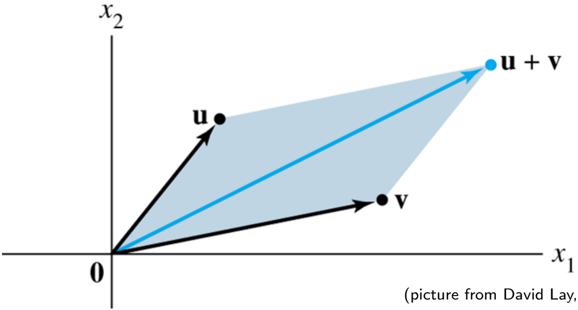
Lecture 19: More Geometry

MAT2040 Linear Algebra

Geometric Interpretation of Vector Addition

(Recall from Lecture 9!)



(picture from David Lay, Linear Algebra.)

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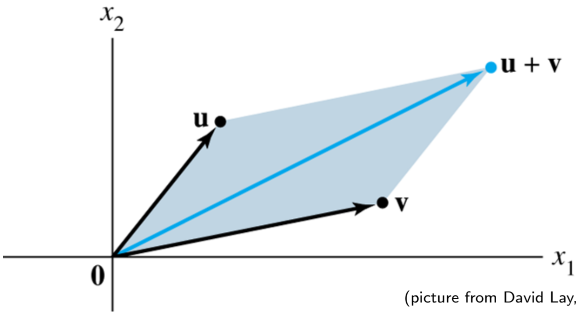

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Coming lectures: more geometry! Length, distance, **orthogonality**, angles! (In $\mathbb{R}^n$!).

Definition 19.1

Let **u** and **v** be two vectors in $\mathbb{R}^n$. The **inner product** of **u** and **v** is

$$\mathbf{u}^T\mathbf{v} = u_1v_1 + u_2v_2 + \cdots + u_nv_n.$$

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(This (and a lot of this lecture) can be generalized to other vector spaces, but we will only talk about $\mathbb{R}^n$ here.)

Example 19.2

Compute the inner product of $\begin{bmatrix} 1 \\ -7 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}$.

Note that the inner product is a matrix times a vector, for which we know all kinds of algebraic rules, for instance:

- $(\mathbf{u}+\mathbf{v})^T\mathbf{w} = \mathbf{u}^T\mathbf{w} + \mathbf{v}^T\mathbf{w}$
- $(t\mathbf{u})^T\mathbf{v} = t(\mathbf{u}^T\mathbf{v})$

for any $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, $t \in \mathbb{R}$.

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Also, from the definition is it not hard to see that

- $\mathbf{u}^T \mathbf{v} = \mathbf{v}^T \mathbf{u}$
- $\mathbf{u}^T \mathbf{u} \geq 0$, and $\mathbf{u}^T \mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$.

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The **(Euclidean) length** of the vector $\mathbf{x} \in \mathbb{R}^n$ is

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Note: in this definition we interpret the vector $\mathbf{x}$ as a line segment (geometrically).

Example 19.4

Let $\mathbf{x} = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}$. Find a **unit vector** in the same direction as $\mathbf{x}$.

(A **unit vector** is a vector with length equal to 1.)

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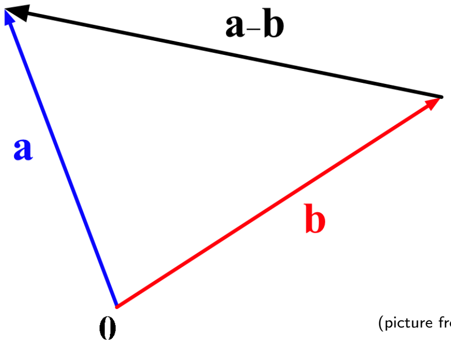
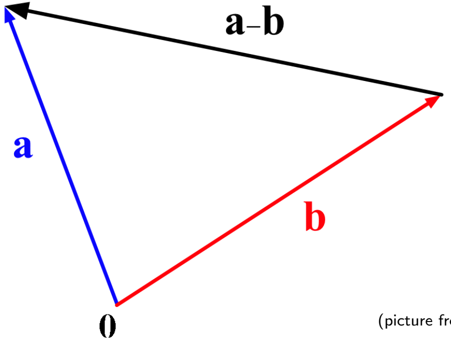
In general, what should be the distance between $\mathbf{a}$ and $\mathbf{b}$?



(picture from `stackoverflow.com`.)



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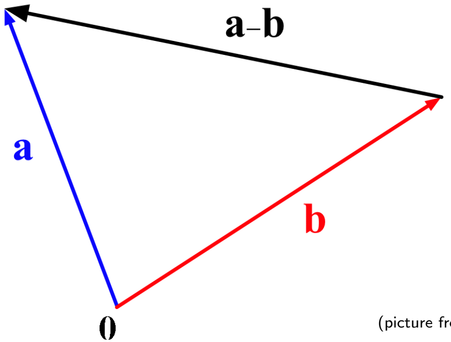
Definition 19.5

The **distance** between $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^n$ is the number $\|\mathbf{a}-\mathbf{b}\|$.



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Definition 19.5

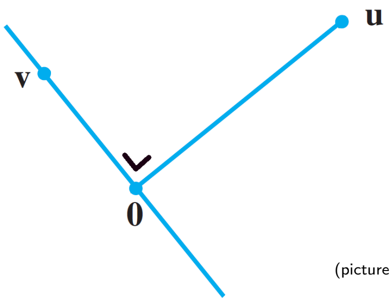
The **distance** between **a** and **b** in $\mathbb{R}^n$ is the number $\|\mathbf{a} - \mathbf{b}\|$.

Note: here we interpret **a** and **b** as points (geometrically), and $\mathbf{a} - \mathbf{b}$ as a line segment.

Example 19.6

What is the distance between $\mathbf{x} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$?

Othogonality in $\mathbb{R}^2$



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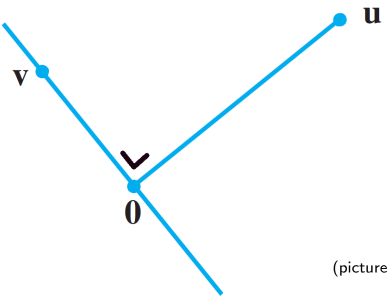

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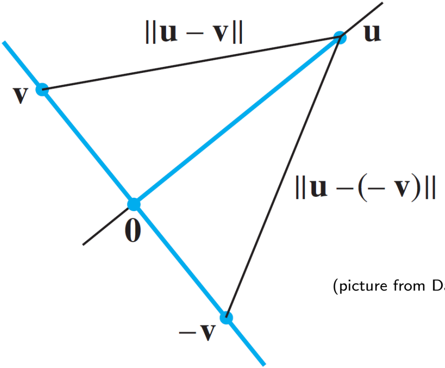
Can we generalize this?

Othogonality in $\mathbb{R}^2$



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Intuition: **u** and **v** are orthogonal exactly when
$\|\mathbf{u} - \mathbf{v}\| = \|\mathbf{u} - (-\mathbf{v})\|$.

Definition 19.7

Vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ are **orthogonal** if $\mathbf{u}^T\mathbf{v} = 0$.

Example 19.8

$\mathbf{0} \in \mathbb{R}^n$ is orthogonal to every vector $\mathbf{x} \in \mathbb{R}^n$.

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Example 19.9

Find all vectors in $\mathbb{R}^2$ that are orthogonal to $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$. Check your answer with a picture.

Theorem 19.10 (Pythagorean Theorem)

Two vectors $\mathbf{u}$ *and* $\mathbf{v}$ *in* $\mathbb{R}^n$ *are orthogonal, if and only if* $\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$.

Angle between two vectors $\mathbf{x}$ and $\mathbf{y}$?

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Fact 19.11

The **angle** ϑ *between two nonzero vectors* $\mathbf{x}$ *and* $\mathbf{y} \in \mathbb{R}^n$ *is given by*

$$\cos\vartheta = \frac{\mathbf{x}^T\mathbf{y}}{\|\mathbf{x}\|\,\|\mathbf{y}\|}.$$

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The **vector projection** *of* $\mathbf{x}$ *onto* $\mathbf{y}$ *is*

$$\mathbf{p} = a\frac{1}{\|\mathbf{y}\|}\mathbf{y} = \frac{\mathbf{x}^T\mathbf{y}}{\|\mathbf{y}\|^2}\mathbf{y}.$$

Example 19.12

Find the angle between $\mathbf{x} = \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$.

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What is the scalar projection of $\mathbf{x}$ onto $\mathbf{y}$?

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Find the angle between $\mathbf{x} = \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$.

What is the scalar projection of $\mathbf{x}$ onto $\mathbf{y}$?

What is the vector projection of $\mathbf{x}$ onto $\mathbf{y}$?

Corollary 19.13 (Cauchy-Schwarz Inequality)

For $\mathbf{x}$ *and* $\mathbf{y} \in \mathbb{R}^n$*, the following inequality holds:*

$$|\mathbf{x}^T\mathbf{y}| \leq \|\mathbf{x}\|\|\mathbf{y}\|.$$

The inequality holds with equality if and only if one of the vectors is zero, or one vector is a multiple of the other.

Back to orthogonality — important concept in Linear Algebra!

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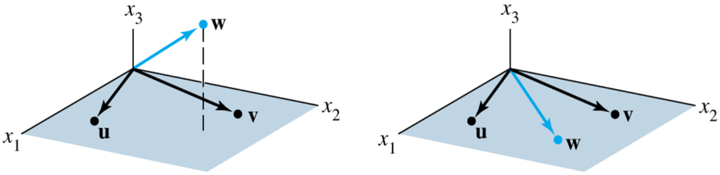
Recall from Lecture 10:



Linearly independent, $\mathbf{w}$ *not* in Span$\{\mathbf{u}, \mathbf{v}\}$

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Theorem 19.14 (Orthogonality implies linear independence)

Let $\mathcal{U} = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ *be a set of linearly independent vectors in* $\mathbb{R}^n$*. Suppose* $\mathbf{x} \in \mathbb{R}^n$ *is orthogonal to* $\mathbf{u}_i$ *for all* $i = 1, 2, \ldots, p$ *and* $\mathbf{x} \neq \mathbf{0}$*.*
Then $\mathcal{U} \cup \{\mathbf{x}\} = \{\mathbf{x}, \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ *is linearly independent.*

Theorem 19.14 (Orthogonality implies linear independence)

Let $\mathcal{U} = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ be a set of linearly independent vectors in $\mathbb{R}^n$, and $\mathcal{V} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a set of linearly independent vectors in $\mathbb{R}^n$, so that $\mathbf{u}_i$ is orthogonal to $\mathbf{v}_j$ for all $i = 1, 2, \ldots, p$ and $j = 1, 2, \ldots, k$.
Then $\mathcal{U} \cup \mathcal{V}$ is linearly independent.

Example 19.15

Suppose $\mathcal{U} = \left\{ \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Check that $\mathbf{x} = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}$ is orthogonal to both vectors in $\mathcal{U}$, and that $\mathcal{U} \cup \{\mathbf{x}\}$ is linearly independent.

Definition 19.16

Two subspaces X and Y of $\mathbb{R}^n$ are said to be **orthogonal** if $\mathbf{x}^T\mathbf{y} = 0$ for every pair of vectors $\mathbf{x} \in X$ and $\mathbf{y} \in Y$.

Notation: We write $X \perp Y$ if X and Y are orthogonal subspaces.

Definition 19.17

Given a subspace X of $\mathbb{R}^n$. The **orthogonal complement of** X, denoted by $X^\perp$, is the set of all vectors in $\mathbb{R}^n$ that are orthogonal to all vectors in X:

$$X^\perp = \{\mathbf{y} \in \mathbb{R}^n : \mathbf{x}^T\mathbf{y} = 0 \text{ for all } \mathbf{x} \in X\}.$$

Example 19.18

Let $X, Y \subseteq \mathbb{R}^4$, where $X = \text{Span}\{\mathbf{e}_1\}$ and

$$Y = \text{Span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 5 \end{bmatrix} \right\}.$$

Are X and Y orthogonal subspaces?

Is Y the orthogonal complement of X?

Example 19.19

Let X be a subspace of $\mathbb{R}^n$. Can $\mathbf{x} \in X$ be in $X^\perp$?

Theorem 19.20 (Fundamental Subspaces Theorem)

For any $m \times n$ matrix A,

$$\operatorname{Null} A = (\operatorname{Row} A)^{\perp}.$$

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Corollary 19.21

For any $m \times n$ matrix A,

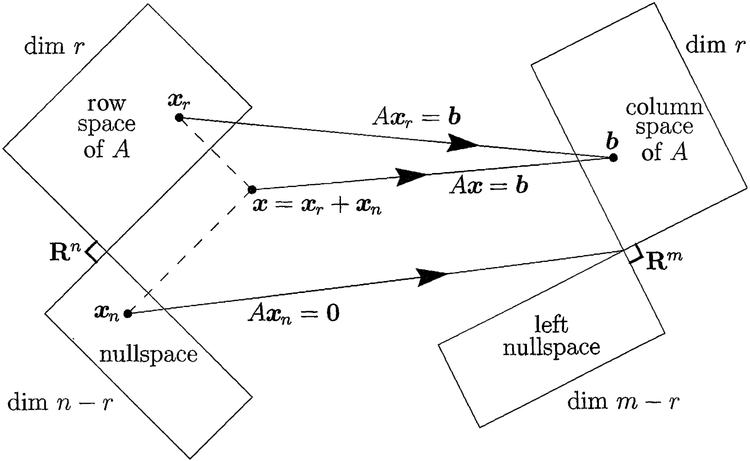
$$\operatorname{Null} A^T = (\operatorname{Row} A^T)^{\perp} = (\operatorname{Col} A)^{\perp}.$$

Corollary 19.22

If S is a subspace of $\mathbb{R}^n$ then $\dim S + \dim S^{\perp} = n$.

Theorem 19.23 (Orthogonal Decomposition Theorem)

If $\mathbf{x} \in \mathbb{R}^n$ *and* S *is a subspace of* $\mathbb{R}^n$, *then* $\mathbf{x}$ *can be* **uniquely** *expressed as* $\mathbf{x} = \hat{\mathbf{x}} + \mathbf{z}$, *where* $\hat{\mathbf{x}} \in S$ *and* $\mathbf{z} \in S^{\perp}$.



(picture from Gilbert Strang, Linear Algebra and Its Applications.)