Lecture 23: Eigenvalues and Eigenvectors

MAT2040 Linear Algebra

Recall: We want to find decomposition $A = PDP^{-1}$ where D is diagonal.

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We want to know what vectors are just **scaled** by a linear transformation $A\mathbf{x}$ (and the scaling factor).

Definition 22.1

Let A be an $n \times n$ (square!) matrix.
A scalar λ is called an **eigenvalue** of A if there exists a nonzero vector $\mathbf{x}$ so that $A\mathbf{x} = \lambda\mathbf{x}$. The vector $\mathbf{x}$ is called an **eigenvector** corresponding to eigenvalue λ.

- Finding eigenvalues of A is equivalent to finding roots of the characteristic equation $\det(A - \lambda I) = 0$ for matrix A.

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- Finding all eigenvectors belonging to a particular eigenvalue $\bar{\lambda}$ is equivalent to finding $\text{Null}(A - \bar{\lambda} I)$
(except that $\mathbf{0}$ is never an eigenvector).

  $\text{Null}(A - \bar{\lambda} I)$ is known as the **eigenspace** belonging to eigenvalue $\bar{\lambda}$.

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We will call matrices with such decomposition **diagonalizable**.

Remarks

1. If A is diagonalizable, then the column vectors of the diagonalizing matrix P are eigenvectors of A and the diagonal elements of D are the corresponding eigenvalues of A.
2. The diagonalizing matrix P is not unique. Reordering the columns of a given diagonalizing matrix P or multiplying them by nonzero scalars will produce a new diagonalizing matrix.

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Example 23.2

Recall the Fibonacci numbers: $f_0 = 0$, $f_1 = 1$, $f_k = f_{k-1} + f_{k-2}$ for $k = 2, 3, \ldots$.

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Write the recursive relation in matrix-vector form, and find an expression for f_k as $k \to \infty$.

Theorem 23.3

If A is $n \times n$ and A has n distinct eigenvalues, then A is diagonalizable.

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Remark

Statement in theorem is **not** an equivalence: if the eigenvalues are not distinct, then A may or may not be diagonalizable depending on whether A has n linearly independent eigenvectors.

Example 23.4

Is $\begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$ diagonalizable?

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Is I diagonalizable?

Example 23.5

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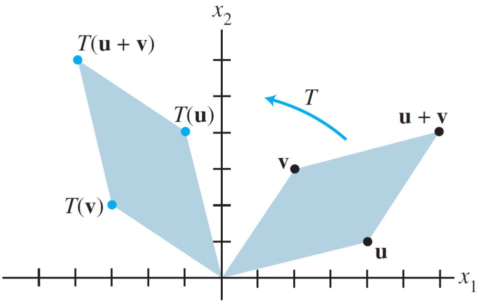

(picture from David Lay, Linear Algebra.)

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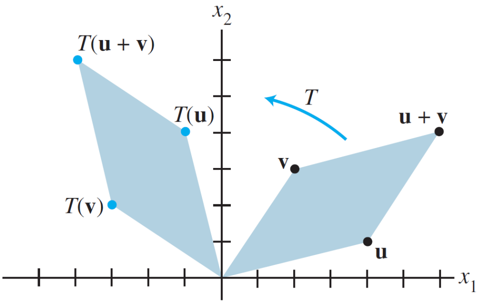

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What can the eigenvectors of A be? Find the eigenvalues of the matrix A.

Example 23.6

Let $A = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$.

Find the eigenvalues and eigenvectors of A.

A matrix that is not diagonalizable, is said to be **defective**.

Note that the characteristic equation of an $n \times n$ matrix A is a polynomial of degree n, and has therefore n roots, provided complex roots are allowed.

In other words, we can write

$$\det(A - \lambda I) = (\lambda_1 - \lambda)(\lambda_2 - \lambda)\cdots(\lambda_n - \lambda)$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A.

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For $\lambda = 0$: $\det A = \lambda_1\lambda_2\cdots\lambda_n$.

Lemma 23.7

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Corollary 23.8

A square matrix A is not invertible if and only if it has an eigenvalue of 0.

Grouping the eigenvalues by value, we can write

$$\det(A - \lambda I) = (\lambda_1 - \lambda)^{m_1}(\lambda_2 - \lambda)^{m_2} \cdots (\lambda_p - \lambda)^{m_p}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_p$ are the **distinct** eigenvalues of A and m_i is called the **algebraic** multiplicity of eigenvalue λ_i.

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Theorem 23.9

A matrix is diagonalizable if and only if the algebraic multiplicity of every eigenvalue is equal to the geometric multiplicity.

(No proof.)

Definition 23.10

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Lemma 23.13

Let A and B be $n \times n$ matrices. The trace of AB and the trace of BA are equal.

Example 23.14

Suppose the trace of A is 6, and $\det A = 8$. What are the eigenvalues of A?

Main points of this lecture

- $A = PDP^{-1}$ exists if A has n linearly independent eigenvectors (we say A is "diagonalizable")
- Eigenvalues may be complex, even if A has only real entries (Note that complex eigenvalues come in pairs if A has only real entries!)
- Sum of eigenvalues of A is equal to the trace of A
- Product of eigenvalues of A is equal to the determinant.

Theorem 23.15 (Spectral Theorem)

If A is a symmetric (real) matrix, then A is diagonalizable, all of A's eigenvalues are real, and the eigenvectors of A are mutually orthogonal.

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Proof of orthogonality of eigenvectors for special case where all eigenvalues of A are distinct.

The eigenvectors of a symmetric (real) matrix A are orthogonal, so we can scale these so that they form an orthonormal basis for $\mathbb{R}^n$, say $\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n$.

By the Spectral Theorem we now know that a symmetric (real) matrix A can be written as

$$A = Q\Lambda Q^T$$

where the columns of Q are the orthonormal basis for $\mathbb{R}^n$ and Λ is a diagonal matrix with the eigenvalues of A on the diagonal.

This decomposition is known as the **eigendecomposition** of A.