4.3: Solving non-homogeneous system — using Laplace transform

① Solve the system of equations

$$\frac{dx}{dt} + 5x - 2y = t$$

$$\frac{dy}{dt} + 2x + y = 0 \qquad \text{with } x(0) = 0,\ y(0) = 0.$$

Sol: Taking Laplace transform, we get

$$L\{x'(t)\} + 5L\{x(t)\} - 2L\{y(t)\} = L\{t\}$$

$$\Rightarrow \{sL\{x(t)\} - \cancel{x(0)}\} + 5L\{x(t)\} - 2L\{y(t)\} = \frac{1}{s^2}$$

$$\Rightarrow (s+5)\,L\{x(t)\} - 2L\{y(t)\} = \frac{1}{s^2} \longrightarrow \text{(i)} \quad (\text{since } x(0) = 0)$$

and

$$L\{y'(t)\} + 2L\{x(t)\} + L\{y(t)\} = L(0)$$

$$\Rightarrow \big(sL\{y(t)\} - y(0)\big) + 2L\{x(t)\} + L\{y(t)\} = 0$$

$$\Rightarrow (s+1)\,L\{y(t)\} + 2L\{x(t)\} = 0 \longrightarrow \text{(ii)} \quad (\text{since } y(0) = 0)$$

By solving (i) & (ii), we get

$$L\{x(t)\} = \frac{s+1}{s^2(s+3)^2} \Rightarrow L\{x(t)\} = \frac{1}{27s} + \frac{1}{9s^2} - \frac{1}{27(s+3)} - \frac{2}{9(s+3)^2}$$

and $$L\{y(t)\} = \frac{-2}{s^2(s+3)^2} \Rightarrow L\{y(t)\} = \frac{4}{27s} - \frac{2}{9s^2} - \frac{4}{27(s+3)} - \frac{2}{9(s+3)^2}$$

and hence $$x(t) = \frac{1}{27} + \frac{t}{9} - \frac{e^{-3t}}{27} - \frac{2}{9}te^{-3t},$$

and $$y(t) = \frac{4}{27} - \frac{2t}{9} - \frac{4}{27}e^{-3t} - \frac{2}{9}te^{-3t}.$$

② solve $x'(t) - 4y(t) + 4e^{t} = 0$

$y'(t) - x(t) - e^{t} = 0$

with $x(0) = 1$, $y(0) = 1$ using Laplace transform.

<u>Answer:</u> $x(t) =$

$$y(t) = e^{t} + \left(\frac{1}{4}\right)e^{2t} - \left(\frac{1}{4}\right)e^{-2t}.$$

③ solve

$x'(t) - x(t) - 2y(t) = 3$

$y'(t) - 2x(t) - y(t) = 0$

with $x = 0$, $y(0) = 2$ using Laplace transform.