Assignment 6

Hand-in Evaluation Deadline: 5:00 pm, December 9th
In-class Evaluation: L1: 2:40 pm - 2:50 pm, December 13th
L2: 9:40 am - 9:50 am, December 13th

> The material in lectures may differ between {L1, L2} on the one hand and {L3, L4} on the other, and therefore the homework assignment will differ for {L1, L2} and {L3, L4}.
>
> It is therefore **not** advisable to go to lecture L3 or L4 for the in-class homework evaluation if you attend L1 or L2!

1. (a) Consider the least squares problem $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \\ 1 & 1 \\ -1 & 0 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 4 \\ 1 \\ 0 \\ -2 \end{bmatrix}.$$

   i. Is there a unique least squares solution to $A\mathbf{x} = \mathbf{b}$? Explain.
   ii. Find all least squares solutions to $A\mathbf{x} = \mathbf{b}$.
   iii. Find the orthogonal projection $\hat{\mathbf{b}}$ of $\mathbf{b}$ onto $\operatorname{Col} A$.
   iv. Find the orthogonal projection $\tilde{\mathbf{b}}$ of $\mathbf{b}$ onto $\operatorname{Null} A^T$.

   (b) Let $B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 1 & 1 & 2 \\ -1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1, & \mathbf{a}_2, & \mathbf{a}_1 + \mathbf{a}_2 \end{bmatrix}$, where $\mathbf{a}_i$ are the columns of the matrix A above, and $\mathbf{b}$ is as defined above.

   i. Is there a unique least squares solution to $B\mathbf{x} = \mathbf{b}$? Explain.
   ii. Find all least squares solutions to $B\mathbf{x} = \mathbf{b}$.
   iii. Find the orthogonal projection $\hat{\mathbf{b}}$ of $\mathbf{b}$ onto $\operatorname{Col} B$.
   iv. Find the orthogonal projection $\tilde{\mathbf{b}}$ of $\mathbf{b}$ onto $\operatorname{Null} B^T$.

   (c) Let $C = \begin{bmatrix} 1 & 2 & 4 \\ 2 & -1 & 1 \\ 1 & 1 & 0 \\ -1 & 0 & -2 \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1, & \mathbf{a}_2, & \mathbf{b} \end{bmatrix}$, where $\mathbf{b}$ is as above.

   i. Is there a unique least squares solution to $C\mathbf{x} = \mathbf{b}$? Explain.
   ii. Find all least squares solutions to $C\mathbf{x} = \mathbf{b}$.
   iii. Find the orthogonal projection $\hat{\mathbf{b}}$ of $\mathbf{b}$ onto $\operatorname{Col} C$.
   iv. Find the orthogonal projection $\tilde{\mathbf{b}}$ of $\mathbf{b}$ onto $\operatorname{Null} C^T$.

2. Note that $\begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = 1 + 0 - 1 = 0.$

   Find the orthogonal projection of $\mathbf{x} = \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}$ on Span $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\}$.

3. (a) Find the best least squares fit by a linear function to the data

x	-1	0	1	2
y	0	1	3	9

   (b) Plot your linear function from part (a) along with the data on a coordinate system.

   (c) Find the best least squares fit by a quadratic polynomial.

   (d) Sketch your quadratic function from part (c) along with the data on a coordinate system.

4. (a) Find the projection matrix for projecting a vector on Span$\{\mathbf{a}\}$ for a nonzero vector $\mathbf{a}$.

   (b) How does this compare with the expression in Theorem 21.3? Explain why this makes sense.

5. Let P be the projection matrix for finding the orthogonal projection on Col A. Show that $P^2 = P$, and explain why that makes sense intuitively.

6. Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 12 \\ 6 \\ 18 \end{bmatrix}$.

   (a) Use the Gramm-Schmidt Process to find an orthonormal basis for the column space of A.

   (b) Find a QR-factorization of A.

   (c) Solve the least squares problem $A\mathbf{x} = \mathbf{b}$.

7. Let U and V be $n \times n$ (square) matrices with orthonormal columns. Show that UV has orthonormal columns.

8. Let A be an $m \times n$ matrix of rank n and let $\mathbf{b} \in \mathbb{R}^m$. Show that if Q and R are the matrices derived from applying the Gramm-Schmidt Process to the column vectors of A and

$$\hat{\mathbf{b}} = c_1\mathbf{q}_1 + c_2\mathbf{q}_2 + \cdots + c_n\mathbf{q}_n$$

   is the orthogonal projection of $\mathbf{b}$ onto Col A, then

   (a) $\mathbf{c} = Q^T\mathbf{b}$

   (b) $\hat{\mathbf{b}} = QQ^T\mathbf{b}$

   (c) $QQ^T = A(A^TA)^{-1}A^T$.

9. Let S be a subspace of $\mathbb{R}^n$.

   Show that the mapping $\mathbf{x} \mapsto \hat{\mathbf{x}}$, where $\hat{\mathbf{x}}$ is the orthogonal projection of $\mathbf{x}$ onto S, is a linear transformation.

10. Prove that $(S^{\perp})^{\perp} = S$ for any subspace S of $\mathbb{R}^n$.

11. Find the eigenvectors and corresponding eigenspaces of $A = \begin{bmatrix} 6 & -4 \\ 3 & -1 \end{bmatrix}$.

12. Show that the eigenvalues of a triangular matrix are the diagonal elements of the matrix.

13. Let A be an $n \times n$ matrix. Prove that A is singular if and only if 0 is an eigenvalue of A.

14. Let T be transformation that rotates points in $\mathbb{R}^3$ about some line through the origin. T is a linear transformation (you do not have to prove this). Without writing A, find an eigenvalue of A and describe the corresponding eigenspace.

15. Suppose Q and R are $n \times n$ matrices, and Q is invertible. Show that the matrices $A = QR$ and $B = RQ$ are similar.

    (This is the basis for the *QR-algorithm*, a numerically stable method for computing the eigenvalues and eigenvectors of a matrix: You iteratively find the QR-decomposition of A, then redefine A to be RQ, and repeat.)