## ④ Multiplication with $t^n$ :-

If $L\{f(t)\} = F(s)$, then

$$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}\left[F(s)\right] \; ; \; n = 1, 2, 3, \ldots$$

Ex (i) We know that $L\{\cos t\} = \frac{s}{s^2+1} = F(s)$

Therefore, $L\{t \cos t\} = (-1)\frac{d}{ds}\left[F(s)\right]$

$$= -\frac{d}{ds}\left[\frac{s}{s^2+1}\right]$$

$$= -\left[\frac{(s^2+1)1 - 2s(s)}{(s^2+1)^2}\right]$$

$$= \frac{s^2-1}{(s^2+1)^2}$$

(ii) We know that $L\{e^{2t}\} = \frac{1}{s-2} = F(s)$

Therefore, $L\{t^2 e^{2t}\} = (-1)^2 \frac{d^2}{ds^2}\left[F(s)\right]$

$$= \frac{d^2}{ds^2}\left[\frac{1}{s-2}\right]$$

$$= \frac{d}{ds}\left[-\frac{1}{(s-2)^2}\right]$$

$$= \frac{2}{(s-2)^3}$$

## (5) Division by 't' :-

If $L\{f(t)\} = F(s)$, then

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)\, ds$$

Ex (i) we have $L\{\sin t\} = \frac{1}{s^2+1} = F(s)$

Therefore, $L\left\{\frac{\sin t}{t}\right\} = \int_s^\infty F(s)\, ds$

$$= \int_s^\infty \frac{1}{s^2+1}\, ds$$

$$= \left[\tan^{-1}(s)\right]_s^\infty$$

$$= \frac{\pi}{2} - \tan^{-1}(s)$$

(ii) we have $L\{1-\cos t\} = L\{1\} - L\{\cos t\}$

$$= \frac{1}{s} - \frac{s}{s^2+1} = F(s)$$

Therefore, $L\left\{\frac{1-\cos t}{t}\right\} = \int_s^\infty F(s)\, ds$

$$= \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+1}\right) ds$$

$$= \left[\log s - \frac{1}{2}\log(s^2+1)\right]_s^\infty$$

$$= \left\{\log\left[\frac{s}{\sqrt{s^2+1}}\right]\right\}_s^\infty$$

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$$= \left\{ \log\left[\frac{1}{\sqrt{1+\frac{1}{s^2}}}\right]\right\}_s^{\infty}$$

$$= \log 1 - \log\left[\frac{1}{\sqrt{1+\frac{1}{s^2}}}\right]$$

$$= 0 - \log\left[\frac{s}{\sqrt{s^2+1}}\right]$$

$$= \log\left[\frac{\sqrt{s^2+1}}{s}\right]$$

(6) Laplace Transform of derivatives:

If $L\{f(t)\} = F(s)$, then

$$L\left\{\frac{d^n}{dt^n}(f(t))\right\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - s^{n-3} f''(0) - \cdots - f^{(n-1)}(0) \; ; \; n = 1, 2, 3, \ldots$$

Note (i) $L\{f'(t)\} = sF(s) - f(0)$

(ii) $L\{f''(t)\} = s^2 F(s) - s f(0) - f'(0)$

(iii) $L\{f'''(t)\} = s^3 F(s) - s^2 f(0) - s f'(0) - f''(0)$

Ex: Let $f(t) = \sin\sqrt{t}$,

then $L\{f(t)\} = L\{\sin\sqrt{t}\} = \frac{\sqrt{\pi}}{2s^{3/2}} \cdot e^{-1/4s} = F(s)$

and $f'(t) = \frac{\cos\sqrt{t}}{2\sqrt{t}}$.

Now $L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = 2L\{f'(t)\}$

$$= 2\left(sF(s) - f(0)\right)$$

$$= 2\left(s\frac{\sqrt{\pi}}{2s^{3/2}} e^{-1/4s} - 0\right)$$

$$= \sqrt{\frac{\pi}{s}}\, e^{-1/4s}$$

$$\therefore L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = \sqrt{\frac{\pi}{s}} \cdot e^{-1/4s}$$

## (7) Laplace Transform of integrals :-

If $L\{f(t)\} = F(s)$

then $L\left\{\int_0^t f(t)\,dt\right\} = \frac{1}{s} F(s)$

Note: (i) $L\left\{\int_0^t \left(\int_0^t f(t)\,dt\right)dt\right\} = \frac{1}{s^2} F(s)$

(ii) $L\left\{\int_0^t \left(\int_0^t \left(\int_0^t f(t)\,dt\right)dt\right)dt\right\} = \frac{1}{s^3} F(s)$.

Ex(i) We have $L\{t\cos t\} = \dfrac{s^2-1}{(s^2+1)^2} = F(s)$

Now $L\left\{\int_0^t t\cos t\, dt\right\} = \dfrac{1}{s} F(s)$

$$= \frac{1}{s}\,\frac{s^2-1}{(s^2+1)^2}$$

(ii) We have $L\{t\sin t\} = \dfrac{2s}{(s^2+1)^2}$

So, $L\{e^{-t}\, t\sin t\} = \dfrac{2(s+1)}{[(s+1)^2+1]^2} = F(s).$

Therefore, $L\left\{\int_0^t e^{-t}\, t\sin t\, dt\right\} = \dfrac{1}{s} F(s)$

$$= \frac{1}{s}\cdot\frac{2(s+1)}{[(s+1)^2+1]^2}$$

Exercise : Evaluate the following

① $L\{\sin\sqrt{t}\}$ ② $L\{t^2\sin t\}$ ③ $L\{e^{-2t}\, t\cos 2t\}$

④ $L\left\{\dfrac{1-\cos 2t}{t}\right\}$ ⑤ $L\left\{\dfrac{e^{-2t}-e^{-3t}}{t}\right\}$ ⑥ $L\left\{\dfrac{\sin 2t\cos t}{t}\right\}$

⑦ $L\left\{\dfrac{\sin t}{t}\right\}$ ⑧ $L\left\{\dfrac{1}{2\sqrt{t}}\right\}$ ⑨ $L\left\{\int_0^t e^{2t}\, t\cos t\, dt\right\}$

⑩ $L\left\{\int_0^t\left(\int_0^t t^2 e^{-3t}\, dt\right) dt\right\}$

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