UNIT STEP FUNCTION :-

(or (Heaviside's function)

The unit step function is denoted by $u(t-a)$ (or, $H(t-a)$) and defined as $u(t-a)=\begin{cases} 1 & \text{if } t>a \\ 0 & \text{if } t<a \end{cases}$

where $a>0$.

This is called unit step function at $t=a$

1. Express the following functions interms of Unit step functions and hence find their Laplace transforms.

(i) $$f(t) = \begin{cases} e^t & \text{if } 0<t<1 \\ 1 & \text{if } t>1 \end{cases}$$

(ii) $$f(t) = \begin{cases} \cos t & \text{if } 0<t<\pi/2 \\ 0 & \text{if } \frac{\pi}{2}<t<\pi \\ 1 & \text{if } t>\pi \end{cases}$$

Sol. (i) We can express the given function as interms of unit step function

$$f(t) = e^t\left(u(t-0)-u(t-1)\right)+1\left(u(t-1)\right)$$

Now,

$$L\left\{f(t)\right\} = \int_0^\infty e^{-st}\left\{e^t\left(u(t-0)-u(t-1)\right)+1\left(u(t-1)\right)\right\}dt$$

$$= \int_0^1 e^{-st}\, e^t \, dt + \int_1^\infty e^{-st} \cdot 1 \, dt$$

$$= \int_0^1 e^{-(s-1)t} \, dt + \left(\frac{e^{-st}}{-s}\right)_1^\infty$$

$$= \left(\frac{e^{-(s-1)t}}{-(s-1)}\right)_0^1 + \left(\frac{e^{-st}}{-s}\right)_1^\infty \qquad (s>1)$$

$$= \frac{-1}{(s-1)}\left(e^{-(s-1)} - e^0\right) - \frac{1}{s}\left(e^{-\infty} - e^{-s}\right)$$

$$= \frac{-1}{(s-1)}\left(e^{-(s-1)} - 1\right) + \frac{e^{-s}}{s}$$

Second Shifting theorem :

If $L\left\{f(t)\right\} = F(s)$ and $g(t) = \begin{cases} f(t-a) & \text{if } t>a \\ 0 & \text{if } t<a \end{cases}$ $(a>0)$

then $L\left\{g(t)\right\} = e^{-as} . F(s)$

Note: we can write $g(t)$ as $g(t) = f(t-a)u(t-a) + 0.u(t-a)$

(or) $\boxed{g(t) = f(t-a)u(t-a)}$

Examples

① Evaluate $L\{g(t)\}$, where

$$g(t) = \begin{cases} e^{-(t-2)} & \text{if } t > 2 \\ 0 & \text{if } t < 2 \end{cases}$$

Sol. Let $f(t) = e^{-t}$

Then $L\{f(t)\} = L\{e^{-t}\} = \frac{1}{s+1} = F(s)$

Here $a = 2$

By the second shifting theorem, we have

$$L\{g(t)\} = e^{-as} F(s)$$

Therefore, $L\{g(t)\} = e^{-2s} \frac{1}{s+1}$

② Evaluate $L\{(t-1)^2 u(t-1)\}$

Sol. Let $f(t) = t^2$

Then $L\{f(t)\} = L\{t^2\} = \frac{2}{s^3} = F(s)$

Here $a = 1$

By the second shifting theorem, we have

$$L\left[f(t-a)\, u(t-a)\right] = e^{-as} F(s)$$

Therefore, $L\{(t-1)^2 u(t-1)\} = e^{-s} \cdot \frac{2}{s^3}$

(3) Evaluate $L\left\{ e^{2t}\, u(t-2) \right\}$

Sol. Let $f(t) = e^{2t}$

Then, $L\{f(t)\} = L\{e^{2t}\} = \frac{1}{s-2} = F(s)$

Here $a = 2$

By the second shifting theorem, we have

$L\left\{ f(t-2)u(t-2) \right\} = e^{-2s} \cdot \frac{1}{s-2}$

Therefore, $L\left\{ f(t-2)\, u(t-2) \right\} = L\left\{ e^{2(t-2)} u(t-2) \right\}$

$= e^{-4} L\left\{ e^{2t}\, u(t-2) \right\}$

Hence, $L\left\{ e^{2t} u(t-2) \right\} = e^{+4}\ L\left\{ f(t-2)\, u(t-2) \right\}$

$= e^{+4} \cdot e^{-2s} \frac{1}{s-2}$

or, $L\left\{ e^{2t} \cdot u(t-2) \right\} = e^{4}\, L\left\{ e^{2(t-2)} \cdot u(t-2) \right\}$

$= e^{4}\, L\left\{ f(t-2)\, u(t-2) \right\}$

$= e^{4}\, e^{-2s} \cdot \frac{1}{s-2}$

Exercise: Evaluate the following

(i) $L\left\{ e^{-t}\, u(t-1) \right\}$ (ii) $L\left\{ g(t) \right\}$, where

$$g(t) = \begin{cases} \cos(t-\pi/3) & ;\ t > \pi/3 \\ 0 & ;\ t < \pi/3. \end{cases}$$

Unit Impulse function (or, Dirac delta function)

The Unit impulse function is Considered as the Limiting form of the following function

$$f_\epsilon(t-a) = \begin{cases} \frac{1}{\epsilon} & \text{if } a \le t \le a+\epsilon \\ 0 & \text{otherwise} \end{cases}$$

Now, $\delta(t-a) = \underset{\epsilon \to 0}{\mathrm{lt}}\ f_\epsilon(t-a)$

$$L\{f_\epsilon(t-a)\} = \int_0^\infty e^{-st} f_\epsilon(t-a)\, dt$$

$$= \int_a^{a+\epsilon} e^{-st}\ \frac{1}{\epsilon}\, dt$$

$$= \frac{1}{\epsilon}\left(\frac{e^{-st}}{-s}\right)_a^{a+\epsilon}$$

$$= \frac{-1}{s\epsilon}\left[e^{-s(a+\epsilon)} - e^{-as}\right]$$

$$= \frac{e^{-as}(1-e^{-s\epsilon})}{s\epsilon}$$

So $L\{\delta(t-a)\} = L\left\{\underset{\epsilon \to 0}{\mathrm{lt}}\ f_\epsilon(t-a)\right\}$

$$= \underset{\epsilon \to 0}{\mathrm{lt}}\ L\{f_\epsilon(t-a)\}$$

$$= \underset{\epsilon \to 0}{\mathrm{lt}}\ \frac{e^{-as}(1-e^{-s\epsilon})}{s\epsilon}$$

$$= e^{-as} \lim_{\epsilon \to 0} \frac{s e^{-s\epsilon}}{s} \quad \text{(L'Hospital's rule)}$$

$$= e^{-as} \cdot 1.$$

Therefore $L\{\delta(t-a)\} = e^{-as}$.

We define $\delta(t-a) = \begin{cases} \infty & \text{if } t = a \\ 0 & \text{if } t \neq a \end{cases}$

Such that $\int_0^\infty \delta(t-a)\,dt = 1 \quad (a > 0)$.

Note: we have

$$\int_0^\infty e^{-st}\,\delta(t-a)\,dt = L\{\delta(t-a)\}$$

$$= e^{-as}$$

Taking $s = 0$, we get

$$\boxed{\int_0^\infty \delta(t-a)\,dt = 1}$$

Evaluation of Integrals :

① Evaluate $\int_0^\infty e^{-t}\left(\frac{1-\cos t}{t}\right) dt$

<u>Sol</u>. we have $\int_0^\infty e^{-st} f(t)\, dt = L\{f(t)\}$

so, $\int_0^\infty e^{-st}\left(\frac{1-\cos t}{t}\right) dt = L\left\{\frac{1-\cos t}{t}\right\}$ — *

we know that $L\left\{\frac{1-\cos t}{t}\right\} = \log\left(\frac{\sqrt{s^2+1}}{s}\right)$

Therefore, $\int_0^\infty e^{-st}\left(\frac{1-\cos t}{t}\right) dt = \log\left(\frac{\sqrt{s^2+1}}{s}\right)$

Taking $s=1$, we get

$$\int_0^\infty e^{-t}\left(\frac{1-\cos t}{t}\right) dt = \log\sqrt{2}.$$

② Evaluate $\int_0^\infty \left(\frac{e^{-t}-e^{-2t}}{t}\right) dt$

<u>Sol</u>. we have $\int_0^\infty e^{-st} f(t)\, dt = L\{f(t)\}$

so $\int_0^\infty e^{-st}\left(\frac{e^{-t}-e^{-2t}}{t}\right) dt = L\left\{\frac{e^{-t}-e^{-2t}}{t}\right\}$ — *

we know that $L\left\{\frac{e^{-t}-e^{-2t}}{t}\right\} = \log\left(\frac{s+2}{s+1}\right)$

Therefore, $\int_0^\infty e^{-st}\left(\frac{e^{-t}-e^{-2t}}{t}\right) dt = \log\left(\frac{s+2}{s+1}\right)$

Taking $s=0$, we get $\int_0^\infty \left(\frac{e^{-t}-e^{-2t}}{t}\right) dt = \log 2$.

③ Evaluate $\int_0^{\infty} e^{-2t}\, t\sin t\, dt$

Sol. We have $\int_0^{\infty} e^{-st} f(t)\, dt = L\{f(t)\}$

So, $\int_0^{\infty} e^{-st}\, t\sin t\, dt = L\{t\sin t\}$ ——— $*$

We know that $L\{t\sin t\} = \frac{2s}{(s^2+1)^2}$

Therefore, from ($*$), we have

$$\int_0^{\infty} e^{-st}\, t\sin t\, dt = \frac{2s}{(s^2+1)^2}$$

Taking $s = 2$, we get

$$\int_0^{\infty} e^{-2t}\, t\sin t\, dt = \frac{4}{25}.$$

Exercise : Evaluate the following

(i) $\int_0^{\infty} e^{t}\, t\cos t\, dt$ (ii) $\int_0^{\infty} e^{-2t}\, u(t-3)\, dt$

(iii) $\int_0^{\infty} e^{-t}\, t\sin 2t\cos t\, dt$ (iv) $\int_0^{\infty} e^{-3t}\, t^2\, \delta(t-1)\, dt.$

(v) $\int_0^{\infty} e^{-t}\left(\frac{e^{-t}-\cos t}{t}\right) dt$