Chapter 9

LAPLACE TRANSFORMS

9.1 INTRODUCTION

The knowledge of Laplace transforms is an essential part of mathematics required by engineers and scientists. The 'Laplace Transfrom' is an excellent tool for solving linear differential equations with given initial values of an unknown function and its derivatives without the necessity of first finding the general solution (complementary function + particular integral) and then evaluating from it the particular solution satisfying the given conditions. This technique is useful to solve some partial differential equations as well. This is a powerful tool in diverse fields of engineering.

9.2 DEFINITION

Let $f(t)$ be a function defined for all positive values of t. Then the Laplace transform of $f(t)$, denoted by $L\{f(t)\}$ or $\bar{f}(s)$ is defined by

$$L\{f(t)\} = \bar{f}(s) = \int_0^{\infty} e^{-st} f(t)\, dt \qquad ...(1)$$

provided that the integral exists. Here the parameter s is a real or complex number.

The relation (1) can also be written as $f(t) = L^{-1}\{\bar{f}(s)\}$

In such a case, the function $f(t)$ is said to be *inverse Laplace transform* of $\bar{f}(s)$. The symbol L which transform $f(t)$ into $\bar{f}(s)$ can be called the Laplace transform operator. The symbol L^{-1} which transforms $\bar{f}(s)$ to $f(t)$ can be called the inverse Laplace transform operator.

9.3 SUFFICIENT CONDITIONS FOR THE EXISTENCE OF THE LAPLACE TRANSFORM OF A FUNCTION

While finding the Laplace transforms of elementary functions, it can be noticed that the integral exists under certain conditions, such as $s > 0$ or $s > a$ etc., In general, the function $f(t)$ must satisfy the following conditions for the existence of the Laplace transform.

(*i*) The function $f(t)$ must be piece-wise continuous or sectionally continuous in any limited interval $0 < a \le t \le b$.

(*ii*) The function $f(t)$ is of exponential order.

9.4 DEFINITIONS

1. Piece-wise Continuous Function

A function $f(t)$ is said to be piece-wise (or sectionally) continuous over the closed interval $[a, b]$ if it is defined on that interval and is such that the interval can be divided into a finite number of subintervals, in each of which $f(t)$ is continuous and has both right and left hand limits at every end point of the subinterval.

e.g. : (i) The function $f(t)=\begin{cases} t^2, 0<t<5 \\ 2t+3, t>5 \end{cases}$

is sectionally continuous for $t>0$.

(ii) The function $f(t)=\frac{1}{t}$ is not sectionally continuous in any interval containing $t=0$.

2. Functions of Exponential Order

A function $f(t)$ is said to be of exponential order a if

$$\lim_{t\to\infty} e^{-at} f(t) = \text{a finite quantity}$$

i.e. for a given positive number T, there exists a real number $M>0$ such that

$$|e^{-at} f(t)| < M, \forall\, t \ge T$$

or $$|f(t)| < Me^{at}, \forall\, t \ge T$$

For example, $f(t)=t^2$, $\sin at$, e^{at} etc. are all of exponential order and also continuous. But $f(t)=e^{t^2}$ is not of exponential order and as such its Laplace transform does not exist.

General Properties of Laplace Transform

A very important property is that the laplace transformation is a linear operator, just as differentiation and integration.

9.5 LINEARITY PROPERTY

Theorem. If $L[f(t)]=\bar{f}(s)$ and $L[g(t)]=\bar{g}(s)$ then

$$L[c_1 f(t)+c_2 g(t)] = c_1 L[f(t)] + c_2 L[g(t)] = c_1\bar{f}(s)+c_2\bar{g}(s);$$ where c_1 and c_2 are constants.

Proof. By definition,

$$L[c_1 f(t)+c_2 g(t)\} = \int_0^\infty e^{-st}[c_1 f(t)+c_2 g(t)]dt = \int_0^\infty e^{-st} c_1 f(t)dt + \int_0^\infty e^{-st} c_2 g(t)dt$$

$$= c_1\int_0^\infty e^{-st} f(t)\, dt + c_2\int_0^\infty e^{-st} g(t)dt$$

$$= c_1 L[f(t)] + c_2 L[g(t)] = c_1\bar{f}(s)+c_2\bar{g}(s)$$

The above result can easily be generalized to more than two functions.

Hence the Laplace transform of the sum of two or more functions of t is the sum of the Laplace transforms of the separate functions.

9.6 LAPLACE TRANSFORMS OF ELEMENTARY FUNCTIONS

Elementary functions include Algebraic and transcendental functions. From the definition and by ordinary integration, we obtain the following results.

1. $L\{k\} = \frac{k}{s} (s>0)$, where k is a constant.

Proof: By definition

$$L\{k\} = \int_0^\infty e^{-st}\cdot k\, dt = k\int_0^\infty e^{-st}\cdot dt = k\left(\frac{e^{-st}}{-s}\right)_0^\infty = -\frac{k}{s}(e^{-\infty}-1)$$

$$= -\frac{k}{s}(0-1)[\because e^{-\infty}=0] = \frac{k}{s},\ \text{if } s>0$$

Note: The above Laplace transform does not exist for $s \le 0$. It follows

(*i*) For $k=0$, $L\{0\}=0$ (*ii*) For $k=1$, $L\{1\}=\frac{1}{s^2}$, $s>0$.

To Prove the result (*ii*), proceed as below.

2. $L\{t\}=\frac{1}{s^2}$

Proof: $L\{t\}=\int_0^\infty e^{-st}.t\,dt=\left[t\left(\frac{e^{-st}}{-s}\right)-(1)\left(\frac{e^{-st}}{(-s)^2}\right)\right]_0^\infty=\frac{1}{s^2}$, if $s>0$

3. $L\{t^n\}=\frac{n!}{s^{n+1}}$ where n is a positive integer

Proof: $L\{t^n\}=\int_0^\infty e^{-st}.t^n dt=\int_0^\infty t^n d\left(\frac{e^{-st}}{-s}\right)=\left[t^n\left(\frac{e^{-st}}{-s}\right)\right]_0^\infty-\int_0^\infty \frac{e^{-st}}{-s}\cdot nt^{n-1}dt$, by parts

$$=\frac{n}{s}\int_0^\infty e^{-st}t^{n-1}dt \quad [\because \text{ first portion} = 0 \text{ at both limits}]$$

$$=\frac{n}{s}L\{t^{n-1}\}$$

Similarly $L\{t^{n-1}\}=\frac{n-1}{s}L\{t^{n-2}\}$

$$L\{t^{n-2}\}=\frac{n-2}{s}L\{t^{n-3}\}$$

By repeatedly applying this, we get

$$L\{t^n\}=\frac{n}{s}\cdot\frac{n-1}{s}\cdot\frac{n-2}{s}\cdots\frac{2}{s}\cdot\frac{1}{s}L\{t^{n-n}\}=\frac{n!}{s^n}L\{1\}=\frac{n!}{s^n}\cdot\frac{1}{s}=\frac{n!}{s^{n+1}}$$

Definition: *Gamma Function.* If $n>0$ then the Gamma function $\Gamma(n)$ is defined by

$$\Gamma(n)=\int_0^\infty e^{-x}x^{n-1}\,dx$$

The following are some important properties of the Gamma function.

1. $\Gamma(n+1)=n\Gamma(n)$, if $n>0$
2. $\Gamma(n+1)=n!$, if n is a positive integer
3. $\Gamma(1)=1$, $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$, $\Gamma\left(-\frac{1}{2}\right)=-2\sqrt{\pi}$
4. $\Gamma(0),\Gamma(-1),\Gamma(-2),\Gamma(-3),\ldots$ are all not defined.

Note: $L\{t^n\}$ can also be expressed in terms of Gamma function.

We have $L\{t^n\}=\int_0^\infty e^{-st}t^n\,dt$

Putting $x=st$., we get

$$L\{t^n\}=\frac{1}{s^{n+1}}\int_0^\infty e^{-x}x^n\,dx=\frac{1}{s^{n+1}}\int_0^\infty e^{-x}x^{(n+1)-1}dx=\frac{\Gamma(n+1)}{s^{n+1}}, \text{ when } n>-1$$

If n is a positive integer, $\Gamma(n+1) = n!$ in particular.

Hence $L\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}} = \frac{n!}{s^{n+1}}$ where n is a positive integer.

4. $L\{e^{at}\} = \frac{1}{s-a}$, $(s-a>0)$

Proof: $L\{e^{at}\} = \int_0^\infty e^{-st} e^{at}\, dt = \int_0^\infty e^{-(s-a)t}\, dt = \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty = \frac{-e^{-\infty}}{s-a} + \frac{e^{-0}}{s-a} = \frac{1}{s-a}$, if $s>a$

Note: Similarly $L\{e^{-at}\} = \frac{1}{s+a}$, if $s>a$

5. $L\{\sinh at\} = \frac{a}{s^2-a^2}$, if $s>|a|$

Proof: Using the linearity property of the Laplace transform, we have

$$L(\sinh at\} = L\left\{\frac{e^{at}-e^{-at}}{2}\right\} = \frac{1}{2}\left[L\{e^{at}\} - L\{e^{-at}\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right] = \frac{a}{s^2-a^2}$$

6. $L\{\cosh at\} = \frac{s}{s^2-a^2}$, $Re(s)>a$

Here for the existence of Laplace Transform we require $s>a$ or more precisely $Re(s)>a$.

7. $L\{\sin at\} = \frac{a}{s^2+a^2}$, if $s>0$

Proof: $L\{\sin at\} = \int_0^\infty e^{-st} \sin at\, dt = \left[\frac{e^{-st}}{s^2+a^2}(-s\sin at - a\cos at)\right]_0^\infty = \frac{a}{s^2+a^2}$

[Using $\int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$]

8. Similarly $L\{\cos at\} = \frac{s}{s^2+a^2}$, if $s>0$

Alternate method to find L{sin at} and L{cos at}

We know that $L\{e^{at}\} = \frac{1}{s-a}$

Replacing a by ia, we get $L\{e^{iat}\} = \frac{1}{s-ia} = \frac{s+ia}{(s-ia)(s+ia)}$

i.e. $L\{\cos at + i\sin at\} = \frac{s+ia}{s^2+a^2}$

Equating the real and imaginary parts on both sides, we have

$$L\{\cos at\} = \frac{s}{s^2+a^2} \text{ and } L\{\sin at\} = \frac{a}{s^2+a^2}$$

EXAMPLES

Example 1 : Find the Laplace transforms of:

(i) $(t^2+1)^2$ (ii) $\dfrac{e^{-at}-1}{a}$ (iii) $\sin 2t \cos t$ (iv) $\cosh^2 2t$

(v) $\cos^3 2t$ (vi) $(\sin t + \cos t)^2$ (vii) $\cos t \cdot \cos 2t \cdot \cos 3t$ (viii) $\sinh^3 2t$

Solution : (i) Here $f(t) = (t^2+1)^2 = t^4 + 2t^2 + 1$

$$\therefore L\{(t^2+1)^2\} = L\{t^4+2t^2+1\} = L\{t^4\} + 2L\{t^2\} + L\{1\}$$

$$= \frac{4!}{s^5} + 2.\frac{2!}{s^3} + \frac{1}{s} = \frac{24}{s^5} + \frac{4}{s^3} + \frac{1}{s} = \frac{1}{s^5}(s^4+4s^2+24), s > 0$$

(ii) $$L\left\{\frac{e^{-at}-1}{a}\right\} = \frac{1}{a}L\{e^{-at}-1\} = \frac{1}{a}\left[L\{e^{-at}\} - L\{1\}\right] = \frac{1}{a}\left[\frac{1}{s+a} - \frac{1}{s}\right] = \frac{-1}{s(s+a)}$$

(iii) Since $\sin 2t \cos t = \frac{1}{2}(2\sin 2t \cos t) = \frac{1}{2}(\sin 3t + \sin t)$

$$\therefore \quad L\{\sin 2t \cos t\} = L\left\{\frac{1}{2}(\sin 3t + \sin t)\right\} = \frac{1}{2}\left[L\{\sin 3t\} + L\{\sin t\}\right]$$

$$= \frac{1}{2}\left[\frac{3}{s^2+9} + \frac{1}{s^2+1}\right] = \frac{2(s^2+3)}{(s^2+1)(s^2+9)}$$

(iv) Since $\cosh^2 2t = \frac{1}{2}(1+\cosh 4t)$

$$\therefore \ L\{\cosh^2 2t\} = \frac{1}{2}\left[L\{1\} + L\{\cosh 4t\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2-16}\right] = \frac{s^2-8}{s(s^2-16)}$$

(v) Since $\cos 6t = \cos 3(2t) = 4\cos^3 2t - 3\cos 2t$

$$\therefore \quad \cos^3 2t = \frac{1}{4}(3\cos 2t + \cos 6t)$$

Hence $$L\{\cos^3 2t\} = L\left\{\frac{1}{4}(3\cos 2t + \cos 6t)\right\} = \frac{3}{4}L\{\cos 2t\} + \frac{1}{4}L\{\cos 6t\}$$

$$= \frac{3}{4}.\frac{s}{s^2+4} + \frac{1}{4}.\frac{s}{s^2+36} = \frac{s}{4}\left(\frac{3}{s^2+4} + \frac{1}{s^2+36}\right)$$

$$= \frac{s}{4}\left[\frac{4s^2+112}{(s^2+4)(s^2+36)}\right] = \frac{s(s^2+28)}{(s^2+4)(s^2+36)}$$

(vi) Since $(\sin t + \cos t)^2 = \sin^2 t + \cos^2 t + 2\sin t \cos t = 1 + \sin 2t$

$$\therefore L\{(\sin t + \cos t)^2\} = L\{1 + \sin 2t\} = \frac{1}{s} + \frac{2}{s^2+4} = \frac{s^2+2s+4}{s(s^2+4)}$$

(*vii*) $\cos t\cos 2t\cos 3t = \frac{1}{2}\cos t(2\cos 2t\cos 3t) = \frac{1}{2}\cos t(\cos 5t+\cos t)$

$$= \frac{1}{2}(\cos t\cos 5t+\cos^2 t) = \frac{1}{4}(2\cos t\cos 5t+2\cos^2 t)$$

$$= \frac{1}{4}[(\cos 6t+\cos 4t)+(1+\cos 2t)] = \frac{1}{4}[1+\cos 2t+\cos 4t+\cos 6t]$$

$$\therefore L\{\cos t\cos 2t\cos 3t\} = \frac{1}{4}\left[\frac{1}{s}+\frac{s}{s^2+4}+\frac{s}{s^2+16}+\frac{s}{s^2+36}\right]$$

(*viii*) We know that

$$\sinh x = \frac{e^x-e^{-x}}{2} \qquad \therefore \quad \sinh 2t = \frac{e^{2t}-e^{-2t}}{2}$$

Hence $L\{\sinh^3 2t\} = L\left\{\left(\frac{e^{2t}-e^{-2t}}{2}\right)^3\right\} = L\left\{\frac{1}{8}(e^{6t}-3e^{2t}+3e^{-2t}-e^{-6t})\right\}$

$$= \frac{1}{8}.\frac{1}{s-6}-\frac{3}{8}.\frac{1}{s-2}+\frac{3}{8}.\frac{1}{s+2}-\frac{1}{8}.\frac{1}{s+6}$$

$$= \frac{1}{8}\left(\frac{1}{s-6}-\frac{1}{s+6}\right)-\frac{3}{8}\left(\frac{1}{s-2}-\frac{1}{s+2}\right)$$

$$= \frac{1}{8}\left(\frac{12}{s^2-36}-\frac{12}{s^2-4}\right) = \frac{48}{(s^2-4)(s^2-36)}$$

Example 2 : Find the Laplace transform of the following:

(*i*) $e^{2t}+4t^3-2\sin 3t+3\cos 3t$ **[JNTU 2003, 2003S (Set No. 1)]**

(*ii*) $(\sin t-\cos t)^3$ (*iii*) $f(t)=|t-1|+|t+1|,\ t\geq 0$

(*iv*) $\sin(wt+\alpha)$ **[JNTU 1995S]** (*v*) $\sin 2t\cos 3t$ (*vi*) $\cosh^3 2t$

Solution :

(*i*) $L\{e^{2t}+4t^3-2\sin 3t+3\cos 3t\} = L(e^{2t})+4L(t^3)-2L(\sin 3t)+3L(\cos 3t)$

$$= \frac{1}{s-2}+4.\frac{3!}{s^4}-2.\frac{3}{s^2+9}+3.\frac{s}{s^2+9}$$

$$= \frac{1}{s-2}+\frac{24}{s^4}-\frac{6}{s^2+9}+\frac{3s}{s^2+9} = \frac{1}{s-2}+\frac{24}{s^4}+\frac{3(s-2)}{s^2+9}$$

(*ii*) $(\sin t-\cos t)^3 = \sin^3 t-\cos^3 t-3\sin^2 t\cos t+3\sin t\cos^2 t$

$$= \sin^3 t-\cos^3 t-3(1-\cos^2 t)\cos t+3\sin t(1-\sin^2 t)$$

$$= 3(\sin t-\cos t)-2(\sin^3 t-\cos^3 t)$$

$$= 3(\sin t-\cos t)-2\left[\frac{1}{4}(3\sin t-\sin 3t)-\frac{1}{4}(\cos 3t+3\cos t)\right]$$

$$= \frac{1}{2}(3\sin t-3\cos t+\sin 3t+\cos 3t)$$

$$\therefore \quad L\{(\sin t-\cos t)^3\} = \frac{1}{2}\left(\frac{3}{s^2+1}-\frac{3s}{s^2+1}+\frac{9}{s^2+9}+\frac{s}{s^2+9}\right) = \frac{1}{2}\left[\frac{3(1-s)}{s^2+1}+\frac{s+9}{s^2+9}\right]$$

(iii) We know that

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

When $0 < t < 1$, $|t-1| = 1-t$

Now $t > 0 \Rightarrow t+1 > 1 > 0$

Also $|t+1| = t+1$

$\therefore \quad f(t) = (1-t) + (t+1) = 2$, when $0 < t < 1$

When $t > 1$, $|t-1| = t-1$ $(\because t-1 > 0)$

Also $|t+1| = t+1$ $(\because t+1 > 2 > 0)$

$\therefore f(t) = (t-1) + (t+1) = 2t$, when $t > 1$

Thus, $$L\{f(t)\} = \int_0^{\infty} f(t)\, e^{-st}\, dt = \int_0^1 2e^{-st}\, dt + \int_1^{\infty} 2t\, e^{-st}\, dt$$

$$= 2\left(\frac{e^{-st}}{-s}\right)_0^1 + 2\left[t\left(\frac{e^{-st}}{-s}\right) - \left(\frac{e^{-st}}{s^2}\right)\right]_1^{\infty}$$

$$= 2\left(\frac{e^{-s}}{-s} + \frac{1}{s}\right) + 2\left(\frac{e^{-s}}{s} + \frac{e^{-s}}{s^2}\right) = \frac{2}{s}\left(1 + \frac{e^{-s}}{s}\right)$$

(iv) Since $\sin(A+B) = \sin A \cos B + \cos A \sin B$, therefore,

$$L\{\sin(wt+\alpha)\} = L\{\sin wt \cos\alpha + \cos wt \sin\alpha\} = \cos\alpha . L(\sin wt) + \sin\alpha . L(\cos wt)$$

$$= \cos\alpha . \frac{w}{s^2+w^2} + \sin\alpha . \frac{s}{s^2+w^2} = \frac{1}{s^2+w^2}(w\cos\alpha + s.\sin\alpha)$$

(v) We have

$$\sin 2t \cos 3t = \frac{1}{2}(2\sin 2t \cos 3t) = \frac{1}{2}[\sin 5t + \sin(-t)]$$

$$\therefore L(\sin 2t \cos 3t) = L\left\{\frac{\sin 5t - \sin t}{2}\right\} = \frac{1}{2}[L(\sin 5t) - L(\sin t)]$$

$$= \frac{1}{2}\left[\frac{5}{s^2+25} - \frac{1}{s^2+1}\right] = \frac{2(s^2-5)}{(s^2+25)(s^2+1)}$$

(vi) We have

$$\cosh 3A = 4\cosh^3 A - 3\cosh A$$

$$\therefore \quad \cosh 6t = 4\cosh^3 2t - 3\cosh 2t$$

$$\Rightarrow \cosh^3 2t = \frac{\cosh 6t + 3\cosh 2t}{4}$$

$$\therefore L(\cosh^3 2t) = \frac{1}{4}[L\{\cosh 6t\} + 3L\{\cosh 2t\}] = \frac{1}{4}\left[\frac{s}{s^2-36} + \frac{3.s}{s^2-4}\right]$$

$$= \frac{s}{4}\left(\frac{1}{s^2-36} + \frac{3}{s^2-4}\right) = \frac{s(s^2-28)}{(s^2-4)(s^2-36)}$$

9.7 FIRST TRANSLATION (OR) FIRST SHIFTING THEOREM

If $L\{f(t)\} = \bar{f}(s)$, then $L\{e^{at} f(t)\} = \bar{f}(s-a), s-a>0$

Proof: By definition,

$$L\{e^{at} f(t)\} = \int_0^\infty e^{-st} e^{at} f(t)\,dt = \int_0^\infty e^{-(s-a)t} f(t)\,dt$$

$$= \int_0^\infty e^{-ut} f(t)\,dt, \text{ where } u = s-a$$

$$= \bar{f}(u) = \bar{f}(s-a)$$

Hence $L\{e^{at} f(t)\} = [L\{f(t)\}]_{\text{change } s \text{ to } s-a}$

Corollary 1: Using the above theorem, we have $L\{e^{-at} f(t)\} = \bar{f}(s+a), (s+a)>0$

and $L\{e^{-at} f(t)\} = [L\{f(t)\}]_{\text{change } s \text{ to } s+a}$

Corollary 2: As an application of this theorem, we obtain the following results.

(i) $L\{e^{at} \cdot t^n\} = [L(t^n)]_{s\to s-a} = \left[\dfrac{n!}{s^{n+1}}\right]_{\text{change } s \text{ to } s-a} = \dfrac{n!}{(s-a)^{n+1}}$

(ii) $L\{e^{at} \sin bt\} = [L(\sin bt)]_{s\to s-a} = \left(\dfrac{b}{s^2+b^2}\right)_{s\to s-a} = \dfrac{b}{(s-a)^2+b^2}$, **[JNTU 2003, 2003S (Set No. 1)]**

(iii) $L\{e^{at} \cos bt\} = \dfrac{s-a}{(s-a)^2+b^2}$, **[JNTU 2003, 2003S (Set No. 1)]**

(iv) $L\{e^{at}\sinh bt\} = [L(\sin bt)]_{s\to s-a} = \left(\frac{b}{s^2+b^2}\right)\frac{b}{(s-a)^2+b^2}$, [JNTU 2003, 2003S (Set No. 2)]

(v) $L\{e^{at}\cosh bt\} = \frac{s-a}{(s-a)^2-b^2}$ [JNTU 2003, 2003S (Set No. 2)]

(vi) $L\left\{e^{-at}t^n\right\} = \frac{n!}{(s+a)^{n+1}}$

(vii) $L\left\{e^{-at}\sin bt\right\} = \frac{b}{(s+a)^2+b^2}$

SOLVED EXAMPLES

Example 1 : (i) Find $L\{(t+3)^2 e^t\}$ (ii) $L\{e^{-t}\cos^2 t\}$

Solution : (i) We have $L\{(t+3)^2\} = L\{t^2+6t+9\} = \frac{2}{s^3}+\frac{6}{s^2}+\frac{9}{s} = \bar{f}(s)$

By First Shifting theorem,

$$L\{e^t(t+3)^2\} = \bar{f}(s-1) = \left[\frac{2}{s^3}+\frac{6}{s^2}+\frac{9}{s}\right]_{\text{change } s \text{ to } s-1}$$

$$= \frac{2}{(s-1)^3}+\frac{6}{(s-1)^2}+\frac{9}{s-1} = \frac{9s^2-12s+5}{(s-1)^3}$$

(ii) We have

$$L\{\cos^2 t\} = L\left\{\frac{1+\cos 2t}{2}\right\} = \frac{1}{2}[L\{1\}+L\{\cos 2t\}] = \frac{1}{2}\left[\frac{1}{s}+\frac{s}{s^2+2^2}\right]$$

Using First Shifting Theorem,

$$L\{e^{-t}\cos^2 t\} = \frac{1}{2}\left[\frac{1}{s}+\frac{s}{s^2+4}\right]_{s\to s+1} = \frac{1}{2}\left[\frac{1}{s+1}+\frac{s+1}{(s+1)^2+4}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s+1}+\frac{s+1}{s^2+2s+5}\right]$$

Example 2 : Find the Laplace transform of

(i) $e^{-t}\cos 2t$ (ii) $te^{2t}\sin 3t$

(iii) $e^{-3t}(2\cos 5t - 3\sin 5t)$ [JNTU 1997, 2004, 2007S (Set No. 4)]

(iv) $e^{-t}(3\sin 2t - 5\cosh 2t)$ [JNTU 2002S]

(v) $e^{-at}\sinh bt$ [JNTU 2003S (Set No. 2)]

(vi) $e^{-at}\cosh bt$

Solution : We have

(i) $L\{\cos 2t\} = \frac{s}{s^2+4}$

Now applying First Shifting theorem, we get

$$L\{e^{-t}\cos 2t\} = [L\{\cos 2t\}]_{\text{change } s \text{ to } s+1} = \left[\frac{s}{s^2+4}\right]_{\text{change } s \text{ to } s+1} = \frac{s+1}{(s+1)^2+4} = \frac{s+1}{s^2+2s+5}$$

(*ii*) Since $L\{t\} = \frac{1}{s^2}$, we have $L\{te^{i3t}\} = \frac{1}{(s-3i)^2} = \frac{(s+3i)^2}{(s-3i)^2(s+3i)^2}$

or $$L\{t\cos 3t + i\, t\sin 3t\} = \frac{(s^2-9)+i6s}{(s^2+9)^2}$$

Equating imaginary parts on both sides, we have $L\{t\sin 3t\} = \frac{6s}{(s^2+9)^2}$

Now applying First Shifting theorem, we have

$$L\{e^{2t}t\sin 3t\} = [L\{t\sin 3t\}]_{\text{change } s \text{ to } s-2}$$

$$= \left[\frac{6s}{(s^2+9)^2}\right]_{\text{change } s \text{ to } s-2} = \frac{6(s-2)}{[(s-2)^2+9]^2} = \frac{6(s-2)}{(s^2-4s+13)^2}$$

(*iii*) We have

$$L(2\cos 5t - 3\sin 5t) = \frac{2s}{s^2+25} - \frac{3(5)}{s^2+25} = \frac{2s-15}{s^2+25}$$

Now applying First Shifting theorem, we have

$$L\{e^{-3t}(2\cos 5t - 3\sin 5t)\} = \left(\frac{2s-15}{s^2+25}\right)_{\text{change } s \text{ to } s+3} = \frac{2(s+3)-15}{(s+3)^2+25} = \frac{2s-9}{s^2+6s+34}$$

(*iv*) We have

$$L(3\sin 2t - 5\cosh 2t) = \frac{3(2)}{s^2+4} - \frac{5s}{s^2-4} = \frac{6}{s^2+4} - \frac{5s}{s^2-4}$$

Using First Shifting theorem,

$$L[e^{-t}(3\sin 2t - 5\cosh 2t)] = \left(\frac{6}{s^2+4} - \frac{5s}{s^2-4}\right)_{\text{change } s \text{ to } s+1}$$

$$= \frac{6}{(s+1)^2+4} - \frac{5(s+1)}{(s+1)^2-4} = \frac{6}{s^2+2s+5} - \frac{5(s+1)}{s^2+2s-3}$$

(*v*) We know that $L(\sinh bt) = \frac{b}{s^2-b^2}$, $s > |b|$

Now applying First Shifting theorem, we have

$$L\{e^{-at}\sinh bt\} = \left(\frac{b}{s^2-b^2}\right)_{\text{change } s \text{ to } s+a} = \frac{b}{(s+a)^2-b^2}$$

(*vi*) We know that $L(\cosh bt) = \frac{s}{s^2-b^2}$, $s > |b|$

Now applying First Shifting theorem, we have

$$L\{e^{-at}\cosh bt\} = \left(\frac{s}{s^2-b^2}\right)_{\text{change } s \text{ to } s+a} = \frac{s+a}{(s+a)^2-b^2}$$