Module 3.2

Method of Undetermined coefficients for finding Particular Integral (P.I) of a linear differential equation $f(D)y = Q(x)$.

S.No:	Form of $Q(x)$	Trail Solution y^* for P.I
1.	$a_n x^n$ (or) $a_0 + a_1 x + \dots + a_n x^n$	$A_0 + A_1 x + \dots + A_n x^n$
2.	$b e^{ax}$	$A e^{ax}$
3.	$a_n x^n e^{ax}$ (or) $e^{ax}(a_0 + a_1 x + \dots + a x^n)$	$e^{ax}(A_0 + A_1 x + \dots + A_n x^n)$
4.	$p \sin ax$ (or) $q \cos ax$ (or) $p \sin ax + q \cos ax$	$A \sin ax + B \cos ax$
5.	$p e^{bx} \sin ax$ (or) $q e^{bx} \cos ax$ (or) $e^{bx}(p \sin ax + q \cos ax)$	$e^{bx}(A \sin ax + B \cos ax)$
6.	$a_n x^n \sin ax$ (or) $a_n x^n \cos ax$ (or) $(a_0 + a_1 x + \dots + a_n x^n) \sin ax$ (or) $(a_0 + a_1 x + \dots + a_n x^n) \cos ax$	$\boxed{(A_0 + A_1 x + \dots + A_n x^n) \sin ax + (B_0 + B_1 x + \dots + B_n x^n) \cos ax}$

1. Solve $(D^2 - 2D + 3)y = x^3 + \sin x$

<u>Sol:</u> Given differential equation is

$$(D^2 - 2D + 3)y = x^3 + \sin x \longrightarrow (1)$$

The A.E is $m^2 - 2m + 3 = 0$

$$\Rightarrow m = 1 \pm i\sqrt{2}$$

$$\therefore y_c = e^x(c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x)$$

Let the trail solution for P.I. be

$$y^* = (A_0 + A_1 x + A_2 x^2 + A_3 x^3) + (A_4 \cos x + A_5 \sin x)$$

Then, from (1), we get

$$D^2 y^* - 2Dy^* + 3y^* = x^3 + \sin x$$

$$\Rightarrow (2A_2 + 6A_3 x - A_4 \cos x - A_5 \sin x)$$
$$-2(A_1 + 2A_2 x + 3A_3 x^2 - A_4 \sin x + A_5 \cos x)$$
$$+3(A_0 + A_1 x + A_2 x^2 + A_3 x^3 + A_4 \cos x + A_5 \sin x)$$
$$= x^3 + \sin x$$

$$\Rightarrow (3A_0 - 2A_1 + 2A_2) + (6A_3 - 4A_2 + 3A_1)x$$
$$+ (3A_2 - 6A_3)x^2 + 3A_3 x^3 + 2(A_4 + A_5)\sin x$$
$$+ 2(A_4 - A_5)\cos x = x^3 + \sin x$$

which implies that

$3A_0 - 2A_1 + 2A_2 = 0$; $6A_3 - 4A_2 + 3A_1 = 0$

$3A_2 - 6A_3 = 0$; $3A_3 = 1$; $2(A_4 - A_5) = 0$;

$2(A_4 + A_5) = 1$

by solving, we get

$A_0 = -\frac{8}{27}, A_1 = \frac{2}{9}, A_2 = \frac{2}{3}, A_3 = \frac{1}{3},$

$A_4 = \frac{1}{4}$ and $A_5 = \frac{1}{4}$.

$$\therefore \quad y_p = P.I = -\frac{8}{27} + \frac{2}{9}x + \frac{2}{3}x^2 + \frac{1}{3}x^3 + \frac{1}{4}\sin x + \frac{1}{4}\cos x$$

Hence,

$$y = y_c + y_p = e^x(c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x) + \frac{1}{27}(9x^3 + 8x^2 + 6x - 8) + \frac{1}{4}(\sin x + \cos x)$$

(2) Solve: $(D^2+4)y = x^2 \cos 2x$

Hint: $y_c = c_1 \cos 2x + c_2 \sin 2x$

Trail solution for P.I. is

$$y^* = (A_0 + A_1 x + A_2 x^2)\sin 2x + (B_0 + B_1 x + B_2 x^2)\cos 2x$$

(3) solve: $(D^2+1)y = x \sin x$

Hint. $y^* = (A_0 + A_1 x)\sin x + (B_0 + B_1 x)\cos x$

(4) Solve: $(D^2-1)\,y = e^x \cos 2x$

Hint: $y^* = e^x (A \cos 2x + B \sin 2x)$

Method of Variation of Parameters :

Consider a second order linear diff. equation with constant co-efficients

$$\frac{d^2y}{dx^2} + k_1 \frac{dy}{dx} + k_2 y = R(x) \longrightarrow (1)$$

where k_1, k_2 are arbitrary constants.

Suppose that C.F of (1) is

$$y_c = c_1 u(x) + c_2 v(x).$$

Then $y_p = A(x)\, u(x) + B(x)\, v(x)$,

where $A(x) = -\int \frac{R(x)\, v(x)}{uv' - vu'}\, dx$

and $B(x) = \int \frac{R(x)\, u(x)}{uv' - vu'}\, dx$ (here $uv' - vu' \neq 0$)

1. Apply the method of variation of parameters to solve $y'' + y = \operatorname{cosec} x$.

Sol: Operator form of the given diff. eqn is $(D^2+1)y = \operatorname{cosec} x$, where $D \equiv \frac{d}{dx}$ $\longrightarrow$ (1)

The A.E. is $m^2+1=0 \Rightarrow m = \pm i$

$\therefore\ y_c = c_1 \cos x + c_2 \sin x$

Let $y_p = A(x)\, u(x) + B(x)\, v(x)$,

where $u(x) = \cos x$ and $v(x) = \sin x$

Clearly, $uv' - vu' = 1 \neq 0$

Here $R(x) = \operatorname{cosec} x$.

Now, $A(x) = -\int \frac{R(x)v(x)}{uv'-vu'}\, dx = -x$

and $B(x) = \int \frac{R(x)\, u(x)}{uv'-vu'}\, dx = \log|\sin x|$

$\therefore\ y_p = -x\cos x + (\log|\sin x|)\cdot \sin x$

Hence the general solution of (1) is

$y = y_c + y_p$

$= c_1 \cos x + c_2 \sin x - x\cos x + (\log|\sin x|)\sin x.$

Solve (1) $(D^2+a^2)\,y = \tan ax$

(2) $y'' + 4y = 4\sec^2 2x$

(3) $(D^2 - 2D)\,y = e^x \cdot \sin x$

using variation of parameters method.

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