1. Augmented matrix is

(i)

$$\left[\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 0 & a_1 \\ 0 & 1 & -1 & 0 & 0 & a_2 \\ 0 & 0 & 1 & -1 & 0 & a_3 \\ 0 & 0 & 0 & 1 & -1 & a_4 \\ -1 & 0 & 0 & 0 & 1 & a_5 \end{array}\right] \xrightarrow{R_5 \to R_1 + R_5} \left[\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 0 & a_1 \\ 0 & 1 & -1 & 0 & 0 & a_2 \\ 0 & 0 & 1 & -1 & 0 & a_3 \\ 0 & 0 & 0 & 1 & -1 & a_4 \\ 0 & -1 & 0 & 0 & 1 & a_1+a_5 \end{array}\right] \xrightarrow{R_5 \to R_2 + R_5}$$

1

$$\left[\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 0 & a_1 \\ 0 & 1 & -1 & 0 & 0 & a_2 \\ 0 & 0 & 1 & -1 & 0 & a_3 \\ 0 & 0 & 0 & 1 & -1 & a_4 \\ 0 & 0 & -1 & 0 & 1 & a_2+a_1+a_5 \end{array}\right] \xrightarrow{R_5 \to R_3 + R_5} \left[\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 0 & a_1 \\ 0 & 1 & -1 & 0 & 0 & a_2 \\ 0 & 0 & 1 & -1 & 0 & a_3 \\ 0 & 0 & 0 & 1 & -1 & a_4 \\ 0 & 0 & 0 & -1 & 1 & a_1+a_2+a_3+a_5 \end{array}\right] \xrightarrow{R_5 \to R_4 + R_5}$$

$$\left[\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 0 & a_1 \\ 0 & 1 & -1 & 0 & 0 & a_2 \\ 0 & 0 & 1 & -1 & 0 & a_3 \\ 0 & 0 & 0 & 1 & -1 & a_4 \\ 0 & 0 & 0 & 0 & 0 & \sum_{i=1}^{5} a_i \end{array}\right]$$

4

The linear system is consistent if and only if the last column is not a pivot, i.e., $\sum_{i=1}^{5} a_i = a_1 + a_2 + a_3 + a_4 + \cdots = 0$ 1

(ii) when $a_1 + a_2 + a_3 + a_4 + a_5 = 0$

$$\left[\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 0 & a_1 \\ 0 & 1 & -1 & 0 & 0 & a_2 \\ 0 & 0 & 1 & -1 & 0 & a_3 \\ 0 & 0 & 0 & 1 & -1 & a_4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_3 \to R_4 + R_3} \left[\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 0 & a_1 \\ 0 & 1 & -1 & 0 & 0 & a_2 \\ 0 & 0 & 1 & 0 & -1 & a_3+a_4 \\ 0 & 0 & 0 & 1 & -1 & a_4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_2 \to R_3 + R_2}$$

$$\left[\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 0 & a_1 \\ 0 & 1 & 0 & 0 & -1 & a_2+a_3+a_4 \\ 0 & 0 & 1 & 0 & -1 & a_3+a_4 \\ 0 & 0 & 0 & 1 & -1 & a_4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 \to R_2 + R_1} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -1 & a_1+a_2+a_3+a_4 \\ 0 & 1 & 0 & 0 & -1 & a_2+a_3+a_4 \\ 0 & 0 & 1 & 0 & -1 & a_3+a_4 \\ 0 & 0 & 0 & 1 & -1 & a_4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

· The equivalent linear system is

$$\begin{cases} x_1 - x_5 = a_1 + a_2 + a_3 + a_4 \\ x_2 - x_5 = a_2 + a_3 + a_4 \\ x_3 - x_5 = a_3 + a_4 \\ x_4 - x_5 = a_4 \end{cases} \quad ② \quad \Rightarrow \quad \begin{cases} x_1 = a_1 + a_2 + a_3 + a_4 + x_5 \\ x_2 = a_2 + a_3 + a_4 + x_5 \\ x_3 = a_3 + a_4 + x_5 \\ x_4 = a_4 + x_5 \end{cases} \quad ①$$

The solution in vector form is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} a_1+a_2+a_3+a_4+x_5 \\ a_2+a_3+a_4+x_5 \\ a_3+a_4+x_5 \\ a_4+x_5 \\ x_5 \end{pmatrix} = \begin{pmatrix} a_1+a_2+a_3+a_4 \\ a_2+a_3+a_4 \\ a_3+a_4 \\ a_4 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad ①$$

$x_5 \in \mathbb{R}$.

~~Or Direct RREF Method~~

Remark: one can get the RREF directly in (i).

to obtain the RREF of the augmented matrix is

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -1 & a_1+a_2+a_3+a_4 \\ 0 & 1 & 0 & 0 & -1 & a_2+a_3+a_4 \\ 0 & 0 & 1 & 0 & -1 & a_3+a_4 \\ 0 & 0 & 0 & 1 & -1 & a_4 \\ 0 & 0 & 0 & 0 & 0 & \sum_{i=1}^{5} a_i \end{array}\right] \quad ⑤ \quad ,$$

Thus, the linear system is consistent $\Leftrightarrow \sum_{i=1}^{5} a_i = 0$ ①

And the solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} a_1+a_2+a_3+a_4+x_5 \\ a_2+a_3+a_4+x_5 \\ a_3+a_4+x_5 \\ a_4+x_5 \\ x_5 \end{pmatrix} ③ = \begin{pmatrix} a_1+a_2+a_3+a_4 \\ a_2+a_3+a_4 \\ a_3+a_4 \\ a_4 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad x_5 \in \mathbb{R} \quad ①$$

2.

$$AB+B=2A$$

$$(A+I)B=2A$$

$$A+I=\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 3 \\ 3 & 4 & 6 \end{bmatrix} \quad 2A=\begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 6 \\ 6 & 8 & 10 \end{bmatrix}$$ ①

Method 1:

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 2 & 2 & 6 \\ 3 & 4 & 6 & 6 & 8 & 10 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 2 & 2 & 6 \\ 2 & 1 & 1 & 2 & 2 & 2 \\ 3 & 4 & 6 & 6 & 8 & 10 \end{array}\right]$$ ①

$$\xrightarrow[R_3 \to -3R_1+R_3]{R_2 \to -2R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 2 & 2 & 6 \\ 0 & -3 & -5 & -2 & -2 & -10 \\ 0 & -2 & -3 & 0 & 2 & -8 \end{array}\right] \xrightarrow{R_2 \to -\frac{1}{3}R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 2 & 2 & 6 \\ 0 & 1 & \frac{5}{3} & \frac{2}{3} & \frac{2}{3} & \frac{10}{3} \\ 0 & -2 & -3 & 0 & 2 & -8 \end{array}\right]$$

$$\xrightarrow{R_3 \to 2R_2+R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 2 & 2 & 6 \\ 0 & 1 & \frac{5}{3} & \frac{2}{3} & \frac{2}{3} & \frac{10}{3} \\ 0 & 0 & \frac{1}{3} & \frac{4}{3} & \frac{10}{3} & -\frac{4}{3} \end{array}\right] \xrightarrow{R_3 \to 3R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 2 & 2 & 6 \\ 0 & 1 & \frac{5}{3} & \frac{2}{3} & \frac{2}{3} & \frac{10}{3} \\ 0 & 0 & 1 & 4 & 10 & -4 \end{array}\right]$$

$$\xrightarrow[R_1 \to -3R_3+R_1]{R_2 \to -\frac{5}{3}R_3+R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -10 & -28 & 18 \\ 0 & 1 & 0 & -6 & -16 & 10 \\ 0 & 0 & 1 & 4 & 10 & -4 \end{array}\right] \xrightarrow{R_1 \to -2R_2+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 4 & -2 \\ 0 & 1 & 0 & -6 & -16 & 10 \\ 0 & 0 & 1 & 4 & 10 & -4 \end{array}\right]$$ ⑤

Thus $$B=\begin{bmatrix} 2 & 4 & -2 \\ -6 & -16 & 10 \\ 4 & 10 & -4 \end{bmatrix}$$ ①

Method 2 $\quad A+I=\begin{bmatrix}2&1&1\\1&2&3\\3&4&6\end{bmatrix}$ 1 , $2A=\begin{bmatrix}2&2&2\\2&2&6\\6&8&10\end{bmatrix}$

First compute $(A+I)^{-1}$,

$$\left[\begin{array}{ccc|ccc}2&1&1&1&0&0\\1&2&3&0&1&0\\3&4&6&0&0&1\end{array}\right] \text{①} \xrightarrow{R_1\leftrightarrow R_2} \left[\begin{array}{ccc|ccc}1&2&3&0&1&0\\2&1&1&1&0&0\\3&4&6&0&0&1\end{array}\right]$$

$$\xrightarrow[R_3\to -3R_1+R_3]{R_2\to -2R_1+R_2} \left[\begin{array}{ccc|ccc}1&2&3&0&1&0\\0&-3&-5&1&-2&0\\0&-2&-3&0&-3&1\end{array}\right] \xrightarrow{R_2\to -\frac{1}{3}R_2} \left[\begin{array}{ccc|ccc}1&2&3&0&1&0\\0&1&\frac{5}{3}&-\frac{1}{3}&\frac{2}{3}&0\\0&-2&-3&0&-3&1\end{array}\right]$$

$$\xrightarrow[R_3\to 2R_2+R_3]{R_1\to -2R_2+R_1} \left[\begin{array}{ccc|ccc}1&0&-\frac{1}{3}&\frac{2}{3}&-\frac{1}{3}&0\\0&1&\frac{5}{3}&-\frac{1}{3}&\frac{2}{3}&0\\0&0&+\frac{1}{3}&-\frac{2}{3}&-\frac{5}{3}&1\end{array}\right] \xrightarrow{R_3\to -\frac{1}{3}R_3} \left[\begin{array}{ccc|ccc}1&0&-\frac{1}{3}&\frac{2}{3}&-\frac{1}{3}&0\\0&1&\frac{5}{3}&-\frac{1}{3}&\frac{2}{3}&0\\0&0&1&-2&-5&3\end{array}\right]$$

$$\xrightarrow[R_1\to \frac{1}{3}R_3+R_1]{R_2\to -\frac{5}{3}R_3+R_2} \left[\begin{array}{ccc|ccc}1&0&0&0&-2&1\\0&1&0&3&9&-5\\0&0&1&-2&-5&3\end{array}\right] \text{④}$$

$$(A+I)^{-1}=\begin{bmatrix}0&-2&1\\3&9&5\\-2&-5&3\end{bmatrix}$$

$$B=(A+I)^{-1}\cdot 2A=\begin{bmatrix}0&-2&1\\3&9&5\\-2&-5&3\end{bmatrix}\begin{bmatrix}2&2&2\\2&2&6\\6&8&10\end{bmatrix}$$

$$=\begin{bmatrix}2&4&-2\\-6&-16&10\\4&10&-4\end{bmatrix} \text{②}$$

3. (i) Method 1

$$\begin{vmatrix} a & 2 & 2 & 2 \\ 2 & a & 2 & 2 \\ 2 & 2 & a & 2 \\ 2 & 2 & 2 & a \end{vmatrix} \xrightarrow[\substack{R_1 \to R_3+R_1 \\ R_1 \to R_4+R_1}]{R_1 \to R_2+R_1} \begin{vmatrix} a+6 & a+6 & a+6 & a+6 \\ 2 & a & 2 & 2 \\ 2 & 2 & a & 2 \\ 2 & 2 & 2 & a \end{vmatrix}.$$

$$= (a+6)\begin{vmatrix} 1 & 1 & 1 & 1 \\ 2 & a & 2 & 2 \\ 2 & 2 & a & 2 \\ 2 & 2 & 2 & a \end{vmatrix} \xrightarrow[\substack{R_3 \to -2R_1+R_3 \\ R_4 \to -2R_1+R_4}]{R_2 \to -2R_1+R_2} (a+6)\begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & a-2 & 0 & 0 \\ 0 & 0 & a-2 & 0 \\ 0 & 0 & 0 & a-2 \end{vmatrix}$$

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$$= (a+6)(a-2)^3$$ 1

(2) Method 2

$$\begin{vmatrix} a & 2 & 2 & 2 \\ 2 & a & 2 & 2 \\ 2 & 2 & a & 2 \\ 2 & 2 & 2 & a \end{vmatrix} \xrightarrow{R_1 \leftrightarrow R_2} -\begin{vmatrix} 2 & a & 2 & 2 \\ a & 2 & 2 & 2 \\ 2 & 2 & a & 2 \\ 2 & 2 & 2 & a \end{vmatrix} \xrightarrow[\substack{R_3 \to -R_1+R_3 \\ R_4 \to -R_1+R_4}]{R_2 \to -R_1+R_2}$$

$$-\begin{vmatrix} 2 & a & 2 & 2 \\ a-2 & 2-a & 0 & 0 \\ 0 & 2-a & a-2 & 0 \\ 0 & 2-a & 0 & a-2 \end{vmatrix}$$ 2 $$= -(a-2)^3\begin{vmatrix} 2 & a & 2 & 2 \\ 1 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{vmatrix} = -(a-2)^3\Bigg[$$

$$2\begin{vmatrix} -1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} + (-1)^{1+2}\begin{vmatrix} a & 2 & 2 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix}\Bigg)$$

$$= -(a-2)^3\left((-2) + (-1)\begin{vmatrix} a & 2 & 2 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix}\right) = (a-2)^3\left(2 + \begin{vmatrix} a & 2 & 2 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix}\right)$$

$$= (a-2)^3 \left(2 + \begin{vmatrix} a+4 & 2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \right)$$ ③

$$= (a-2)^3 \left(2 + (a+4)\cdot 1 \cdot 1 \right)$$

$$= (a-2)^3 (a+6)$$ ①

(ii) $\quad A$ is singular $\Leftrightarrow \det(A) = 0$

$\Leftrightarrow (a-2)^3 (a+6) = 0$

$\Leftrightarrow a = 2$ ① or $a = -6$ ①.

Remark:

If student compute $\det(A) = (a-2)^3(a+6)$ correctly then student directly say A is singular when $a=2$ or $a=-6$ i.e. $\det(A)=0$.

In this case, should give the full mark

Please check every step carefully, even the final result is wrong, you still need to check intermediate steps and gives the partial marks.

4. (i) Method 1: Proof: $\forall \alpha, \beta \in \mathbb{R}, \ \forall A, B \in M_2(\mathbb{R})$

$$L(\alpha A+\beta B) = (\alpha A+\beta B)^T + 2(\alpha A+\beta B)$$
$$= \alpha(A^T+2A) + \beta(B^T+2B)$$
$$= \alpha L(A) + \beta L(B) \quad 4$$

Thus L is a linear transformation.

(i) Method 2: $\forall \alpha \in \mathbb{R}, \ \forall A, B \in M_2(\mathbb{R})$

① $$L(A+B) = (A+B)^T + 2(A+B)$$
$$= A^T + 2A + B^T + 2B$$
$$= L(A) + L(B) \quad 2$$

② $$L(\alpha A) = (\alpha A)^T + 2(\alpha A)$$
$$= \alpha(A^T+2A)$$
$$= \alpha L(A) \quad 2$$

(ii) $\forall A \in \ker(L) = \{B \in M_2(\mathbb{R}) \mid L(B) = 0\}$

$\therefore L(A) = 0$ i.e. $A^T + 2A = 0$ (0 is zero matrix)

suppose $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $A^T + 2A = \begin{bmatrix} a & c \\ b & d \end{bmatrix} + \begin{bmatrix} 2a & 2b \\ 2c & 2d \end{bmatrix}$

$$= \begin{bmatrix} 3a & 2b+c \\ b+2c & 3d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Thus $\begin{cases} 3a = 0 \\ b+2c = 0 \\ 2b+c = 0 \\ 3d = 0 \end{cases} \Rightarrow \begin{cases} a = 0 \\ b = 0 \\ c = 0 \\ d = 0 \end{cases}$ 2

$\therefore \ker(L) = \{0\}$ ①

$\forall\ S = \begin{bmatrix} x & y \\ z & w \end{bmatrix} \in M_2(\mathbb{R})$, Let $A \in M_2(\mathbb{R})$ $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

s.t. $L(A) = A^T + 2A = S$, i.e.,

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} + 2\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} x & y \\ z & w \end{bmatrix} \Rightarrow \begin{bmatrix} 3a & c+2b \\ b+2c & 3d \end{bmatrix} = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$$

$$\Rightarrow \begin{cases} 3a = x \\ 2b + c = y \\ b + 2c = z \\ 3d = w \end{cases} \Rightarrow \begin{cases} a = \frac{x}{3} \\ b = \frac{2y-z}{3} \\ c = \frac{2z-y}{3} \\ d = \frac{w}{3} \end{cases}$$ ②, Thus

$\forall\ S = \begin{bmatrix} x & y \\ z & w \end{bmatrix} \in M_2(\mathbb{R})$, $\exists\ A = \begin{bmatrix} \frac{x}{3} & \frac{2y-z}{3} \\ \frac{2z-y}{3} & \frac{w}{3} \end{bmatrix}$, s.t.

$L(A) = S$, Thus the range of L is $M_2(\mathbb{R})$ ①

(iii) L is both injective ① and surjective ①

since $\ker(L) = \{0\}$ and range of L is $M_2(\mathbb{R})$

5. (i) Augmented matrix is

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 3 & 4 \end{array}\right] \xrightarrow[R_3 \to -R_1+R_3]{R_2 \to -R_1+R_2} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & 3 \end{array}\right] \xrightarrow[R_3 \to -2R_2+R_3]{R_1 \to -R_2+R_1} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{array}\right]$$

The last column is a pivot.

$$\xrightarrow{R_2 \to -R_3+R_2} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$ 2

The linear system is inconsiste

(ii) The normal equation is

$A^TA\underset{\sim}{x} = A^T\underset{\sim}{b}$ 1

$$A^TA = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix}$$ 1

$$A^T\underset{\sim}{b} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 7 \\ 17 \end{bmatrix}$$ 1

$$\left[\begin{array}{cc|c} 3 & 6 & 7 \\ 6 & 14 & 17 \end{array}\right] \xrightarrow{R_1 \to \frac{1}{3}R_1} \left[\begin{array}{cc|c} 1 & 2 & \frac{7}{3} \\ 6 & 14 & 17 \end{array}\right]$$

$$\xrightarrow{R_2 \to -6R_1+R_2} \left[\begin{array}{cc|c} 1 & 2 & \frac{7}{3} \\ 0 & 2 & 3 \end{array}\right] \xrightarrow{R_2 \to \frac{1}{2}R_2} \left[\begin{array}{cc|c} 1 & 2 & \frac{7}{3} \\ 0 & 1 & \frac{3}{2} \end{array}\right]$$

$$\xrightarrow{R_1 \to -2R_2+R_1} \left[\begin{array}{cc|c} 1 & 0 & -\frac{2}{3} \\ 0 & 1 & \frac{3}{2} \end{array}\right]$$ 3

The least-square Soluti is $\hat{x} = \begin{pmatrix} -\frac{2}{3} \\ \frac{3}{2} \end{pmatrix}$

(iii) compute $(A^TA)^{-1}$

$$\left[\begin{array}{cc|cc} 3 & 6 & 1 & 0 \\ 6 & 14 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \to \frac{1}{3}R_1} \left[\begin{array}{cc|cc} 1 & 2 & \frac{1}{3} & 0 \\ 6 & 14 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \to -6R_1+R_2}$$

$$\left[\begin{array}{cc|cc} 1 & 2 & \frac{1}{3} & 0 \\ 0 & 2 & -2 & 1 \end{array}\right] \xrightarrow{R_2 \to \frac{1}{2}R_2} \left[\begin{array}{cc|cc} 1 & 2 & \frac{1}{3} & 0 \\ 0 & 1 & -1 & \frac{1}{2} \end{array}\right] \xrightarrow{R_1 \to -2R_2+R_1}$$

$$\left[\begin{array}{cc|cc} 1 & 0 & \frac{7}{3} & -1 \\ 0 & 1 & -1 & \frac{1}{2} \end{array}\right] \qquad (A^TA)^{-1} = \begin{bmatrix} \frac{7}{3} & -1 \\ -1 & \frac{1}{2} \end{bmatrix}$$

2

projection matrix P is

$$P = A(A^TA)^{-1}A^T = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} \frac{7}{3} & -1 \\ -1 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{4}{3} & -\frac{1}{2} \\ \frac{1}{3} & 0 \\ -\frac{2}{3} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} \frac{5}{6} & \frac{1}{3} & -\frac{1}{6} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{6} & \frac{1}{3} & \frac{5}{6} \end{pmatrix}$$

2

projection vector is

$$\underset{\sim}{p} = A\hat{\underset{\sim}{x}} = A(A^TA)^{-1}A^T\underset{\sim}{b} = P\underset{\sim}{b} = \begin{pmatrix} \frac{5}{6} & \frac{1}{3} & -\frac{1}{6} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{6} & \frac{1}{3} & \frac{5}{6} \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} \frac{5}{6} \\ \frac{7}{3} \\ \frac{23}{6} \end{pmatrix}$$

2

(iv). The remaining vector is

$$\underset{\sim}{b} - \underset{\sim}{p} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} - \begin{pmatrix} \frac{5}{6} \\ \frac{7}{3} \\ \frac{23}{6} \end{pmatrix} = \begin{pmatrix} \frac{1}{6} \\ -\frac{1}{3} \\ \frac{1}{6} \end{pmatrix} \perp C(A),$$

1

The distance between $\underset{\sim}{b}$ and $C(A)$ is

$$\|\underset{\sim}{b} - \underset{\sim}{p}\| = \sqrt{(\tfrac{1}{6})^2 + (\tfrac{1}{3})^2 + (-\tfrac{1}{6})^2} = \frac{1}{\sqrt{6}}$$

1

6.

(i) $$\begin{bmatrix} 1 & -1 & 4 \\ 1 & 2 & -2 \\ 1 & 2 & 2 \\ 1 & -1 & 6 \end{bmatrix} \xrightarrow[\substack{R_3 \to -R_1 + R_3 \\ R_4 \to -R_1 + R_4}]{R_2 \to -R_1 + R_2} \begin{bmatrix} 1 & -1 & 4 \\ 0 & 3 & -6 \\ 0 & 3 & -2 \\ 0 & 0 & -4 \end{bmatrix} \xrightarrow{R_2 \to \frac{1}{3} R_2} \begin{bmatrix} 1 & -1 & 4 \\ 0 & 1 & -2 \\ 0 & 3 & -2 \\ 0 & 0 & -4 \end{bmatrix} \xrightarrow[R_3 \to -3R_2 + R_3]{R_1 \to R_2 + R_1} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 4 \\ 0 & 0 & -4 \end{bmatrix}$$

$$\xrightarrow{R_3 \to \frac{1}{3} R_3} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & -4 \end{bmatrix} \xrightarrow[\substack{R_2 \to 2R_3 + R_1 \\ R_4 \to 4R_3 + R_4}]{R_1 \to -2R_3 + R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$ 2

Three pivots

$\therefore$ $\underset{\sim}{a_1}, \underset{\sim}{a_2}, \underset{\sim}{a_3}$ are linearly independent.

(ii) step 1:

$$\underset{\sim}{q_1} = \frac{a_1}{\|a_1\|} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$ 1

step 2:

$$\underset{\sim}{p_1} = \langle \underset{\sim}{a_2}, \underset{\sim}{q_1} \rangle \underset{\sim}{q_1} = \left((-1, 2, 2, -1) \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \right) \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

$$\underset{\sim}{r_1} = \underset{\sim}{a_2} - \underset{\sim}{p_1} = \begin{pmatrix} -1 \\ 2 \\ 2 \\ -1 \end{pmatrix} - \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} -\frac{3}{2} \\ \frac{3}{2} \\ \frac{3}{2} \\ -\frac{3}{2} \end{pmatrix}, \quad \underset{\sim}{q_2} = \frac{r_1}{\|r_1\|} = \frac{1}{3}\begin{pmatrix} -\frac{3}{2} \\ \frac{3}{2} \\ \frac{3}{2} \\ -\frac{3}{2} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$$ 3

step 3:

$$\underset{\sim}{p_2} = \langle \underset{\sim}{a_3}, \underset{\sim}{q_1} \rangle \underset{\sim}{q_1} + \langle \underset{\sim}{a_3}, \underset{\sim}{q_2} \rangle \underset{\sim}{q_2}$$

$$= \left((4, -2, 2, 0) \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \right) \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} + \left((4, -2, 2, 0) \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} \right) \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - 2\begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 2 \end{pmatrix}$$

$$\underset{\sim}{r_2} = \underset{\sim}{a_3} - \underset{\sim}{p_2} = \begin{pmatrix} 4 \\ -2 \\ 2 \\ 0 \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 2 \\ -2 \end{pmatrix}$$

$$\underset{\sim}{q_3} = \frac{r_2}{\|r_2\|} = \frac{1}{4}\begin{pmatrix} 2 \\ -2 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$$ 4

(iii) $R = Q^T A$ $\quad A = \begin{bmatrix} 1 & -1 & 4 \\ 1 & 2 & -2 \\ 1 & 2 & 2 \\ 1 & -1 & 0 \end{bmatrix}$ $\quad Q = [q_1, q_2, q_3] = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$ 1

$$= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & -1 & 4 \\ 1 & 2 & -2 \\ 1 & 2 & 2 \\ 1 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & 2 \\ 0 & 3 & -2 \\ 0 & 0 & 4 \end{bmatrix}$$ 3

then

$A = QR$

7. (i) Characteristic polynomial $p(\lambda)$ is

$$p(\lambda)=\begin{vmatrix}2-\lambda & -1 & 1\\ -1 & 2-\lambda & -1\\ 1 & -1 & 2-\lambda\end{vmatrix} \overset{R_1\Leftrightarrow R_3}{=} -\begin{vmatrix}-1 & 2-\lambda & -1\\ 2-\lambda & -1 & 1\\ 1 & -1 & 2-\lambda\end{vmatrix} = \begin{vmatrix}1 & \lambda-2 & 1\\ 2-\lambda & -1 & 1\\ 1 & -1 & 2-\lambda\end{vmatrix}$$

①

$$\underset{R_3\to -R_1+R_3}{\overset{R_2\to(\lambda-2)R_1+R_2}{=}} \begin{vmatrix}1 & \lambda-2 & 1\\ 0 & -1+(\lambda-2)^2 & -1+\lambda\\ 0 & 1-\lambda & 1-\lambda\end{vmatrix} = \begin{vmatrix}(\lambda-2)^2-1 & \lambda-1\\ 1-\lambda & 1-\lambda\end{vmatrix} = (1-\lambda)\left((\lambda-2)^2-1\right)+(\lambda-1)^2$$

$$= (1-\lambda)\left[(\lambda-2)^2-1+1-\lambda\right] = (1-\lambda)(\lambda^2-5\lambda+4) = (1-\lambda)(\lambda-1)(\lambda-4)$$
$$= -(\lambda-1)^2(\lambda-4)$$ ②

Let $p(\lambda)=0$, then $\lambda_1=\lambda_2=1$, $\lambda_3=4$ ① Since all eigenvalues are positive, A is positive definite. ①

(ii)

① $\lambda=1$ $\quad A-\lambda I=\begin{bmatrix}1 & -1 & 1\\ -1 & 1 & -1\\ 1 & -1 & 1\end{bmatrix} \xrightarrow[R_3\to -R_3+R_1]{R_2\to R_2+R_1} \begin{bmatrix}1 & -1 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$

$$(A-\lambda I)\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix} \Leftrightarrow x_1-x_2+x_3=0 \Leftrightarrow \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}x_2-x_3\\x_2\\x_3\end{pmatrix}=x_2\begin{pmatrix}1\\1\\0\end{pmatrix}+x_3\begin{pmatrix}-1\\0\\1\end{pmatrix}$$

eigenspace w.r.t. to $\lambda=1$ is $\qquad x_2, x_3\in\mathbb{R}$. ①

② $\quad \text{span}\left(\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}-1\\0\\1\end{pmatrix}\right)$

Using Gram-Schmidt process for $\underbrace{\begin{pmatrix}1\\1\\0\end{pmatrix}}_{a_1}, \underbrace{\begin{pmatrix}-1\\0\\1\end{pmatrix}}_{a_2}$, we hav

$$q_1=\frac{a_1}{\|a_1\|}=\begin{pmatrix}\frac{1}{\sqrt2}\\ \frac{1}{\sqrt2}\\ 0\end{pmatrix}, \quad p_1=\langle a_2, q_1\rangle q_1=\left((-1,0,1)\begin{pmatrix}\frac{1}{\sqrt2}\\ \frac{1}{\sqrt2}\\ 0\end{pmatrix}\right)\begin{pmatrix}\frac{1}{\sqrt2}\\ \frac{1}{\sqrt2}\\ 0\end{pmatrix}=\begin{pmatrix}-\frac12\\ -\frac12\\ 0\end{pmatrix}$$

$$r_1=a_2-p_1=\begin{pmatrix}-1\\0\\1\end{pmatrix}-\begin{pmatrix}-\frac12\\ -\frac12\\ 0\end{pmatrix}=\begin{pmatrix}-\frac12\\ \frac12\\ 1\end{pmatrix}$$

$$q_2=\frac{r_1}{\|r_1\|}=\frac{1}{\sqrt{\frac32}}\begin{pmatrix}-\frac12\\ \frac12\\ 1\end{pmatrix}=\begin{pmatrix}-\frac{1}{\sqrt6}\\ \frac{1}{\sqrt6}\\ \frac{2}{\sqrt6}\end{pmatrix}$$ ②

② $\lambda=4$

$$A-\lambda I=\begin{pmatrix}-2 & -1 & 1\\ -1 & -2 & -1\\ 1 & -1 & -2\end{pmatrix}\xrightarrow{R_1\leftrightarrow R_3}\begin{pmatrix}1 & -1 & -2\\ -1 & -2 & -1\\ -2 & -1 & 1\end{pmatrix}\xrightarrow[R_3\to 2R_1+R_3]{R_2\to R_1+R_2}$$

$$\begin{pmatrix}1 & -1 & -2\\ 0 & -3 & -3\\ 0 & -3 & -3\end{pmatrix}\xrightarrow{R_2\to -\frac{1}{3}R_2}\begin{pmatrix}1 & -1 & -2\\ 0 & 1 & 1\\ 0 & -3 & -3\end{pmatrix}\xrightarrow[R_3\to 3R_2+R_3]{R_1\to R_2+R_1}\begin{pmatrix}1 & 0 & -1\\ 0 & 1 & 1\\ 0 & 0 & 0\end{pmatrix}$$

eigenspace w.r.t. $\lambda=4$ is

$$\operatorname{span}\left(\begin{pmatrix}1\\ -1\\ 1\end{pmatrix}\right)$$

$$\underset{\sim}{q_3}=\frac{1}{\sqrt{3}}\begin{pmatrix}1\\ -1\\ 1\end{pmatrix}=\begin{pmatrix}\frac{1}{\sqrt{3}}\\ -\frac{1}{\sqrt{3}}\\ \frac{1}{\sqrt{3}}\end{pmatrix}$$ ②

let $U=[\underset{\sim}{q_1},\underset{\sim}{q_2},\underset{\sim}{q_3}]=\begin{bmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}}\\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}}\end{bmatrix}$

U is orthogonal matrix ①

then $AU=U\Lambda$, $\Lambda=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 4\end{bmatrix}=\operatorname{diag}(1,1,4)$

Thus $U^TAU=U^{-1}AU=\Lambda$

Remark: U is not unique, please check carefully, another U is given in previous page. The key point here is to check U is orthogonal matrix and $A\underset{\sim}{q_i}=\lambda_i\underset{\sim}{q_i}$, where $i=1,2,3$ and $U=[\underset{\sim}{q_1}\underset{\sim}{q_2}\underset{\sim}{q_3}]$ $\underset{\sim}{q_i}$ are orthonormal, $\underset{\sim}{q_i}$ is the eigenvector w.r.t. λ_i $(i=1,2,3)$

Another U

7. $\lambda = 1$ $\quad (A - \lambda I)\cancel{x} \Rightarrow x_1 - x_2 + x_3 = 0 \Rightarrow x_3 = -x_1 + x_2$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ -x_1 + x_2 \end{pmatrix} = x_1 \underbrace{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}_{= a_1} + x_2 \underbrace{\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}}_{= a_2}$$

$$q_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad p_1 = \langle a_2, q_1 \rangle q_1 = -\frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix}$$

$$r_1 = a_2 - p_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} -\frac{1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 1 \\ \frac{1}{2} \end{pmatrix}$$

$$q_2 = \frac{r_1}{\|r_1\|} = \frac{1}{\sqrt{\frac{3}{2}}} \begin{pmatrix} \frac{1}{2} \\ 1 \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix}$$

$$q_3 = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}$$

$$U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix} \qquad U^T A U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

7. $\lambda = 1.$ $(A-\lambda I)\underset{\sim}{x} = \underset{\sim}{0} \Rightarrow x_2 = x_1 + x_3$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_1 + x_3 \\ x_3 \end{pmatrix} = x_1 \underbrace{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}_{\underset{\sim}{a_1}} + x_3 \underbrace{\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}}_{\underset{\sim}{a_2}}$$

$$\underset{\sim}{q_1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} \quad \underset{\sim}{p_1} = \langle \underset{\sim}{a_2}, \underset{\sim}{q_1} \rangle \underset{\sim}{q_1} = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix}$$

$$\underset{\sim}{r_1} = \underset{\sim}{a_2} - \underset{\sim}{q_1} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \end{pmatrix}$$

$$\underset{\sim}{q_2} = \frac{\underset{\sim}{r_1}}{\|\underset{\sim}{r_1}\|} = \frac{1}{\sqrt{\frac{3}{2}}} \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \end{pmatrix} = \frac{2}{\sqrt{6}} \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{pmatrix}$$

$\lambda = 4,$ $(A-\lambda I)\underset{\sim}{x} = \underset{\sim}{0},$ eigenvector is $(1,1,1)^T$.

$$\underset{\sim}{q_3} = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}$$

$$U = [\underset{\sim}{q_1}, \underset{\sim}{q_2}, \underset{\sim}{q_3}] = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$U^T A U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

8. (i) Suppose λ is the eigenvalue of A, there is an eigenvector $\underset{\sim}{x}$ ($\underset{\sim}{x} \neq \underset{\sim}{0}$) s.t. $A\underset{\sim}{x} = \lambda \underset{\sim}{x}$ 1 ($\underset{\sim}{x}$ is the corresponding eigenvector of λ). Then

$$A^2\underset{\sim}{x} = A(\lambda\underset{\sim}{x}) = \lambda(A\underset{\sim}{x}) = \lambda^2\underset{\sim}{x}$$
$$A^3\underset{\sim}{x} = A(\lambda^2\underset{\sim}{x}) = \lambda^2(A\underset{\sim}{x}) = \lambda^3\underset{\sim}{x}$$

$\therefore$ $A^3 + 2A^2 = 3A$ gives $A^3\underset{\sim}{x} + 2A^2\underset{\sim}{x} = 3A\underset{\sim}{x}$, and this yields $\lambda^3\underset{\sim}{x} + 2\lambda^2\underset{\sim}{x} = 3\lambda\underset{\sim}{x}$ $\Rightarrow$ $(\lambda^3 + 2\lambda^2 - 3\lambda)\underset{\sim}{x} = \underset{\sim}{0}$ 3

since $\underset{\sim}{x} \neq \underset{\sim}{0}$, one has $\lambda^3 + 2\lambda^2 - 3\lambda = 0$

$$\Rightarrow \lambda(\lambda^2 + 2\lambda - 3) = 0$$
$$\Rightarrow \lambda(\lambda+3)(\lambda-1) = 0$$
$$\Rightarrow \lambda_1 = 0,\ \lambda_2 = -3,\ \lambda_3 = +1$$ 2

The possible eigenvalues are $0, -3, +1$.

(ii) since $A \in M_n(\mathbb{R})$, thus $r(A) \leq n$, 1

If $r(A) = n$, then A is invertible 1
since the column vectors of A are linearly independent

From $A^3 + 2A^2 - 3A = 0$, one has $A(A^2 + 2A - 3I_n) = 0$

$\Rightarrow$ $A^{-1}A(A^2 + 2A - 3I_n) = 0$ $\Rightarrow$ $A^2 + 2A - 3I_n = 0$

which is a contradiction! with condition 2 Thus $r(A) < n$.

9(i) By spectral theorem, there is an orthogonal matrix U, such that

$$U^{-1} A U = \Lambda = \mathrm{diag}(l_1, l_2, \cdots l_n) \quad ① \quad (*)$$

($l_1, l_2, \cdots l_n$ are eigenvalues of A)

Since A is positive definite, all eigenvalue $l_1, l_2, \cdots l_n$ are positive real numbers. ①

(*) gives $A = U \Lambda U^{-1} = U \Lambda U^T$

($U^{-1} = U^T$ since U is an orthogonal matrix ①)

Let $\Lambda_1 = \mathrm{diag}(l_1^{\frac{1}{2}}, l_2^{\frac{1}{2}}, \cdots l_n^{\frac{1}{2}})$ ①

then $A = U \Lambda_1 \Lambda_1 U^T = U \Lambda_1 U^T U \Lambda_1 U^T \quad (U^T U = I)$ ①

Now let $B = U \Lambda_1 U^T$, then $A = B \cdot B = B^2$ ①

9 (ii) since $\begin{bmatrix} I_n & 0 \\ -A & I_n \end{bmatrix}\begin{bmatrix} I_n & B \\ A & I_n \end{bmatrix} = \begin{bmatrix} I_n & B \\ 0 & I_n-AB \end{bmatrix}$ 1

Thus

$$\det\left(\begin{bmatrix} I_n & 0 \\ -A & I_n \end{bmatrix}\right) \det\left(\begin{bmatrix} I_n & B \\ A & I_n \end{bmatrix}\right) = \det\left(\begin{bmatrix} I_n & B \\ 0 & I_n-AB \end{bmatrix}\right)$$

$$\det\left(\begin{bmatrix} I_n & 0 \\ -A & I_n \end{bmatrix}\right) = 1 \qquad \det\left(\begin{bmatrix} I_n & B \\ 0 & I_n-AB \end{bmatrix}\right) = \det(I_n)\det(I_n-AB)$$

$$= \det(I_n-AB)$$

$$\therefore \det\left(\begin{bmatrix} I_n & B \\ A & I_n \end{bmatrix}\right) = \det(I_n-AB) \quad (1)$$ 1

since $\begin{bmatrix} I_n & -B \\ 0 & I_n \end{bmatrix}\begin{bmatrix} I_n & B \\ A & I_n \end{bmatrix} = \begin{bmatrix} I_n-BA & 0 \\ A & I_n \end{bmatrix}$ 1

Thus $\det\left(\begin{bmatrix} I_n & -B \\ 0 & I_n \end{bmatrix}\right) \det\left(\begin{bmatrix} I_n & B \\ A & I_n \end{bmatrix}\right) = \det\left(\begin{bmatrix} I_n-BA & 0 \\ A & I_n \end{bmatrix}\right)$

$$\det\left(\begin{bmatrix} I_n & -B \\ 0 & I_n \end{bmatrix}\right) = 1, \quad \det\left(\begin{bmatrix} I_n-BA & 0 \\ A & I_n \end{bmatrix}\right) = \det(I_n-BA)\det(I_n)$$

$$= \det(I_n-BA)$$

$$\therefore \det\left(\begin{bmatrix} I_n & B \\ A & I_n \end{bmatrix}\right) = \det(I_n-BA) \quad (2)$$ 1

combining (1) and (2), one has

$$\det(I_n-AB) = \det(I_n-BA)$$

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Other Method.

$$\begin{bmatrix} I_n & -B \\ 0 & I_n \end{bmatrix}\begin{bmatrix} I_n & B \\ A & I_n \end{bmatrix} = \begin{bmatrix} I_n - BA & 0 \\ A & I_n \end{bmatrix} \quad 1 \quad \Rightarrow \det\begin{pmatrix} I_n & B \\ A & I_n \end{pmatrix} = \det(I_n - BA) \quad 1$$

$$\begin{bmatrix} I_n & B \\ A & I_n \end{bmatrix}\begin{bmatrix} I_n & -B \\ 0 & I_n \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ A_n & I_n - AB \end{bmatrix} \quad 1 \quad \Rightarrow \det\begin{pmatrix} I_n & B \\ A & I_n \end{pmatrix} = \det(I_n - AB) \quad 1$$

other Method

$$\begin{pmatrix} I_n & -A \\ 0 & I_n \end{pmatrix}\begin{bmatrix} I_n & A \\ B & I_n \end{bmatrix} = \begin{bmatrix} I_n - AB & 0 \\ B & I_n \end{bmatrix} \quad 1$$

$$\Rightarrow \det\begin{pmatrix} I_n & A \\ B & I_n \end{pmatrix} = \det(I_n - AB) \quad 1$$

$$\begin{bmatrix} I_n & A \\ B & I_n \end{bmatrix}\begin{bmatrix} I_n & -A \\ 0 & I_n \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ B & I_n - BA \end{bmatrix} \quad 1 \quad \Rightarrow \det\begin{pmatrix} I_n & A \\ B & I_n \end{pmatrix} = \det(I_n - BA) \quad 1$$

other Method.

$$\begin{bmatrix} I_n & 0 \\ -A & I_n \end{bmatrix}\begin{bmatrix} I_n & B \\ A & I_n \end{bmatrix} = \begin{bmatrix} I_n & B \\ 0 & I_n - AB \end{bmatrix} \quad 1 \quad \Rightarrow \det\begin{pmatrix} I_n & B \\ A & I_n \end{pmatrix} = \det(I_n - AB) \quad 1$$

$$\begin{bmatrix} I_n & B \\ A & I_n \end{bmatrix}\begin{bmatrix} I_n & 0 \\ -A & I_n \end{bmatrix} = \begin{pmatrix} I_n - BA & B \\ 0 & I_n \end{pmatrix} \quad 1 \quad \Rightarrow \det\begin{pmatrix} I_n & B \\ A & I_n \end{pmatrix} = \det(I_n - BA) \quad 1$$