Applications of Linear Differential Equations with Constant Coeff's. (LDWCC)

Electrical Circuit

The formation of Differential Equation for an electric circuit depends upon the following laws.

(i) $i = \dfrac{dq}{dt}$

(ii) Voltage drop across resistance $(R) = Ri$

(iii) Voltage drop across inductance $(L) = L\dfrac{di}{dt}$

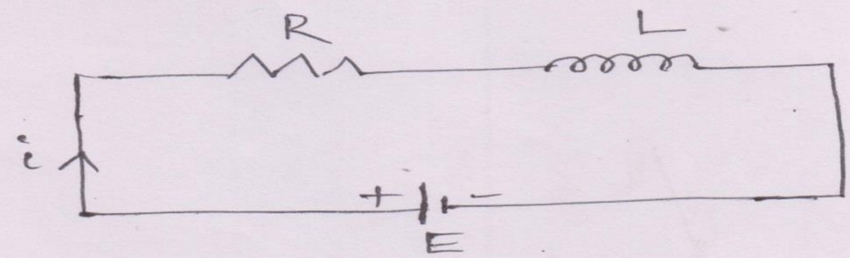
(iv) Voltage drop across Capacitance $(C) = \dfrac{q}{C}$

Kirchoff's Law

Voltage Law : The algebraic sum of the voltage drop around any closed circuit is equal to the resultant electromotive force in the circuit.

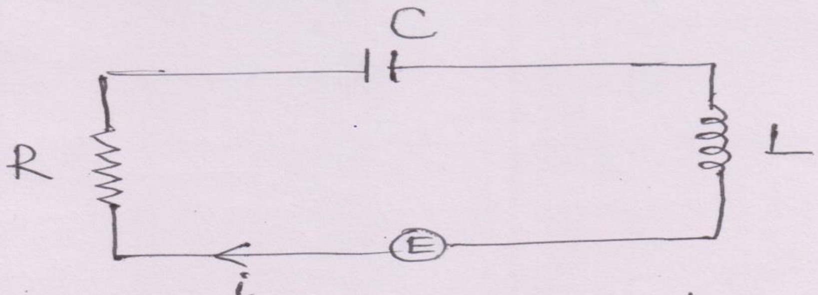
Current Law ; At a junction (or) nodes, Current coming is equal to current going.

L-R Series circuit



by voltage Law, $Ri + L\frac{di}{dt} = E$

L-R-C Circuit



by voltage Law, $Ri + L\frac{di}{dt} + \frac{q}{c} = E$

(or) $L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = E \quad (\because i = \frac{dq}{dt})$

① An electric circuit consists of an inductance 0.1 henry, a resistance of 20 ohms and a condenser of capacitance 25 micro-farads. Find the charge q and the current i at any time 't', given that at $t=0$, $q = 0.05$ coulomb, $i=0$ when $t=0$

Sol: Given $L = 0.1\ h$, $R = 20\ \Omega$,

$C = 25\ m.f = 25 \times 10^{-6}$ farad

The D.E. for the given data is

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = 0 \qquad \text{①}$$

It's operator form is $\left[D^2 + \left(\frac{R}{L}\right)D + \frac{1}{LC}\right]q = 0 \qquad \text{②}$

(or) $$\left[D^2 + \left(\frac{20}{0.1}\right)D + \frac{1}{(0.1)(25\times10^{-6})}\right]q = 0$$

Clearly $q_c = e^{-100t}\left[c_1 \cos(100\sqrt{39}\,t) + c_2 \sin(100\sqrt{39}\,t)\right]$

$\therefore q = q_c = e^{-100t}\left[c_1 \cos(100\sqrt{39}\,t) + c_2 \sin(100\sqrt{39}\,t)\right]$

when $t=0$; $q = 0.05$ & $i = 0$, then $c_1 = 0.05$
$c_2 = 0.008$

$\therefore q = e^{-100t}\left[0.05 \cos(624.5t) + 0.08 \sin(624.5t)\right]$

(2) An inductor of 2 henries, resistor of 16 ohms and capacitor of 0.02 farads are connected in series with a battery of electromotive force $E = 100 \sin 3t$. At $t = 0$, the charge on the capacitor and current in the circuit are zero. Find the charge and current.

Sol: Given $L = 2\,h$; $R = 16\,\Omega$; $C = 0.02$ farad

$E = 100 \sin 3t$.

The D.E. for the given data is

$$L \frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = E$$

(or) It's operator form is $\left[D^2 + \frac{R}{L} D + \frac{1}{LC}\right] q = \frac{E}{L}$ ——— (1)

(or) $$\left[D^2 + \left(\frac{16}{2}\right) D + \left(\frac{1}{2 \times 0.02}\right)\right] q = \frac{100 \sin 3t}{2}$$

Clearly $q_c = e^{-4t}\left[c_1 \cos 3t + c_2 \sin 3t\right]$

$$q_p = \frac{25}{52}\left[2 \sin 3t - 3 \cos 3t\right]$$

$\therefore q = q_c + q_p = e^{-4t}\left[c_1 \cos 3t + c_2 \sin 3t\right]$
$+ \frac{25}{52}\left[2 \sin 3t - 3 \cos 3t\right]$

$$i = \frac{dq}{dt} = -4e^{-4t}(c_1 \cos 3t + c_2 \sin 3t)$$

$$- 3e^{-4t}(c_1 \sin 3t - c_2 \cos 3t)$$

$$+ \frac{25}{52}(6\cos 3t + 9\sin 3t)$$

when $t=0$, $q=0$ & $i=0$

$$\therefore \; c_1 = 75/52 \;\; ; \;\; c_2 = 50/52$$

$$\therefore q = \frac{25}{52} e^{-4t}\left[3\cos 3t + 2\sin 3t\right]$$

$$+ \frac{25}{52}\left[2\sin 3t - 3\cos 3t\right]$$

$$i = \frac{75}{52}\left[2\cos 3t + 3\sin 3t\right]$$

$$- \frac{25}{52} e^{-4t}\left[17\sin 3t + 6\cos 3t\right]$$

Exercise :-

1. For an electric circuit with $L = 0.05$ henry, $R = 20$ ohms and $C = 100 \times 10^{-6}$ farad, the applied e.m.f is 100 volts. prove that the charge q at time 't' is given by
$$q(t) = 0.01 - e^{-200t}\left[0.01\cos(400t) + 0.02\sin(400t)\right]$$
if initially $q = 0$ and $i = 0$

2. An alternating E.M.F $E\sin pt$ is applied to a circuit at $t=0$. Given the equation for the current i as $L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{i}{c} = pE\cos pt$.
Find the current i when $CR^2 > 4L$.