(7) (i) $\frac{1}{8}\left[1-e^{-2t}(2t^2+2t+1)\right]$ (ii) $\frac{1}{a^4}+\frac{e^{at}}{6a^4}(a^3t^3-3a^2t^2+6at-6)]$

(8) $\frac{1}{5}\left[1-e^{-2t}(2\sin t+\cos t)\right]$ (9) (i) $\frac{1}{4}(e^{-2t}+2t-1)$ (ii) $\frac{1}{a^2}(e^{at}-at-1)$

(10) (i) $\frac{1}{\omega^2}\left(t-\frac{1}{\omega}\sin\omega t\right)$ (ii) $\frac{1}{a^3}(\sinh at-at)$ (iii) $\sinh 2t-2t$

(11) (i) $t(1+e^{-t})$ (ii) $\frac{1}{4}(1-2t-\cos 2t+\sin 2t)$ (12) $\frac{1}{8}(e^{2t}-2t^2-2t-1)$

9.29 CONVOLUTION

Convolution is useful for obtaining Inverse Laplace Transform of a product of two transforms and solving ordinary differential equations.

Definition : Let $f(t)$ and $g(t)$ be two functions defined for $t>0$. We define

$$f(t)*g(t)=\int_0^t f(u)g(t-u)du$$

assuming that the integral on the right hand side exists.

$f(t)*g(t)$ is called the convolution product of $f(t)$ and $g(t)$.

It can be proved that

(i) Convolution product is commutative. *i.e.* $f(t)*g(t)=g(t)*f(t)$

(ii) Convolution product is associative . *i.e.* $f(t)*(g(t)*h(t))=(f(t)*g(t))*h(t)$

(iii) $f(t)*0=0*f(t)=0$

Note : In general $1*f(t)\neq f(t)$

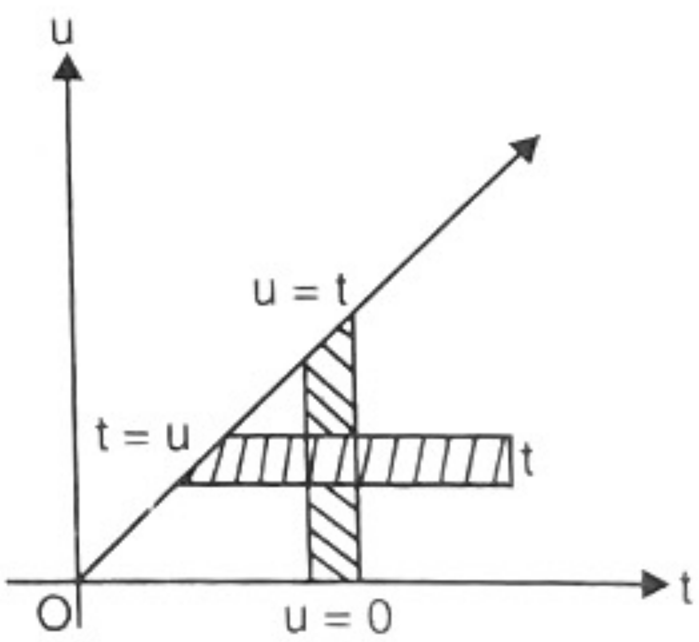
Convolution Theorem : If $L\{f(t)\}=\bar{f}(s)$ and $L\{g(t)\}=g(s)$ then $L\{f(t)*g(t)\}=f(s)\cdot g(s)$

or $L^{-1}\{\bar{f}(s)\cdot\bar{g}(s)\}=f(t)*g(t)$ **[JNTU 2008S (Set No. 3)]**

Proof: Let $\phi(t)=f(t)*g(t)=\int_0^t f(u)g(t-u)\,du$. Then

$$L\{\phi(t)\}=\int_0^\infty e^{-st}\left\{\int_0^t f(u)g(t-u)du\right\}dt=\int_0^\infty\int_0^t e^{-st}f(u)g(t-u)du\,dt$$

The double integral is considered within the region enclosed by the lines $u=0$ and $u=t$.



On changing the order of integration, we get

$$L\{\phi(t)\}=\int_0^\infty\int_u^\infty e^{-st}f(u)g(t-u)dt\,du$$

$$=\int_0^\infty e^{-su}f(u)\left\{\int_u^\infty e^{-s(t-u)}g(t-u)dt\right\}du$$

$$= \int_0^\infty e^{-su} f(u) \left\{ \int_0^\infty e^{-sv} g(v)dv \right\} du, \text{ on putting } t-u=v$$

$$= \int_0^\infty e^{-su} f(u)\{\bar{g}(s)\}du = \bar{g}(s)\int_0^\infty e^{-su} f(u)du = \bar{g}(s).\ \bar{f}(s)$$

$\therefore L\{\phi(t)\} = \bar{f}(s).\bar{g}(s)$ or $\phi(t) = L^{-1}\{\bar{f}(s).\bar{g}(s)\}$ or $f(t) * g(t) = L^{-1}\{\bar{f}(s).\bar{g}(s)\}$

Hence the theorem follows.

SOLVED EXAMPLES

Example 1 : Using Convolution theorem, find (*i*) $L^{-1}\left\{\frac{1}{(s+a)(s+b)}\right\}$ (*ii*) $L^{-1}\left\{\frac{1}{s(s^2+4)}\right\}$

Solution : (*i*) Let $\bar{f}(s) = \frac{1}{s+a}$ and $\bar{g}(s) = \frac{1}{s+b}$. Then

$$f(t) = L^{-1}\{\bar{f}(s)\} = L^{-1}\left\{\frac{1}{s+a}\right\} = e^{-at}$$

and $$g(t) = L^{-1}\{\bar{g}(s)\} = L^{-1}\left\{\frac{1}{s+b}\right\} = e^{-bt}$$

$\therefore$ By Convolution theorem,

$$L^{-1}\left\{\frac{1}{(s+a)(s+b)}\right\} = L^{-1}\left\{\frac{1}{s+a}\cdot\frac{1}{s+b}\right\} = L^{-1}\{\bar{f}(s)\cdot\bar{g}(s)\}$$

$$= f(t)*g(t) = \int_0^t f(u)g(t-u)du$$

$$= \int_0^t e^{-au}\cdot e^{-b(t-u)}du = e^{-bt}\int_0^t e^{-(a-b)u}du$$

$$= e^{-bt}\left[\frac{e^{-(a-b)u}}{-(a-b)}\right]_0^t = -\frac{1}{a-b}e^{-bt}\left[e^{-(a-b)t}-1\right]$$

$$= \frac{1}{b-a}(e^{-at}-e^{-bt})$$

(*ii*) Let $\bar{f}(s) = \frac{1}{s}$ and $\bar{g}(s) = \frac{1}{s^2+4}$. Then

$$f(t) = L^{-1}\left\{\frac{1}{s}\right\} = 1 \text{ and } g(t) = L^{-1}\left\{\frac{1}{s^2+2^2}\right\} = \frac{1}{2}\sin 2t$$

Applying Convolution theorem,

$$L^{-1}\left\{\frac{1}{s(s^2+4)}\right\} = L^{-1}\left\{\frac{1}{s}\cdot\frac{1}{s^2+4}\right\} = L^{-1}\{\bar{f}(s)\cdot\bar{g}(s)\}$$

$$= f(t)*g(t) = \int_0^t f(u)g(t-u)du = \int_0^t 1\cdot\frac{1}{2}\sin 2(t-u)du$$

$$= \frac{1}{2}\int_0^t \sin 2(t-u)du = \frac{1}{2}\left[\frac{-\cos 2(t-u)}{-2}\right]_0^t$$

$$= \frac{1}{4}(\cos 0-\cos 2t) = \frac{1}{4}(1-\cos 2t)$$

Example 2 : Using Convolution theorem, evaluate $L^{-1}\left\{\frac{1}{s(s^2+2s+2)}\right\}$

Solution : Since $f(t)=L^{-1}\left\{\frac{1}{s}\right\}=1$ and

$$g(t) = L^{-1}\left\{\frac{1}{s^2+2s+2}\right\} = L^{-1}\left\{\frac{1}{(s+1)^2+1}\right\} = e^{-t}L^{-1}\left\{\frac{1}{s^2+1}\right\} = e^{-t}\sin t$$

$\therefore$ By Convolution theorem, we get

$$L^{-1}\left\{\frac{1}{s(s^2+2s+2)}\right\} = L^{-1}\left\{\frac{1}{s}\cdot\frac{1}{s^2+2s+2}\right\} = f(t)*g(t) = g(t)*f(t)$$

$$= \int_0^t g(u)f(t-u)du = \int_0^t e^{-u}\sin u\cdot 1\, du = \int_0^t e^{-u}\sin u\, du$$

$$= \left[\frac{e^{-u}}{1+1}(-\sin u-\cos u)\right]_0^t = -\frac{1}{2}\left[e^{-u}(\sin u+\cos u)\right]_0^t$$

$$= -\frac{1}{2}\left[e^{-t}(\sin t+\cos t)-1\cdot(0+1)\right] = \frac{1}{2}\left[1-e^{t}(\sin t+\cos t)\right]$$

Example 3 : Using Convolution theorem, find $L^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\}$

Solution : Let $\bar{f}(s)=\frac{1}{s^2+a^2}$ and $\bar{g}(s)=\frac{1}{s^2+a^2}$. Then

$$f(t) = L^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{1}{a}\sin at \text{ and } g(t) = L^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{1}{a}\sin at$$

$$\therefore\ L^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\} = L^{-1}\left\{\frac{1}{s^2+a^2}\cdot\frac{1}{s^2+a^2}\right\} = L^{-1}\{\bar{f}(s)\cdot\bar{g}(s)\} = f(t)*g(t)$$

$$= \int_0^t f(u)g(t-u)du = \int_0^t \frac{1}{a}\sin au\cdot\frac{1}{a}\sin a(t-u)du$$

$$= \frac{1}{2a^2}\int_0^t 2\sin au\ \sin\ (at-au)\ du$$

$$= f(t)*g(t) = \int_0^t f(u)g(t-u)du = \int_0^t 1\cdot\frac{1}{2}\sin 2(t-u)du$$

$$= \frac{1}{2}\int_0^t \sin 2(t-u)du = \frac{1}{2}\left[\frac{-\cos 2(t-u)}{-2}\right]_0^t$$

$$= \frac{1}{4}(\cos 0 - \cos 2t) = \frac{1}{4}(1-\cos 2t)$$

Example 2 : Using Convolution theorem, evaluate $L^{-1}\left\{\frac{1}{s(s^2+2s+2)}\right\}$

Solution : Since $f(t) = L^{-1}\left\{\frac{1}{s}\right\} = 1$ and

$$g(t) = L^{-1}\left\{\frac{1}{s^2+2s+2}\right\} = L^{-1}\left\{\frac{1}{(s+1)^2+1}\right\} = e^{-t}L^{-1}\left\{\frac{1}{s^2+1}\right\} = e^{-t}\sin t$$

$\therefore$ By Convolution theorem, we get

$$L^{-1}\left\{\frac{1}{s(s^2+2s+2)}\right\} = L^{-1}\left\{\frac{1}{s}\cdot\frac{1}{s^2+2s+2}\right\} = f(t)*g(t) = g(t)*f(t)$$

$$= \int_0^t g(u)f(t-u)du = \int_0^t e^{-u}\sin u\cdot 1\, du = \int_0^t e^{-u}\sin u\, du$$

$$= \left[\frac{e^{-u}}{1+1}(-\sin u - \cos u)\right]_0^t = -\frac{1}{2}\left[e^{-u}(\sin u + \cos u)\right]_0^t$$

$$= -\frac{1}{2}\left[e^{-t}(\sin t + \cos t) - 1\cdot(0+1)\right] = \frac{1}{2}\left[1 - e^{t}(\sin t + \cos t)\right]$$

Example 3 : Using Convolution theorem, find $L^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\}$

Solution : Let $\bar{f}(s) = \frac{1}{s^2+a^2}$ and $\bar{g}(s) = \frac{1}{s^2+a^2}$. Then

$$f(t) = L^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{1}{a}\sin at \text{ and } g(t) = L^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{1}{a}\sin at$$

$$\therefore\quad L^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\} = L^{-1}\left\{\frac{1}{s^2+a^2}\cdot\frac{1}{s^2+a^2}\right\} = L^{-1}\{\bar{f}(s)\cdot\bar{g}(s)\} = f(t)*g(t)$$

$$= \int_0^t f(u)g(t-u)du = \int_0^t \frac{1}{a}\sin au\cdot\frac{1}{a}\sin a(t-u)du$$

$$= \frac{1}{2a^2}\int_0^t 2\sin au\ \sin\ (at-au)\ du$$

$$= \frac{1}{2a^2}\int_0^t [\cos\ (2au-at)-\cos\ at]\,du$$

$$= \frac{1}{2a^2}\left[\frac{\sin(2au-at)}{2a}-\cos at\cdot u\right]_0^t$$

$$= \frac{1}{2a^2}\left[\frac{1}{2a}\sin at - t\cos at + \frac{1}{2a}\sin at\right] = \frac{1}{2a^3}(\sin\ at - at\ \cos\ at)$$

Example 4 : Apply Convolution theorem to evaluate $L^{-1}\left\{\frac{1}{(s-2)(s+2)^2}\right\}$

Solution : Let $\bar{f}(s)=\frac{1}{s-2}$ and $\bar{g}(s)=\frac{1}{(s+2)^2}$ so that

$$f(t)=L^{-1}\left\{\frac{1}{s-2}\right\} = e^{2t} \text{ and}$$

$$g(t)=L^{-1}\left\{\frac{1}{(s+2)^2}\right\} = e^{-2t}L^{-1}\left\{\frac{1}{s^2}\right\}=te^{-2t}$$

$\therefore$ By Convolution theorem,

$$L^{-1}\left\{\frac{1}{(s-2)(s+2)^2}\right\} = L^{-1}\left\{\frac{1}{s-2}\cdot\frac{1}{(s+2)^2}\right\} = L^{-1}\{\bar{f}(s)\cdot\bar{g}(s)\} = f(t)*g(t)$$

$$= \int_0^t f(u)g(t-u)du = \int_0^t e^{2u}(t-u)e^{-2(t-u)}du$$

$$= e^{-2t}\int_0^t e^{4u}(t-u)\cdot du = e^{-2t}\left[t\int_0^t e^{4u}du \int_0^t ue^{4u}du\right]$$

$$= e^{-2t}\left[t\left(\frac{e^{4u}}{4}\right)_0^t - \left\{u\cdot\frac{e^{4u}}{4}-1\cdot\frac{e^{4u}}{16}\right\}_0^t\right]$$

$$= e^{-2t}\left[\frac{t}{4}(e^{4t}-1)-\left\{\frac{t}{4}e^{4t}-\frac{1}{16}e^{4t}-0+\frac{1}{16}\right\}\right]$$

$$= e^{-2t}\left[-\frac{t}{4}+\frac{1}{16}e^{4t}-\frac{1}{16}\right] = \frac{e^{-2t}}{16}(-4t+e^{4t}-1)$$

$$= \frac{1}{16}\left[e^{2t}-(4t+1)e^{-2t}\right]$$

Alternative method : Let $\bar{f}(s)=\frac{1}{(s+2)^2}$ and $\bar{g}(s)=\frac{1}{s-2}$.

Then $f(t) = e^{-2t}.t$ and $g(t) = e^{2t}$

By convolution Theorem,

$$L^{-1}\left\{\frac{1}{(s-2)+(s+2)^2}\right\} = \int_0^t e^{-2u}.u.e^{2(t-u)}du$$

$$= e^{2t}\int_0^t u\,e^{-4u}du = e^{2t}\left[u\left(\frac{e^{-4u}}{-4}\right) - 1.\left(\frac{e^{-4u}}{16}\right)\right]_0^t$$

$$= \frac{1}{16}[e^{2t} - (4t+1)\,e^{-2t}]$$

Example 5 : Using the Convolution theorem, find

(i) $L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}$ **[JNTU 1997 S, 1998, (A) June 2010 (Set No. 2)]**

(ii) $L^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\}$

(iii) $L^{-1}\left\{\frac{1}{s^2(s+1)^2}\right\}$ (iv) $L^{-1}\left\{\frac{1}{s(s+1)(s+2)}\right\}$ *[JNTU 2003]*

Solution : (i) $L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = L^{-1}\left\{\frac{s}{s^2+a^2}.\frac{1}{s^2+a^2}\right\}$

Let $\bar{f}(s) = \frac{s}{s^2+a^2}$ and $\bar{g}(s) = \frac{1}{s^2+a^2}$. Then

$$L^{-1}\{\bar{f}(s)\} = L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at = f(t), \text{ say}$$

and $$L^{-1}\{\bar{g}(s)\} = L^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{1}{a}\sin at = g(t), \text{ say}$$

∴ By the Convolution theorem, we have

$$L^{-1}\left\{\frac{s}{(s^2+a)^2}\right\} = (\cos at)*\left(\frac{1}{a}\sin at\right) = \frac{1}{a}\int_0^t \cos au\sin a(t-u)du$$

$$= \frac{1}{2a}\int_0^t [\sin(au+at-au) - \sin(au-at+au)]\,du$$

$$= \frac{1}{2a}\int_0^t [\sin at - \sin(2au-at)]\,du = \frac{1}{2a}\left[\sin at.u + \frac{1}{2a}\cos(2au-at)\right]_0^t$$

$$= \frac{1}{2a}\left[t\sin at + \frac{1}{2a}\cos at - \frac{1}{2a}\cos at\right] = \frac{t}{2a}\sin at$$

Note : Taking $a = 1$, the above problem becomes $L^{-1}\left[\frac{s}{(s^2+1)^2}\right] = \frac{t}{2}\sin t$

(*ii*) $L^{-1}\left\{\dfrac{s^2}{(s^2+a^2)(s^2+b^2)}\right\} = L^{-1}\left\{\dfrac{s}{s^2+a^2}\cdot\dfrac{s}{s^2+b^2}\right\}$

Let $\bar{f}(s) = \dfrac{s}{s^2+a^2}$ and $\bar{g}(s) = \dfrac{s}{s^2+b^2}$

Then $f(t) = \cos at$ and $g(t) = \cos bt$

$$\therefore\ L^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\} = \cos at * \cos bt$$

$$= \int_0^t \cos au \cdot \cos b(t-u)du = \frac{1}{2}\int_0^t 2\cos au \cdot \cos b(t-u)du$$

$$= \frac{1}{2}\int_0^t [\cos(au+bt-bu)+\cos(au-bt+bu)]du$$

$$= \frac{1}{2}\int_0^t \{\cos[(a-b)u+bt]+\cos[(a+b)u-bt]\}\,du$$

$$= \frac{1}{2}\left[\frac{\sin\{(a-b)u+bt\}}{a-b}+\frac{\sin\{(a+b)u-bt\}}{a+b}\right]_0^t$$

$$= \frac{1}{2}\left[\frac{1}{a-b}(\sin at-\sin bt)+\frac{1}{a+b}(\sin at+\sin bt)\right]$$

$$= \frac{1}{2}\left[\sin at\left(\frac{1}{a-b}-\frac{1}{a+b}\right)+\sin bt\left(\frac{1}{a+b}-\frac{1}{a-b}\right)\right] = \frac{a\sin at-b\sin bt}{a^2-b^2}$$

Note : (*i*) Putting $a = 2$ and $b = 3$ in the above problem, we obtain

$$L^{-1}\left\{\frac{s^2}{(s^2+4)(s^2+9)}\right\} = -\frac{1}{5}(2\sin 2t-3\sin 3t)$$ **[JNTU 2006, 2006S, (A) 2010 (Set No.1)**

(*ii*) Putting $a = 2$ and $b = 5$ in the above problem, we obtain

$$L^{-1}\left\{\frac{s^2}{(s^2+4)(s^2+25)}\right\} = \frac{2\sin 2t-5\sin 5t}{2^2-5^2} = \frac{1}{21}(5\sin 5t-2\sin 2t)$$

[JNTU Aug. 2008S, (K) May 2010 (Set No.4

(*iii*) Since $L^{-1}\left\{\dfrac{1}{s^2}\right\} = t$ and $L^{-1}\left\{\dfrac{1}{(s+1)^2}\right\} = e^{-t}L^{-1}\left\{\dfrac{1}{s^2}\right\} = te^{-t}$,

$\therefore$ By Convolution theorem, we get

$$L^{-1}\left\{\frac{1}{(s+1)^2}\cdot\frac{1}{s^2}\right\} = \int_0^t u\,e^{-u}(t-u)\,du = t\int_0^t u\,e^{-u}du - \int_0^t u^2e^{-u}du$$

$$= t[-(t+1)e^{-t}+1]-[-e^{-t}(t^2+2t+2)+2]$$

$$= -t^2e^{-t}-t\,e^{-t}+t+t^2e^{-t}+2te^{-t}+2e^{-t}-2$$

$$= t(e^{-t}+1)+2(e^{-t}-1)$$

(iv) $\dfrac{1}{s(s+1)(s+2)} = \dfrac{1}{s(s+1)} \cdot \dfrac{1}{s+2}$

Consider $\dfrac{1}{s(s+1)} = \dfrac{1}{s} \cdot \dfrac{1}{s+1}$

Since $L^{-1}\left\{\dfrac{1}{s}\right\} = 1$ and $L^{-1}\left\{\dfrac{1}{s+1}\right\} = e^{-t}$,

$\therefore$ By Convolution theorem, we get

$$L^{-1}\left\{\frac{1}{s(s+1)}\right\} = \int_0^t 1.e^{-u}\,du = -\left(e^{-u}\right)_0^t = 1 - e^{-t}. \text{ Also } L^{-1}\left\{\frac{1}{s+2}\right\} = e^{-2t}$$

Using convolution theorem again, we get

$$L^{-1}\left\{\frac{1}{s(s+1)(s+2)}\right\} = L^{-1}\left\{\frac{1}{s(s+1)} \cdot \frac{1}{s+2}\right\} = \int_0^t e^{-2(t-u)}.(1-e^{-u})\,du$$

$$= e^{-2t}\int_0^t (e^{2u} - e^{u})\,du = e^{-2t}\left(\frac{e^{2u}}{2} - e^{u}\right)_0^t$$

$$= e^{-2t}\left(\frac{e^{2t}}{2} - e^{t} - \frac{1}{2} + 1\right) = e^{-2t}\left(\frac{e^{2t}}{2} - e^{t} + \frac{1}{2}\right) = \frac{1}{2} + \frac{1}{2}e^{-2t} - e^{-t}$$

Example 6 : Using Laplace transform, solve $y(t) = 1 - e^{-t} + \int_0^t y(t-u)\,\sin u\,du$.

[JNTU June 2008 (Set No. 2)]

Solution : Given integral equation can be written as

$y(t) = 1 - e^{-t} + y(t) * \sin t$, using definition of convolution

Taking the Laplace Transform of both the sides, we have

$$L\{y(t)\} = L\{1\} - L\{e^{-t}\} + L\{y(t) * \sin t\}$$

$$= \frac{1}{s} - \frac{1}{s+1} + L\{y(t)\} \cdot L\{\sin t\}, \text{ using convolution theorem.}$$

$$= \frac{1}{s(s+1)} + L\{y(t)\} \cdot \frac{1}{s^2+1}$$

$$\Rightarrow \quad \left(1 - \frac{1}{s^2+1}\right) L\{y(t)\} = \frac{1}{s(s+1)}$$

$$\Rightarrow \quad L\{y(t)\} = \frac{s^2+1}{s^3(s+1)}$$

$$\therefore \quad y(t) = L^{-1}\left\{\frac{s^2+1}{s^3(s+1)}\right\} = L^{-1}\left\{\frac{1}{s(s+1)} + \frac{1}{s^3(s+1)}\right\}$$