Lecture 24: Symmetric Matrices

MAT2040 Linear Algebra

Theorem 24.1

If A is a symmetric (real) matrix, then any two eigenvectors of A from different eigenspaces are orthogonal.

Definition 24.2

A is **orthogonally diagonalizable** if $A = Q\Lambda Q^T$ for some matrix Q with orthonormal columns (so $Q^{-1} = Q^T$), and Λ a diagonal matrix.

Example 24.3

Suppose A is orthogonally diagonalizable. What can we say about about A^T?

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Converse is true as well!

Theorem 24.4

A is orthogonally diagonalizable if and only if A is a symmetric matrix.

(No proof.)

Theorem 24.5 (Spectral Theorem)

If A is a symmetric (real) matrix, then A is diagonalizable, all of A's eigenvalues are real, and the eigenvectors of A are mutually orthogonal.

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In other words, we can write A as

$$A = Q\Lambda Q^T$$

where Q is a matrix with orthonormal columns, and Λ is a diagonal matrix.

(No proof.)

Corollary 24.6 (Spectral Decomposition)

If A is a symmetric (real) $n \times n$ matrix, then

$$A = \lambda_1 \mathbf{q}_1 \mathbf{q}_1^T + \lambda_2 \mathbf{q}_2 \mathbf{q}_2^T + \cdots + \lambda_n \mathbf{q}_n \mathbf{q}_n^T$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the (real) eigenvalues of A, and $\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n$ the corresponding eigenvectors (that form an orthonormal set).

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Note: Each term is a rank one matrix. In fact, $\mathbf{q}_i \mathbf{q}_i^T$ is the projection matrix onto $\text{Span}\{\mathbf{q}_i\}$.

Example 24.7

Construct a spectral decomposition of A where A is a matrix with the following orthogonal decomposition

$$A = \begin{bmatrix} -6 & 12 \\ 12 & 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{bmatrix} \begin{bmatrix} 10 & 0 \\ 0 & -15 \end{bmatrix} \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{bmatrix}.$$

Spectral decomposition is for symmetric (square) matrices. Can we find a similar decomposition for general (rectangular) matrices?

Spectral decomposition is for symmetric (square) matrices. Can we find a similar decomposition for general (rectangular) matrices?

Yes! For $m \times n$ matrix A, the decomposition we will aim for is

$$A = U\Sigma V^T,$$

where U is an $m \times m$ matrix with orthonormal columns, Σ is an $m \times n$ matrix with only nonzero entries in position (i, i) for $i = 1, 2, \ldots, \min\{m, n\}$, and V is an $n \times n$ matrix with orthonormal columns.

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Note that A^TA is a symmetric $n \times n$ matrix, and AA^T is a symmetric $m \times m$ matrix.

We will use the eigendecompositions of A^TA and AA^T to find the matrices U, Σ and V.

By the spectral decomposition theorem we know we can write

$$A^T A = Q_1 \Lambda_1 Q_1^T$$

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Also: if $A = U\Sigma V^T$ then

$A^T A = (U\Sigma V^T)^T (U\Sigma V^T) = V\Sigma^T U^T U\Sigma V^T = V\Sigma^T \Sigma V^T.$
and $AA^T = (U\Sigma V^T)(U\Sigma V^T)^T = U\Sigma V^T V\Sigma^T U^T = U\Sigma\Sigma^T U^T.$

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So **if** we somehow can find a Σ so that $\Sigma^T \Sigma = \Lambda_1$ and $\Sigma\Sigma^T = \Lambda_2$, then we can set $V = Q_1$ and $U = Q_2$ and have the decomposition we want!

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Let's think about the eigenvalues of A^TA and AA^T:

Eigenvalues and eigenvectors of A^TA are pairs of λ and nonzero $\mathbf{x}$ so that $A^TA\mathbf{x} = \lambda\mathbf{x}$.

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Same reasoning holds for eigenvalues of AA^T.

All eigenvalues of $A^T A$ and AA^T are nonnegative.

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Let's think about the eigenvalues of $A^T A$ and AA^T:

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AA^T and $A^T A$ have the same positive eigenvalues.

Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > \lambda_{r-1} = 0$ be the positive eigenvalues of $A^T A$ and AA^T.

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$$\text{Define } \Sigma = \left[\begin{array}{cccc|c} \sqrt{\lambda_1} & & & & \\ & \sqrt{\lambda_2} & & & \\ & & \ddots & & 0 \\ & & & \sqrt{\lambda_p} & \\ \hline & & 0 & & 0 \end{array}\right]$$

so that Σ is an $m \times n$ matrix.

Then we have the decompositions

$$A^T A = Q_1 \Lambda_1 Q_1^T$$

and

$$AA^T = Q_2 \Lambda_2 Q_2^T.$$

for $\Lambda_1 = \Sigma^T \Sigma$ and $\Lambda_2 = \Sigma\Sigma^T$!

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If $Q = \begin{bmatrix} \mathbf{b}_1, & \mathbf{b}_2, & \ldots, & \mathbf{b}_n \end{bmatrix}$ is a matrix with orthonormal columns, then $\tilde{Q} = \begin{bmatrix} \pm\mathbf{b}_1, & \pm\mathbf{b}_2, & \ldots, & \pm\mathbf{b}_n \end{bmatrix}$ is also a matrix with orthonormal columns for any choice of pluses and minuses, and the columns are still eigenvectors corresponding to the diagonal elements in Λ in order!

This is easy to fix, though. Note that $A = U\Sigma V^T$ implies $AV = U\Sigma$.

So can choose $V = Q_1$ for **any** Q_1 so that $A^T A = Q_1 \Lambda_1 Q_1^T$, and then find U from

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We can find U column by column: $\sigma_i \mathbf{u}_i = A\mathbf{v}_i$ for $i = 1, 2, \ldots, r$.

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We can find U column by column: $\sigma_i\mathbf{u}_i = A\mathbf{v}_i$ for $i = 1, 2, \ldots, r$.

(The other columns of U can be chosen arbitrarily as long as U's columns form an orthonormal basis.)

Theorem 24.8 (Singular Value Decomposition)

Any $m \times n$ matrix A can be written as

$$A = U\Sigma V^T$$

where U is an $m \times m$ matrix with orthonormal columns, Σ is an $m \times n$ matrix with only nonzero entries in position (i, i) for $i = 1, 2, \ldots, \min\{m, n\}$, and V is an $n \times n$ matrix with orthonormal columns.

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The $\min\{m, n\}$ entries in Σ in position (i, i) (for $i = 1, 2, \ldots, n$) are called the **singular values** of A.

Finding SVD of A:

- Find eigendecomposition of $A^T A = Q_1 \Lambda_1 Q_1^T$.
  Reorder eigenvalues (diagonal elements in Λ) in nonincreasing order, and order the eigenvectors (columns of Q_1) in the same way.
- $V = Q_1$.
- $$\Sigma = \left[\begin{array}{cccc|c} \sqrt{\lambda_1} & & & & \\ & \sqrt{\lambda_2} & & & 0 \\ & & \ddots & & \\ & & & \sqrt{\lambda_p} & \\ \hline & & 0 & & 0 \end{array}\right]$$
- Construct U as follows:
  $\mathbf{u}_i = \frac{1}{\sqrt{\lambda_i}} A \mathbf{v}_i$ for $i = 1, 2, \ldots, r$, where λ_r is the smallest nonzero eigenvalue of $A^T A$.
  Finish the orthonormal basis arbitrarily.

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$A = U\Sigma V^T$ implies $AV = U\Sigma$, which means that the first r columns of U are in $\operatorname{Col} A$ (where r is the number of nonzero singular values).

$A = U\Sigma V^T$ also implies $U^T A = \Sigma V^T$, or $A^T U = V\Sigma$ which means that the first r columns of V are in $\operatorname{Row} A$.

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where $\mathbf{u}_i$ is the ith column of U, and r is the number of nonzero singular values.

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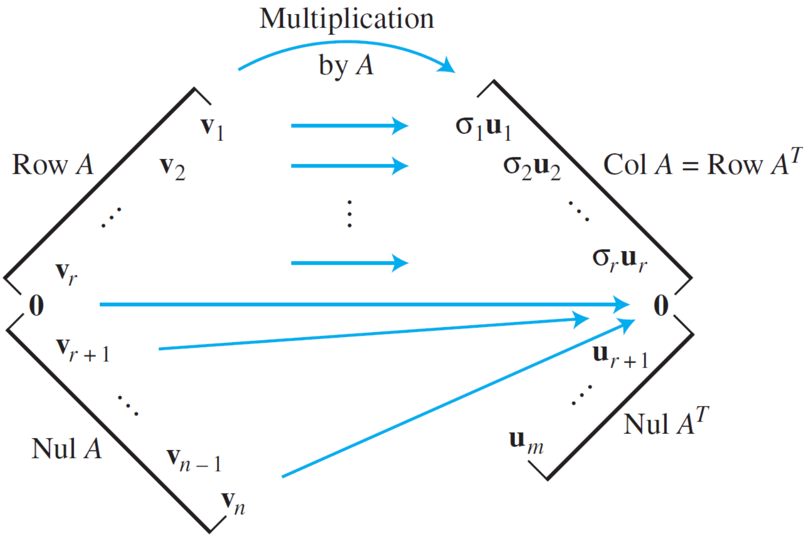
This decomposition is telling is **exactly** where every vector in Row A is mapped, and how it is scaled!!! (Using orthonormal bases!)



(picture from David Lay, Linear Algebra.)

Theorem 24.9 (Rank one decomposition)

Any $m \times n$ matrix A can be written as $A = U\Sigma V^T$, and thus as

$$A = \sigma_1 \mathbf{u}_1 \mathbf{v}_1^T + \sigma_2 \mathbf{u}_2 \mathbf{v}_2^T + \cdots + \sigma_r \mathbf{u}_r \mathbf{v}_r^T$$

where r is the number of positive singular values of A.
(Note that $r = \operatorname{rank} A$!)

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Note: Each term is a rank one matrix.