Lecture 18: Change of Basis and Transformations

MAT2040 Linear Algebra

We will now combine coordinate transformations (change of basis) and linear transformations!

Theorem 17.12

Let V be a vector space with basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ and W be a vector space with basis $\mathcal{C} = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m\}$.
If $T(\mathbf{x})$ from V to W is a linear transformation then there exists an $m \times n$ matrix A so that

$$[T(\mathbf{x})]_{\mathcal{C}} = A[\mathbf{x}]_{\mathcal{B}}.$$

In fact,

$$A = \left[[T(\mathbf{v}_1)]_{\mathcal{C}}, \quad [T(\mathbf{v}_2)]_{\mathcal{C}}, \quad \ldots, [T(\mathbf{v}_n)]_{\mathcal{C}} \right].$$

A is called the **matrix for T relative to the bases $\mathcal{B}$ and $\mathcal{C}$.**

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If $T(\mathbf{x})$ from V to $\cancel{W}$ V is a linear transformation then there exists an $\cancel{m \times n}$ $n \times n$ matrix A so that

$$[T(\mathbf{x})]_{\cancel{\mathcal{C}}\mathcal{B}} = A[\mathbf{x}]_{\mathcal{B}}.$$

In fact,

$$A = \left[[T(\mathbf{v}_1)]_{\cancel{\mathcal{C}}\mathcal{B}}, \quad [T(\mathbf{v}_2)]_{\cancel{\mathcal{C}}\mathcal{B}}, \quad \ldots, [T(\mathbf{v}_n)]_{\cancel{\mathcal{C}}\mathcal{B}} \right].$$

A is called the **matrix for T relative to** $\mathcal{B}$ ~~**the bases $\mathcal{B}$ and $\mathcal{C}$**~~**.**

Theorem 18.1

Let V be a vector space with basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$.
If $T(\mathbf{x})$ from V to V is a linear transformation then there exists an $n \times n$ matrix A so that

$$[T(\mathbf{x})]_{\mathcal{B}} = A[\mathbf{x}]_{\mathcal{B}}.$$

In fact,

$$A = \left[[T(\mathbf{v}_1)]_{\mathcal{B}}, \quad [T(\mathbf{v}_2)]_{\mathcal{B}}, \quad \ldots, [T(\mathbf{v}_n)]_{\mathcal{B}} \right].$$

A is called the **matrix for T relative to $\mathcal{B}$ (the $\mathcal{B}$-matrix for T).**

This is useful, because often linear transformations are easier to understand (work with) in a certain basis.

Example 18.2

Consider $T(\mathbf{x}) = A\mathbf{x}$ where $A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$.

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What is the $\mathcal{B}$-matrix for T for $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\}$?

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What is the $\mathcal{B}$-matrix for T for $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\}$?

Express A in terms of the $\mathcal{B}$-matrix for T.

$$
\begin{tikzcd}
\mathbf{x} \arrow[r, "\text{Multiplication}", "\text{by } A"'] \arrow[d, "\text{Multiplication by } P^{-1}"'] & A\mathbf{x} \\
{[\mathbf{x}]_{\mathcal{B}}} \arrow[r, "\text{Multiplication}", "\text{by } C"'] & {[A\mathbf{x}]_{\mathcal{B}}} \arrow[u, "\text{Multiplication by } P"']
\end{tikzcd}
$$

(picture from David Lay, Linear Algebra.)

Definition 18.3

The $n \times n$ matrices A and B are **similar matrices** if there exists an invertible $n \times n$ matrix P so that

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The intuition is that A and B can be interpreted as matrices corresponding to the same linear transformation, just expressed in different bases.

Example 18.4 (Preview)

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How can we find such a basis?

Suppose $A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$.