Exercises 10

These used to be called "Fundamental Questions". Do not hand the solutions to these exercises in; they are just to make sure you can have some practice with the current material.

1. Let Q be an $n \times n$ matrix with orthonormal columns. Show that $\det Q = \pm 1$.

2. A is the matrix with eigenvalue $\lambda_1 = 5$ and corresponding eigenvector $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and $\lambda_1 = -1$ and corresponding eigenvector $\mathbf{x}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

   (a) Write down A explicitly.

   (b) Write down A^{10} explicitly (scalars to the power 10 in your answer are fine).

3. Let $A = \begin{bmatrix} 8 & 20 \\ 4 & -12 \end{bmatrix}$ and $B = \begin{bmatrix} -14 & 36 \\ 1 & 10 \end{bmatrix}$.

   (a) What are $\operatorname{trace} A$ and $\det A$?

   (b) What are $\operatorname{trace} B$ and $\det B$?

   (c) Do A and B have the same eigenvalues? Explain your answer.

4. Let $A = \begin{bmatrix} \sqrt{6} & 1 \\ 0 & \sqrt{6} \end{bmatrix}$.

   (a) Give all distinct eigenvalues of A with their algebraic and geometric multiplicities.

   (b) Is A diagonalizable?

   (c) Find a singular value decomposition of A.

5. Let $\mathbf{u} \in \mathbb{R}^m$ and $\mathbf{v} \in \mathbb{R}^n$.

   (a) What is the size of $\mathbf{u}\mathbf{v}^T$?

   (b) What is the rank of $\mathbf{u}\mathbf{v}^T$?

   (c) Give an orthonormal basis for $\operatorname{Span}\{\mathbf{u}\}$.

   (d) Give a singular value decomposition of $\mathbf{u}\mathbf{v}^T$.

6. Let $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

   (a) Find a singular value decomposition of A.

   (b) Recall that $\mathbf{x} \mapsto A\mathbf{x}$ is the linear transformation rotating every vector 90° about the origin.
   Explain that the singular value decomposition you found makes intuitive sense.

7. Suppose $A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & -1 \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} 7 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}$.

(a) What is the size of A?

(b) What is $\operatorname{rank} A$?

(c) What is $\operatorname{Col} A$?

(d) What is $\operatorname{Row} A$?

(e) What is $\operatorname{Null} A$?

(f) What is $\operatorname{Null} A^T$?

(g) What is $A\mathbf{e}_1$?

(h) What is $A \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$?

(i) Find $\mathbf{y}$ so that $\|A\mathbf{y}\|$ is maximized.

(j) Give the best rank-1 approximation of A.