Exercises 9

These used to be called "Fundamental Questions". Do not hand the solutions to these exercises in; they are just to make sure you can have some practice with the current material.

1. Let $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \\ -2 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 4 \\ 0 \\ 4 \\ 0 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix}$, and $\mathbf{u}_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

   Which pairs are orthogonal among the vectors $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$, and $\mathbf{u}_4$?

2. Let $\mathbf{x} = \begin{bmatrix} \sqrt{3} & 1 & 3 \end{bmatrix}$, and $\mathbf{y} = \begin{bmatrix} 2 & 1 & 2 \end{bmatrix}$.

   (a) What is the (Euclidean) length of $\mathbf{x}$, and the (Euclidean) length of $\mathbf{y}$?

   (b) What is the (Euclidean) distance between $\mathbf{x}$ and $\mathbf{y}$?

   (c) What is the angle between $\mathbf{x}$ and $\mathbf{y}$?

   (d) What is the scalar projection of $\mathbf{x}$ onto $\mathbf{y}$?

   (e) What is the vector projection of $\mathbf{x}$ onto $\mathbf{y}$?

3. Let A be an $n \times n$ matrix, and $\mathbf{x}$ and $\mathbf{y}$ be vectors so that $A\mathbf{x} = \mathbf{0}$ and $A^T\mathbf{y} = 5\mathbf{y}$. Show that $\mathbf{x}$ and $\mathbf{y}$ are orthogonal.

4. Suppose $S = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 3 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 2 \\ -5 \end{bmatrix} \right\}$.

   (a) Find two vectors that span $S^\perp$.

   (b) This is the same as solving $A\mathbf{x} = \mathbf{0}$ for which matrix A?

5. Find the orthogonal projection of $\mathbf{x} = \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}$ on $\operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\}$.

6. Find the least squares solution to

$$\begin{bmatrix} -1 & 2 \\ 2 & -3 \\ -1 & 3 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}.$$

7. Let $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$. Give all vectors $\mathbf{y}$ for which the Cauchy-Schwarz inequality for $\mathbf{x}$ and $\mathbf{y}$ holds with equality.

8. Prove the Pythagorean Theorem. (Don't forget it is an "if and only if"!)

9. Explain why the following statement is false: If a subspace V is orthogonal to subspace W then $V^{\perp}$ is orthogonal to $W^{\perp}$.

10. Let S be a subspace of $\mathbb{R}^n$. Prove that $S^{\perp}$ is a subspace.

11. Suppose $\mathcal{U} = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$, and $\mathcal{V} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$, and $\mathbf{u}_i \cdot \mathbf{v}_j = 0$ for all $i = 1, 2, \ldots, p$ and $j = 1, 2, \ldots, k$.

    Show that $\operatorname{Span} \mathcal{U}$ and $\operatorname{Span} \mathcal{V}$ are orthogonal subspaces.