4.2: Solutions of PDEs Using Laplace Transform

Given a function $u(x,t)$ defined for all $t>0$ and assumed to be bounded, we can apply Laplace transform in t considering x as a parameter.

$$L\{u(x,t)\} = \int_0^\infty e^{-st}\, u(x,t)\,dt \equiv U(x,s).$$

Also
$$L\{u_t(x,t)\} = \int_0^\infty e^{-st}\, u_t(x,t)\,dt$$

$$= e^{-st}\, u(x,t)\Big]_0^\infty + s\int_0^\infty e^{-st}\, u(x,t)\,dt$$

$$= 0 - u(x,0) + s\,U(x,s)$$

$$\therefore\ L\{u_t(x,t)\} = s\,U(x,s) - u(x,0)$$

and
$$L\{u_x(x,t)\} = \int_0^\infty e^{-st}\, u_x(x,t)\,dt \equiv U_x(x,s)$$

Solved Problems:

1. Solve $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial t} = x \quad (x>0,\ t>0)$
   with $u(0,t) = 0\ (t>0)$ and $(u(x,0) = 0\ (x>$

Sol: Given pde is $u_x(x,t) + u_t(x,t) = x \quad \longrightarrow (1)$

Taking Laplace transform on both sides, we get

$$L\{u_x(x,t)\} + L\{u_t(x,t)\} = L\{x\}$$

$$\Rightarrow U_x(x,s) + s\,U(x,s) - u(x,0) = x\,L\{1\}$$

$$\Rightarrow \frac{dU(x,s)}{dx} + s.U(x,s) = \frac{x}{s} \longrightarrow (2)$$

This is a linear differential equation with constant Co-efficients.

operator form of (2) is

$$(D+s)\,U(x,s) = \frac{x}{s}, \quad D \equiv \frac{d}{dt}.$$

The A.E. is $m+s=0 \Rightarrow m=-s$

$$\therefore C.F = c\,e^{-sx}$$

Let $U^*(x,s) = A + Bx$ be the trail solution of P.I. of (2).

Then $(D+s)(A+Bx) = \frac{x}{s}$

$$\Rightarrow s\left[\boxed{B} + s(A+Bx)\right] = x$$

$$\Rightarrow \quad s^2 A + Bs = 0 \text{ and } s^2 B = 1$$

$$\Rightarrow A = -\frac{1}{s^3} \text{ and } B = \frac{1}{s^2}$$

$$\therefore \text{ P.I } = -\frac{1}{s^3} + \frac{x}{s^2}$$

and hence $\quad U(x,s) = c e^{-sx} + \frac{x}{s^2} - \frac{1}{s^3}$

Finally, $\quad 0 = U(0,s) \quad \left(U(0,s) = L\{u(0,t)\} = L\{0\} = 0 \right)$

$$\Rightarrow \quad 0 = c - \frac{1}{s^3} \Rightarrow c = \frac{1}{s^3}$$

Hence $\quad U(x,s) = \frac{e^{-sx}}{s^3} + \frac{x}{s^2} - \frac{1}{s^3}$

and hence

$$L^{-1}(U(x,s)) = L^{-1}\left\{e^{-sx} \cdot \frac{1}{s^3}\right\} + L^{-1}\left\{\frac{x}{s^2}\right\} - L^{-1}\left\{\frac{1}{s^3}\right\}$$

$$\Rightarrow u(x,t) = \frac{1}{2} H(t-x)(t-x)^2 + x t - \frac{t^2}{2}$$

2. Solve $\quad \frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} + u = 0$

with $u(0,t) = 0 \ (t > 0)$

and $\quad u(x,0) = \sin x \ (x > 0)$.

Sol: Given pde is

$$u_x(x,t) + u_t(x,t) + u(x,t) = 0 \longrightarrow (1)$$

Taking Laplace Transform on both sides, we get

$$L\{u_x(x,t)\} + L\{u_t(x,t)\} + L\{u(x,t)\} = L\{0\}$$

$$\Rightarrow U_x(x,s) + \left(sU(x,s) - u(x,0)\right) + U(x,s) = 0$$

$$\Rightarrow \frac{d}{dt}U(x,s) + (s+1)\,U(x,s) = \sin x \longrightarrow (2)$$

This is a linear differential equation with constant co-efficients.

operator form of (2) is

$$\left(D + (s+1)\right)U(x,s) = \sin x, \quad \left(D \equiv \frac{d}{dt}\right)$$

The A.E. is $m + (s+1) = 0$

$$\Rightarrow m = -(s+1)$$

$$\therefore C.F = c\,e^{-(s+1)x}$$

let $U^*(x,s) = A\cos x + B\sin x$ be the trail solution of P.I of (2).

Then $D(A\cos x + B\sin x) + (s+1)(A\cos x + B\sin x)$

$$= \sin x$$

$$\Rightarrow -A\sin x + B\cos x + (s+1)A\cos x + (s+1)B\sin x$$

$$= \sin x$$

$$\Rightarrow -A+(s+1)B = 1 \quad \text{and} \quad B+(s+1)A = 0$$

$$\Rightarrow A = \frac{-1}{s^2+2s+2} \quad \text{and} \quad B = \frac{s+1}{s^2+2s+2}$$

$$\therefore PI = \frac{-\cos x}{s^2+2s+2} + \frac{(s+1)\sin x}{s^2+2s+2}$$

and hence

$$U(x,s) = c\,e^{-(s+1)x} + \frac{(s+1)\sin x}{s^2+2s+2} - \frac{\cos x}{s^2+2s+2}$$

Now, $U(0,s) = 0$

$$\Rightarrow c + 0 - \frac{1}{s^2+2s+2} = 0 \Rightarrow c = \frac{1}{s^2+2s+2}$$

$$\text{So, } U(x,s) = \frac{1}{s^2+2s+2}\,e^{-(s+1)x} + \frac{(s+1)(\sin x)}{s^2+2s+2} - \frac{\cos x}{s^2+2s+2}$$

Taking the inverse Laplace Transform, we get

$$L^{-1}\{U(x,s)\} = L^{-1}\left\{e^{-(s+1)x} \cdot \frac{1}{s^2+2s+2}\right\} + L^{-1}\left\{(\sin x)\frac{(s+1)}{s^2+2s+2}\right\} - L^{-1}\left\{(\cos x)\frac{1}{s^2+2s+2}\right\}$$

$$\Rightarrow u(x,t) = L^{-1}\left\{ e^{-(s+1)x} \frac{1}{(s+1)^2+1} \right\}$$
$$+ (\sin x)\, L^{-1}\left\{ \frac{s+1}{(s+1)^2+1} \right\}$$
$$- (\cos x)\, L^{-1}\left\{ \frac{1}{(s+1)^2+1} \right\}$$

Thus,

$$u(x,t) = e^{-x} L^{-1}\left\{ e^{-sx}\, L\{e^{-t}\sin t\} \right\}$$
$$+ (\sin x)\, L^{-1}\left\{ L\{e^{-t}\cos t\} \right\}$$
$$- (\cos x)\, L^{-1}\left\{ L\{e^{-t}\sin t\} \right\}$$

$$= e^{-x} H(t-x)\, e^{-(t-x)} \sin(t-x)$$
$$+ (\sin x)\, e^{-t}\cos t - (\cos x)\, e^{-t}\sin t$$

$$= e^{-t}\left[H(t-x)\sin(t-x) + \sin x \cdot \cos t - \cos x \sin t \right]$$

$$= e^{-t}\left[H(t-x)\sin(t-x) + \sin(x-t) \right]$$

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