Lecture 13: Sensitivity Analysis

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Announcement

- Homework 4 due next Wednesday (10/24)

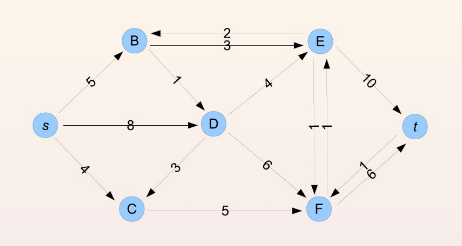
Recap: Duality Theory

- Construct the dual problem
- Weak duality theorem/strong duality theorem
- Complementarity conditions
- Interpret the dual problem
  1. The production planning problem
  2. The multi-firm alliance problem
  3. The transportation problem
  4. The alternative systems problem

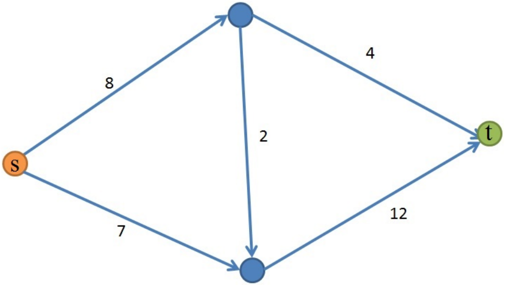
One More Example — Maximum Flow Problem

The maximum flow problem can be described as follows:

- Given a directed, weighted graph $G = (V, E)$ and a pair of nodes s and t (V is the set of nodes, E is the set of edges)
- One can think this as a traffic network
- There is an edge capacity w_{ij} on each edge
- Question: What is the largest amount of flow one can send from s to t, subject to the capacity constraints?



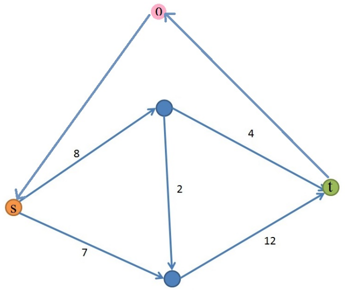
Homework Example



One Transformation

Assume there is an imaginary node o, with edges (o, s) and (t, o). There is no capacity constraint on those edges

- The problem becomes a closed system. One wants to maximize the flow from o to s, which we denote by Δ.



LP Formulation

Using this transformation, we can write down the LP formulation. Let x_{ij} denote the amount of flow across edge (i, j).

$$
\begin{array}{lcl}
\text{maximize}_{\mathbf{x},\Delta} & \Delta & \\
\text{subject to} & \sum_{j:(j,i)\in E} x_{ji} - \sum_{j:(i,j)\in E} x_{ij} = 0, & \forall i \neq s, t \\
& \sum_{j:(j,s)\in E} x_{js} - \sum_{j:(s,j)\in E} x_{sj} + \Delta = 0 & \\
& \sum_{j:(j,t)\in E} x_{jt} - \sum_{j:(t,j)\in E} x_{tj} - \Delta = 0 & \\
& x_{ij} \leq w_{ij}, & \forall (i,j) \in E \\
& x_{ij} \geq 0, & \forall (i,j) \in E
\end{array}
$$

- The first constraint is the flow balancing constraints for all nodes other than s and t
- The second (third, resp.) constraint is the flow balancing constraints for node s (t, resp.)

Dual of the Maximum Flow Problem

We construct the dual problem:

$$\begin{aligned} \text{minimize} \quad & \sum_{(i,j)\in E} w_{ij} z_{ij} \\ \text{subject to} \quad & z_{ij} \geq y_i - y_j, \qquad\qquad \forall (i,j) \in E \\ & y_s - y_t = 1 \\ & z_{ij} \geq 0 \end{aligned}$$

What does the dual problem mean?

First assume all y's are 0 or 1. Then

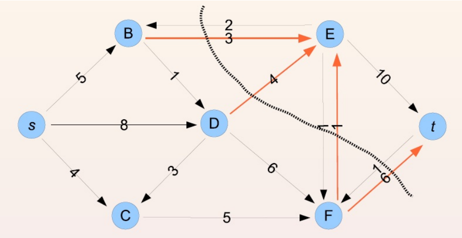
- We assign a label (0 or 1) to each node, 1 to s and 0 to t.
- If i has a larger label than j for $(i,j) \in E$, there is a cost w_{ij}.

Interpretation of the Dual

The dual problem is equivalent to finding a subset S of vertices containing s but not t, that minimizes the weight of the cut, i.e.

$$\sum_{i \in S, j \notin S} w_{ij}$$

- This is called the min-cut problem



Strong Duality Says..

Theorem

The maximum flow of the network is equal to the smallest cut size of any subset S of vertices.

Corollary

If the value of a flow is equal to the value of some cut, then both are optimal.

- One can view the min-cut as the bottleneck of the network.
- The maximum flow that can be sent through this network is equal to the tightest bottleneck of this network.

This is one classical example of dual problems: maximum flow versus minimum cut.

Sensitivity Analysis

One important question when studying LP is as follows:

- How do the optimal solution and the optimal value change when the input changes?

This type of problems is called the *Sensitivity Analysis* of LP.

- We first study this question from a local perspective, and then globally

Local Sensitivity

Consider the standard LP:

$$\begin{aligned} \text{minimize}_{\mathbf{x}} \quad & \mathbf{c}^T\mathbf{x} \\ \text{s.t.} \quad & A\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq 0 \end{aligned}$$

Define the optimal value by V.

- Given A and $\mathbf{c}$ fixed, V can be viewed as a function of $\mathbf{b}$: $V(\mathbf{b})$

Theorem

If the dual has a unique optimal solution $\mathbf{y}^*$, *then* $\nabla V(\mathbf{b}) = \mathbf{y}^*$.

- If the dual optimal solution is not unique (or is unbounded or infeasible), then the gradient does not exist.
- If one changes b_i by a small amount Δb_i, then the change of the objective value will be $\Delta b_i y_i^*$

Explanation

We know that the optimal value V is also the optimal value of the dual problem:

$$\begin{aligned} \text{maximize}_{\mathbf{y}} \quad & \mathbf{b}^T\mathbf{y} \\ \text{s.t.} \quad & A^T\mathbf{y} \leq \mathbf{c} \end{aligned}$$

i.e., $V(\mathbf{b}) = \mathbf{b}^T\mathbf{y}^*$.

If we change $\mathbf{b}$ by a small amount $\Delta\mathbf{b}$, such that the optimal solution does not change, then the change to V must be $\Delta\mathbf{b}^T\mathbf{y}^*$.

Local Sensitivity

Similarly, given A and $\mathbf{b}$ fixed, V can be viewed as a function of $\mathbf{c}$.

Theorem

If the primal problem has a unique optimal solution $\mathbf{x}^*$, then $\nabla V(\mathbf{c}) = \mathbf{x}^*$.

If one changes c_i by a small amount Δc_i, then the change of the objective value will be $\Delta c_i x_i^*$

- Reason: If we change $\mathbf{c}$ by a small amount $\Delta\mathbf{c}$, such that the optimal solution does not change, then the change to V must be $\Delta\mathbf{c}^T\mathbf{x}^*$.

The above results also hold for inequality constraints (or maximization problem) such as follows:

$$
\begin{array}{rl}
\text{maximize}_{\mathbf{x}} & \mathbf{c}^T\mathbf{x} \\
\text{s.t.} & A\mathbf{x} \le \mathbf{b} \\
 & \mathbf{x} \ge 0
\end{array}
$$

We have:

1. If the dual has a unique optimal solution $\mathbf{y}^*$, then $\nabla V(\mathbf{b}) = \mathbf{y}^*$
2. If the primal has a unique optimal solution $\mathbf{x}^*$, then $\nabla V(\mathbf{c}) = \mathbf{x}^*$

- To see why this must be true, one can add a slack variable and transform it back to the standard form and then one can use the earlier result.

Example (Production Planning)

$$\begin{array}{rrrl} \text{maximize} & x_1 & +2x_2 & \\ \text{subject to} & x_1 & & \leq 100 \\ & & 2x_2 & \leq 200 \\ & x_1 & +x_2 & \leq 150 \\ & x_1, & x_2 & \geq 0 \end{array}$$

The optimal solution is $\mathbf{x}^* = (50, 100)$ with optimal value 250.

The dual problem is

$$\begin{array}{rrrrl} \text{minimize} & 100y_1 & +200y_2 & +150y_3 & \\ \text{subject to} & y_1 & & +y_3 & \geq 1 \\ & & 2y_2 & +y_3 & \geq 2 \\ & y_1, & y_2, & y_3 & \geq 0 \end{array}$$

The optimal solution is $\mathbf{y}^* = (0, 0.5, 1)$ with optimal value 250.

Example Continued

$$\begin{array}{lllll} \text{maximize} & x_1 & +2x_2 & & \\ \text{subject to} & x_1 & & \leq & 100 \\ & & 2x_2 & \leq & 200 \\ & x_1 & +x_2 & \leq & 150 \\ & x_1, & x_2 & \geq & 0 \end{array}$$

The optimal solution is $\mathbf{x}^* = (50, 100)$ with optimal value 250. The dual optimal solution is $\mathbf{y}^* = (0, 0.5, 1)$

1. What would be the optimal value if we now have 202 units of resource 2?

- It will change by $\Delta b_2 y_2^* = 1$. Therefore, the optimal value would be 251

Example Continued

$$\begin{array}{rrrl} \text{maximize} & x_1 & +2x_2 & \\ \text{subject to} & x_1 & & \leq 100 \\ & & 2x_2 & \leq 200 \\ & x_1 & +x_2 & \leq 150 \\ & x_1, & x_2 & \geq 0 \end{array}$$

The optimal solution is $\mathbf{x}^* = (50, 100)$ with optimal value 250. The dual optimal solution is $\mathbf{y}^* = (0, 0.5, 1)$

2. What would be the optimal value if we now have 99 units of resource 1?

- It will change by $\Delta b_1 y_1^* = 0$. Therefore, the optimal value would be unchanged.

Example Continued

$$\begin{array}{lrrl} \text{maximize} & x_1 & +2x_2 & \\ \text{subject to} & x_1 & & \leq 100 \\ & & 2x_2 & \leq 200 \\ & x_1 & +x_2 & \leq 150 \\ & x_1, & x_2 & \geq 0 \end{array}$$

The optimal solution is $\mathbf{x}^* = (50, 100)$ with optimal value 250. The dual optimal solution is $\mathbf{y}^* = (0, 0.5, 1)$

3. What would be the optimal value if the profit of product 1 becomes 1.02?

- It will increase by $\Delta c_1 x_1^* = 1$. Therefore, the optimal value would be 251

Example Continued

$$
\begin{array}{lrrl}
\text{maximize} & x_1 & +2x_2 & \\
\text{subject to} & x_1 & & \leq 100 \\
 & & 2x_2 & \leq 200 \\
 & x_1 & +x_2 & \leq 150 \\
 & x_1, & x_2 & \geq 0
\end{array}
$$

The optimal solution is $\mathbf{x}^* = (50, 100)$ with optimal value 250. The dual optimal solution is $\mathbf{y}^* = (0, 0.5, 1)$

4. What would be the optimal value if the profit of product 2 becomes 1.97?

- It will decrease by $\Delta c_2 x_2^* = -3$. Therefore, the optimal value would be 247

Another Property

$$\begin{aligned} \text{maximize}_{\mathbf{x}} \quad & \mathbf{c}^T\mathbf{x} \\ \text{s.t.} \quad & A\mathbf{x} \le \mathbf{b} \\ & \mathbf{x} \ge 0 \end{aligned}$$

If at optimal $\mathbf{x}^*$, $\mathbf{a}_i^T\mathbf{x}^* < b_i$, then what happens if we change b_i?

- By the complementarity conditions, the corresponding dual variable y_i^* must be 0.
- Therefore, changing the right-hand-side of an inactive constraint by a small amount won't affect the optimal value (also the optimal solution)
- Intuition: If a resource is already redundant, then adding or reducing a small amount wouldn't matter

Shadow Prices

Recall that

- $\nabla V(\mathbf{b}) = \mathbf{y}^*$, where $\mathbf{y}^*$ is the optimal dual solution

We call $\mathbf{y}^*$ the shadow prices of $\mathbf{b}$.

- In the production example, the shadow price of a resource corresponds to the increment of profit if there is one unit more of that resource (locally)
- Therefore, it can be viewed as the *unit value* or *unit fair price* for that resource
- Remember we come up with the same explanation when discussing its dual problem

Caveat

The above analysis is *local*, meaning that it can only deal with small changes.

- Basically, it is valid as long as the optimal basis does not change.
- Otherwise, it may not be true.

For example, in the production planning problem, if the amount of resource 1 reduces to 0, then the optimal solution will be $(0, 100)$, with optimal value 200 (reduced by 50). This difference would be different from $\Delta b_1 y_1^* = 0$

- We want to study what ranges of changes belong to *small* changes.
- This will be the *global sensitivity analysis*.