Applications of Linear Differential equations

Spring Mass System

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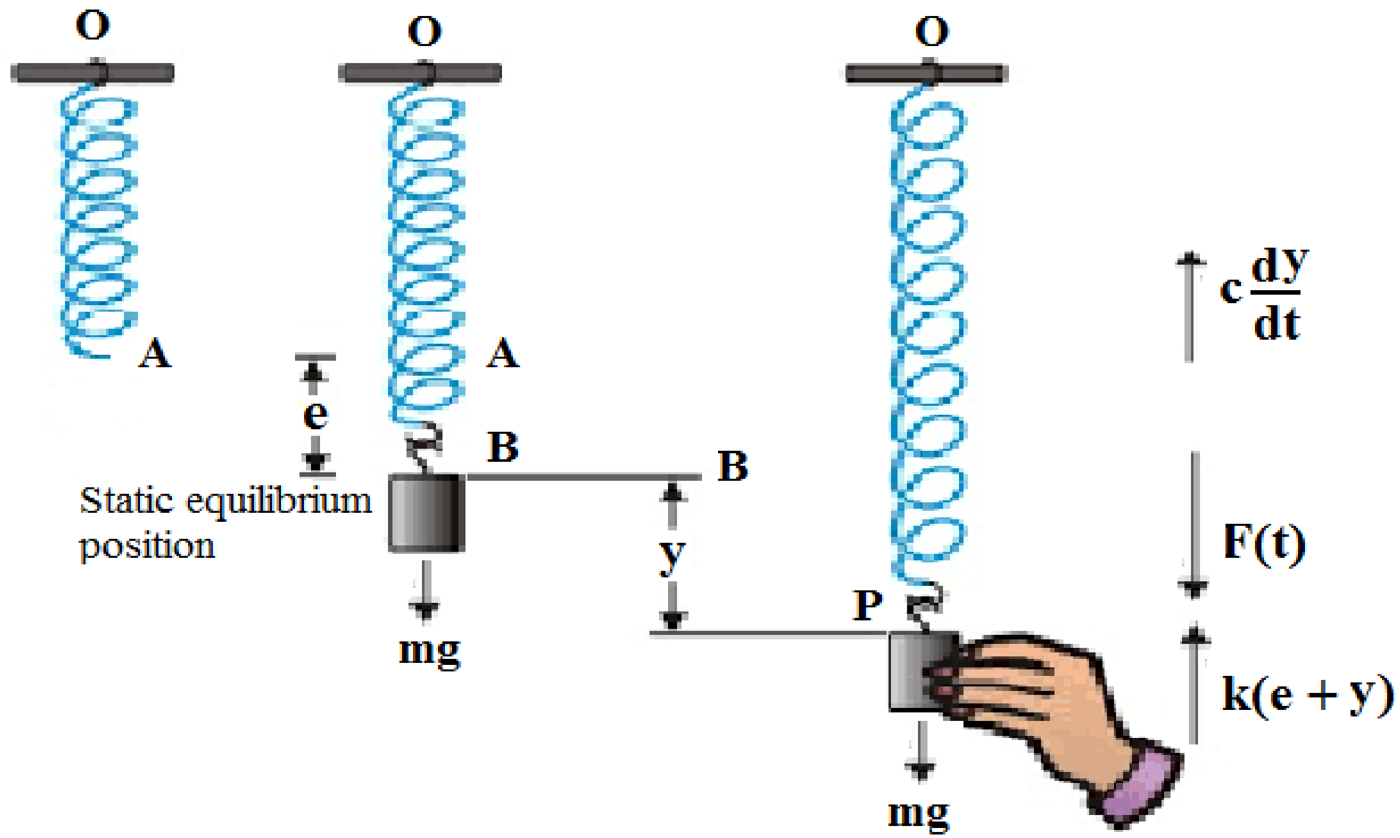
Mass-spring system



Oscillations of a Spring

Forces acting on the mass

- Gravitational force $mg = ke$
- Restoring force $k(e + y)$
  (due to the displacement of the spring)
- Damping (frictional) force $c\dfrac{dy}{dt}$
- External applied force $F(t)$

Net force acting on the mass

$$
\begin{aligned}
F_{net}(t) &= mg - k(e + \mathbf{y}) - c\frac{d\mathbf{y}}{dt} + F(t) \\
&= ke - ke - ky - c\frac{dy}{dt} + F(t) \\
&= -ky - c\frac{d\mathbf{y}}{dt} + F(t)
\end{aligned}
$$

From Newton's Second Law of Motion

$$F_{net} = ma = my''(t)$$

For our spring-mass system

$$m\frac{d^2y}{dt^2} = -ky - c\frac{dy}{dt} + F(t)$$

$$\boxed{m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + ky = F(t)}$$

1. Undamped Free Vibrations

$$my''(t) + \cancel{cy'(t)} + ky(t) = \cancel{F(t)}$$

no damping

no external force

$$my''(t) + ky(t) = 0$$

$$y''(t) + \omega_0^2 y(t) = 0 \qquad \omega_0^2 = \frac{k}{m}$$

$$\cos\omega_0 t, \quad \sin\omega_0 t$$ (particular solutions)

$$y(t) = A\cos\omega_0 t + B\sin\omega_0 t$$ (general solution)

A and *B* are arbitary constants determined from initial conditions $y(0) = y_0$ and $y'(0) = y'_0$

1. Undamped Free Vibrations

$$T = \frac{2\pi}{\omega_0} = 2\pi\left(\frac{m}{k}\right)^{1/2}$$ **Period** of motion

$$\omega_0 = \sqrt{\frac{k}{m}}$$ **Natural frequency** of the vibration

$$R = \sqrt{A^2 + B^2}$$ **Amplitude** (constant in time)

2. Damped Free Vibrations

$$my''(t) + cy'(t) + ky(t) = \cancel{F(t)}$$

no external force

$$my''(t) + cy'(t) + ky(t) = 0$$

Auxiliary equation is

$$mr^2 + cr + k = 0$$

2. Damped Free Vibrations

$$mr^2 + cr + k = 0$$

Solutions to auxiliary equation are:

$$r_1, r_2 = \frac{-c \pm \sqrt{c^2 - 4km}}{2m} = \frac{c}{2m}\left(-1 \pm \sqrt{1 - \frac{4km}{c^2}}\right)$$

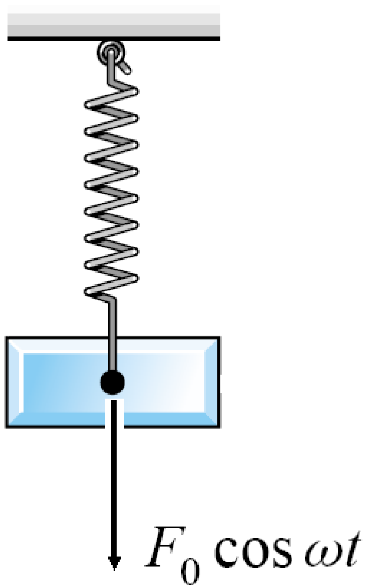
$$c^2 - 4km > 0, \quad y = Ae^{r_1 t} + Be^{r_2 t}$$ over damped

$$c^2 - 4km = 0, \quad y = (A + Bt)e^{-ct/2m}$$ critically damped

$$c^2 - 4km < 0, \quad y = e^{-ct/2m}(A\cos\mu t + B\sin\mu t)$$

Oscillatory or under damped

3. Forced Vibrations (Undamped)



$$my''(t) + cy'(t) + ky(t) = \underbrace{F(t)}_{F_0 \cos \omega t}$$

Periodic external force:

$$my''(t) + \cancel{cy'(t)} + ky(t) = F_0 \cos \omega t$$

no damping

$$my''(t) + ky(t) = F_0 \cos \omega t$$

4. Forced Vibrations (damped)

If the external force F(t) is applied and damping force is present then equation of motion of the spring is

$$\boxed{m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + ky = F(t)}$$

Determine the solutions of a mechanical system with weight 8 lb, stiffness constant 4 lb/ft, damping force is 2 times instantaneous velocity and external force is $8 \sin 4t$. The initial conditions are $y = 1$ and $y' = 0$.

Solution:

The equation of motion of the spring is

$$m \frac{d^2y}{dt^2} + c \frac{dy}{dt} + k y = F(t)$$

Weight = 8 lb => mg = 8 lb => m = 8/32 = ¼, k = 4 lb/ft

Hence equation of motion is

$$\frac{1}{4} \frac{d^2y}{dt^2} + 2 \frac{dy}{dt} + 4y = 8 \sin 4t$$

Roots of the auxiliary equation are: - 4, - 4

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Roots of the auxiliary equation are: - 4, - 4

Hence C. F. is $y_c(t) = (A\,t + B)\,e^{-4t}$

P.I. is $y_p(t) = -\cos 4t$

Complete solution is $y(t) = (A\,t + B)\,e^{-4t} - \cos 4t$

when $t = 0$, $y = 1 \Rightarrow B = 2$

when $t = 0$, $y' = 0 \Rightarrow A = 8$

Hence $y(t) = (8\,t + 2)\,e^{-4t} - \cos 4t$