CHAPTER 2

LINEAR DIFFERENTIAL EQUATIONS OF SECOND AND HIGHER ORDER

2.1 DEFINITION

An equation of the form $\frac{d^n y}{dx^n} + P_1(x)\frac{d^{n-1}y}{dx^{n-1}} + P_2(x)\frac{d^{n-2}y}{dx^{n-2}} + \dots + P_n(x)y = Q(x)$ where $P_1(x), P_2(x), \dots, P_n(x)$ and $Q(x)$ are all continuous and real valued functions of x is called a linear differential equation of order n.

2.2 LINEAR DIFFERENTIAL EQUATION WITH CONSTANT COEFFICIENTS

Def. An equation of the form $\frac{d^n y}{dx^n} + P_1\frac{d^{n-1}y}{dx^{n-1}} + \dots + P_n y = Q(x)$ ...(1)

where $P_1, P_2, \dots, P_n$ are real constants and $Q(x)$ is a continuous function of x is called an ordinary linear equation of order n with constant coefficients. We now state a theorem without proof.

Theorem 1: If y_1 and y_2 are two solutions of the equation

$$\frac{d^n y}{dx^n} + P_1\frac{d^{n-1}y}{dx^{n-1}} + P_2\frac{d^{n-2}y}{dx^{n-2}} + \dots + P_n y = 0 \qquad \dots(1)$$

then $c_1y_1 + c_2y_2$ is also its solution, where c_1 and c_2 are constants.

The general solution of a differential equation of nth order contains n arbitrary constants.

If $y_1, y_2, \dots, y_n$ are n independent solutions of (1) then $c_1y_1 + c_2y_2 + \dots + c_ny_n$ is the most general solution of (1). Let us denote this with u.

If $y = v$ is any particular solution of $\frac{d^n y}{dx^n} + P_1\frac{d^{n-1}y}{dx^{n-1}} + P_2\frac{d^{n-2}y}{dx^{n-2}} + \dots + P_n y = Q$

then $y = u + v$ is the most general solution of the above equation. The part 'u' is called the "Complementary Function" (C.F.) and the part v is called the Particular Intergral (P.I.) of (1). The complete solution of (1) is given by y = C.F. + P.I.

1. Operator D

Let us denote $\frac{d}{dx}, \frac{d^2}{dx^2}, \frac{d^3}{dx^3}, \dots$ with $D, D^2, D^3, \dots$ so that $Dy = \frac{d}{dx}(y)$, $D^2y = \frac{d^2}{dx^2}(y)$, $D^3y = \frac{d^3}{dx^3}(y)\dots$. The equation (1) can now be written in the symbolic form as

$$\left(D^n + P_1D^{n-1} + P_2D^{n-2} + \dots + P_n\right)y = Q(x) \ (i.e.)\ f(D)y = Q(x)$$

where $f(D) = D^n + P_1D^{n-1} + P_2D^{n-2} + \dots + P_n$ is a polynomial in D. The symbol D stands for the operation of differentiation.

It can be seen that this operator D or more generally, the above $f(D)$ follows the usual algebra (with the understanding that the use of the operator is interpreted properly).

2. To find the General solution (complementary function) of $f(D)y = 0$

The algebraic equation $f(m) = 0$ (*i.e.*) $m^n + P_1 m^{n-1} + P_2 m^{n-2} + \ldots + P_n = 0$ where P_1, P are real constants, is called the auxiliary equation (A.E.) of $f(D)y = 0$. Since the A.E., $f(m$ a polynomial equation of degree n, it will have n roots $m_1, m_2, \ldots, m_n$.

Case (*i*). *Let α be a real root of $f(m) = 0$ and that α be non repeated.* Then $ce^{\alpha x}$, wh an arbitrary constant, is the corresponding part of the complementary function.

If $m_1, m_2, \ldots, m_n$ are all real and distinct then the solution is $c_1 e^{m_1 x} + c_2 e^{m_2 x} + \ldots + c_n e^{m_n}$

Case (*ii*). *Let α be a real root of $f(m) = 0$ which is repeated r times.* $f(m) = (m - \alpha)^r Q(m)$ where $Q(\alpha) \neq 0$.

Then the corresponding part of the complementary function is $(c_1 + c_2 x + c_3 x^2 + \ldots + c_r x^{r}$

Case (*iii*). *Let $\alpha + i\beta$ be a non-repeated complex root of $f(m) = 0$.*

Then $\alpha - i\beta$ is also a non-repeated complex root of $f(m) = 0$. Then the corresponding p the complementary function is $e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$

Note : If $\alpha + i\beta$ and $\alpha - i\beta$ are repeated twice and the remaining roots of $f(m) = 0$ ar and distinct, then the solution is $e^{\alpha x}\left[(c_1 + c_2 x)\cos \beta x + (c_3 + c_4 x)\sin \beta x\right] + c_5 e^{m_5 x} + \ldots + c_n e^{m}$

Case (*iv*). If $(\alpha + i\beta)$ is a root repeated r times, then $\alpha - i\beta$ is also a root repeated r times corresponding part of the complementary function is given by

$$e^{\alpha x}(c_1 \cos \beta x + d_1 \sin \beta x) + x e^{\alpha x}(c_2 \cos \beta x + d_2 \sin \beta x) \ldots + x^{r-1} e^{\alpha x}(c_r \cos \beta x + d_r \sin \beta$$

or $e^{\alpha x}[(c_1 + c_2 x + c_3 x^2 + \ldots + c_r x^{r-1})\cos \beta x + (d_1 + d_2 x + d_3 x^2 + \ldots + d_r x^{r-1})\sin \beta x]$

Table 8.1

S.No.	Roots of A.E. $f(m) = 0$	C.F. (Complementary Function)
1.	$m_1, m_2, m_3, \ldots, m_n$ are real and distinct.	$c_1 e^{m_1 x} + c_2 e^{m_2 x} + \ldots + c_n e^{m_n x}$
2.	$m_1, m_1, m_3, \ldots, m_n$ (*i.e.*, two roots are real and equal and rest are real and different).	$(c_1 + c_2 x)e^{m_1 x} + c_3 e^{m_3 x} + \ldots + c_n e^{m}$
3.	$m_1, m_1, m_1, m_4, \ldots m_n$ (*i.e.*, three roots are real and equal and rest are real and different).	$(c_1 + c_2 x + c_3 x^2)e^{m_1 x} + c_4 e^{m_4 x} + \ldots + c_n e^{m_n x}$
4.	Two roots of A.E. are complex say $\alpha + i\beta$ and $\alpha - i\beta$ and the remaining roots are real and different.	$e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x) + c_3 e^{m_3 x} + \ldots + c_n e^{m_n x}$
5.	A pair of conjugate complex roots $\alpha \pm i\beta$ are repeated twice and the remaining roots are real and different.	$e^{\alpha x}[(c_1 + c_2 x)\cos \beta x + (c_3 + c_4 x) \sin \beta x] + c_5 e^{m_5 x} + \ldots + c_n e^{m_n x}$
6.	A pair of conjugate complex roots $\alpha \pm i\beta$ are repeated thrice and the remaining roots are real and different.	$e^{\alpha x}[(c_1 + c_2 x + c_3 x^2)\cos \beta x + (c_4 + c_5 x + c_6 x^2)\sin \beta x] + c_7 e^{m_7 x} + c_8 e^{m_8 x} + \ldots + c_n e^{m_n x}$

EXAMPLES

Example 1: Solve $\dfrac{d^2y}{dx^2} - a^2y = 0,\ a \neq 0$

Solution: Given equation in the operator form is

$$(D^2 - a^2)y = 0 \qquad \ldots(1)$$

Let $f(D) = D^2 - a^2$. Then the AE is $f(m) = 0$

$\Rightarrow m^2 - a^2 = 0 \quad \therefore\ m = \pm a$.

The roots are real and different.

$\therefore$ The general solution of (1) is $y = c_1 e^{ax} + c_2 e^{-ax}$

where c_1, c_2 are arbitrary constants.

Note: The above solution can be also written as $y = c_1 \cosh ax + c_2 \sinh ax$

Example 2: Solve $\dfrac{d^3y}{dx^3} - 9\dfrac{d^2y}{dx^2} + 23\dfrac{dy}{dx} - 15y = 0$

Solution : Given equation in the operator form is $(D^3 - 9D^2 + 23D - 15)y = 0 \qquad \ldots(1)$

Let $f(D) \equiv D^3 - 9D^2 + 23D - 15$

Auxiliary equation is $f(m) = 0 \Rightarrow m^3 - 9m^2 + 23m - 15 = 0 \qquad \ldots(2)$

$\Rightarrow (m-1)(m-3)(m-5) = 0$

The roots are 1, 3, 5. The roots are real and different and hence the general solution is

$$y = c_1 e^x + c_2 e^{3x} + c_3 e^{5x}$$

where c_1, c_2, c_3 are arbitrary constants.

Example 3: Solve $\dfrac{d^3x}{dt^3} - 2\dfrac{d^2x}{dt^2} - 3\dfrac{dx}{dt} = 0$

Solution: Given equation can be written as $(D^3 - 2D^2 - 3D)x = 0 \qquad \ldots(1)$

where $D \equiv \dfrac{d}{dt}$. Let $f(D) = D^3 - 2D^2 - 3D$

Auxiliary equation is $m^3 - 2m^2 - 3m = 0 \qquad \ldots(2)$

$\Rightarrow m(m^2 - 2m - 3) = 0 \Rightarrow m(m-3)(m+1) = 0$

The roots are $m = 0, 3,$ and -1. The general solution of (1) is $x = c_1 + c_2 e^{3t} + c_3 e^{-t}$

Example 4: Solve $\dfrac{d^3y}{dx^3} - 3\dfrac{dy}{dx} + 2y = 0$

Solution: Given equation in the operator form is $(D^3 - 3D + 2)y = 0 \qquad \ldots(1)$

Let $f(D) = D^3 - 3D + 2$

The AE is $f(m) = 0 \Rightarrow m^3 - 3m + 2 = 0 \qquad \ldots(2)$

$\Rightarrow (m-1)(m^2 + m - 2) = 0$

$\Rightarrow (m-1)(m-1)(m+2) = 0$

The roots of (2) are $m = 1, 1, -2$

Since two roots of $f(m) = 0$ are equal, the general solution of (1) is $y = (c_1 + c_2x)\, e^x + c_3 e^{-2x}$

Example 5: Solve $\left(D^4 - 2D^3 - 3D^2 + 4D + 4\right)y = 0$

Solution: Given equation is $\left(D^4 - 2D^3 - 3D^2 + 4D + 4\right)y = 0 \qquad \ldots(1)$

Let $f(D) = D^4 - 2D^3 - 3D^2 + 4D + 4$

The AE is $f(m)=0$ (*i.e.*) $m^4-2m^3-3m^2+4m+4=0$

$$\Rightarrow (m+1)(m^3-3m^2+4)=0 \Rightarrow (m+1)(m+1)(m^2-4m+4)=0$$

$$\Rightarrow (m+1)^2(m-2)^2=0$$

The roots are $m=-1,-1,2,2$. Hence the general solution of (1) is

$$y=(c_1+c_2x)e^{-x}+(c_3+c_4x)\,e^{2x}$$

Example 6: Solve $\dfrac{d^2y}{dx^2}+\dfrac{dy}{dx}+y=0$

Solution: Given equation in operator form is $(D^2+D+1)y=0$

Let $f(D)=D^2+D+1$

A.E. is $f(m)=0$ (*i.e.*) $m^2+m+1=0 \Rightarrow m=\dfrac{-1\pm\sqrt{1-4}}{2}=\dfrac{-1\pm i\sqrt{3}}{2}$

The roots are $m=\dfrac{-1+i\sqrt{3}}{2}$ and $m=\dfrac{-1-i\sqrt{3}}{2}$.

$\therefore$ The general solution of (1) is $y=e^{\frac{-x}{2}}\left(c_1\cos\dfrac{x\sqrt{3}}{2}+c_2\sin\dfrac{x\sqrt{3}}{2}\right)$

Example 7: Solve $(D^4+8D^2+16)y=0$

Solution: Given equation is $(D^4+8D^2+16)y=0$ ...(1)

Let $f(D)=D^4+8D^2+16$

The AE is $f(m)=0$ (*i.e.*) $m^4+8m^2+16=0$ ...(2)

$$\Rightarrow (m^2+4)^2=0 \Rightarrow (m-2i)^2(m+2i)^2=0$$

The roots of (2) are $m=2i,\ 2i,\ -2i,\ -2i$ where $2i,\ -2i,$ are occurring twice.

$\therefore$ The general solution of (1) is $y=(c_1+c_2x)\cos 2x+(c_3+c_4x)\sin 2x$

Note: If $\alpha+\sqrt{\beta}$ is a real irrational root of $f(m)=0$, $\alpha-\sqrt{\beta}$ is also a root of the equation. The part of the complementary function corresponding to these roots can also be put in the form

$$e^{\alpha x}\left(c_1\cosh\sqrt{\beta}\,x+d_1\sinh\sqrt{\beta}\,x\right)$$

Example 8: Solve $(D^3-14D+8)\,y=0$

Solution: Given equation is $D^3-14D+8=0$ ...(1)

Let $f(D)=D^3-14D+8$

AE is $f(m)=0$ *i.e.* $m^3-14m+8=0$

$$\Rightarrow (m+4)(m^2-4m+2)=0$$

$$\therefore m=-4 \text{ and } m=2\pm\sqrt{2}$$

The general solution of (1) is $y=c_1e^{-4x}+e^{2x}\left[c_2\cosh(x\sqrt{2})+c_3\sinh(x\sqrt{2})\right]$

Example 9: Find the general solution of (*i*) $y''+2y'=0$ **[JNTU 2001]**

(*ii*) Solve $y''+6y'+9y=0,\ y(0)=-4,\ y'(0)=14.$

Solution : (*i*) Given equation in the standard form is $(D^2+2D)y=0$

The A.E. is $m^2+2m=0$

i.e., $m(m+2) = 0 \Rightarrow m = 0, -2,$

$\therefore$ The general solution is

$$y = c_1 e^{0.x} + c_2 e^{-2x} = c_1 + c_2 e^{-2x}$$

where c_1, c_2 are constants.

(*ii*) Given equation in the standard form is $(D^2 + 6D + 9)y = 0$

The A.E. is $(m+3)^2 = 0$

$\therefore$ $m = -3, -3$

$\therefore$ The general solution is $y = (c_1 + c_2 x) e^{-3x}$. ... (1)

Diff. w.r.t. x,

$$y' = (c_1 + c_2 x)(-3e^{-3x}) + e^{-3x}(c_2)$$

Given $y'(0) = 14$

$\therefore$ $14 = -3c_1 + c_2$ ... (2)

and $y(0) = -4$

$\therefore$ $-4 = c_1$ ... (3)

From (2) and (3), we get $c_1 = -4$, $c_2 = 2$

Substituting the values of c_1 and c_2 in (1), we get $y = (-4 + 2x) e^{-3x}$.

Example 10 : Solve $y'' - y' - 2y = 0$ **[JNTU 2000S]**

Solution : Given D.E. can be written in operator form as $(D^2 - D - 2)y = 0$

Auxiliary equation is $f(m) = 0$

$\Rightarrow m^2 - m - 2 = 0 \Rightarrow m^2 - 2m + m - 2 \Rightarrow (m+1)(m-2) = 0$

$\Rightarrow m = 2, -1$

Roots are real and different

General solution is $y = c_1 e^{2x} + c_2 e^{-x}$ where c_1 and c_2 are constants.

Example 11 : Solve $y'' + y' - 2y = 0$. $y(0) = 4$, $y'(0) = 1$ **[JNTU 2000S]**

Solution : Given D.E. can be written in operator form as $(D^2 + D - 2)y = 0$ ... (1)

Auxiliary equation is $f(m) = 0$

$\Rightarrow$ $m^2 + m - 2 = 0$ ... (2)

$\Rightarrow$ $m^2 + 2m - m - 2 = 0 \Rightarrow (m-1)(m+2) = 0$

$\Rightarrow$ $m = 1, -2$

$\therefore$ Roots are real and different.

General solution is $y = c_1 e^{x} + c_2 e^{-2x}$ where c_1 and c_2 are constants ... (3)

Differentiating (1) w.r.t. to 'x'

$$y' = c_1 e^{x} - 2c_2 e^{-2x} \quad \text{... (4)}$$

Using the data $y(0) = 4$, $y'(0) = 1$ we get

$$c_1 + c_2 = 4$$
$$c_1 - 2c_2 = 1$$

Solving these equations, we get $c_1 = 3$ and $c_2 = 1$

$\therefore$ The solution is $y = 3e^{x} - 2e^{-2x}$

Example 12 : Solve $4y''' + 4y'' + y' = 0$ **[JNTU 2003]**

Solution : Writing in operator form $(4D^3 + 4D^2 + D)y = 0$

A.E. is $4m^3 + 4m^2 + m = 0 \Rightarrow m(4m^2 + 4m + 1) = 0$

$\Rightarrow$ $m(2m+1)^2 = 0$