12. $y = A\,x + \dfrac{B}{x} + e^x - \dfrac{e^x}{x}$

13. $y = A\cos x + B\sin x - \dfrac{\cos x \cdot \tan^2 x}{2} + \sin x \tan x$

14. $y = A\cos x + B\sin x - x\cos x - 1 + \sin x \cdot \log(1 + \sin x)$

15. $y = A + B\cos x + C\sin x - \log(\operatorname{cosec} x + \cot x) - \cos x \log \sin x - x\sin x$

16. $y = A\,e^{-2x} + B\,xe^{-2x} - e^{-2x}\sin x$

17. $y = A\,e^{-3x} + B\,xe^{-3x} + x(\log x - 1)\,e^{-3x}$

18. $y = A\cos 3x + B\sin 3x + 3x\cos 3x - \sin 3x + \sin 3x \cdot \log(\sec 3x)$

19. $y = A\,e^x + Be^{2x} - e^x + (e^{2x} - e^x)\log(1 + e^{-x})$

20. $y = A\cos x + B\sin x - \cos x\log(\sin x) - x\sin x - \sin x\cot x$

21. $y = e^{-x}(A\cos x + B\sin x) + \dfrac{1}{2}e^{-x} \cdot \sin x\tan x$

22. $y = A\,e^x\cos x + Be^x\sin x - e^x\cos x\log(\sec x + \tan x)$

4.5 METHOD OF UNDETERMINED COEFFICIENTS

To find the particular integral of $f(D)y = g(x)$, we assume a trial solution containing unknown constants which are determined by substitution in the given equation. The trial solution to be assumed in each case, depends on the form of $g(x)$. The trial solution may be chosen in the following way.

$g(x)$	trial solution
$e^{\alpha x}$	$a_1e^{\alpha x}$ if α is not a root of the auxiliary equation $a_1xe^{\alpha x}$ if α is a root of the auxiliary equation. $a_1x^2e^{\alpha x}$ if α, α are the roots of the auxiliary equation.
$\sin \alpha x$ (or) $\cos \alpha x$	$a_1\cos \alpha x + a_2\sin \alpha x$ if $\pm i\alpha$ are not roots of the auxiliary equation. $a_1x\cos \alpha x + a_2\,x\sin \alpha x$ if $\pm i\alpha$ are the roots of the auxiliary equation.
$b_0x^n + b_1x^{n-1} + \ldots + b_n$	$a_0x^n + a_1x^{n-1} + \ldots a_n$ if 0 is not a root of the auxiliary equation. $x(a_0x^n + a_1x^{n-1} + \ldots + a_n)$ if 0 is a root of the auxiliary equation. $x^2(a_0x^n + a_1x^{n-1} + \ldots + a_n)$ if 0, 0 are the roots of the auxiliary equation.
$e^x\cos x$ (or) $e^x\sin x$	$e^x(a_1\cos x + a_2\sin x)$

Remark: This method fails, when $g(x) = \tan x$ or $\sec x$ since the number of terms obtained by differentiating $g(x) = \tan x$ or $\sec x$ is infinite. This method holds so long as no term in the trial solution appears in the complimentary function. If any term of the trial solution appears in the complimentary function we multiply this trial solution by the lowest positive integral power of x which is large enough so that none of the terms which are then present appear in the complimentary function.

SOLVED PROBLEMS

EXAMPLE 1: Solve $(D^2 - 6D + 9)\, y = 4e^x$ by the method of undetermined coefficients.

Solution: The auxiliary equation is $m^2 - 6m + 9 = 0 \Rightarrow m = 3, 3$

So the complimentary function is C.F $= (C_1 x + C_2)\, e^{3x}$

Assume the particular integral as

$$y = ae^x$$

$$y' = ae^x$$

$$y'' = ae^x$$

Substituting in the given equation, we get

$$ae^x - 6ae^x + 9ae^x = 4e^x$$

$$\Rightarrow 4ae^x = 4e^x$$

$$\Rightarrow a = 1$$

$$\therefore \text{P.I} = e^x$$

Hence the solution is $y = (C_1 x + C_2)\, e^{3x} + e^x$

EXAMPLE 2: Solve $(D^2 + 2D + 4)\, y = 4x^2 + \dfrac{6}{e^x}$

Solution: The auxiliary equation is $m^2 + 2m + 4 = 0$

$$\Rightarrow m = -1 \pm \sqrt{3}i$$

So the complimentary function is

$$\text{C.F} = e^{-x}\,(C_1 \cos \sqrt{3}x + C_2 \sin \sqrt{3}x)$$

Assume the particular integral as

$$y = a_1 x^2 + a_2 x + a_3 + a_4 e^{-x}$$

Substituting in the given equation, we get

$$2a_1 + a_4 e^{-x} + 2\,(2a_1 x + a_2 - a_4 e^{-x}) + 4\,(a_1 x^2 + a_2 x + a_3 + a_4 e^{-x}) = 4x^2 + 6e^{-x}$$

$$\Rightarrow 4a_1x^2 + (4a_1 + 4a_2)\,x + (2a_1 + 2a_2 + 4a_3) + 3a_4e^{-x} = 4x^2 + 6e^{-x}$$

$$\Rightarrow a_1 = 1,\ a_2 = -1,\ a_3 = 0,\ a_4 = 2$$

So the particular integral is P.I $= x^2 - x + 2e^{-x}$.

Hence the solution is

$$y = e^{-x}\,(C_1\cos\sqrt{3}x + C_2\sin\sqrt{3}x) + x^2 - x + 2e^{-x}$$

EXAMPLE 3: Solve by the method of undetermined coefficients $(D^2 + 1)\,y = 2\sin x$

Solution: The auxiliary equation is $m^2 + 1 = 0 \Rightarrow m = \pm i$. So the complimentary function is C.F $= a_1\cos x + a_2\sin x$. Since $\pm i$ are the roots of the auxiliary equation, we assume the particular integral as

$$y = x\,(a_1\cos x + a_2\sin x)$$

So the given equation becomes

$$(2a_2 - a_1x)\cos x - (2a_1 + a_2x)\sin x + a_1x\cos x + a_2x\sin x = 2\sin x$$

$$\Rightarrow 2a_2\cos x - 2a_1\sin x = 2\sin x$$

$$\Rightarrow a_1 = -1,\ a_2 = 0$$

So the particular integral is P.I $= -x\cos x$

Hence the solution is $y = a_1\cos x + a_2\sin x - x\cos x$

EXAMPLE 4: Solve $(D^2 - 1)\,y = 2e^x$ by the method of undetermined coefficients.

Solution: The auxiliary equation is $m^2 - 1 = 0 \Rightarrow m = \pm 1$. So the complimentary function is C.F $= C_1e^x + C_2e^{-x}$. Since 1 is a root of the auxiliary equation, let us assume the particular integral as $y = axe^x$.

Then the given equation becomes

$$a\,(xe^x + 2e^x) - axe^x = 2e^x$$

$$\Rightarrow 2ae^x = 2e^x$$

$$\Rightarrow a = 1$$

So the particular integral is P.I $= xe^x$. Hence the solution is

$$y = C_1e^x + C_2e^{-x} + xe^x$$

EXAMPLE 5: Solve $(D^2 + 4D + 4)\,y = 2e^{-2t}$ by the method of undetermined coefficients.

Solution: The auxiliary equation is $m^2 + 4m + 4 = 0 \Rightarrow m = -2, -2$.

So the complimentary function is

$$\text{C.F} = (C_1t + C_2)\, e^{-2t}.$$

Since $-2, -2$ are the roots of auxiliary equation, let us assume the particular integral as

$$y = at^2 e^{-2t}$$

The given equation becomes

$$(4at^2e^{-2t} - 8ate^{-2t} + 2ae^{-2t}) + 4a\,(-2t^2e^{-2t} + 2te^{-2t}) + 4at^2\, e^{-2t} = 2e^{-2t}$$

$$2ae^{-2t} = 2e^{-2t} \Rightarrow a = 1.$$

So the particular integral is P.I $= t^2e^{-2t}$. Hence the solution is

$$y = (C_1t + C_2)\, e^{-2t} + t^2e^{-2t}$$

EXAMPLE 6: Solve $(D^2 + D - 1)\,x = 13\sin 2t$ by the method of undetermined coefficients

Solution: The auxiliary equation is $m^2 + m - 1 = 0 \Rightarrow m = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$. So the complimentary function is

$$\text{C.F} = e^{-\frac{t}{2}}\left(C_1\cos\frac{\sqrt{3}}{2}t + C_2\sin\frac{\sqrt{3}}{2}t \right)$$

Assume the particular integral as $x = a_1\cos 2t + a_2\sin 2t$

Then the given equation becomes

$$-4a_1\cos 2t - 4a_2\sin 2t - 2a_1\sin 2t + 2a_2\sin 2t + a_1\cos 2t + a_2\sin 2t = 13\sin 2t$$

$$\Rightarrow -3a_1 + 2a_2 = 0$$

$$-2a_1 - 3a_2 = 13$$

$$\Rightarrow a_1 = -2,\ a_2 = -3$$

So the particular integral is P.I $= -2\cos 2t - 3\sin 2t$. Hence the solution is

$$x = e^{-t/2}\left(C_1\cos\frac{\sqrt{3}}{2}t + C_2\sin\frac{\sqrt{3}}{2}t \right) - 2\cos 2t - 3\sin 2t$$

EXAMPLE 7: Solve $(D^2 + 16)\,y = 8\cos 4x$ by the method of undetermined coefficients.

Solution: The auxiliary equation is $m^2 + 16 = 0 \Rightarrow m = \pm 4i$. So the complementary function is C.F $= C_1\cos 4x + C_2\sin 4x$

Assume the particular integral as $y = x\,(a_1\cos 4x + a_2\sin 4x)$

Then the given equation becomes,

$$\cos 4x\,(8a_2 - 16a_1x) + \sin 4x\,(-8a_1 - 16a_2x) + 16xa_1 \cos 4x + 16x\,a_2 \sin 4x = 8\cos 4x$$

$$\Rightarrow \quad -8a_1 = 0\,;\; 8a_2 = 8$$

$$\Rightarrow \quad a_1 = 0\,;\; a_2 = 1$$

So the particular integral is P.I $= x\sin 4x$. Hence the solution is

$$y = C_1\cos 4x + C_2\sin 4x + x\sin 4x.$$

EXAMPLE 8: Solve $(D^2 + 4)\,y = 8 + 8x^2$ by the method of undetermined coefficients.

Solution: The auxiliary equation is $m^2 + 4 = 0 \Rightarrow m = \pm 2i$.

So the complimentary function is

$$\text{C.F} = C_1\cos 2x + C_2\sin 2x$$

Assume the particular integral as $y = a_1x^2 + a_2x + a_3$

Then the given equation becomes

$$2a_1 + 4a_1x^2 + 4a_2x + 4a_3 = 8 + 8x^2$$

$$\Rightarrow 4a_1 = 8 \Rightarrow a_1 = 2$$

$$4a_2 = 0 \Rightarrow a_2 = 0$$

$$2a_1 + 4a_3 = 8 \Rightarrow a_3 = 1$$

So the particular integral is P.I $= 2x^2 + 1$

Hence is the solution is

$$y = C_1\cos 2x + C_2\sin 2x + 2x^2 + 1$$

EXAMPLE 9: Solve $(D^3 - D)\,y = 2x$ by the method of undetermined coefficients.

Solution: The auxiliary equation is $m^3 - m = 0 \Rightarrow m = 0, 1, -1$

So the complimentary function is C.F $= C_1 + C_2e^x + C_3e^{-x}$

Since 0 is a root of the auxiliary equation, let us assume the particular integral as $y = (a_1x + a_2)\,x$. Then the given equation becomes

$$0 - 2a_1x - a_2 = 2x$$

$$\Rightarrow a_1 = -1,\; a_2 = 0$$

So the particular integral is P.I $= -x^2$

Hence the solution is $y = C_1 + C_2e^x + C_3e^{-x} - x^2$

EXAMPLE 10: Solve $(D$
efficients.

Solution: The auxiliary eq

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Since 0,0 are the roots
integral as

The given equation bec

$(120a_1x + 24a_2) - 2\,(60a$

Solving, we get $a_1 = 1\,;\, a$

So the particular integral

P.I

Hence the solution is

$y = (C_1 + C_2x) + (C_3 + C_4x)$

EXAMPLE 11: Solve $(D^3 + D$
ficients.

Solution: The auxiliary equati

So the complimentary fun

Let us assume the particul

Then the given differential

$(4a_2 - 2a_1)\,e^x \cos x + (-$

EXAMPLE 10: Solve $(D^4 - 2D^3 + D^2)\,y = 20x^3$ by the method of undetermined coefficients.

Solution: The auxiliary equation is $m^4 - 2m^3 + m^2 = 0 \Rightarrow m = 0, 0, 1, 1.$

So the complimentary function is

$$\text{C.F} = (C_1 + C_2x) + (C_3 + C_4x)\, e^x$$

Since 0,0 are the roots of the auxiliary equation, let as assume that the particular integral as

$$y = x^2\,(a_1x^3 + a_2x^2 + a_3x + a_4)$$

The given equation becomes

$$(120a_1x + 24a_2) - 2\,(60a_1x^2 + 24a_2x + 6a_3) + (20a_1x^3 + 12a_2x^2 + 6a_3x + 2a_4) = 20x^3$$

$$\Rightarrow 20a_1 = 20$$

$$-120a_1 + 12a_2 = 0$$

$$120a_1 - 48a_2 + 6a_3 = 0$$

$$24a_2 - 12a_3 + 2a_4 = 0$$

Solving, we get $a_1 = 1$; $a_2 = 10$, $a_3 = 60$, $a_4 = 240$.

So the particular integral is

$$\text{P.I} = x^2\,(x^3 + 10x^2 + 60x + 240)$$

Hence the solution is

$$y = (C_1 + C_2x) + (C_3 + C_4x)\, e^x + x^5 + 10x^4 + 60x^3 + 240x^2$$

EXAMPLE 11: Solve $(D^3 + D^2)\,y = 10e^x \cos x$ by the method of undetermined coefficients.

Solution: The auxiliary equation is $m^3 + m^2 = 0 \Rightarrow m = 0, 0, -1.$

So the complimentary function is

$$\text{C.F} = C_1 + C_2x + C_3e^{-x}$$

Let us assume the particular integral be

$$y = e^x\,(a_1\cos x + a_2\sin x)$$

Then the given differential equation becomes

$$(4a_2 - 2a_1)\, e^x \cos x + (-4a_1 - 2a_2) \sin x = 10e^x \cos x$$

$$-2a_1 + 4a_2 = 10$$

$$-4a_1 - 2a_2 = 0$$

Solving we get, $a_1 = -1, a_2 = 2$. So the particular integral is

$$\text{P.I} = e^x(-\cos x + 2\sin x)$$

Hence the solution is

$$y = C_1 + C_2 x + C_3 e^{-x} - e^x \cos x + 2e^x \sin x$$

EXAMPLE 12: Solve $(D^2 - 3D)\,y = e^{3x} + \sin x$ by the method of undetermined co-efficients.

Solution: The auxiliary equation is $m^2 - 3m = 0 \Rightarrow m = 0, 3$.

So the complimentary function is C.F $= C_1 + C_2 e^{3x}$

Since 0 is a root of the auxiliary equation, let us assume the particular integral as

$$y = a_1 x e^{3x} + a_2 \sin x + a_3 \cos x$$

Then the given equation becomes

$$6a_1 e^{3x} + 9a_1 x e^{3x} - a_2 \sin x - a_3 \cos x - 3a_1 e^{3x} - 9a_1 x e^{3x} - 3a_2 \cos x + 3a_3 \sin x = e^{3x} + \sin x$$

$$\Rightarrow 3a_1 = 1$$

$$-a_2 + 3a_3 = 1$$

$$-3a_2 - a_3 = 0$$

$$\Rightarrow a_1 = \frac{1}{3},\ a_2 = -\frac{1}{10},\ a_3 = \frac{3}{10}$$

So the particular integral is

$$\text{P.I} = \frac{1}{3} x e^{3x} - \frac{1}{10}\sin x + \frac{3}{10}\cos x$$

Hence the solution is

$$y = C_1 + C_2 e^{3x} + \frac{1}{3} x e^{3x} - \frac{1}{10}\sin x + \frac{3}{10}\cos x$$

EXAMPLE 13: Solve by the method of undetermined coefficients, $(D^2 - 1)\,y = e^{3x}\cos 2x - e^{2x}\sin 3x$

Solution: The auxiliary equation is $m^2 - 1 = 0 \Rightarrow m = \pm 1$. So the complimentary function is

$$\text{C.F} = C_1 e^x + C_2 e^{-x}$$

Let us assume that the particular integral be

$$y = e^{3x}(a_1 \cos 2x + a_2 \sin 2x) - e^{2x}(a_3 \cos 3x + a_4 \sin 3x)$$

Then the given equation becomes

$$e^{3x}\{(4a_1 + 12a_2)\cos 2x + (4a_2 - 12a_1)\sin 2x\}$$

$$- e^{2x}\{(12a_4 - 6a_3)\cos 3x - (6a_4 + 12a_3)\sin 3x\} = e^{3x}\cos 2x - e^{2x}\sin 3x$$

$$\Rightarrow 4a_1 + 12a_2 = 1,\ 4a_2 - 12a_1 = 0$$

$$12a_4 - 6a_3 = 0,\ 6a_4 - 12a_3 = -1$$

$$\Rightarrow a_1 = \frac{1}{40},\ a_2 = \frac{3}{40},\ a_3 = \frac{-1}{15},\ a_4 = \frac{-1}{30}$$

So the particular integral is

$$\text{P.I} = \frac{1}{40}e^{3x}(\cos 2x + 3\sin 2x) + \frac{1}{30}e^{2x}(2\cos 3x + \sin 3x)$$

Hence the solution is

$$y = C_1 e^{x} + C_2 e^{-x} + \frac{1}{40}e^{3x}(\cos 2x + 3\sin 2x) + \frac{1}{30}e^{2x}(2\cos 3x + \sin 3x)$$

PROBLEMS FOR PRACTICE

Solve by the method of undetermined coefficients

1. $(D^2 - 3D + 2)\,y = 2\,(x^2 + e^x)$
2. $(D^2 + 4)\,y = 4\sin 2x$