9.29 CONVOLUTION

Convolution is useful for obtaining Inverse Laplace Transform of a product of two transforms and solving ordinary differential equations.

Definition : Let $f(t)$ and $g(t)$ be two functions defined for $t > 0$. We define

$$f(t) * g(t) = \int_0^t f(u) g(t-u) du$$

assuming that the integral on the right hand side exists.

$f(t) * g(t)$ is called the convolution product of $f(t)$ and $g(t)$.

It can be proved that

(*i*) Convolution product is commutative *i.e.* $f(t)*g(t)=g(t)*f(t)$

(*ii*) Convolution product is associative *i.e.* $f(t)*(g(t)*h(t))=(f(t)*g(t))*h(t)$

(*iii*) $f(t)*0=0*f(t)=0$

Note : In general $1*f(t)\neq f(t)$

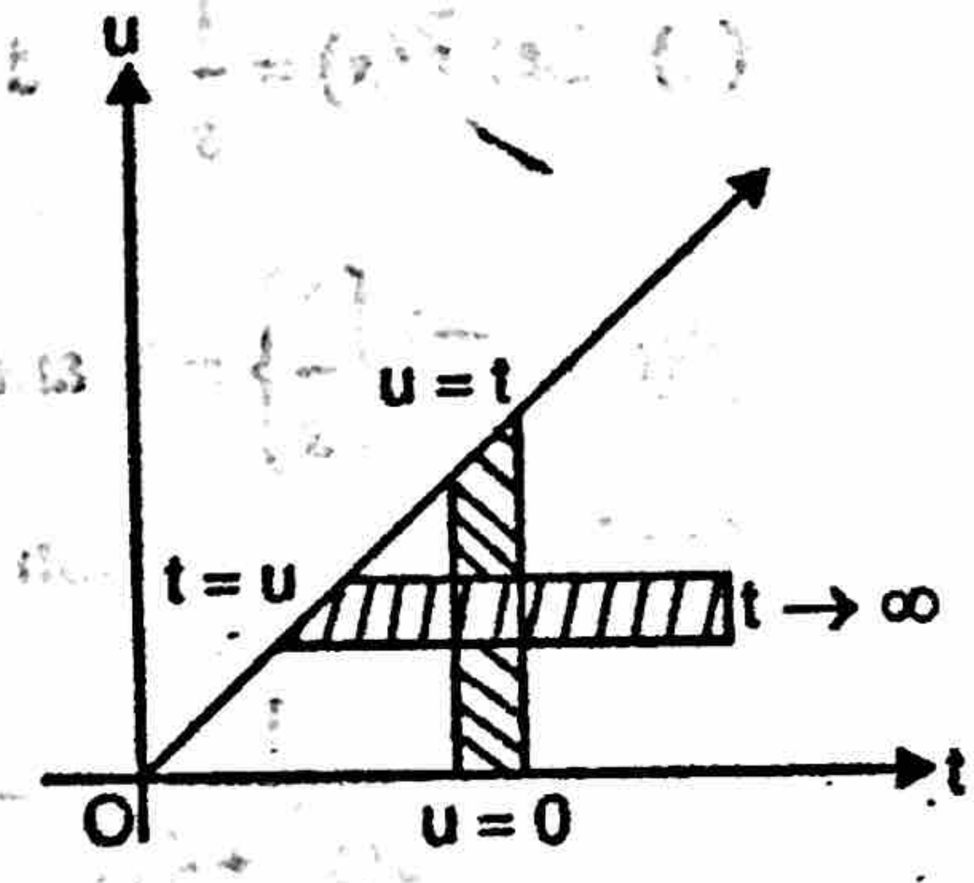
Convolution Theorem : If $L\{f(t)\}=\bar{f}(s)$ and $L\{g(t)\}=\bar{g}(s)$ then $L\{f(t)*g(t)\}=\bar{f}(s)\cdot\bar{g}(s)$

or $L^{-1}\{\bar{f}(s)\cdot\bar{g}(s)\}=f(t)*g(t)$ **[JNTU 2008S (Set No. 3)]**

Proof: Let $\phi(t)=\int_0^t f(u)g(t-u)du$. Then

$$L\{\phi(t)\}=\int_0^\infty e^{-st}\left\{\int_0^t f(u)g(t-u)du\right\}dt=\int_0^\infty\int_0^t e^{-st}f(u)g(t-u)du\,dt$$

The double integral is considered within the region enclosed by the lines $u=0$ and $u=t$. On changing the order of integration, we get



$$L\{\phi(t)\}=\int_0^\infty\int_u^\infty e^{-st}f(u)g(t-u)dt\,du$$

$$=\int_0^\infty e^{-su}f(u)\left\{\int_u^\infty e^{-s(t-u)}g(t-u)dt\right\}du$$

$$=\int_0^\infty e^{-su}f(u)\left\{\int_0^\infty e^{-sv}g(v)dv\right\}du,\text{ on putting } t-u=v$$

$$=\int_0^\infty e^{-su}f(u)\{\bar{g}(s)\}du=\bar{g}(s)\int_0^\infty e^{-su}f(u)du=\bar{g}(s).\bar{f}(s)$$

$\therefore L\{\phi(t)\}=\bar{f}(s).\bar{g}(s)$.

Hence the theorem follows.

SOLVED EXAMPLES

Example 1 : Using Convolution theorem, find (*i*) $L^{-1}\left\{\frac{1}{(s+a)(s+b)}\right\}$ (*ii*) $L^{-1}\left\{\frac{1}{s(s^2+4)}\right\}$

<u>**Solution :**</u> (*i*) Let $\bar{f}(s)=\frac{1}{s+a}$ and $\bar{g}(s)=\frac{1}{s+b}$. Then

$$f(t)=L^{-1}\{\bar{f}(s)\}=L^{-1}\left\{\frac{1}{s+a}\right\}=e^{-at}$$

and $\quad g(t) = L^{-1}\{\bar{g}(s)\} = L^{-1}\left\{\dfrac{1}{s+b}\right\} = e^{-bt}$

$\therefore$ By Convolution theorem,

$$L^{-1}\left\{\frac{1}{(s+a)(s+b)}\right\} = L^{-1}\left\{\frac{1}{s+a}\cdot\frac{1}{s+b}\right\} = L^{-1}\{\bar{f}(s)\cdot\bar{g}(s)\}$$

$$= f(t) * g(t) = \int_0^t f(u)g(t-u)du$$

$$= \int_0^t e^{-au}\cdot e^{-b(t-u)}du = e^{-bt}\int_0^t e^{-(a-b)u}du$$

$$= e^{-bt}\left[\frac{e^{-(a-b)u}}{-(a-b)}\right]_0^t = -\frac{1}{a-b}e^{-bt}\left[e^{-(a-b)t}-1\right]$$

$$= \frac{1}{b-a}(e^{-at}-e^{-bt})$$

(ii) Let $\bar{f}(s) = \dfrac{1}{s}$ and $\bar{g}(s) = \dfrac{1}{s^2+4}$. Then

$$f(t) = L^{-1}\left\{\frac{1}{s}\right\} = 1 \text{ and } g(t) = L^{-1}\left\{\frac{1}{s^2+2^2}\right\} = \frac{1}{2}\sin 2t$$

Applying Convolution theorem,

$$L^{-1}\left\{\frac{1}{s(s^2+4)}\right\} = L^{-1}\left\{\frac{1}{s}\cdot\frac{1}{s^2+4}\right\} = L^{-1}\{\bar{f}(s)\cdot\bar{g}(s)\}$$

$$= f(t) * g(t) = \int_0^t f(u)g(t-u)du = \int_0^t 1\cdot\frac{1}{2}\sin 2(t-u)du$$

$$= \frac{1}{2}\int_0^t \sin 2(t-u)du = \frac{1}{2}\left[\frac{-\cos 2(t-u)}{-2}\right]_0^t$$

$$= \frac{1}{4}(\cos 0 - \cos 2t) = \frac{1}{4}(1-\cos 2t)$$

Example 2 : Using Convolution theorem, evaluate $L^{-1}\left\{\dfrac{1}{s(s^2+2s+2)}\right\}$

Solution : Since $f(t) = L^{-1}\left\{\dfrac{1}{s}\right\} = 1$ and

$$g(t) = L^{-1}\left\{\frac{1}{s^2+2s+2}\right\} = L^{-1}\left\{\frac{1}{(s+1)^2+1}\right\} = e^{-t}L^{-1}\left\{\frac{1}{s^2+1}\right\} = e^{-t}\sin t$$

∴ By Convolution theorem, we get

$$L^{-1}\left\{\frac{1}{s(s^2+2s+2)}\right\} = L^{-1}\left\{\frac{1}{s}\cdot\frac{1}{s^2+2s+2}\right\} = f(t)*g(t) = g(t)*f(t)$$

$$= \int_0^t g(u)f(t-u)du = \int_0^t e^{-u}\sin u\cdot 1\,du = \int_0^t e^{-u}\sin u\,du$$

$$= \left[\frac{e^{-u}}{1+1}(-\sin u-\cos u)\right]_0^t = -\frac{1}{2}\left[e^{-u}(\sin u+\cos u)\right]_0^t$$

$$= -\frac{1}{2}\left[e^{-t}(\sin t+\cos t)-1\cdot(0+1)\right] = \frac{1}{2}\left[1-e^{t}(\sin t+\cos t)\right]$$

Example 3 : Using Convolution theorem, find $L^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\}$

Solution : Let $\bar{f}(s)=\frac{1}{s^2+a^2}$ and $\bar{g}(s)=\frac{1}{s^2+a^2}$. Then

$$f(t)=L^{-1}\left\{\frac{1}{s^2+a^2}\right\}=\frac{1}{a}\sin at \text{ and}$$

$$g(t)=L^{-1}\left\{\frac{1}{s^2+a^2}\right\}=\frac{1}{a}\sin at$$

$$\therefore\ L^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\} = L^{-1}\left\{\frac{1}{s^2+a^2}\cdot\frac{1}{s^2+a^2}\right\} = L^{-1}\{\bar{f}(s)\cdot\bar{g}(s)\} = f(t)*g(t)$$

$$= \int_0^t f(u)g(t-u)du = \int_0^t \frac{1}{a}\sin au\cdot\frac{1}{a}\sin a(t-u)du$$

$$= \frac{1}{2a^2}\int_0^t 2\sin au\,.\sin\,(at-au)\,du$$ cos (A-B)-cos(A+B)

$$= \frac{1}{2a^2}\int_0^t[\cos\,(2au-at)-\cos\,at]\,du$$

$$= \frac{1}{2a^2}\left[\frac{\sin(2au-at)}{2a}-\cos at\cdot u\right]_0^t$$

$$= \frac{1}{2a^2}\left[\frac{1}{\cancel{2a}}\sin at-t\cos at+\frac{1}{\cancel{2a}}\sin at\right]$$

$$= \frac{1}{2a^3}($$ [redacted] $$-at\ \cos\ at)$$

Example 5 : Using the Convolution theorem, find

(i) $L^{-1}\left\{\dfrac{s}{(s^2+a^2)^2}\right\}$ *[JNTU 1997 S, 1998]*

(ii) $L^{-1}\left\{\dfrac{s^2}{(s^2+a^2)(s^2+b^2)}\right\}$

(iii) $L^{-1}\left\{\dfrac{1}{s^2(s+1)^2}\right\}$

(iv) $L^{-1}\left\{\dfrac{1}{s(s+1)(s+2)}\right\}$ *[JNTU 2003]*

<u>**Solution :**</u> (i) $L^{-1}\left\{\dfrac{s}{(s^2+a^2)^2}\right\} = L^{-1}\left\{\dfrac{s}{s^2+a^2}\cdot\dfrac{1}{s^2+a^2}\right\}$

Let $\quad \bar{f}(s) = \dfrac{s}{s^2+a^2}$ and $\bar{g}(s) = \dfrac{1}{s^2+a^2}$. Then

$$L^{-1}\{\bar{f}(s)\} = L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at = f(t), \text{ say}$$

and $$L^{-1}\{\bar{g}(s)\} = L^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{1}{a}\sin at = g(t), \text{ say}$$

$\therefore$ By the Convolution theorem, we have

$$L^{-1}\left\{\frac{s}{(s^2+a)^2}\right\} = (\cos at)*\left(\frac{1}{a}\sin at\right) = \frac{1}{a}\int_0^t \cos au \sin a(t-u)\,du$$

$$= \frac{1}{2a}\int_0^t [\sin(au+at-au) - \sin(au-at+au)]\,du$$

$$= \frac{1}{2a}\int_0^t [\sin at - \sin(2au-at)]\,du = \frac{1}{2a}\left[\sin at.u + \frac{1}{2a}\cos(2au-at)\right]_0^t$$

$$= \frac{1}{2a}\left[t\sin at + \frac{1}{2a}\cos at - \frac{1}{2a}\cos at\right] = \frac{t}{2a}\sin at$$

Note : Taking $a = 1$, the above problem becomes $L^{-1}\left[\frac{s}{(s^2+1)^2}\right] = \frac{t}{2}\sin t$

(ii) $$L^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\} = L^{-1}\left\{\frac{s}{s^2+a^2}\cdot\frac{s}{s^2+b^2}\right\}$$

Let $\bar{f}(s) = \frac{s}{s^2+a^2}$ and $\bar{g}(s) = \frac{s}{s^2+b^2}$

Then $f(t) = \cos at$ and $g(t) = \cos bt$

$$\therefore\ L^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\} = \cos at * \cos bt$$

$$= \int_0^t \cos au \cdot \cos b(t-u)\,du = \frac{1}{2}\int_0^t 2\cos au \cdot \cos b(t-u)\,du$$

$$= \frac{1}{2}\int_0^t [\cos(au+bt-bu) + \cos(au-bt+bu)]\,du$$

$$= \frac{1}{2}\int_0^t \{\cos[(a-b)u+bt] + \cos[(a+b)u-bt]\}\,du$$

$$= \frac{1}{2}\left[\frac{\sin\{(a-b)u+bt\}}{a-b} + \frac{\sin\{(a+b)u-bt\}}{a+b}\right]_0^t$$

$$= \frac{1}{2}\left[\frac{1}{a-b}(\sin at - \sin bt) + \frac{1}{a+b}(\sin at + \sin bt)\right]$$

$$= \frac{1}{2}\left[\sin at\left(\frac{1}{a-b} - \frac{1}{a+b}\right) + \sin bt\left(\frac{1}{a+b} - \frac{1}{a-b}\right)\right] = \frac{a\sin at - b\sin bt}{a^2-b^2}$$

Note : (*i*) Putting $a = 2$ and $b = 3$ in the above problem, we obtain

$$L^{-1}\left\{\frac{s^2}{(s^2+4)(s^2+9)}\right\} = -\frac{1}{5}(2\sin 2t - 3\sin 3t)$$ **[JNTU 2006, 2006S (Set No.3)]**

(*ii*) Putting $a = 2$ and $b = 5$ in the above problem, we obtain

$$L^{-1}\left\{\frac{s^2}{(s^2+4)(s^2+25)}\right\} = \frac{2\sin 2t - 5\sin 5t}{2^2 - 5^2} = \frac{1}{21}(5\sin 5t - 2\sin 2t)$$

[JNTU Aug. 2008S (Set No.4)]

(*iii*) Since $L^{-1}\left\{\frac{1}{s^2}\right\} = t$ and $L^{-1}\left\{\frac{1}{(s+1)^2}\right\} = e^{-t} L^{-1}\left\{\frac{1}{s^2}\right\} = te^{-t}$,

$\therefore$ By Convolution theorem, we get

$$L^{-1}\left\{\frac{1}{(s+1)^2}\cdot\frac{1}{s^2}\right\} = \int_0^t u\,e^{-u}(t-u)\,du = t\int_0^t u\,e^{-u}\,du - \int_0^t u^2 e^{-u}\,du$$

$$= t[-(t+1)e^{-t}+1] - [-e^{-t}(t^2+2t+2)+2]$$

$$= -t^2e^{-t} - te^{-t} + t + t^2e^{-t} + 2te^{-t} + 2e^{-t} - 2$$

$$= t(2e^{-t}+1) + 2(e^{-t}-1)$$

(*iv*) $\dfrac{1}{s(s+1)(s+2)} = \dfrac{1}{s(s+1)}\cdot\dfrac{1}{s+2}$

Consider $\dfrac{1}{s(s+1)} = \dfrac{1}{s}\cdot\dfrac{1}{s+1}$

Since $L^{-1}\left\{\frac{1}{s}\right\} = 1$ and $L^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t}$,

$\therefore$ By Convolution theorem, we get

$$L^{-1}\left\{\frac{1}{s(s+1)}\right\} = \int_0^t 1.e^{-u}\,du = -\left(e^{-u}\right)_0^t = 1 - e^{-t}. \text{ Also } L^{-1}\left\{\frac{1}{s+2}\right\} = e^{-2t}$$

Using convolution theorem again, we get

$$L^{-1}\left\{\frac{1}{s(s+1)(s+2)}\right\} = L^{-1}\left\{\frac{1}{s(s+1)}\cdot\frac{1}{s+2}\right\} = \int_0^t e^{-2(t-u)}.(1-e^{-u})\,du$$

$$= e^{-2t}\int_0^t (e^{2u} - e^{u})\,du = e^{-2t}\left(\frac{e^{2u}}{2} - e^{u}\right)_0^t$$

$$= e^{-2t}\left(\frac{e^{2t}}{2} - e^{t} - \frac{1}{2} + 1\right) = e^{-2t}\left(\frac{e^{2t}}{2} - e^{t} + \frac{1}{2}\right) = \frac{1}{2} + \frac{1}{2}e^{-2t} - e^{-t}$$

Chapter 10

APPLICATION TO DIFFERENTIAL EQUATIONS

10.1 SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS:

Ordinary linear differential equations with constant coefficients can be easily solved by the Laplace transform method, without the necessity of first finding the general solution and then evaluating the arbitrary constants. This method is, in general, shorter than our earlier methods and is especially suitable to obtain the solution of linear non-homogeneous ordinary differential equations with constant coefficients.

Let us consider a linear differential equation with constant coefficients

$$\frac{d^n y}{dt^n}+a_1\frac{d^{n-1}y}{dt^{n-1}}+a_2\frac{d^{n-2}y}{dt^{n-2}}+....+a_{n-1}\frac{dy}{dt}+a_n y=F(t) \quad ...(1)$$

where $F(t)$ is a function of independent variable t.

Let $y(0)=c_0, y'(0)=c_1, ..., y^{n-1}(0)=c_{n-1}$ ...(2)

be the given initial or boundary conditions, where $c_0, c_1, c_2, ..., c_{n-1}$ are constants.

If $a_1, a_2, ..., a_n$ are constants, then we use

$$L\{f^{(n)}(t)\}=s^n L\{f(t)\}-s^{n-1}f(0)-s^{n-2}f'(0)-....... -f^{n-1}(0) \quad ...(3)$$

Taking Laplace transform of both sides of (1) and applying (3) and using conditions (2) we obtain an algebraic equation known as "subsidiary equation", from which $\bar{y}(s)=L\{y(t)\}$ is obtained. The required solution $y(t)$ is obtained by taking the inverse Laplace transform of $\bar{y}(s)$.

Working Rule to Solve Differential Equation by Laplace Transform Method.

Step 1. Take the Laplace transform of both sides of the given differential equation.

Step 2. Use the formula

(i) $L\{y'(t)\}=s\bar{y}(s)-y(0)$

(ii) $L\{y''(t)\}=s^2\bar{y}(s)-s\cdot y(0)-y'(0)$

(iii) $L\{y'''(t)\}=s^3\bar{y}(s)-s^2\cdot y(0)-s\cdot y'(0)-y''(0)$

Step 3. Replace $y(0)$, $y'(0)$, $y''(0)$ with the given initial conditions.

Step 4. Transpose the terms with minus signs to the right.

Step 5. Divide by the coefficient of $\bar{y}$, getting $\bar{y}$ as a known function of s.

Step 6. Resolve this function of s (obtained in step 5) into partial fractions.

Step 7. Take the Inverse $L.T.$ of $\bar{y}$ obtained in step 5. This gives y as a function of t which is the required solution of the given equation satisfying the given initial conditions.

SOLVED EXAMPLES

Example 1 : Using Laplace transform method, solve [JNTU 2002S]

$(D^2+1)y = 6\cos 2t, t>0$ If $y=3, Dy=1$ when $t=0$.

Solution : Given equation is $y''+y=6\cos 2t$

Taking Laplace transform of this equation

$$L\{y''\}+L\{y\}=6L\{\cos 2t\}$$

i.e. $$s^2L\{y\}-sy(0)-y'(0)+L\{y\}=6.\frac{s}{s^2+4}$$

Using the given conditions $y=3, Dy=1$ at $t=0$. we get

$$(s^2+1)L\{y\}-3s-1=\frac{6s}{s^2+4}$$

or $$L\{y\}=\frac{6s}{(s^2+4)(s^2+1)}+\frac{3s}{s^2+1}+\frac{1}{s^2+1}$$

$$=2\left[\frac{s}{s^2+1}-\frac{s}{s^2+4}\right]+\frac{3s}{s^2+1}+\frac{1}{s^2+1}=\frac{5s}{s^2+1}-\frac{2s}{s^2+4}+\frac{1}{s^2+1}$$

Taking Inverse Laplace transform, we get

$$y=5L^{-1}\left\{\frac{s}{s^2+1}\right\}-2L^{-1}\left\{\frac{s}{s^2+4}\right\}+L^{-1}\left\{\frac{1}{s^2+1}\right\}=5\cos t-2\cos 2t+\sin t$$

Example 2 : Using Laplace transform, solve

$\frac{d^2y}{dt^2}+2\frac{dy}{dt}+5y=e^{-t}\sin t$, given that $y(0)=0,\ y'(0)=1$.

[JNTU 2003, 2006S, 2007S (Set No.1)]

Solution : Given equation can be written as $y''+2y'+5y=e^{-t}\sin t$

Taking the Laplace transform of both sides of the given differential equation, and using the given conditions, we have

$$L\{y''\}+2\cdot L\{y'\}+5\cdot L\{y\}=L\{e^{-t}\sin t\}$$

i.e. $$\left[s^2L\{y\}-sy(0)-y'(0)\right]+2\left[sL\{y\}-y(0)\right]+5L\{y\}=\frac{1}{(s+1)^2+1}$$

or $s^2L\{y\}-1+2sL\{y\}+5L\{y\}=\frac{1}{s^2+2s+2}$ or $(s^2+2s+5)L\{y\}=\frac{1}{s^2+2s+2}+1$

or $$L\{y\}=\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}$$

$$\therefore\ y=L^{-1}\left\{\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}\right\}=L^{-1}\left\{\frac{1/3}{s^2+2s+2}+\frac{2/3}{s^2+2s+5}\right\}, \quad \text{...(1)}$$

On resolving into partial fractions

$$=\frac{1}{3}L^{-1}\left\{\frac{1}{(s+1)^2+1}\right\}+\frac{2}{3}L^{-1}\left\{\frac{1}{(s+1)^2+4}\right\}$$

$$=\frac{1}{3}e^{-t}L^{-1}\left\{\frac{1}{s^2+1}\right\}+\frac{2}{3}e^{-t}L^{-1}\left\{\frac{1}{s^2+4}\right\}, \text{ by First Shifting theorem}$$

$$=\frac{e^{-t}}{3}(\sin t+\sin 2t)$$

Example 3 : Using Laplace transform method, solve

$(D^2+1)y=\sin t\sin 2t, t>0$, if $y=1, Dy=0$ when $t=0$

Solution : Given equation is $y''+y=\sin t\cdot\sin 2t$

Taking the Laplace transform of both sides, we get

$$L\{y''\}+L\{y\}=\frac{1}{2}L\{2\sin t\cdot\sin 2t\}$$

i.e. $s^2L\{y\}-sy(0)-y'(0)+L\{y\}=\frac{1}{2}\left[L\{\cos t-\cos 3t\}\right]$

Using the given conditions, it reduces to

$$s^2L\{y\}-s+L\{y\}=\frac{s}{2(s^2+1)}-\frac{s}{2(s^2+9)}$$

i.e. $$(s^2+1)L\{y\}=s+\frac{s}{2(s^2+1)}-\frac{s}{2(s^2+9)}$$

or $$L\{y\}=\frac{s}{s^2+1}+\frac{s}{2(s^2+1)^2}-\frac{s}{2(s^2+1)(s^2+9)}$$

$\therefore$ $$y=L^{-1}\left\{\frac{s}{s^2+1}\right\}+\frac{1}{2}L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\}-\frac{1}{16}L^{-1}\left\{\frac{s}{s^2+1}-\frac{s}{s^2+9}\right\}$$

$$=\cos t+\frac{1}{4}t\sin t-\frac{1}{16}\cos t+\frac{1}{16}\cos 3t \quad \left[\because L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}=\frac{t}{2a}\sin at\right]$$

$$=\frac{15}{16}\cos t+\frac{t}{4}\sin t+\frac{1}{16}\cos 3t$$

Example 4 : Solve by Laplace transform