Lecture 21: Orthogonality and Bases

MAT2040 Linear Algebra

Recall coordinates:

Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_p\}$ be a basis for a subspace W of $\mathbb{R}^n$, and $P = \begin{bmatrix} \mathbf{b}_1, & \mathbf{b}_2, & \dots, & \mathbf{b}_p \end{bmatrix}$.

To compute the $\mathcal{B}$-coordinates of $\mathbf{x} \in W$ we have to solve the system of linear equations

$$P[\mathbf{x}]_{\mathcal{B}} = \mathbf{x}.$$

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A lot of work in general.

What if $\mathbf{b}_i$ is **orthogonal** to $\mathbf{b}_j$ for every $i \neq j$? (The set $\mathcal{B}$ is then called an **orthogonal set**.)

Theorem 21.1 (Coordinates with respect to Orthogonal Basis)

Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_p\}$ *be a basis for a subspace* W *of* $\mathbb{R}^n$*, that is also an orthogonal set. For every* $\mathbf{x} \in W$ *the weights in the linear combination*

$$\mathbf{x} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_p\mathbf{b}_p$$

are given by

$$c_i = \frac{\mathbf{x} \cdot \mathbf{b}_i}{\mathbf{b}_i \cdot \mathbf{b}_i}.$$

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$$c_i = \frac{\mathbf{x} \cdot \mathbf{b}_i}{\mathbf{b}_i \cdot \mathbf{b}_i}.$$

We will call such $\mathcal{B}$ an **orthogonal basis** for W.

Example 21.2

Let $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ -1 \\ 0 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} \right\}$ and $\mathbf{x} = \begin{bmatrix} 4 \\ -7 \\ -2 \\ 15 \end{bmatrix}$.

Compute $[\mathbf{x}]_{\mathcal{B}}$.

Theorem 21.3 (Projection on Subspace with Orthogonal Basis)

Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_p\}$ be an orthogonal basis for a subspace W of $\mathbb{R}^n$. Let $\mathbf{x} \in \mathbb{R}^n$, and let $\hat{\mathbf{x}}$ be the orthogonal projection of $\mathbf{x}$ on W.

Then $\hat{\mathbf{x}}$ is given by

$$\hat{\mathbf{x}} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_p\mathbf{b}_p$$

where

$$c_i = \frac{\mathbf{x} \cdot \mathbf{b}_i}{\mathbf{b}_i \cdot \mathbf{b}_i}.$$

Recall:

Theorem 19.23 (Orthogonal Decomposition Theorem)

If $\mathbf{x} \in \mathbb{R}^n$ *and* S *is a subspace of* $\mathbb{R}^n$*, then* $\mathbf{x}$ *can be* **uniquely** *expressed as* $\mathbf{x} = \hat{\mathbf{x}} + \mathbf{z}$*, where* $\hat{\mathbf{x}} \in S$ *and* $\mathbf{z} \in S^{\perp}$*.*

Example 21.4

Let $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ -1 \\ 0 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} \right\}$ and $\mathbf{x} = \begin{bmatrix} 6 \\ 2 \\ 2 \\ 1 \end{bmatrix}$.

Compute the orthogonal projection of $\mathbf{x}$ on $\operatorname{Span} \mathcal{B}$.

This would be even nicer if $\mathbf{b}_i \cdot \mathbf{b}_i = 1$ for every $i = 1, 2, \ldots, p$.

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Definition 21.5

An orthogonal set $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_p\}$ where $\|\mathbf{b}_i\| = 1$ or all $i = 1, 2, \ldots, p$ is called an **orthonormal set**.

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An orthogonal set $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_p\}$ where $\|\mathbf{b}_i\| = 1$ or all $i = 1, 2, \ldots, p$ is called an **orthonormal set**.

An orthonormal set $\mathcal{B}$ that is also a basis, will be referred to as an **orthonormal basis**.

Theorem 21.6 (Coordinates with respect to and Projection on Subspace with Orthonormal Basis)

Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_p\}$ be an orthonormal basis for a subspace W of $\mathbb{R}^n$. For every $\mathbf{x} \in W$ the weights in the linear combination

$$\mathbf{x} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_p\mathbf{b}_p$$

are given by

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$$\mathbf{x} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_p\mathbf{b}_p$$

are given by

$$c_i = \mathbf{x} \cdot \mathbf{b}_i.$$

For every $\mathbf{y} \in \mathbb{R}^n$ *the orthogonal projection of* $\mathbf{y}$ *on* W *is given by*

$$\hat{\mathbf{y}} = t_1\mathbf{b}_1 + t_2\mathbf{b}_2 + \cdots + t_p\mathbf{b}_p$$

where

$$t_i = \mathbf{y} \cdot \mathbf{b}_i.$$

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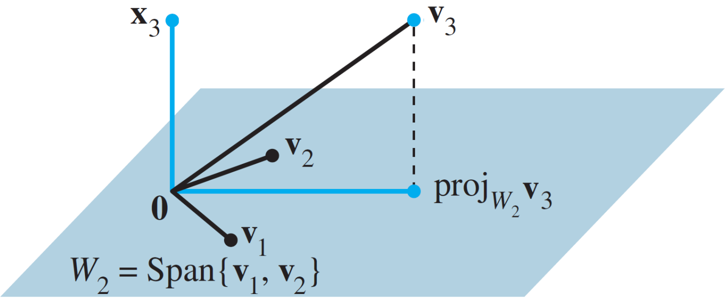
I hear you thinking: "This is very exciting, but do these nice orthonormal bases exist for any subspace? And if they exist, how can we find them?"

Idea of how to change basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ to orthogonal basis:



(picture from David Lay, Linear Algebra.)

$$\mathbf{x}_3 = \mathbf{v}_3 - \text{proj}_{W_2}\mathbf{v}_3$$

$\mathbf{x}_3$ is orthogonal to both $\mathbf{v}_1$ and $\mathbf{v}_2$!

Note that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{x}_3\}$ are linearly independent vectors in $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and must therefore be a basis for this subspace.

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We can therefore use $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{x}_3\}$ instead of $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ as a basis, and then repeat this process for $\mathbf{v}_2$ (for example).

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(And making an orthogonal basis orthonormal can be done by just scaling the vectors, to make them have unit length.)

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Calculations are easy, only requiring the calculation of inner products — see Theorem 21.6!

The Gram-Schmidt Process

Input: a basis $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_p\}$ for a subspace W.

Output: an orthonormal basis $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ for subspace W.

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Step 1: $\mathbf{u}_1 = \dfrac{1}{\|\mathbf{x}_1\|}\mathbf{x}_1$.

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Step 1: $\mathbf{u}_1 = \dfrac{1}{\|\mathbf{x}_1\|}\mathbf{x}_1$.

Step 2: $\mathbf{u}_2 = \dfrac{1}{\|\mathbf{y}_2\|}\mathbf{y}_2$ where $\mathbf{y}_2 = \mathbf{x}_2 - (\mathbf{x}_2 \cdot \mathbf{u}_1)\mathbf{u}_1$.

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$\hat{\mathbf{x}}_2 = (\mathbf{x}_2 \cdot \mathbf{u}_1)\mathbf{u}_1$ is the orthogonal projection of $\mathbf{x}_2$ on Span$\{\mathbf{u}_1\}$ (Theorem 21.6).

We therefore want to add $\mathbf{y}_2 = \mathbf{x}_2 - \hat{\mathbf{x}}_2$ to the basis — but we have to make it unit length.

Note that $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthonormal basis for Span$\{\mathbf{x}_1, \mathbf{x}_2\}$.

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Step 3: $\mathbf{u}_3 = \dfrac{1}{\|\mathbf{y}_3\|}\mathbf{y}_3$ where $\mathbf{y}_3 = \mathbf{x}_3 - (\mathbf{x}_3 \cdot \mathbf{u}_1)\mathbf{u}_1 - (\mathbf{x}_3 \cdot \mathbf{u}_2)\mathbf{u}_2$.

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Input: a basis $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_p\}$ for a subspace W.

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$\hat{\mathbf{x}}_3 = (\mathbf{x}_3 \cdot \mathbf{u}_1)\mathbf{u}_1 + (\mathbf{x}_3 \cdot \mathbf{u}_2)\mathbf{u}_2$ is the orthogonal projection of $\mathbf{x}_3$ on $\mathrm{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$ (Theorem 21.6).

We therefore want to add $\mathbf{y}_3 = \mathbf{x}_3 - \hat{\mathbf{x}}_3$ to the basis — after scaling. Note that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthonormal basis.

The Gram-Schmidt Process

Input: a basis $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_p\}$ for a subspace W.

Output: an orthonormal basis $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ for subspace W.

Step k: $\mathbf{u}_k = \dfrac{1}{\|\mathbf{y}_k\|}\mathbf{y}_k$ where

$$\mathbf{y}_k = \mathbf{x}_k - (\mathbf{x}_k \cdot \mathbf{u}_1)\mathbf{u}_1 - (\mathbf{x}_k \cdot \mathbf{u}_2)\mathbf{u}_2 - \cdots - (\mathbf{x}_k \cdot \mathbf{u}_{k-1})\mathbf{u}_{k-1}.$$

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$\hat{\mathbf{x}}_k = (\mathbf{x}_k \cdot \mathbf{u}_1)\mathbf{u}_1 + (\mathbf{x}_k \cdot \mathbf{u}_2)\mathbf{u}_2 + \cdots + (\mathbf{x}_k \cdot \mathbf{u}_{k-1})\mathbf{u}_{k-1}$ is the orthogonal projection of $\mathbf{x}_3$ on $\mathrm{Span}\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_{k-1}\}$ (Theorem 21.6).

We therefore want to add $\mathbf{y}_k = \mathbf{x}_k - \hat{\mathbf{x}}_k$ to the basis — but we have to make it unit length. Note that $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k\}$ is an orthonormal basis for $\mathrm{Span}\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\}$.

Theorem 21.7

Let $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_p\}$ be a basis for a subspace W.
The Gram-Schmidt Process produces an orthonormal basis $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ for subspace W after p steps. The vector $\mathbf{u}_k$ is computed in step k as:

$$\mathbf{u}_k = \frac{1}{\|\mathbf{y}_k\|}\mathbf{y}_k, \text{ where}$$

$$\mathbf{y}_k = \mathbf{x}_k - (\mathbf{x}_k \cdot \mathbf{u}_1)\mathbf{u}_1 - (\mathbf{x}_k \cdot \mathbf{u}_2)\mathbf{u}_2 - \cdots - (\mathbf{x}_k \cdot \mathbf{u}_{k-1})\mathbf{u}_{k-1}.$$

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($\mathbf{y}_k = \mathbf{x}_k - \hat{\mathbf{x}}_k$, where $\hat{\mathbf{x}}_k$ is the orthogonal projection of $\mathbf{x}_k$ on $\text{Span}\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{k-1}\}$.)

Example 21.8

Find an orthonormal basis for Col A, where

$$A = \begin{bmatrix} 1 & -1 & 4 \\ 1 & 4 & -2 \\ 1 & 4 & 2 \\ 1 & -1 & 0 \end{bmatrix}.$$

Let's try to understand the Gram-Schmidt Process a little better when we perform the Gram-Schmidt Process on the columns of a matrix A.

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It (iteratively) defines

$$\mathbf{u}_k = \lambda_k(\mathbf{a}_k - (\mathbf{a}_k \cdot \mathbf{u}_1)\mathbf{u}_1 - (\mathbf{a}_k \cdot \mathbf{u}_2)\mathbf{u}_2 - \cdots - (\mathbf{a}_k \cdot \mathbf{u}_{k-1})\mathbf{u}_{k-1}),$$

where λ_k is some (positive) real number (making sure $\|\mathbf{u}_k\| = 1$).

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where λ_k is some (positive) real number (making sure $\|\mathbf{u}_k\| = 1$).

Note that we can also solve this equation for $\mathbf{a}_k$, and we see that we can interpret this equation as "$\mathbf{a}_k$ can be written as a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$".

If we let Q be the matrix with columns equal to the vectors in the orthonormal basis, then the Gram-Schmidt Process tell us that column i of A can be written as a linear combination of the first i columns of Q!

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I.e., for every $i = 1, 2, \ldots, n$ there exist r_{ij} for $j = 1, 2, \ldots, i$ so that

$$\mathbf{a}_i = r_{1i}\mathbf{q}_1 + r_{2i}\mathbf{q}_2 + \cdots + r_{ii}\mathbf{q}_i.$$

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$$\mathbf{a}_i = r_{1i}\mathbf{q}_1 + r_{2i}\mathbf{q}_2 + \cdots + r_{ii}\mathbf{q}_i.$$

In matrix times vector notation...

$$\mathbf{a}_i = r_{1i}\mathbf{q}_1 + r_{2i}\mathbf{q}_2 + \cdots + r_{ii}\mathbf{q}_i = Q \begin{bmatrix} r_{1i} \\ r_{2i} \\ \vdots \\ r_{ii} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

And therefore

$$A = \begin{bmatrix} \mathbf{a}_1, & \mathbf{a}_2, & \ldots, & \mathbf{a}_n \end{bmatrix} = Q \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ 0 & r_{22} & \cdots & r_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & r_{nn} \end{bmatrix}.$$

Theorem 21.9

Let A be an $m \times n$ matrix of rank n. The Gram-Schmidt process can be used to find a decomposition $A = QR$, where Q is an $m \times n$ matrix with orthonormal column vectors that are a basis for $\operatorname{Col} A$, and R is an upper triangular $n \times n$ matrix, with positive diagonal entries.

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In fact, $r_{11} = \|\mathbf{a}_1\|$ and $r_{ii} = \|\mathbf{y}_i\|$ for all $i = 2, \ldots, n$ (where $\mathbf{y}_i$ as in the description of the algorithm), and

$$r_{ij} = \mathbf{a}_j \cdot \mathbf{u}_i$$

for all $i = 1, 2, \ldots, j-1$ and $j = 2, 3, \ldots, n$.

Example 21.10

Find a QR-decomposition of

$$A = \begin{bmatrix} 1 & -1 & 4 \\ 1 & 4 & -2 \\ 1 & 4 & 2 \\ 1 & -1 & 0 \end{bmatrix}.$$

Theorem 21.11

Let A be an $m \times n$ matrix of rank n. The least squares solution of $A\mathbf{x} = \mathbf{b}$ is the same as the solution to

$$R\mathbf{x} = Q^T\mathbf{b}.$$

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Note that you can use back substitution to find the solution, because R is upper triangular!

Example 21.12

Find the least squares solution to $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & -1 & 4 \\ 1 & 4 & -2 \\ 1 & 4 & 2 \\ 1 & -1 & 0 \end{bmatrix}$$

and

$$\mathbf{b} = \begin{bmatrix} 0 \\ 2 \\ 2 \\ 2 \end{bmatrix}.$$

Theorem 21.13

If Q is a matrix with orthonormal columns, then $Q^T Q = I$.

Summarizing:

- First decomposition you learned: $A = LU$.
  - L is lower triangular, U is upper triangular.
  - Good for solving a system of equations $A\mathbf{x} = \mathbf{b}$.
- Second decomposition: $A = QR$.
  - Q has orthonormal columns, R is upper triangular with positive diagonal.
  - Good for finding least squared solution to (inconsistent) $A\mathbf{x} = \mathbf{b}$, or (equivalently) finding projection of $\mathbf{b}$ on $\operatorname{Col} A$.

Note that Q's columns span $\operatorname{Col} A$.

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We assumed that A's columns are linearly independent (they are the basis of some subspace). What happens (in the Gram-Schmidt process) if this is not the case?