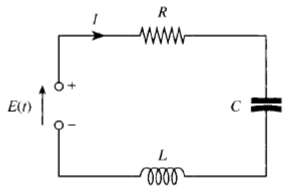
Applications of Differential Equations - Second Order Equations

Series LCR Circuit

Consider a simple electrical circuit shown in the Figure, which consists of a resistor R in ohms; a capacitor C in farads; an inductor L in henries; and an electromotive force (emf) $E(t)$ in volts, usually a battery or a generator, all connected in series. The current I flowing through the circuit is measured in amperes and the charge q on the capacitor is measured in coulombs.



LCR Circuit

By Kirchhoff's law, we

$$RI + L\frac{dI}{dt} + \frac{1}{C}q = E(t). \tag{1}$$

The relationship between q and I is

$$I = \frac{dq}{dt} \qquad \frac{dI}{dt} = \frac{d^2q}{dt^2}. \tag{2}$$

Substituting these values into the above differential equation, we obtain

$$\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{1}{LC}q = \frac{1}{L}E(t). \tag{3}$$

The initial conditions for q are

$$q(0) = q_0, \quad \left.\frac{dq}{dt}\right|_{t=0} = I(0) = I_0. \tag{4}$$

To obtain the differential equation for the current we differentiate the Eq. (1) with respect to time t and then substitute the Eq. (2) directly into the resulting equation to obtain

$$\frac{d^2I}{dt^2} + \frac{R}{L}\frac{dI}{dt} + \frac{LC}{I} = \frac{1}{L}\frac{dE(t)}{dt}. \tag{5}$$

The first initial condition $I(0) = I_0$. The second initial condition is obtained from Eq (1) by solving for $\frac{dI}{dt}$ and then setting $t = 0$. Thus,

$$\left.\frac{dI}{dt}\right|_{t=0} = \frac{1}{L}E(0) - \frac{R}{L}I(0) - \frac{1}{LC}q(0) \tag{6}$$

Problem. An LCR circuit connected in series has $R = 180$ ohms, $C = \frac{1}{280}$ farad, $L = 20$ henries and an applied voltage $E(t) = 10\sin t$. Assuming no initial charge on the capacitor, but an initial current of 1 ampere at $t = 0$ when the voltage is first applied, find the subsequent charge on the capacitor.

Solution. We have $R = 180$ ohms, $C = \frac{1}{280}$ farad, $L = 20$ henries, $E(t) = 10\sin t$ volts and $q_0 = 0$, $I_0 = 1$. So, we have

$$\frac{d^2q}{dt^2} + 9\frac{dq}{dt} + 14q = \frac{1}{2}\sin t.$$

The auxiliary equation is $m^2 + 9m + 14 = 0$ and whose roots are $m = -2, m = -7$. Thus, the homogeneous solution is:

$$q_c = c_1e^{-2t} + c_2e^{-7t}.$$

To find the particular solution, by method of un-determined coefficients we assume a solution of the form,

$$q_p = a\cos t + b\sin t.$$

Substituting this in the DE and comparing the coefficients, we obtain $a = -\frac{9}{500}$ and $b = \frac{13}{500}$. So,

$$q = c_1e^{-2t} + c_2e^{-7t} + \frac{13}{500}\sin t - \frac{9}{500}\cos t.$$

Applying the initial conditions $q(0) = 0$, $I(0) = \left.\frac{dq}{dt}\right|_{t=0} = 1$, we obtain $c_1 = \frac{110}{500}, c_2 = \frac{-101}{500}$. So, the charge on the capacitor at any time t is given by

$$q = \frac{1}{500}\left(110e^{-2t} - 100e^{-7t} + 13\sin t - 9\cos t\right).$$

Problem. An LCR circuit connected in series has $R = 10$ ohms, $C = 10^{-2}$ farad, $L = \frac{1}{2}$ henry, and an applied voltage $E = 12$ volts. Assuming no initial current and no initial charge at $t = 0$ when the voltage is first applied, find the subsequent current in the system.

Solution. The equation for the current is the LCR circuit is given by

$$\frac{d^2I}{dt^2} + \frac{R}{L}\frac{dI}{dt} + \frac{1}{LC}I = \frac{1}{L}\frac{dE(t)}{dt}$$

where $R = 10$ ohms, $C = 10^{-2}$ farad, $L = \frac{1}{2}$ henry, $E(t) = 12$ volts and $I(0) = 0$, $q(0) = 0$. Thus, we have

$$\frac{d^2I}{dt^2} + 20\frac{dI}{dt} + 200I = 0. \quad (\because E(t) = 12)$$

The auxiliary equation is $m^2 + 20m + 200 = 0$, whose roots are $m = -10 \pm 10i$. So, we have

$$I = e^{-10t}\left(c_1\cos 10t + c_2\sin 10t\right).$$

Initial conditions are $I(0) = 0$ and $\left.\frac{dI}{dt}\right|_{t=0} = \frac{1}{L}E(0) - \frac{R}{L}I(0) - \frac{LC}{q}(0) = 24$. Applying these initial conditions, we obtain $c_1 = 0$ and $c_2 = \frac{12}{5}$. So, we have

$$I(t) = \frac{12}{5}e^{-10t}\sin 10t.$$

Exercise Problems.

1. Solve the above problem by first finding the expression for charge on the capacitor at any time t and then solving for current.
2. An LCR circuit connected in series has a resistance of 5 ohms, and inductance of 0.05 henry, a capacitor of 4×10^{-4} farad, and an applied alternating emf of $200\cos 100t$ volts. Find an expression for the current flowing through this circuit if the initial current and the initial charge on the capacitor are both zero.