

Lecture 18: Change of Basis and Transformations

MAT2040 Linear Algebra

We will now combine coordinate transformations (change of basis) and linear transformations!

Theorem 17.12

Let V be a vector space with basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ and W be a vector space with basis $\mathcal{C} = \{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_m\}$.

If $T(\mathbf{x})$ from V to W is a linear transformation then there exists an $m \times n$ matrix A so that

$$[T(\mathbf{x})]_{\mathcal{C}} = A[\mathbf{x}]_{\mathcal{B}}.$$

In fact,

$$A = [[T(\mathbf{v}_1)]_{\mathcal{C}}, [T(\mathbf{v}_2)]_{\mathcal{C}}, \dots, [T(\mathbf{v}_n)]_{\mathcal{C}}].$$

A is called the **matrix for T relative to the bases \mathcal{B} and \mathcal{C}** .

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Theorem 18.1

Let V be a vector space with basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$. If $T(\mathbf{x})$ from V to V is a linear transformation then there exists an $n \times n$ matrix A so that

$$[T(\mathbf{x})]_{\mathcal{B}} = A[\mathbf{x}]_{\mathcal{B}}.$$

In fact,

$$A = \left[[T(\mathbf{v}_1)]_{\mathcal{B}}, [T(\mathbf{v}_2)]_{\mathcal{B}}, \dots, [T(\mathbf{v}_n)]_{\mathcal{B}} \right].$$

A is called the **matrix for T relative to \mathcal{B} (the \mathcal{B} -matrix for T)**.

This is useful, because often linear transformations are easier to understand (work with) in a certain basis.

Example 18.2

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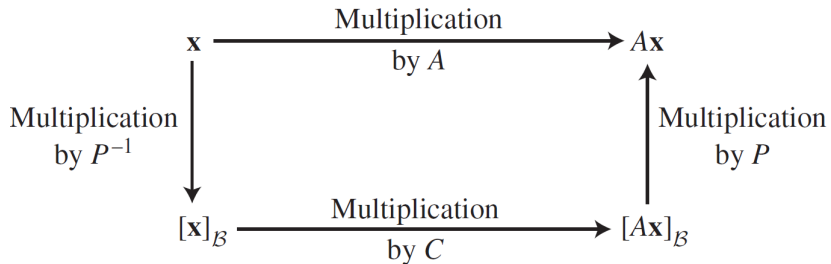
What is the \mathcal{B} -matrix for T for $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\}$?

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Consider $T(\mathbf{x}) = A\mathbf{x}$ where $A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$.

What is the \mathcal{B} -matrix for T for $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\}$?

Express A in terms of the \mathcal{B} -matrix for T .



(picture from David Lay, Linear Algebra.)

Definition 18.3

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The intuition is that A and B can be interpreted as matrices corresponding to the same linear transformation, just expressed in different bases.

Example 18.4 (Preview)

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How can we find such a basis?

Suppose $A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$.