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1.(a). $[\vec{u_1}, \vec{u_2}, \vec{u_3}] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & 3 & 1 \\ 1 & 4 & -1 \\ 1 & 5 & 1 \end{bmatrix} \xrightarrow[\text{Elimination}]{\text{Gaussian}} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

The linear system has no non-trival solution $\Rightarrow \vec{u_1}, \vec{u_2}, \vec{u_3}$ are independent

(b). $[\vec{v_1}, \vec{v_2}, \vec{v_3}, \vec{v_4}, \cancel{\vec{v_5}}] = \begin{bmatrix} 1 & 1 & 1 & 0 \\ -1 & 1 & 2 & 0 \\ 1 & 2 & 1 & 3 \\ -1 & 3 & 1 & 8 \end{bmatrix} \xrightarrow[\text{Elimination}]{\text{Gaussian}} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

we can find a non-zero solution $\begin{bmatrix} 1 \\ -3 \\ 2 \\ 1 \end{bmatrix}$ s.t. $[\vec{v_1}, \vec{v_2}, \vec{v_3}, \vec{v_4}] \begin{bmatrix} 1 \\ -3 \\ 2 \\ 1 \end{bmatrix} = \vec{0}$

$\Rightarrow \vec{v_1}, \vec{v_2}, \vec{v_3}$ and $\vec{v_4}$ are linearly dependent and $\vec{v_1} - 3\vec{v_2} + 2\vec{v_3} + \vec{v_4} = \vec{0}$

2. $\begin{bmatrix} 1 & -2 & 3 & -4 & 5 \\ 0 & 3 & -6 & 3 & 9 \\ -2 & -3 & 1 & 1 & -4 \\ 1 & 4 & -3 & 2 & 7 \end{bmatrix} \xrightarrow[\text{Elimination}]{\text{Gaussian}} \begin{bmatrix} 1 & -2 & 3 & -4 & 5 \\ 0 & 1 & -2 & 1 & 3 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Hence, the maximal linearly independent subset is $\left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ -3 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ -6 \\ 1 \\ -3 \end{bmatrix} \right\}$

3. (a). ~~se~~ set of all 2×2 diagonal matrices is a subspace of $\mathbb{R}^{2\times2}$ since that two arbitrary 2×2 diagonal matrices ~~pto~~ add each other, the product is a 2×2 diagonal matrix, and the scalar multiplication on a arbitrary 2×2 diagonal matrix, the product is still a 2×2 diagonal matrix;

(b). set of all 2×2 triangular matrices is not a subspace of $\mathbb{R}^{2\times2}$ since a lower triangular matrix add a upper triangular matrix is definitely not a triangular matrix

(c). set of all 2×2 lower triangular matrices is a subspace of $\mathbb{R}^{2\times2}$, since an arbitrary lower triangular matrix add another arbitrary lower triangular matrix is a lower triangular matrix; and scaler multiplication on an arbitrary lower triangular matrix is still a lower triangular matrix.

(d). set of all matrices s.t. $a_{12}=1$, is not a subspace of $\mathbb{R}^{2\times2}$, since $a_{12} + a_{12}' = 2 \neq 1$; the add production of 2 arbitrary matrices in set is NOT in set.

(e). set of all matrices s.t. $b_{12}=0$ is a subspace of $\mathbb{R}^{2\times2}$, since add products & scalar multiplication products on arbitrary matrices in set is still matrices with $b_{12}=0$.

(f). The set of all symmetric 2×2 matrices is a subspace of $\mathbb{R}^{2\times2}$

Assume that $A=\begin{bmatrix} x & a_1 \\ a_1 & y \end{bmatrix}$, $B=\begin{bmatrix} w & b_1 \\ b_1 & z \end{bmatrix}$

$\Rightarrow A+B=\begin{bmatrix} x+w & a_1+b_1 \\ a_1+b_1 & y+z \end{bmatrix} \in$ set of all 2×2 symmetric matrices

$\Rightarrow \alpha A=\begin{bmatrix} \alpha x & \alpha a_1 \\ \alpha a_1 & \alpha y \end{bmatrix} \in$ set of all 2×2 symmetric matrices

(g). The set of all sigular 2×2 matrices is not a subspace of $\mathbb{R}^{2\times2}$

Pick 2 singular 2×2 matrices $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}+\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}=\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \notin$ singular matrices.

4. (a). $\left[\begin{array}{ccc|c} 3 & 4 & 4 & 2 \\ -3 & -2 & 0 & 2 \\ 6 & 2 & -4 & 8 \end{array}\right] \xrightarrow{\text{Gaussian Elimination}} \left[\begin{array}{ccc|c} 3 & 4 & 4 & 2 \\ 0 & 2 & 4 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \vec{x}=\begin{bmatrix} \frac{4}{3}v-6 \\ 2-2v \\ v \end{bmatrix}=\begin{bmatrix} -6+\frac{4}{3}v \\ 2-2v \\ v \end{bmatrix}$

$=\begin{bmatrix} -6 \\ 2 \\ 0 \end{bmatrix}+v\begin{bmatrix} \frac{4}{3} \\ -2 \\ 1 \end{bmatrix}$

(b). $\begin{bmatrix} \frac{4}{3}x_3-\frac{5}{3} \\ 2-2x_3 \\ x_3 \end{bmatrix}=\begin{bmatrix} -\frac{3}{5} \\ 2 \\ 0 \end{bmatrix}+v\begin{bmatrix} \frac{4}{3} \\ -2 \\ 1 \end{bmatrix}$, the coefficient matrix not change, and hence the only thing need to change is the start point

5. (a). $\left[\vec{b}_1, \vec{b}_2, \vec{b}_3\right]=\begin{bmatrix} 1 & -3 & 3 \\ 0 & 4 & -6 \\ 0 & 0 & 3 \end{bmatrix} \Longrightarrow$ we can find non-trival solution of this Linear System, and hence the vectors are linear independent

Further more, we can find $[\vec{b}_1, \vec{b}_2, \vec{b}_3]^{-1}$, which means any $\vec{b}$ in $\mathbb{R}^{3\times3}$, we can always find $x=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ to guarantee $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\vec{b}\,[\vec{b}_1, \vec{b}_2, \vec{b}_3]^{-1}$

$\Rightarrow \mathrm{span}(\vec{b}_1, \vec{b}_2, \vec{b}_3)=\mathbb{R}^3$

(b) Suppose that $\begin{bmatrix} -8 \\ 2 \\ 3 \end{bmatrix}=\lambda_1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}+\lambda_2\begin{bmatrix} -3 \\ 4 \\ 0 \end{bmatrix}+\lambda_3\begin{bmatrix} 3 \\ -6 \\ 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -3 & 3 \\ 0 & 4 & -6 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix}=\begin{bmatrix} -8 \\ 2 \\ 3 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix}=\begin{bmatrix} -5 \\ 2 \\ 1 \end{bmatrix}$

(c). $y = \begin{bmatrix} b_1, b_2, b_3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ -10 \\ 9 \end{bmatrix}$

6. (a). For span $\left\{ \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} -3 \\ 3 \\ 6 \end{bmatrix} \right\}$: $\begin{bmatrix} 1 & 2 & -3 \\ -2 & -2 & 3 \\ 2 & 4 & 6 \end{bmatrix} \xrightarrow{\text{Gaussian Elimination}} \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & -\frac{3}{2} \\ 0 & 0 & 1 \end{bmatrix}$

Since the vectors in span ~~is~~ are linearly independent and span$\{v_1, v_2, v_3\} = \mathbb{R}^3$

The dimension of span $\left\{ \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} -3 \\ 3 \\ 6 \end{bmatrix} \right\}$ is 3

(b). span$\{(x-2)(x+2), x^2(x^4-2), x^6-8\} = $ span$\{x^2-4, x^6-2x^2, x^6-8\}$.

We can easily find that $x^6-8 = 2(x^2-4) + (x^6-2x^2)$.

The dimension of span$\{(x-2)(x+2), x^2(x^4-2), x^6-8\}$ is 2.

7. (a). n vectors in set are linearly independent $\Leftrightarrow$ n vectors span $\mathbb{R}^n$ (Theorem 10.13)

$\left\{ \begin{array}{l} \text{vectors are linearly independent} \\ \text{vectors span } \mathbb{R}^n \end{array} \right.$ $\Longrightarrow$ set of n vectors linearly independent form a basis of $\mathbb{R}^n$ (Theorem 12.7)

(b). From (Theorem 10.13.), we can conclude that linearly independence of n ~~a~~ vectors set is equivalent to n vectors span $\mathbb{R}^n$, and therefor we can redirect the statement to 7(a).

8. Proof (1). Linear Independence

$\begin{bmatrix} v_1, v_2, v_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{bmatrix} \xrightarrow{\text{Gaussian Elimination}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow$ we can find a non-trival solution of this linear system

(2) $\{v_1, v_2, v_3\}$ span V

$V \Leftrightarrow \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ -x_1-x_2-x_3 \end{bmatrix} \middle| \; x_1, x_2, x_3 \in \mathbb{R} \right\}$

For any vector v in V, $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ -x_1-x_2-x_3 \end{bmatrix}$ we can always find a vector $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ -x_1-x_2-x_3 \end{bmatrix}$ s.t.

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ -x_1-x_2-x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ -x_1-x_2-x_3 \end{bmatrix} \Rightarrow \{v_1, v_2, v_3\}$ span V

From Definition of ~~basis~~ basis we can conclude that S is a basis of V, $\dim(V)=3$

9- Proof: Suppose that space U has basis $\{u_1, u_2, \cdots u_i\}$, space V has basis $\{v_1, v_2 \cdots v_j\}$

~~we can~~ Note that $\forall \tilde{u} \in U$ can be represented by linear combination of $u_1, u_2 \cdots u_i$

$\forall \tilde{v} \in V$ can be represented by linear combination of $v_1, v_2 \cdots v_j$

$\because U \cap V = \{0\}$

$\Rightarrow \forall \tilde{u} \in U$, ~~$\exists$~~ $\{v_1, v_2 \cdots v_j\}$ cannot find a linear combination to present $\tilde{u}$

$\forall \tilde{v} \in V$, $\{u_1, u_2, \cdots u_j\}$ cannot find a linear combination to present $\tilde{v}$

$\Rightarrow \{v_1, v_2 \cdots v_j\}$ and $\{u_1, u_2, \cdots u_j\}$ are linearly independent

Further more, ~~suppose~~ for an arbitrary matrix A in set $(U+V)$.

A can represented by $\lambda_1 u_1 + \lambda_2 u_2 + \cdots + \lambda_j u_j + \cdots + \lambda_{i+j} v_j$

which means $\{u_1, \cdots u_i, v_1, \cdots v_j\}$ span $\{U+V\}$

$\Rightarrow \{u_1, u_2, \cdots, u_i, v_1, v_2, \cdots v_j\}$ are basis of $U+V$.

$\Rightarrow \dim(U+V) = i+j = \dim(U) + \dim(V)$.

10. Proof: "$\Rightarrow$": Assume that basis $B = \{v_1, v_2, \cdots v_n\}$.

if $\underset{\sim}{w}$ is a linear combination of $\underset{\sim}{u_1}, \underset{\sim}{u_2}, \cdots, \underset{\sim}{u_p}$, then

$$\underset{\sim}{w} = [\underset{\sim}{u_1}, \underset{\sim}{u_2}, \cdots, \underset{\sim}{u_p}]\begin{bmatrix}\lambda_1\\ \lambda_2\\ \vdots\\ \lambda_p\end{bmatrix} \Rightarrow B[\underset{\sim}{w}]_B = [B[\underset{\sim}{u_1}]_B, B[\underset{\sim}{u_2}]_B, \cdots, B[u_p]_B]\begin{bmatrix}\lambda_1\\ \lambda_2\\ \vdots\\ \lambda_p\end{bmatrix}$$

$$\Rightarrow B[\underset{\sim}{w}]_B = B[[\underset{\sim}{u_1}]_B, [\underset{\sim}{u_2}]_B, \cdots [\underset{\sim}{u_p}]_B]\begin{bmatrix}\lambda_1\\ \vdots\\ \lambda_p\end{bmatrix}$$

$$\Rightarrow [\underset{\sim}{w}]_B = [[\underset{\sim}{u_1}]_B, [\underset{\sim}{u_2}]_B, \cdots, [\underset{\sim}{u_p}]_B]\begin{bmatrix}\lambda_1\\ \vdots\\ \lambda_p\end{bmatrix}$$

$\Rightarrow [\underset{\sim}{w}]_B$ is a linear combination of $[\underset{\sim}{u_1}]_B, \cdots [u_p]_B$.

"$\Leftarrow$". Each step above are invertible, which means ~~the~~ $[w_B]$ is a linear combination of $[\underset{\sim}{u_1}]_B, \cdots [\underset{\sim}{u_p}]_B$ implies $\underset{\sim}{w}$ is a linear combination of $\underset{\sim}{u_1}, \underset{\sim}{u_2}, \cdots, \underset{\sim}{u_p}$.

11. Assume that $\underset{\sim}{w} = \begin{bmatrix}w_1\\ w_2\\ w_3\\ \cancel{w_4}\end{bmatrix} = c_1 \underset{\sim}{v_1} + c_2 \underset{\sim}{v_2} = c_3 \underset{\sim}{v_3} + c_4 \underset{\sim}{v_4}$

$$\Rightarrow c_1\begin{bmatrix}1\\-1\\-3\end{bmatrix} + c_2\begin{bmatrix}8\\-9\\6\end{bmatrix} = c_3\begin{bmatrix}-3\\-1\\8\end{bmatrix} + c_4\begin{bmatrix}3\\-5\\4\end{bmatrix}$$

$$\Rightarrow c_1\begin{bmatrix}1\\-1\\-3\end{bmatrix} + c_2\begin{bmatrix}8\\-9\\6\end{bmatrix} + c_3\begin{bmatrix}3\\1\\-8\end{bmatrix} + c_4\begin{bmatrix}-3\\5\\-4\end{bmatrix} = \vec{0} \Rightarrow \begin{bmatrix}1 & 8 & 3 & 3\\ -1 & -9 & 1 & 5\\ -3 & 6 & -8 & -4\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\\c_4\end{bmatrix} = \vec{0}$$

$$\begin{bmatrix}1 & 8 & 3 & -3\\ -1 & -9 & 1 & 5\\ -3 & 6 & -8 & 4\end{bmatrix} \xrightarrow[\text{Elimination}]{\text{Gaussian}} \begin{bmatrix}1 & 8 & 3 & -3\\ 0 & 1 & 4 & -2\\ 0 & 0 & 1 & \frac{-47}{121}\end{bmatrix}$$

Assign $c_4 = 121$, then $c_3 = -47 \Rightarrow w = -47\begin{bmatrix}-3\\-1\\8\end{bmatrix} + 121\begin{bmatrix}3\\-5\\4\end{bmatrix} = \begin{bmatrix}504\\-558\\108\end{bmatrix}$

12.(a). $u_i \in \text{span}(S)\ \forall i = 1, \cdots, n \Rightarrow \exists a_{i1}, a_{i2}, \cdots a_{im}$ s.t. $u_i = a_{i1}v_1 + \cdots + a_{im}v_m$

$\Rightarrow \exists t \in \text{span}(T)$, $t = k_1 u_1 + k_2 u_2 + \cdots + k_n u_n$

$= k_1(a_{11}v_1 + \cdots + a_{1m}v_m) + \cdots + k_n(a_{n1}v_1 + \cdots a_{nm}v_m)$

$= (k_1 a_{11} + k_2 a_{21} \cdots + k_n a_{n1}) v_1 + \cdots + (k_1 a_{1m} + \cdots + k_n a_{nm}) v_m$

$\Rightarrow \text{span}(T) \in \text{span}(S)$

(b). Suppose a vector $t \in \text{span}(T)$, $\Rightarrow t = a_1 u_1 + \cdots + a_n u_n$

$\Rightarrow t = a_1(u_1+3u_2) + (a_2-3a_1)u_2 + \cdots + a_n u_n$, which is a linear combination of $\{u_1+3u_2, u_2, \cdots, u_n\}$

$\Rightarrow span(T) \subseteq span(S)$

Suppose there's a vector $s \in span(S) \Rightarrow s = \cancel{b_1 v_1 + \cdots + b_n v_n}$ $b_1(u_1+3u_2) + b_2 u_2 + \cdots + b_n u_n$

$\Rightarrow s = b_1 u_1 + (3b_1+b_2)u_2 + \cdots + b_n u_n$, which is a linear combination of $\{u_1, \cdots u_n\}$

$\Rightarrow span(S) \subseteq span(T)$

$\Rightarrow span(S) = span(T)$