CIE 6020 Assignment 1

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1. If the base of the logarithm is b, we denote the entropy as $H_b(X)$. Show that $H_b(X) = (\log_b a)H_a(X)$.

Proof:

$$\begin{aligned}
(\log_b a)H_a(X) &= (\log_b a)\sum_{x\in\mathcal{X}} p(x)\log_a p(x) \\
&= \sum_{x\in\mathcal{X}} p(x)(\log_b a)\log_a p(x) \\
&= \sum_{x\in\mathcal{X}} p(x)(\log_b a^{\log_a p(x)}) \\
&= \sum_{x\in\mathcal{X}} p(x)\log_b p(x) \\
&= H_b(X)
\end{aligned}$$

2. *Coin flips.* A fair coin is flipper until the first head occurs. Let X denote the number of flips required.

(a) Find the entropy $H(X)$ in bits. The following expressions may be useful:

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$$

$$\sum_{n=0}^{\infty} nr^n = \frac{r}{(1-r)^2}$$

(b) A random variable X is drawn according to this distribution. Find an "efficient" sequence of yes-no questions of the form, "Is X contained in the set S?" Compare $H(X)$ to the expected number of questions required to determine X.

Answer:

(a): The probability mass function of X: $p_X(n) = P(X = n) = (\frac{1}{2})^{n-1}\frac{1}{2} = (\frac{1}{2})^n$

$$\begin{aligned} H(X) &= -\sum_{i=1}^{\infty} (\frac{1}{2})^i \log(\frac{1}{2}^i) \\ &= -\sum_{i=1}^{\infty} (\frac{1}{2})^i i \log(\frac{1}{2}) \\ &= \sum_{i=1}^{\infty} i(\frac{1}{2})^i \\ &= 2 \end{aligned}$$

(b): Since the pmf of X is exponentially decreasing, one of the reasonable questions for nth question is "Is X = n?". Let Y denote the number of questions need to ask to determine the exact number of flips, then the probability mass function of Y can be given by

$$p_Y(n) = P(X = n | X \geq n) = (1 - \sum_{i=1}^{n-1} p(x))(\frac{1}{2})^n = (\frac{1}{2})^n$$

and therefore, the expectation of Y can by given by

$$E[Y] = \sum_{i=1}^{\infty} i p_Y(i)$$
$$= 2$$
$$= H(X)$$

From the equivalence of $E[Y]$ and $H(X)$ we can infer that this sequence of questions are optimal, since it can be proved that each nth question can get 1 bit information from the set of all possible solutions.

3. *Entropy of functions.* Let X be a random variable taking on a finite number of values. What is the (general) inequality relationship of $H(X)$ and $H(Y)$ if

(a) $Y = 2^X$?

(b) $Y = cos(X)$?

Answer:

(a) Suppose that x's alphabet $\mathcal{X} = (x_1, x_2, ..., x_m)$ and y's alphabet $\mathcal{Y} = (y_1, y_2, ..., y_n)$

For $Y = f(X) = 2^X$, $f : \mathcal{X} \mapsto \mathcal{Y}$ is a one-to-one mapping, and therefore by definition

$$\begin{aligned} H(X) &= -\sum_{x\in\mathcal{X}} p(x) \log p(x) \\ &= -\sum_{y} \sum_{x:f(x)=y} p(x) \log p(x) \\ &= -\sum_{y\in\mathcal{Y}} p(y) \log p(y) \\ &= H(Y) \end{aligned}$$

(b) Suppose that x's alphabet $\mathcal{X} = (x_1, x_2, ..., x_m)$ and y's alphabet $\mathcal{Y} = (y_1, y_2, ..., y_n)$

Intuitively, for $Y = f(X) = cos(X)$, $f : \mathcal{X} \mapsto \mathcal{Y}$ is surjective but not injective

$$
\begin{aligned}
H(X) &= -\sum_{x\in\mathcal{X}} p(x)\log p(x) \\
&= -\sum_{y}\sum_{x:f(x)=y} p(x)\log p(x) \\
&> -\sum_{y}\sum_{x:f(x)=y} p(x)\log p(y) \\
&= -\sum_{y} p(y)\log p(y) \\
&= H(Y)
\end{aligned}
$$

Therefore, $H(X) > H(Y)$ for $Y = cos(X)$

4. What is the minimum value of $H(p_1, ..., p_n) = H(\mathbf{p})$ as $\mathbf{p}$ ranges over the set of n-dimensional probability vectors? Find all $\mathbf{p}$'s that achieve this minimum

Answer: The entropy of $\mathbf{p}$ is given by

$$H(\mathbf{p}) = -\sum_{i=1}^{n} p_i \log p_i \geq 0$$

The equivalence holds that $H(\mathbf{p}) = 0$ iff $p_i = 0$ or $p_i = 1$ for $i = 1, ..., n$.

Hence, $\mathbf{p}$ that achieve this minimum are: {1,0,...,0}, {0,1,...,0},...,{0,0,...,1}.

5. Let X be a discrete random variable. Show that the entropy of a function of X is less than or equal to the entropy of X, i.e., $H(g(X)) \leq H(X)$.

Proof: From the chain rule we can obtain an equivalence that

$$H(X, g(X)) = H(X) + H(g(X)|X) = H(g(X)) + H(X|g(X))$$

Since that function $g(X)$ is determined by X, so intuitively $H(g(X)|X) = 0$

Claim: $H(g(X)|X) = 0$

$$\begin{aligned} H(g(X)|X) &= \sum_{x \in \mathcal{X}} [p(x) \sum p(g(x)|X = x) \log(p(g(x)|X = x))] \\ &= 0 \end{aligned}$$

Hence, $H(X) = H(g(X)) + H(X|g(X))$, and $H(X|g(X)) \geq 0$ with the equivalence holds iff X is a function of $g(X)$. Therefore, $H(X) \geq H(g(X))$

6. Let $p(x, y)$ be given by

Y — X	0	1
0	$\frac{1}{3}$	$\frac{1}{3}$
1	0	$\frac{1}{3}$

Find by definition: (a) $H(X)$, $H(Y)$. (b) $H(X|Y)$, $H(Y|X)$. (c) $H(X, Y)$. (d) $I(X; Y)$. Check that $H(X) + H(Y|X) = H(Y) + H(X|Y)$, and $H(X) - H(X|Y) = H(Y) - H(Y|X)$. Draw a Venn diagram (information diagram) for the quantities in parts (a) through (d).

Answer:

$H(X) = -\sum_{x \in \mathcal{X}} p(x) \log p(x) = -(\frac{1}{3} \log \frac{1}{3} + \frac{2}{3} \log \frac{2}{3}) = \log 3 - \frac{2}{3}$

$H(Y) = -\sum_{y \in \mathcal{Y}} p(y) \log p(y) = -(\frac{1}{3} \log \frac{1}{3} + \frac{2}{3} \log \frac{2}{3}) = \log 3 - \frac{2}{3}$

$H(X|Y) = p_Y(0) H(X|Y = 0) + p_Y(1) H(X|Y = 1) = \frac{2}{3}[-(\frac{1}{2} \log \frac{1}{2} + \frac{1}{2} \log \frac{1}{2})] + \frac{1}{3} * 0 = \frac{2}{3}$

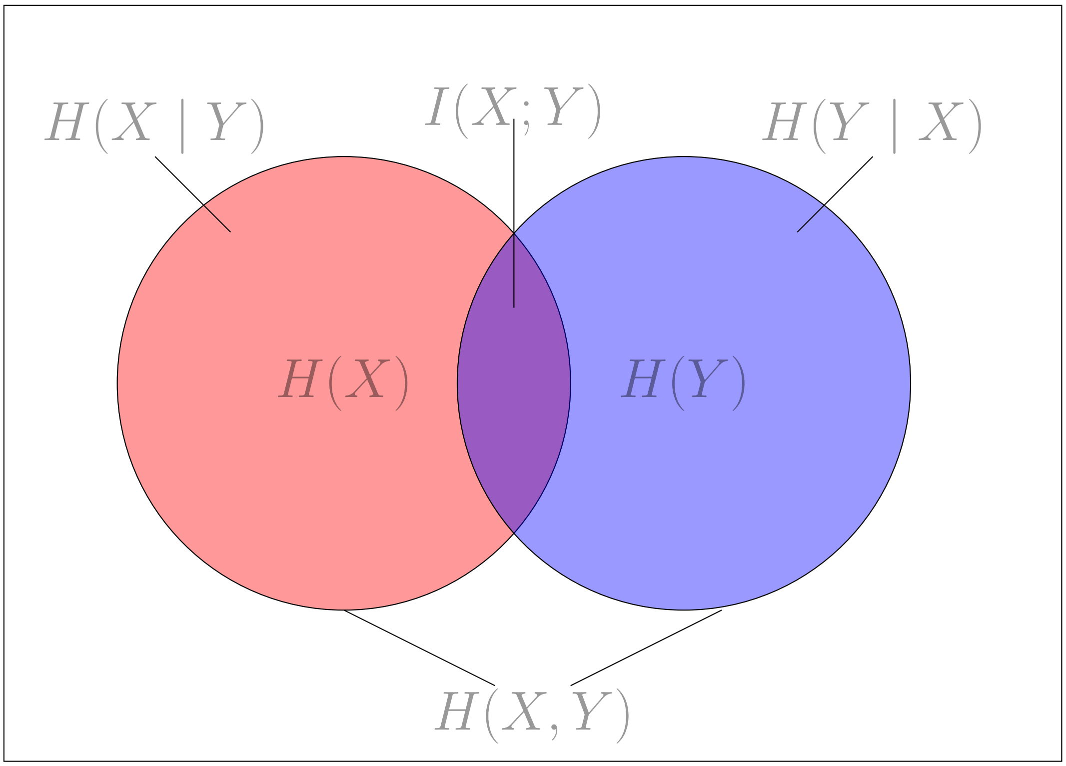
$H(Y|X) = p_X(0) H(Y|X = 0) + p_X(1) H(Y|X = 1) = \frac{1}{3} * 0 + \frac{2}{3}[-(\frac{1}{2} \log \frac{1}{2} + \frac{1}{2} \log \frac{1}{2})] = \frac{2}{3}$

$H(X, Y) = -\sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} p(x, y) \log p(x, y) = -\log \frac{1}{3}$

$I(X; Y) = -\sum_{x,y} p(x, y) \log \frac{p(x,y)}{p(x)p(y)} = \log 3 - \frac{4}{3}$

Check 1: $H(X) + H(Y|X) = \log 3 - \frac{2}{3} + \frac{2}{3} = \log 3 = H(Y) + H(X|Y)$

Check 2: $H(X) - H(X|Y) = \log 3 - \frac{2}{3} - \frac{2}{3} = \log 3 - \frac{4}{3} = H(Y) - H(Y|X)$



7. *Chain rule for conditional entropy.* Show that

$$H(X_1, X_2, ..., X_n \mid Y) = \sum_{i=1}^{n} H(X_i \mid X_1, ..., X_{i-1}, Y)$$

Proof: From the *Chain rule for entropy*, we have

$$H(X_1, X_2, ..., X_n) = \sum_{i=1}^{n} H(X_i \mid X_1, ..., X_{i-1})$$

then for conditional entropy

$$
\begin{aligned}
H(X_1, X_2, ..., X_n \mid Y) &= \sum_{y \in \mathcal{Y}} p(y) H(X_1, ..., X_n \mid Y = y) \\
&= \sum_{y \in \mathcal{Y}} p(y) \sum_{i=1}^{n} H(X_i \mid X_1, ..., X_{i-1}, Y = y) \\
&= \sum_{i=1}^{n} H(X_i \mid X_1, ..., X_{i-1}, Y)
\end{aligned}
$$

8. *Entropy of a sum.* Let X and Y be random variables that take on values $x_1, x_2, ..., x_r$ and $y_1, y_2, ..., y_s$, respectively. Let $Z = X + Y$.

(a) Show that $H(Z|X) = H(Y|X)$. Argue that if X, Y are independent, then $H(Y) \leq H(Z)$ and $H(X) \leq H(Z)$. Thus, the addition of *independent* random variable adds uncertainty.

(b) Give an example of (necessarily dependent) random variables in which $H(X) > H(Z)$ and $H(Y) > H(Z)$.

(c) Under what conditions does $H(Z) = H(X) + H(Y)$?

Proof:

(a)

$$
\begin{aligned}
H(Z|X) &= -\sum_{x \in \mathcal{X}} p(x) H(Z|X = x) \\
&= -\sum_{x \in \mathcal{X}} p(x) \sum_{y \in \mathcal{Y}} p_Y(z - x) \log(p_Y(z - x)) \\
&= -\sum_{x \in \mathcal{X}} p(x) \sum_{y \in \mathcal{Y}} p_Y(y) \log(p_Y(y)) \\
&= H(Y|X)
\end{aligned}
$$

If X and Y are independent, then

$$\begin{aligned}
H(Y|X) &= -\sum_{x\in\mathcal{X}} p(x) \sum_{y\in\mathcal{Y}} p(y)\log p(y) \\
&= H(Y) \\
H(Z|X) &= -\sum_{x\in\mathcal{X}} p(x) \sum_{z\in\mathcal{Z}} p(z|X=x)\log(p(z|X=x)) \\
&\leq -\sum_{x\in\mathcal{X}} p(x) \sum_{z\in\mathcal{Z}} p(z)\log(p(z)) \\
&= H(Z)
\end{aligned}$$

From the equivalence proved above, we have $H(Y) = H(Y|X) = H(Z|X) \leq H(Z)$

And also $H(X) \leq H(Z)$

(b) Flip a coin, if the result is head then let X = 1, if the result is tail then let X = 0. For the random variable Y, Y follows following rules: if X = 1 then let Y = 0, if X = 0 then let Y = 1.

Then, we can calculate that $H(X) = 1$, $H(Y) = 1$, $H(Z) = H(X+Y) = 0$, which meas conditions above give an example of random variables in which $H(X) > H(Z)$ and $H(Y) > H(Z)$.

(c) If the equality holds, then from *chain rule of entropy*, we can obtain that

$$\begin{aligned}
H(X,Y,Z) &= H(Z) + H(X,Y|Z) \\
\rightarrow H(X,Y,Z) &= H(X) + H(Y) + H(X,Y|Z)
\end{aligned}$$

from here intuitively we can obtain that X and Y are independent, and $H(Z) = (H(X) \cup H(Y))$.

claim: If (1):X and Y are independent, (2):$H(Z|X) = I(Z;Y)$ and (3):$H(Z|Y) = I(Z;X)$, then $H(Z) = H(X) + H(Y)$

From (1) it can be obtained that

$$H(X,Y) = H(X) + H(Y)$$

And from (2) and (3) it can be derived that

$$\begin{aligned}
&H(Z|X) = I(Z;Y) \\
&\to H(Z|X) = H(Z) + H(Y) - H(Z,Y) \\
&\to H(Z) = H(X,Z) + H(Y,Z) - H(X) - H(Y)
\end{aligned}$$

where

$$\begin{aligned}
H(X,Z) + H(Y,Z) &= 2H(Z) + H(X) + H(Y) - I(X;Z) - I(Y;Z) \\
&= 2[H(X) + H(Y)]
\end{aligned}$$

Hence, $H(Z) = H(X) + H(Y)$.