Module - 1 Ordinary Differential Equations:

Definition: A differential equation is an equation which contains derivatives, either ordinary derivatives or partial derivatives.

Here are a few examples of differential equations:

1. $\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = x$ (ordinary diff. eqn)

$\begin{pmatrix} \text{order} - 2 \\ \text{degree} - 1 \end{pmatrix}$

2. $\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial u}{\partial y} = 0$ (Partial diff" eqn)

1.1 Second order Linear differential equation with constant co-efficients:

Definition: An equation of the form

$$\frac{d^2y}{dx^2} + k_1\frac{dy}{dx} + k_2 y = Q(x), \longrightarrow (1)$$

where k_1 and k_2 are constants, is called a second order linear differential equation in y with constant co-efficients.

The operator form of differential equation (1)

is $D^2y + K_1 Dy + K_2 y = Q(x)$

or, $(D^2 + K_1 D + K_2) y = Q(x)$, where $D \equiv \frac{d}{dx}$ $\longrightarrow$ (2)

$\downarrow$ (differential operator)

Let $f(D) = D^2 + K_1 D + K_2$, then

equation (2) becomes $f(D)y = Q(x) \longrightarrow$ (3)

and $f(m) = 0$ is the Auxiliary equation (A.E) of $f(D)y = 0$.

The general solution of equation (3)

is $y = \underbrace{y_c}_{\text{Complementary Function (C.F)}} + y_p \rightarrow$ Particular Integral (P.I)

This is the general solution of (1).

<u>Note</u>: (i) If $Q(x) = 0$, then $f(D)y = 0$ is called homogeneous differential equation and the general solution of $f(D)y = 0$

is $\boxed{y = y_c}$

(ii) If $Q(x) \neq 0$, then $f(D)y = Q(x)$ is called non-homogeneous differential equation.

(iii) If the given differential equation is of order n, then its general solution contains n arbitrary constants and all these constants are in y_c only. That means y_p does not contains any arbitrary constant.

Finding C.F. for $f(D)y = Q(x)$

Let $f(D)y = Q(x)$ be the given second order linear differential equation with constants co-efficients.

Case(i) If m_1, m_2 are two distinct $(m_1 \neq m_2)$ real roots of the A.E. $f(m) = 0$,

$$C.F (= y_c) = c_1 e^{m_1 x} + c_2 e^{m_2 x}.$$

Case(ii) If m_1, m_2 are repeated $(m_1 = m_2 = m \text{ say})$ real roots of the A.E. $f(m) = 0$, then

$$C.F (= y_c) = (c_1 + c_2 x) e^{mx}.$$

Case(iii) If $\alpha \pm i\beta$ are the roots of $f(m) = 0$, then $C.F (= y_c) = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x)$

Example Problems:

1. Obtain the general solution of $\frac{d^2y}{dx^2}+6\frac{dy}{dx}+8y=0$

Sol: Operator form of the given differential equation is $(D^2+6D+8)y=0$, where $D\equiv\frac{d}{dx}$.

Let $f(D)=D^2+6D+8$. $\longrightarrow (1)$

The Auxiliary Equation of (1) is $f(m)=0$.

i.e., $m^2+6m+8=0 \Rightarrow (m+2)(m+4)=0$

$\Rightarrow m=-2,-4$.

Therefore, $y_c=c_1e^{-2x}+c_2e^{-4x}$ and hence the general solution of the given differential equation is $y=y_c$

i.e., $\boxed{y=c_1e^{-2x}+c_2e^{-4x}}$

2. Solve $\frac{d^2y}{dt^2}+2\frac{dy}{dt}+2y=0$

Sol: operator form of the given differential equation is $(D^2+2D+2)y=0$, where $D\equiv\frac{d}{dt}$.
$\longrightarrow (1)$

Let $f(D)=D^2+2D+2$. Then the A.E. is $f(m)=0$. i.e., $m^2+2m+2=0$

$\Rightarrow m=\frac{-2\pm\sqrt{4-8}}{2}$

$\Rightarrow m = -1 \pm i$

Therefore, $y_c = e^{-t}(c_1 \cos t + c_2 \sin t)$

Hence the general solution of eqn ① is

$$y = y_c \quad \text{i.e.,} \quad y = e^{-t}(c_1 \cos t + c_2 \sin t).$$

3. Solve $(D^2 + 2D + 1)y = 0$, where $D \equiv \frac{d}{dt}$.

Sol: Let $f(D) \equiv D^2 + 2D + 1$.

Then the A.E. of $f(D)y = 0$ is $f(m) = 0$

i.e., $m^2 + 2m + 1 = 0 \Rightarrow m = -1, -1$

Therefore, $y_c = (c_1 + c_2 t)e^{-t}$

Hence the general solution of the given differential equation is $y = y_c$. i.e., $y = (c_1 + c_2 t)e^{-t}$.

4. Solve $(D^2 + 2D - 3)y = 0$ with $y(0) = 0$, $y'(0) = 1$, where $D \equiv \frac{d}{dx}$

Sol: Given differential equation is

$$(D^2 + 2D - 3)y = 0 \longrightarrow ①$$

Let $f(D) \equiv D^2 + 2D - 3$. Then the A.E. is

$f(m) = 0$. i.e., $m^2 + 2m - 3 = 0$

$\Rightarrow (m+3)(m-1) = 0 \Rightarrow m = 1, -3.$

Therefore, $y_c = c_1 e^{x} + c_2 e^{-3x}$

Hence, $y = c_1 e^{x} + c_2 e^{-3x}$

and $y' = c_1 e^{x} - 3c_2 e^{-3x}$

when $x=0,\ y=0$ we get $c_1+c_2=0 \rightarrow$ (i)

when $x=0,\ y'=1$ we get $c_1-3c_2=1 \rightarrow$ (ii)

By solving (i) and (ii) we get $c_1=\frac{1}{4}$ and $c_2=-\frac{1}{4}$

Thus $y=\frac{1}{4}e^{x}-\frac{1}{4}e^{-3x}$

This is the particular solution of the given differential equation.

Exercise :

Solve the following

1) $\frac{d^2y}{dx^2}+\frac{dy}{dx}-6y=0$ (Answer : $y=c_1e^{2x}+c_2e^{-3x}$)

2) $2\frac{d^2x}{dt^2}+2\frac{dx}{dt}+6x=0$

[given diff. eqn can be written as

$\frac{d^2x}{dt^2}+\frac{dx}{dt}+3x=0$ and $y=e^{-t/2}\left(c_1\cos\frac{\sqrt{11}}{2}t+c_2\sin\frac{\sqrt{11}}{2}t\right)$

3) $(D^2+D+1)y=0$ with $y(0)=0,\ y'(0)=1,\ D\equiv\frac{d}{dx}$

(Answer : $y=\frac{2}{\sqrt{3}}e^{-x/2}\sin\frac{\sqrt{3}x}{2}$

4) $(D^2+5D+6)y=0,\ D\equiv\frac{d}{dx}$ (Answer : $y=c_1e^{-2x}+c_2e^{-3x}$)

5) $(D^2-2D-3)y=0,\ D\equiv\frac{d}{dx}$ (Answer: $y=c_1e^{-x}+c_2e^{3x}$)