Lecture 8: The Simplex Tableau

Zizhuo Wang

Institute of Data and Decision Analytics (iDDA)
Chinese University of Hong Kong, Shenzhen

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Recap: Simplex Method

We have completed our discussion on the algebraic procedure of the simple method.

Start from a BFS $\mathbf{x}$

1. We first compute the reduced costs $\bar{\mathbf{c}}$
   - If no reduced costs is negative, we are already optimal
   - Otherwise choose some j such that $\bar{c}_j < 0$
2. Compute the jth basic direction
$$\mathbf{d} = [-A_B^{-1}A_j; 0; ...; 1; ...; 0]$$
   - If $\mathbf{d} \geq 0$, then the problem is unbounded, i.e., the optimal value is $-\infty$.
   - Otherwise, compute θ^*
3. Define $\mathbf{y} = \mathbf{x} + \theta^*\mathbf{d}$. And repeat these procedures

Degeneracy

Degenerate case happens when there exists basic variable that equals 0.

- To ensure there is no cycling, we could use the smallest index rules (Bland's rule) when choosing the incoming and outgoing basis.
- If the smallest index rule is used, one needs not to worry about the degenerate cases in the simplex method.

Two-Phase Method

The goal of the two-phase method is to find an initial BFS for the original problem.

In the two-phase simplex method, we first solve an auxiliary problem ($\mathbf{e}$ means an all-one vector, $\mathbf{b} \geq 0$).

$$\begin{array}{rc} \text{minimize}_{\mathbf{x},\mathbf{y}} & \mathbf{e}^T\mathbf{y} \\ \text{subject to} & A\mathbf{x} + \mathbf{y} = \mathbf{b} \\ & \mathbf{x}, \mathbf{y} \geq 0 \end{array}$$

- Solving this auxiliary problem will give a BFS for the original problem or tell that the original problem is infeasible

Note that there are other methods of finding initial basic feasible solution, e.g., the big-M method.

Simplex Tableau

Now we have obtained the algebraic procedures for the simplex method.

- We want to have a simpler implementation of it — in particular, we want to avoid explicit matrix inversion in the calculation

We are going to introduce the simplex tableau, which is a practical way to implement the simplex method.

- The simplex tableau maintains a table of numbers
- It visualizes the procedures of the simplex algorithm and facilitates the computation
- After learning the simplex tableau, one should be able to solve small-sized linear optimization problems by hand

Simplex Tableau

The simplex tableau is a table with the following structure when the basis is B (the corresponding basis matrix and objective coefficients are A_B and $\mathbf{c}_B$):

$\mathbf{c}^T - \mathbf{c}_B^T A_B^{-1} A$	$-\mathbf{c}_B^T A_B^{-1} \mathbf{b}$
$A_B^{-1} A$	$A_B^{-1} \mathbf{b}$

In the following, we take a closer look at what each part of the tableau means (and looks like) and how we can update the tableau efficiently in each iteration.

Simplex Tableau

$\mathbf{c}^T - \mathbf{c}_B^T A_B^{-1} A$	$-\mathbf{c}_B^T A_B^{-1} \mathbf{b}$
$A_B^{-1} A$	$A_B^{-1} \mathbf{b}$

The lower part of the tableau can be viewed as a transformation of the constraint $A\mathbf{x} = \mathbf{b}$ to $A_B^{-1} A\mathbf{x} = A_B^{-1}\mathbf{b}$

- It is equivalent to the original constraint

Furthermore, if we write $A = [A_B, A_N]$, then

$$A_B^{-1} A = [I, A_B^{-1} A_N]$$

Therefore, this part must contain an identity matrix.

Also when the basis is B, the current basic feasible solution is

$$\mathbf{x} = [\mathbf{x}_B; \mathbf{x}_N] = [A_B^{-1}\mathbf{b}; 0]$$

Therefore the lower right corner gives the current BFS.

Simplex Tableau

The term

$$\mathbf{c}^T - \mathbf{c}_B^T A_B^{-1} A$$

is the reduced cost at this basis.

- Recall that the reduced costs for basic variables are 0's. Therefore, this part are 0's for the basic indices

Lastly, the term

$$-\mathbf{c}_B^T A_B^{-1} \mathbf{b} = -\mathbf{c}_B^T \mathbf{x}_B$$

is the negative of the objective value at this basis.

Simplex Tableau

Therefore, the simplex tableau should look like (after reordering the columns)

$\mathbf{0}_m$ $\mathbf{c}_N^T - \mathbf{c}_B^T A_B^{-1} A_N$		$-\mathbf{c}_B^T \mathbf{x}_B$
$\mathbf{I}_m$	$A_B^{-1} A_N$	$\mathbf{x}_B$

Here $\mathbf{0}_m$ is a vector of m zeros and $\mathbf{I}_m$ is the m-dimensional identity matrix.

This form of LP is called the *canonical* form.

- The constraint matrix for the basic variables (not necessarily the first m columns) is an identity matrix
- The objective part for the basic variables is zeros

Example

The production problem

$$\begin{array}{rrrrrrl} \text{minimize} & -x_1 & -2x_2 & & & & \\ \text{subject to} & x_1 & & +s_1 & & & = 100 \\ & & 2x_2 & & +s_2 & & = 200 \\ & x_1 & +x_2 & & & +s_3 & = 150 \\ & x_1, & x_2, & s_1, & s_2, & s_3 & \geq 0 \end{array}$$

It is already in the canonical form as follows:

-1	-2	0	0	0	0
1	0	1	0	0	100
0	2	0	1	0	200
1	1	0	0	1	150

Simplex Tableau

The simplex tableau will maintain a canonical form for each iteration until it reaches optimality

- For now we assume that we already have a canonical form to start with (we have discussed how to find the initial basic solution).

Now we want to map each algebraic steps we derived earlier to an operation on the simplex tableau. Here are the key steps:

1. Compute the reduced costs
2. Choose the incoming basis
3. Compute θ^* and choose the outgoing basis
4. Update the tableau with the new set of basis

We call the procedure of transforming from one canonical form (one set of basis) to another *pivoting*

Pivoting Step I: Choose the Incoming Basis

In the canonical form, the reduced costs are simply the coefficients in the top row.

- Therefore, we can choose any column with negative reduced cost to be the incoming basis.

In the production example:

-1	-2	0	0	0	0
1	0	1	0	0	100
0	2	0	1	0	200
1	1	0	0	1	150

We can choose either the first or second column as the entering basis (if we use Bland's rule, then we choose the first one).

Pivoting Step II: Compute θ^* and Choose Outgoing Basis

Assume we have chosen column j as the incoming basis.

Now we need to make sure that the next BFS is still feasible (≥ 0). Recall we use the following method:

$$\theta^* = \min_{d_i < 0, i \in B} \left\{ -\frac{x_i}{d_i} \right\}$$

where x_i is the ith entry of the basic solution, and $\mathbf{d} = -A_B^{-1} A_j$

Therefore, in the simplex tableau, this is to find

$$\theta^* = \min \left\{ \frac{\bar{b}_i}{\bar{A}_{ij}} : \bar{A}_{ij} > 0 \right\}$$

where $\bar{\mathbf{b}}$ is the lower right column and $\bar{A}$ is the lower left part of the tableau.

- This is called the Minimal Ratio Test (MRT)

Pivoting Step II: Compute θ^* and Choose Outgoing Basis

If $\bar{A}_{ij} \le 0$ for all i, then the problem is unbounded.

Otherwise, assume index i achieves the minimum in:

$$\theta^* = \min\left\{\frac{\bar{b}_i}{\bar{A}_{ij}} : \bar{A}_{ij} > 0\right\}$$

- Then the column in the current basis whose ith element is 1 is the outgoing basis. We call row i the *pivot row*.

Example: If we choose column 1 to be the incoming basis

-1	-2	0	0	0	0
1	0	1	0	0	100
0	2	0	1	0	200
1	1	0	0	1	150

Then MRT will choose the third column to be the outgoing basis.

Example Continued

-1	-2	0	0	0	0
1	0	1	0	0	100
0	2	0	1	0	200
1	1	0	0	1	150

What if we choose column 2 to be the incoming basis?

- Then the MRT will choose the fourth column as the outgoing basis (the MRT finds that the second row achieves the minimum ratio. Then we choose the basis whose second row element is 1 to be the outgoing basis, which in this case is column 4)

Some Terms

- We call the incoming column the *pivot column*
- We call the row that achieves the MRT the *pivot row* (determines the outgoing basis)
- The intersection element of the pivot column and the pivot row is called the *pivot element*

Update the Tableau

Assume we have determined the incoming and outgoing basis (pivoting element $\bar{A}_{ij}$).

Then we perform the following two steps

1. Divide each element in the pivot row by the pivot element
2. Add proper multiples (could be negative) of the pivot row (after the first step) to each other rows, including the top row of objective coefficients, such that all other elements in the pivot column become zeros (including the top row)
3. Both operations include the right-hand-side column of $\mathbf{b}$

After this procedure, the new pivot column should be $(0; \ldots; 0; 1; 0; \ldots; 0)$ with 1 at the pivot row.

- The new resulting tableau will still be in a canonical form, however, with the new basis.

Simplex Method in the Tableau

We have shown how to get from one canonical form to another, we then repeat this procedure until we reach optimality.

- When choosing the incoming and the outgoing basis, we use the smallest index rule
- This will guarantee that the simplex iterations will terminate in a finite number of steps

We also attach the index of the basis to the left of the tableau to indicate the current basis (just for clarity).

Example

Consider the production example. The initial simplex tableau is:

B	-1	-2	0	0	0	0
3	1	0	1	0	0	100
4	0	2	0	1	0	200
5	1	1	0	0	1	150

We use the smallest index rule. The pivot column (incoming basis) is the first column, the pivot row is the first row (outgoing basis is column 3), the pivot element is 1 (in red).

- Divide the pivot row by the pivot element
- Add proper multiples of row 1 to other rows (including the top row) such that all other elements in the new pivot column become zero (including the top element)

Example Continued

The tableau becomes:

B	0	-2	1	0	0	100
1	1	0	1	0	0	100
4	0	2	0	1	0	200
5	0	1	-1	0	1	50

- It is not optimal since there is one negative reduced cost
- The only choice for the pivot column is column 2
- Use MRT, the pivot row should be row 3 (outgoing basis is column 5)

Then we apply the same procedure

- Add $2\times$ row 3 to the very top row, and $-2\times$ row 3 to the second row in the constraint

Example Continued

The tableau becomes

B	0	0	-1	0	2	200
1	1	0	1	0	0	100
4	0	0	2	1	-2	100
2	0	1	-1	0	1	50

- It is still not optimal since there is one negative reduced cost
- The only choice for the pivot column is column 3
- Use MRT, the pivot row should be row 2 (outgoing basis is column 4)

We apply the same procedure again

- Divide row 2 by 2, then add $1\times$ row 2 to the very top row, add $-1\times$ row 2 to the first row in the constraint, add $1\times$ row 2 to the last row.

Example Continued..

The tableau becomes:

B						
B	0	0	0	1/2	1	250
1	1	0	0	-1/2	1	50
3	0	0	1	1/2	-1	50
2	0	1	0	1/2	0	100

All the reduced costs are positive now

- Thus it is optimal
- The optimal solution is $(50, 100, 50, 0, 0)$ with optimal value -250.

Another Example

Consider the linear optimization problem:

$$\begin{aligned} \text{minimize} \quad & -10x_1 - 12x_2 - 12x_3 \\ \text{s.t.} \quad & x_1 + 2x_2 + 2x_3 \le 20 \\ & 2x_1 + x_2 + 2x_3 \le 20 \\ & 2x_1 + 2x_2 + x_3 \le 20 \\ & x_1, x_2, x_3 \ge 0 \end{aligned}$$

First, we write down the standard form:

$$\begin{array}{lccccccc} \text{minimize} & -10x_1 & -12x_2 & -12x_3 & & & & \\ \text{s.t.} & x_1 & +2x_2 & +2x_3 & +s_1 & & & = 20 \\ & 2x_1 & +x_2 & +2x_3 & & +s_2 & & = 20 \\ & 2x_1 & +2x_2 & +x_3 & & & +s_3 & = 20 \\ & x_1 & , x_2 & , x_3 & , s_1 & , s_2 & , s_3 & \ge 0 \end{array}$$

Simplex Algorithm: Step I

We write down the initial tableau:

B	-10	-12	-12	0	0	0	0
4	1	2	2	1	0	0	20
5	2	1	2	0	1	0	20
6	2	2	1	0	0	1	20

This is also in a canonical form already.

By the smallest index rule, we choose column 1 to enter the basis. By the minimum ratio test, we have two candidates to leave the basis: 5th column (row 2) or 6th column (row 3).

By the smallest index rule again, we choose 5th column to exit (pivot row is row 2). We then

- Divide 2 to each element in row 2
- Add $10\times$ new row 2 to the top row, $-1\times$ new row 2 to the first constraint row, and $-2\times$ new row 2 to the last row.

Simple Algorithm: Step II

Then the tableau becomes:

B							
B	0	-7	-2	0	5	0	100
4	0	3/2	1	1	-1/2	0	10
1	1	1/2	1	0	1/2	0	10
6	0	1	-1	0	-1	1	0

Column 2 is the pivot column. By MRT, the pivot row is row 3.

- Here we encounter a degeneracy case where the minimal ratio is 0
- It means that in this pivoting, we can't strictly improve the objective value.
- But we can still proceed as normal (no cycle will occur if we use the Bland's rule).
  - We add $7\times$ row 3 to the top row, $-3/2\times$ row 3 to the first constraint row and $-1/2\times$ row 3 to the second constraint row

Simplex Algorithm: Step III

Then the tableau becomes:

B							
B	0	0	-9	0	-2	7	100
4	0	0	5/2	1	1	-3/2	10
1	1	0	3/2	0	1	-1/2	10
2	0	1	-1	0	-1	1	0

We choose column 3 to enter the basis. By MRT, the pivot row is row 1 (column 4 leaving basis)

- We multiply $2/5$ to each number in row 1, then add $9\times$ row 1 to the top row, $-3/2\times$ row 1 to the second constraint row and $1\times$ row 1 to the last row.

Simplex Algorithm: Step IV

Then the tableau becomes:

B	0	0	0	18/5	8/5	8/5	136
3	0	0	1	2/5	2/5	-3/5	4
1	1	0	0	-3/5	2/5	2/5	4
2	0	1	0	2/5	-3/5	2/5	4

This is optimal since all reduced costs are non-negative. The optimal solution is $(4, 4, 4, 0, 0, 0)$ with optimal value -136.