Fundamental Questions 7

1. Let $\mathbf{u}_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix}1\\2\\2\end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix}2\\3\\4\end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix}4\\6\\7\end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix}0\\1\\1\end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix}0\\1\\2\end{bmatrix}$
   (a) Find the transition matrix from $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ to $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.
   (b) If $\mathbf{x} = \mathbf{v}_1 + 2\mathbf{v}_2 - 3\mathbf{v}_3$, determine the coordinate of $\mathbf{x}$ with respect to $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.

2. Let A be a 4×5 matrix and U be the reduced row-echelon form of A. If

$$\mathbf{a}_1 = \begin{bmatrix}2\\1\\-3\\-2\end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix}-1\\2\\3\\1\end{bmatrix}, \quad \mathbf{a}_4 = \begin{bmatrix}1\\-1\\2\\-1\end{bmatrix}, \quad U = \begin{bmatrix}1 & 0 & 2 & 0 & -1\\0 & 1 & 3 & 0 & -2\\0 & 0 & 0 & 1 & 5\\0 & 0 & 0 & 0 & 0\end{bmatrix}.$$

   (a) Find $\mathbf{a}_3$ and $\mathbf{a}_5$.
   (b) Find a basis for $\mathrm{Null}\,(A)$.
   (c) Find a basis for $\mathrm{Col}\,(A)$.

3. Let $E = \{(5,3)^T, (3,2)^T\}$ and let $\mathbf{x} = (1,1)^T, \mathbf{y} = (1,-1)^T$, and $\mathbf{z} = (10,7)^T$. Determine the values of $[\mathbf{x}]_E$, $[\mathbf{y}]_E$, and $[\mathbf{z}]_E$.

4. Given

$$\mathbf{v_1} = \begin{bmatrix}1\\2\end{bmatrix}, \mathbf{v_2} = \begin{bmatrix}2\\3\end{bmatrix}, S = \begin{bmatrix}3 & 5\\1 & -2\end{bmatrix}$$

   find vectors $\mathbf{w}_1$ and $\mathbf{w}_2$ so that S will be the transition matrix from $\{\mathbf{w}_1, \mathbf{w}_2\}$ to $\{\mathbf{v}_1, \mathbf{v}_2\}$.

5. In each of the following, determine the dimension of the subspace of $\mathcal{R}^3$ spanned by the given vectors.
   (a)

$$\begin{bmatrix}1\\-2\\2\end{bmatrix}, \begin{bmatrix}2\\-2\\4\end{bmatrix}, \begin{bmatrix}3\\-3\\6\end{bmatrix}$$

   (b)

$$\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}2\\3\\1\end{bmatrix}$$

   (c)

$$\begin{bmatrix}1\\-1\\2\end{bmatrix}, \begin{bmatrix}-2\\2\\-4\end{bmatrix}, \begin{bmatrix}3\\-2\\5\end{bmatrix}, \begin{bmatrix}2\\-1\\3\end{bmatrix}$$

6. Let $A = \begin{bmatrix} 1 & 2 & 2 & 3 & 1 & 4 \\ 2 & 4 & 5 & 5 & 4 & 9 \\ 3 & 6 & 7 & 8 & 5 & 9 \end{bmatrix}$

(a) Compute the reduced row echelon form U of A. Write the nonpivot columns of U as a linear combination of the pivot columns of U.
(b) Give a basis for $\operatorname{Col} A$. Show that you can write the columns of A as linear combinations of your basis vectors. (Be smart about it — you can do this without solving any system of linear equations.)

7. Let $A = \begin{bmatrix} 1 & 5 & 0 & -1 & 3 & 7 \\ -1 & -5 & 2 & 0 & -8 & -12 \\ -2 & -10 & 2 & 2 & -11 & -19 \\ 1 & 5 & -6 & 2 & 18 & 22 \end{bmatrix}$.

Give a basis for $\operatorname{Col} A$ and $\operatorname{Null} A$.

8. Suppose $\operatorname{Null} A = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 5 \\ 0 \\ -1 \\ 6 \end{bmatrix} \right\}$.

(a) What do you know about the size of A?

(b) Give a linear dependence relation for the columns of A.

9. Given an $m \times n$ matrix A, and an $n \times n$ matrix B.

(a) What are the columns of AB?

(b) Show that $\operatorname{Col} AB \subseteq \operatorname{Col} A$.

(c) Give an example of A and B where the subset inequality is strict, i.e., $\operatorname{Col} AB \neq \operatorname{Col} A$ (you can choose m and n).