Free Oscillations in Mechanical Circuits without Damping

Consider a mass m attached to the end of a spring with spring constant k. Suppose that the mass is released from the initial position $x(0) = x_0$ with a downward velocity $x'(0) = w_0$. Its vertical motion is modeled by the differential equation: $m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + kx = 0$. Note that the resulting motion is a *simple harmonic motion*(SHM).

Example 1. *A mass weighing 4 pounds, attached to a spring whose spring constant is 16 lb/ft, is in the mean position. If the mass is released with a downward velocity 8 ft/s, what is the subsequent vertical displacement? What is the period of simple harmonic motion?*

Solution. Note that $m = w/g = 4/32 = 1/8$ slug and $k = 16$ lb/ft. The vertical displacement of the spring is described by the differential equation $\frac{1}{8}\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 16x = 0$ or $\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 128x = 0$. Its general solution is given by $x = x(t) = A\cos 8\sqrt{2}t + B\sin 8\sqrt{2}t$. Using the initial condition $x(0) = 0$, we see that $A = 0$. Thus $x(t) = B\sin 8\sqrt{2}t$. Differentiating with respect to t and then applying the initial velocity condition $x'(0) = 8$, we find that $8\sqrt{2}B\cos 0 = 8$ or $B = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}\cdot$ Therefore, the subsequent vertical displacement is $x(t) = \frac{\sqrt{2}}{2}\sin(8\sqrt{2}t)$. The period of the motion is $T = \frac{2\pi}{\omega} = \frac{2\pi}{8\sqrt{2}} = \sqrt{\frac{2\pi}{8}}$ seconds.

Self-check Exercises

Exercise 1. A 20-kilogram mass is attached to a spring. Describe the undamped simple harmonic motion of the spring. If the frequency of the motion is $2/\pi$ cps, what is the spring constant k?

Ans. The frequency is $\frac{\omega}{2\pi}$ and hence k = 320 Newtons per meter.

Exercise 2. A mass weighing 24 pounds, attached to the end of a spring, stretches it 4 inches. Initially, the mass is released from rest from a point 3 inches above the equilibrium position. Find the equation of motion.

Ans. The mass is $m = w/g = 24/32 = 3/4$ slug and $k = 72$ lb/ft. The initial conditions are $x(0) = -1/4,\ x'(0) = 0$. The displacement is $x(t) = -\frac{1}{4}\cos 4\sqrt{6}\,t$.

Exercise 3. A force of 400 newtons stretches a spring 2 meters. A mass of 50 kilograms is attached to the end of the spring and is initially released from the equilibrium position with an upward velocity of 10 m/s. Find the equation of motion.

Ans. The mass is $m = 50$ kilograms and $k = \frac{400}{2} = 200$ newtons per meter. The initial conditions are $x(0) = 0,\ x'(0) = -10$.

Free Oscillations in Mechanical Circuits with Damping

Consider a mass m attached to the end of a spring with spring constant k. Suppose that the medium offers a damping force (by immersing the mass in a liquid), which is numerically equal to the instantaneous velocity. If it is released from the initial position $x(0) = x_0$ with a downward velocity $x'(0) = w_0$. Its motion is modeled by the differential equation: $m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \beta\frac{\mathrm{d}x}{\mathrm{d}t} + kx = 0$, where β is the damping constant.

Example 1. *A mass weighing 4 pounds is attached to a spring whose constant is 2 lb/ft. The medium offers a damping force that is numerically equal to the instantaneous velocity. The mass is initially released from a point 1 foot above the equilibrium position with a downward velocity of 8 ft/s. Determine the equation of motion.*

Solution. Note that $m = w/g = 4/32 = 1/8$ slug and The damping constant is $\beta = 1$ and the spring constant $k = 2$ lb/dt. The equation of motion is $m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \beta\frac{\mathrm{d}x}{\mathrm{d}t} + kx = 0$ or $\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 8\frac{\mathrm{d}x}{\mathrm{d}t} + 16x = 0$. Its general solution is $x(t) = (c_1 + c_2 t)e^{-4t}$. Applying the initial conditions $x(0) = -1\ x'(0) = 8$, we get $c_1 = -1$ and $c_2 = 4$. Thus $x(t) = (4t - 1)e^{-4t}$.

Self-check Exercises

Exercise 1. A mass *m* is attached to both a spring (with given spring constant *k*) and a dashpot (with given damping constant *c*). The mass is set in motion with initial position x_0 and initial velocity w_0. Find the position function $x(t)$ and determine whether the motion is overdamped, critically damped, or underdamped:

(a) $m = \frac{1}{2}, c = 3, k = 4, x_0 = 2, w_0 = 0$
(b) $m = 3, c = 30, k = 63, x_0 = 2, w_0 = 2$
(c) $m = 1, c = 8, k = 16, x_0 = 5, w_0 = -10$

Free Oscillations in LC- Circuits (Electrical Circuits without Damping - without an external emf)

The differential equation which describes the charge *q*(*t*) on the capacitor at any time *t* seconds in an LC-series circuit is $L\frac{\mathrm{d}^2q}{\mathrm{d}t^2} + \frac{1}{C}q = 0$. The initial conditions are $q(0) = q_0, q'(0) = i_0$. Note that *L* is the inductance in henry and *C* is the capacitance in farad.

Self-check Exercises

Exercise 1. Find the charge on the capacitor in an LC-series circuit when $L = 1/2$ henry, *C* = 0.01 farad, *q*(0) = 1 coulomb, and *i*(0) = 0 ampere. What is the charge on the capacitor after a long time?

Exercise 2. Find the charge on the capacitor in an LC-series circuit when $L = .02$ henry, *C* = 0.05 farad, *q*(0) = 1.5 coulomb, and *i*(0) = 0 ampere.

Free Oscillations in LRC- Circuits (Electrical Circuits with Damping - without an external emf)

The differential equation which describes the charge *q*(*t*) on the capacitor at any time *t* seconds in an LRC-series circuit is $L\frac{\mathrm{d}^2q}{\mathrm{d}t^2} + R\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{1}{C}q = 0$. The initial conditions are $q(0) = q_0, q'(0) = i_0$. Note that *L* is the inductance in henry, *R* is the resistance in ohm and *C* is the capacitance in farad.

Self-check Exercises

Exercise 1. Find the charge on the capacitor in an LRC-series circuit at *t* = 0.01 seconds, when L = 1/4 henry, R = 20 ohm, C = 1/300 farad, *E*(*t*) = 0 volt, *q*(0) = 4 coulomb, and *i*(0) = 0 ampere. Determine the charge on the capacitor at any time *t* > 0. Is the charge on the capacitor ever equal to zero?

Exercise 2. Find the charge on the capacitor in an LRC-series circuit at *t* = 0.01 second when *L* = 0.05 henry, *R* = 2 ohm, C = 0.0l farad, *E*(t) = 0 volt, *q*(0) = 5 coulomb, and *i*(0) = 0 ampere. Determine the charge on the capacitor at time *t* > 0.

Forced Oscillations in Mechanical Circuits without Damping

Mathematical model: $m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + kx = F(t)$
Initial conditions: $x(0) = x_0, x'(0) = w_0$.

Exercise 1. Suppose that $m = 1, k = 9, F(t) = 80\cos 5t$. Find the vertical displacement $x(t)$, if $x(0) = 0, x'(0) = 0$.

Exercise 2. Suppose that $m = 0.1, k = 9, F(t) = \sin 5t \sin 50t$. Find the vertical displacement $x(t)$, if $x(0) = 0, x'(0) = 0$.

Exercise 3. Solve the following initial value problems related to mass-spring system:

(a) $\frac{d^2x}{dt^2} + 9x = 10\cos 2t, x(0) = 0, x'(0) = 0$

(b) $\frac{d^2x}{dt^2} + 100x = 225\cos 5t + 300\sin 5t, x(0) = 375, x'(0) = 0.$

Forced Oscillations in Mechanical Circuits with Damping

Mathematical model: $m\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + kx = 0 = F(t)$

Initial conditions: $x(0) = x_0,\ x'(0) = w_0.$

Exercise 1. Solve the following initial value problems related to damped mass-spring system:

(a) $\frac{d^2x}{dt^2} + \frac{dx}{dt} + 5x = 10\cos 3t, x(0) = 0, x'(0) = 0$

(b) $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 26x = 600\cos 10t, x(0) = 10, x'(0) = 0.$

Forced Oscillations in LC- Circuits (Electrical Circuits without Damping – under an external emf)

Mathematical model: $L\frac{d^2q}{dt^2} + \frac{1}{C}q = E(t)$

Initial conditions: $q(0) = q_0, q'(0) = i_0.$

Exercise 1. Solve the following initial value problems related to LC-circuits:

(a) $10\frac{d^2q}{dt^2} + 0.02q = 50\sin 2t, q(0) = 0, q'(0) = 0$

(b) $5\frac{d^2q}{dt^2} + 0.005q = 400\sin 100t + 300\cos 100t, q(0) = 2, q'(0) = 0.$

Forced Oscillations in LRC- Circuits (Electrical Circuits with Damping – under an external emf)

Mathematical model: $L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t)$ ... (1)

Initial conditions: $q(0) = q_0, q'(0) = i_0.$ ... (2)

Exercise 1. Find the charge q on the capacitor and the current I in the given LRC-series circuit with L = 5/3 henry, R = 10 ohm, C = 1/30 farad, $E(t)$ = 300 volt, $q(0)$ = 0 coulomb, and $I(0)$ = 0 ampere. Find the maximum charge on the capacitor.

Exercise 2. Find the charge q on the capacitor and the current I in the given LRC-series circuit with L = 5/3 henry, R = 10 ohm, C = 1/30 farad, $E(t)$ = 300 volt, $q(0)$ = 0 coulomb, and $I(0)$ = 0 ampere. Find the maximum charge on the capacitor. Find the steady-state charge and the steady-state current in an LRC-series circuit when L = 1 henry, R = 2 ohm, C = 0.25 farad, and $E(t)$ = 50 cos t volt.

Exercise 3. Solve the following initial value problem (1)-(2) with each of the following sets of values:

(a) $L = 2, R = 16, C = .02, E(t) = 100; q(0) = 5, i(0) = 0$

(b) $L = 2, R = 60, C = .0025, E(t) = 100e^{-10t}; q(0) = 0, i(0) = 0.$

Note: In problems related tofoced oscillations, you may use either the method of undetermined coefficients or the method of variation of parameters.