直线

$$E = \frac{\lambda}{2\pi\epsilon_0 r} \quad (2.1.7)$$

均匀带电平面

$$E = \frac{\sigma}{2\epsilon_0} \quad (2.1.8)$$

Circle:

$$E(x) = \frac{Q}{4\pi\epsilon_0} \left(\frac{L}{(r^2 + L^2)^{3/2}} \right) \quad (\text{考虑分量})$$

Annulus:

$$E_x = \frac{\sigma L}{2\epsilon_0} \left(\frac{1}{\sqrt{R_1^2 + L^2}} - \frac{1}{\sqrt{R_2^2 + L^2}} \right)$$

2.2 导体

静电平衡状态下, 内部 $\rho = 0, \vec{E} = 0$, 全导体电势相同.

$$\vec{n} \cdot \vec{E} = \frac{\sigma}{\epsilon_0} \quad \vec{n} \times \vec{E} = 0$$

导体空腔: 内部无电荷时, 内表面无电荷

2.3 electric dipole

$+q, -q$ 电荷, 间隔 d . 定义 $p = qd$, \vec{p} 为 $-q$ 指向 $+q$ 的向量, 大小为 p . 以两者中点为原点, 有

$$\varphi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{\vec{p} \cdot \vec{r}}{r^3} = -\frac{1}{4\pi\epsilon_0} \vec{p} \cdot \nabla \left(\frac{1}{r} \right) \quad (2.3.1)$$

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \left(\frac{3(\vec{p} \cdot \vec{r})\vec{r}}{r^5} - \frac{\vec{p}}{r^3} \right) \quad (2.3.2)$$

2.4 uniqueness theorem

唯一性定理: 闭合区域 V , 已知电荷分布 $\rho(\vec{x})$, 若在 $S = \partial V$ 上以下两者之一成立:

1. $\varphi|_S$ 给定
2. $\frac{\partial \varphi}{\partial n}|_S$ 给定
3. 导体所带电荷量确定

则 V 内电场分布唯一确定。

证明主要利用 $\nabla^2 \varphi = -\frac{\rho}{\epsilon_0}$ 以及恒等式

$$\int_S u \nabla u \cdot d\vec{S} = \int_V \nabla \cdot (u \nabla u) dV = \int_V (u \nabla^2 u + |\nabla u|^2) dV. \quad (2.4.1)$$

唯一性定理与导体: 闭合区域内有一些导体, 若以上条件均满足且每个导体电荷量或者电势之一确定, 则电场分布唯一。

2.5 image charge

image charge 不在希望求电场的空间中, 且引入后导体边界面电势不变, 从而可以忽视导体, 认为整个空间是 Uniform 的。如此引入 image charge 后, 导体外电场不变。

导体平面: 对称位置放等量反号电荷

导体球: 球外距离球心 d 处有电荷 q , 则在连线上距离球心 $\frac{R^2}{d}$ 处放一个 $q' = -\frac{R}{d}q$ 的电荷, 这两个电荷在球面上每一点产生电势为 0. 后可以根据球的电势或者带电量来在球心放置适量电荷。

point charge inside a spherical cavity - 直接就是 q and q' , 无论外导体的带电量, 因为内壁永远只产生 $-q$ 的电荷量

2.6 capacitor

两极板带电 $Q, -Q$, 面积 A , 距离 d , 电荷全部集中在两个内表面 (考虑两个无限平面)

$$E = \frac{\sigma}{\epsilon_0} = \frac{Q}{Ad} \quad (2.6.1)$$

$$V = Ed = \frac{Qd}{A\epsilon_0}, C = \frac{Q}{V} = \frac{A\epsilon_0}{d} \quad (2.6.2)$$

2.7 complex functions

$f: \mathbb{C} \rightarrow \mathbb{C}$, analytic(解析/可导)。 $f(x+yi) = U(x, y) + iV(x, y)$, U, V 均可作为电势函数。取 V 为电势, U 的差为路径上电场的 flux, 故 U 的等值线为电场方向

$$\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y}, \frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x}$$

取 V 为电势

$$E = -\nabla V = -\frac{\partial V}{\partial x} - i\frac{\partial V}{\partial y} = -\frac{\partial V}{\partial x} - i\frac{\partial U}{\partial x} = (-i)\bar{f}'(z)$$

取 U 为电势

$$E = -\nabla U = -\frac{\partial U}{\partial x} - i\frac{\partial U}{\partial y} = -\bar{f}'(z)$$

2.8 examples

plasma oscillation 等离子体, 几乎电中性, 由 ions(正电)+electrons(负电) 构成。受到扰动时, 电子被推离原位, 形成局部正电局部负电 (正电 ions 质量大, 受影响小)。产生恢复电场, 如此反复形成 oscillation. 电子数密度 n_0 , plasma frequency

$$\omega_p = \sqrt{\frac{q_e^2 n_0}{m_e \epsilon_0}} \quad (2.8.1)$$

Colloidal particles

$$\begin{aligned} \rho &= q_e (n_+ - n_-) \\ &= q_e n_0 (e^{-q_e \varphi(x)/kT} - e^{q_e \varphi(x)/kT}) \\ &= -2n_0 q_e^2 \varphi(x)/kT \end{aligned}$$

$$\frac{d^2 \phi}{dx^2} = \frac{2n_0 q_e^2}{\epsilon_0 kT} \phi(x)$$

$$\phi = \frac{D\sigma}{\epsilon_0} e^{x/D}, D = \sqrt{\frac{\epsilon_0 kT}{2n_0 q_e^2}}$$

if $n_0 \uparrow$ then $D \downarrow$, $\phi \downarrow$ 静电 (排斥) 长程力变为短程力, 范德华力 (吸引) 占据主导, 产生沉淀

electric field of a grid 足够远的地方形成均匀电场

$$\varphi_n = F_n(z) \cos \frac{2\pi n x}{a} \quad n = 1, 2, 3, \dots, \quad \varphi = \sum \varphi_n$$

$$\varphi(x, z) = \underbrace{C - E_0 z}_{\text{均匀项}} + \underbrace{\sum_{n=1}^{\infty} A_n e^{-n\pi z/a} \cos\left(\frac{2n\pi x}{a}\right)}_{\text{衰减项}}$$

注意: 如果是感应电荷, $E_{1n} - E_{2n} = \frac{\sigma_f + \sigma_p}{\epsilon_0} = \frac{\sigma_p}{\epsilon_0}$, 这份差上下均分

唯一性定理要求: $\nabla^2 \phi = -\frac{\rho}{\epsilon_i}$, 满足电势的边界条件, 满足介质的边界条件

2.9 静电能

两个电荷的 electrostatic energy: work to bring them together (from infinity)

$$U = \frac{q_1 q_2}{4\pi\epsilon_0 r_{12}} \quad (2.9.1)$$

electrostatic energy for a charge system:

$$U = \frac{1}{2} \sum_i q_i \varphi_i \quad (2.9.2)$$

φ_i : 除 q_i 外, 其余 charge 在 q_i 处产生的电势 (规定无穷远处电势为 0)

连续情形:

$$U = \frac{1}{2} \int \varphi dq \quad (2.9.3)$$

同时有导体和点电荷时的静电能: (点电荷自能不考虑)

$$U = \frac{1}{2} \left(\sum_i \varphi_i Q_i + \sum_j \varphi'_j q_j \right) \quad (2.9.4)$$

导体电荷量 Q_i , φ_i 为该导体在此体系下具有的电势。点电荷 q_j , φ'_j 为 q_j 之外的带电体、电荷在此处产生的电势。

电场与电能

$$U = \frac{\epsilon_0}{2} \int_{\Omega} E^2 dV \quad (2.9.5)$$

$$U = U_{self} + U_{inter}$$

自能:

$$U_{ball} = \frac{3}{5} \frac{Q^2}{4\pi\epsilon_0 R}, U_{sphere} = \frac{Q^2}{8\pi\epsilon_0 R}$$

电容:

$$U = \frac{1}{2} C V^2$$

2.10 dielectric & polarization

极化率越高, 介质内部电场越小, 导体有无穷介电常数。注意 P, D, σ 有相同的量纲。

$$\rho_{pol} = -\nabla \cdot \mathbf{P}, \sigma_{pol} = \vec{P} \cdot \vec{n}$$

$$\mathbf{P} = (\epsilon_r - 1)\epsilon_0 \mathbf{E}, \quad \mathbf{D} = \epsilon_r \epsilon_0 \mathbf{E}$$

$$\nabla \cdot \mathbf{D} = \nabla \cdot (\epsilon_r \epsilon_0 \mathbf{E}) = \rho_{free}$$

$$\nabla \cdot \mathbf{E} = \frac{\rho_{free} + \rho_{pol}}{\epsilon_0}$$

$$E_{1\tau} = E_{2\tau}, E_{1n} - E_{2n} = \frac{\sigma_f + \sigma_p}{\epsilon_0}, D_{1n} - D_{2n} = \sigma_f$$