CIE6020/MAT3350
Selected Topics in Information Theory

Lecture 18: Fountain Codes

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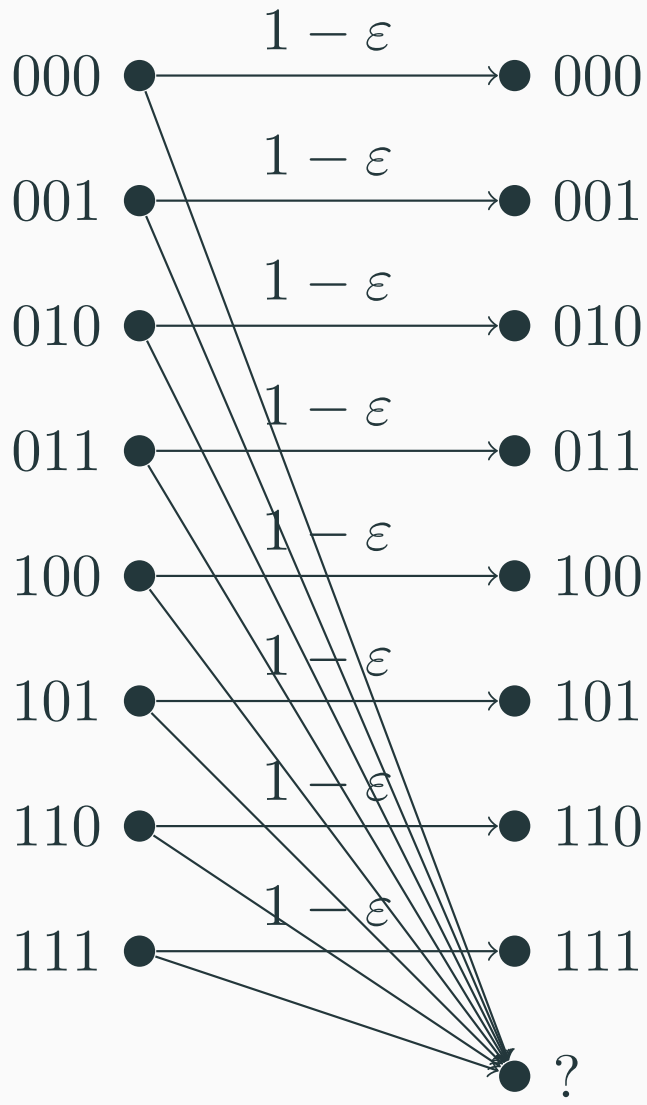
12 April 2019

The Chinese University of Hong Kong, Shenzhen

Coding for Erasure Channels

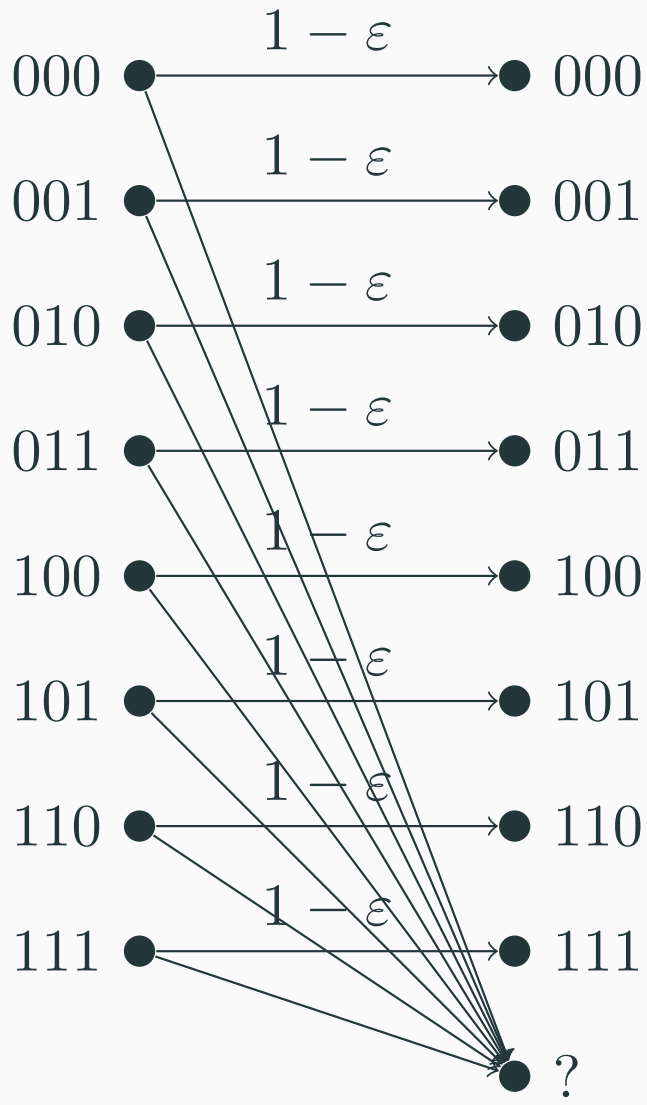
Erasure channel

- Model packet loss in networks (e.g. Internet, wireless networks)
- Capacity: $1 - \varepsilon$ symbol per use
- Solutions:



Erasure channel

- Model packet loss in networks (e.g. Internet, wireless networks)
- Capacity: $1 - \varepsilon$ symbol per use
- Solutions:
  - Retransmission
  - Forward error correction



Retransmission

- Example: TCP, 802.11 MAC, cellular networks
- Achieve capacity
- Require feedbacks

Retransmission

- Example: TCP, 802.11 MAC, cellular networks
- Achieve capacity
- Require feedbacks
- Not good for many scenarios
  1. wireless transmissions
  2. deep-space (satellite), underwater communications
  3. multicast transmissions

Forward error correction

- Capacity achieving without feedbacks
- Reed-Solomon code $(n, k, n-k+1)$
  - Encoding and decoding complexity: $O((n-k)\log n)$ per symbol.
- Can we have better solutions?
  - $O(1)$ complexity (per symbol).
  - Adaptive for different erasure rates/patterns.

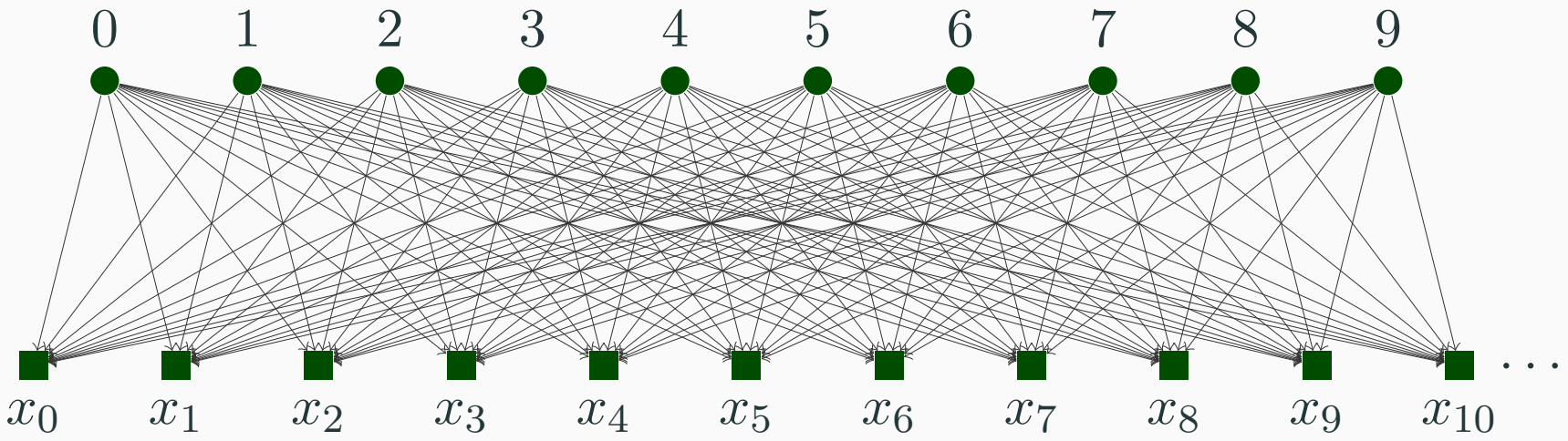
What are fountain codes?

- Transmit a file of k packets: $\mathbf{B} = [b_1, b_2, \ldots, b_k]$ where $b_i \in \mathbb{F}_q^T$
- Encoder generates potentially an infinite number of coded packets
- The file can be recovered from any set of n coded packets, where n is slightly larger than k.
- Also known as *rateless codes*

Rateless Random Linear Codes

Rateless random linear codes

- Encoding: $x_j = \sum_{i=1}^{k} \alpha_{j,i} b_i$.
- Decode from any k coded packets with linearly independent coding vectors.
- Work for any erasure patterns – universal.



Error Probability as a block code of length n

- When n packets are transmitted, the number of received packets N is $B(n, 1-\epsilon)$.
- The received packets $\mathbf{Y} = [y_1, y_2, \ldots y_N]$ is given by $\mathbf{Y} = \mathbf{BA}$.
- The decoding is correct iff $\mathrm{rank}(\mathbf{A}) = k$.
- Error probability

$$\begin{aligned} P_e &= 1 - \Pr\{\mathrm{rank}(\mathbf{A}) = k\} \\ &= 1 - \sum_{j=k}^{n} \binom{n}{j} \epsilon^{n-j} (1-\epsilon)^j \Pr\{\mathrm{rank}(\mathbf{A}) = k | N = j\} \\ &= 1 - \sum_{j=k}^{n} \binom{n}{j} \epsilon^{n-j} (1-\epsilon)^j \zeta_k^j, \end{aligned}$$

where ζ_k^j is the probability that a $k \times j$ totally random matrix has rank k.

Coding overhead (as a rateless code)

- Coding overhead is the number of packets received minus k when decoding.
- Expected coding overhead:

$$\mathrm{CO} = \sum_{i=1}^{\infty} i \Pr\{E_i \cap E'_{i-1}\} = \sum_{i=1}^{\infty} i \zeta_{k-1}^{k,k+i-1}(1 - q^{-1}),$$

  where

  - E_i: the first $k + i$ received packets have rank k.
  - $\zeta_r^{m,n}$: the probability that an $m \times n$ tatally random matrix has rank r.

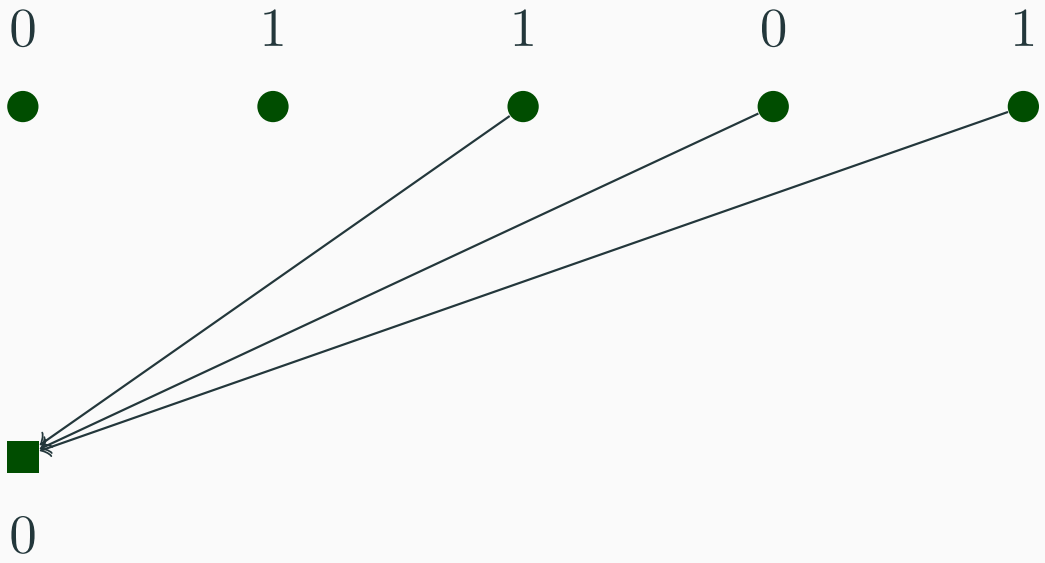
LT codes [Luby 98]

LT codes

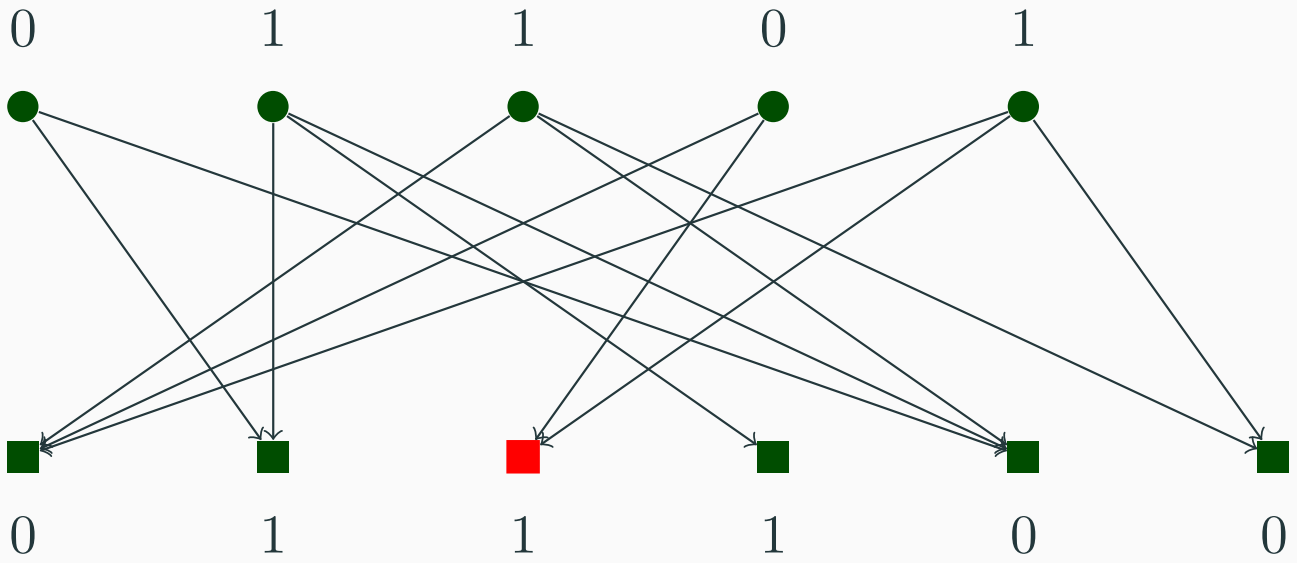
- Sparse encoding
  1. pick a degree d by sampling a degree distribution $\Psi = (\Psi_1, \Psi_2, \ldots, \Psi_K)$.
  2. uniformly at random pick d input packets.
  3. generate a coded packet by linearly combinate of the d input packets.
  4. repeat 1 - 3.
- Belief propogation decoding
  1. find a coded packet with degree one, which recovers the corresponding input packet.
  2. substitute the recovered input packet into the other coded packets that it involves.
  3. repeat 1 - 2 until there is no coded packets with degree one.
- Encoding/decoding complexity: $O(\log K)$ per packet, determined by the average degree $\mathbb{E}[\Psi]$.

Tanner graph of LT codes

Tanner graph of LT codes



Tanner graph of LT codes



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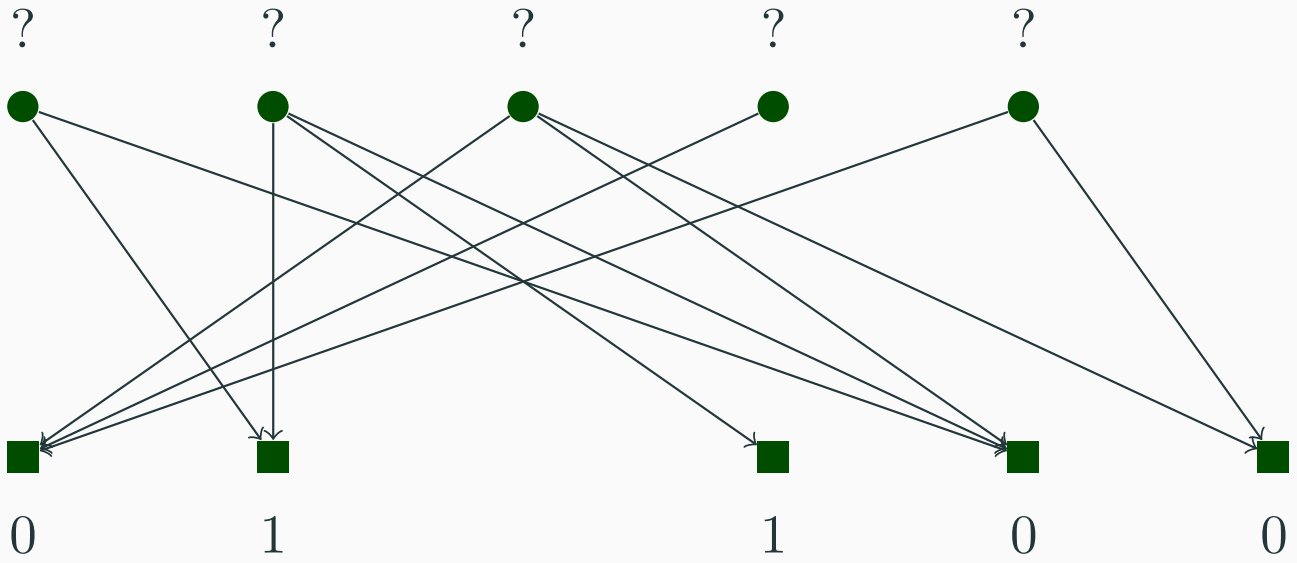

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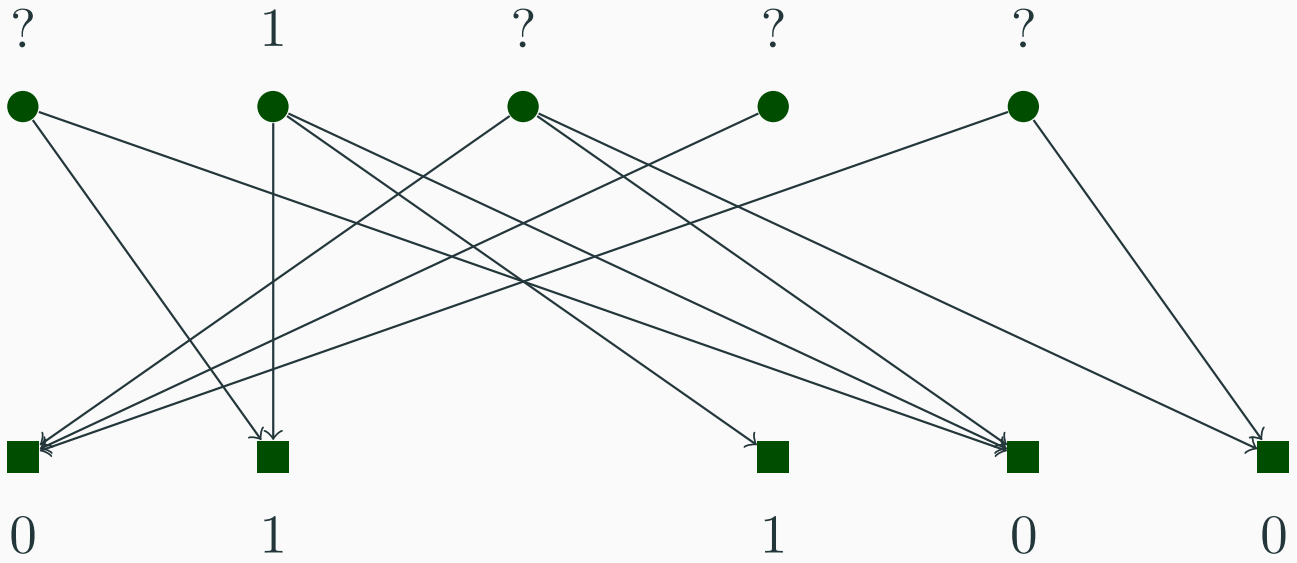

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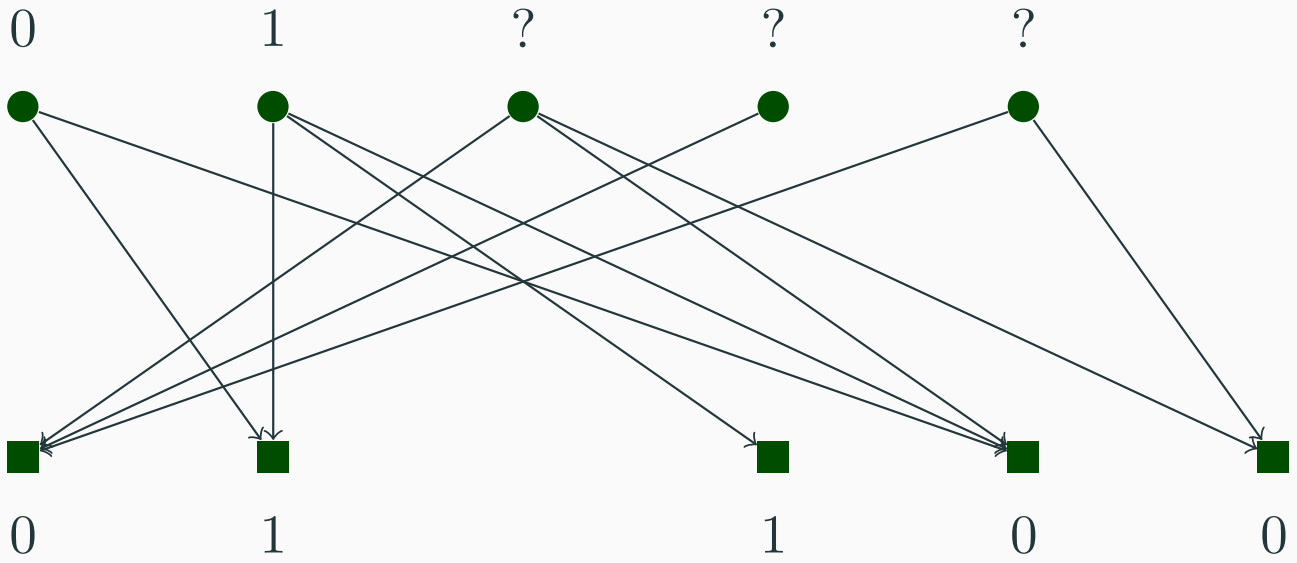

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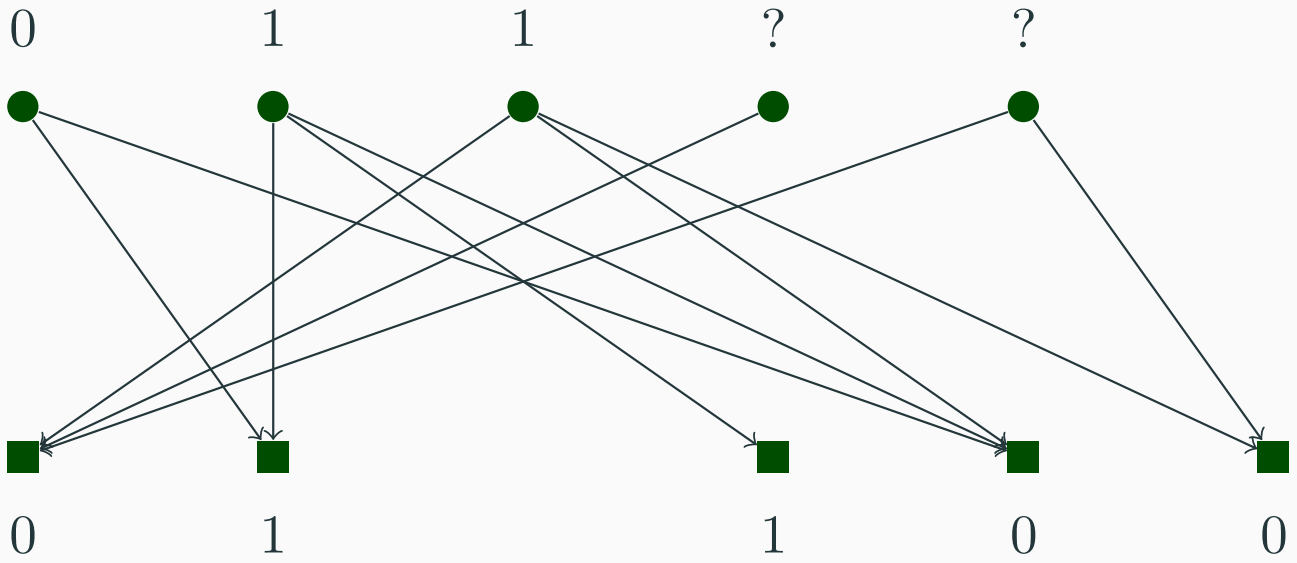

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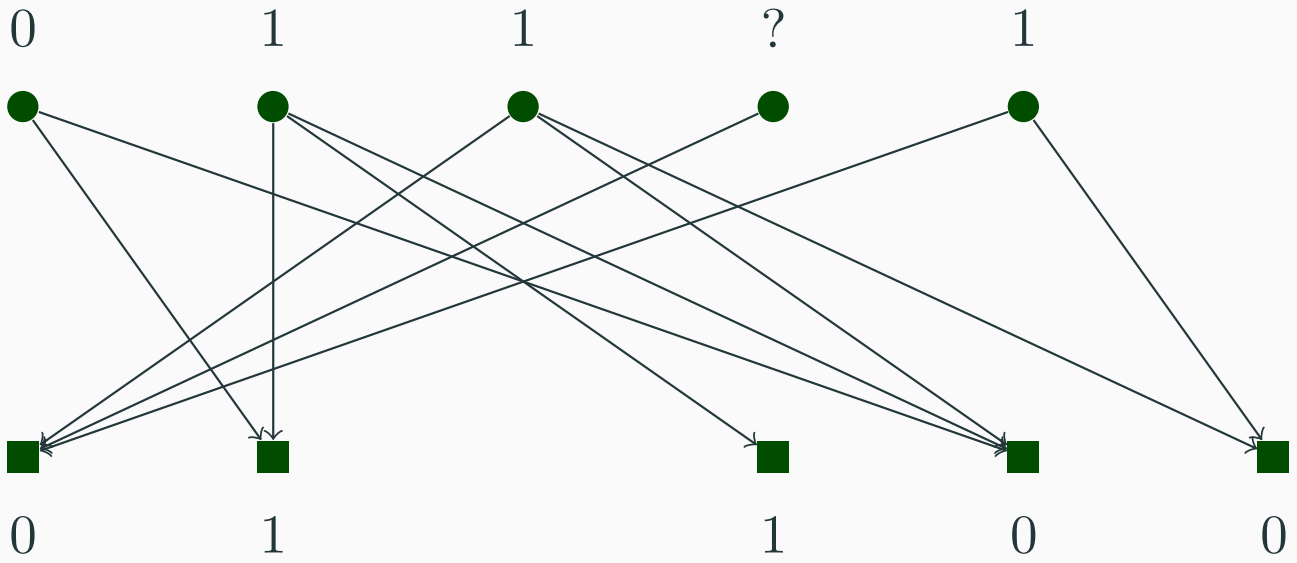

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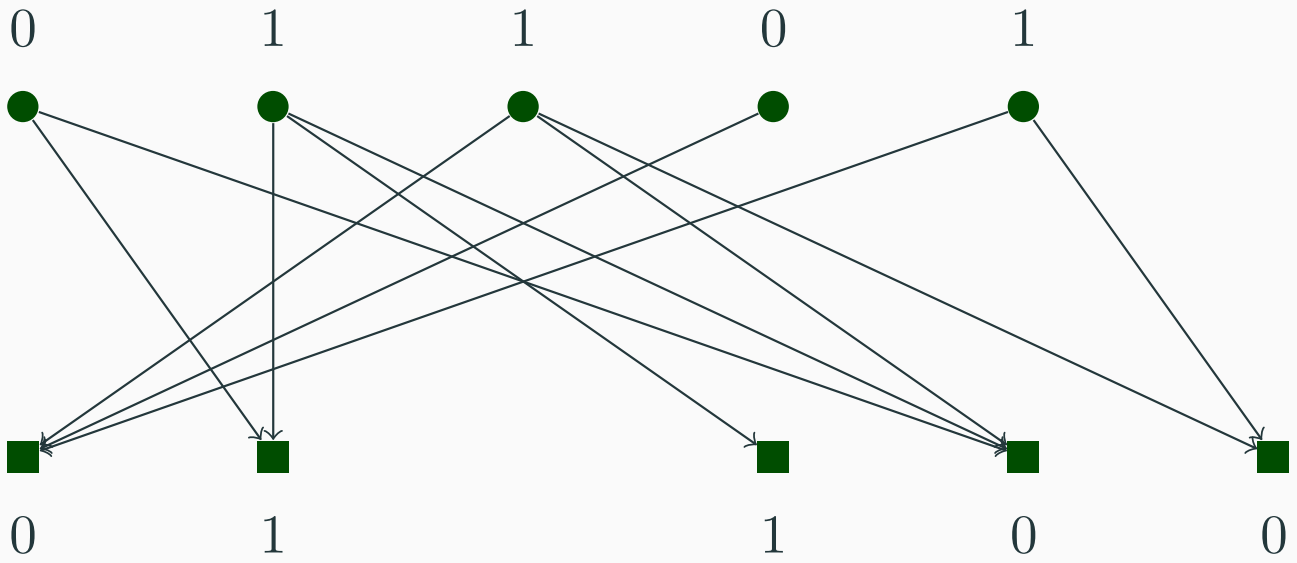


A bound on degree distribution

Theorem
For an LT code with k input packets and n coded packets, if there exists a decoding algorithm with $P_e \leq k^{-c}$, then $\mathbb{E}[\Psi] \geq c' \frac{k}{n} \ln k$.

- So when n is close to k, $\mathbb{E}[\Psi] \geq c' \ln k$.
- Luby showed that there exists a degree distribution such that
  1. $\mathbb{E}[\Psi] = O(\log(k))$,
  2. the BP decoding succeeds with vanishing error probability for n coded packets with $\frac{n-k}{k} \to 0$.

Degree distribution of LT codes

- Ideal Soliton distribution:

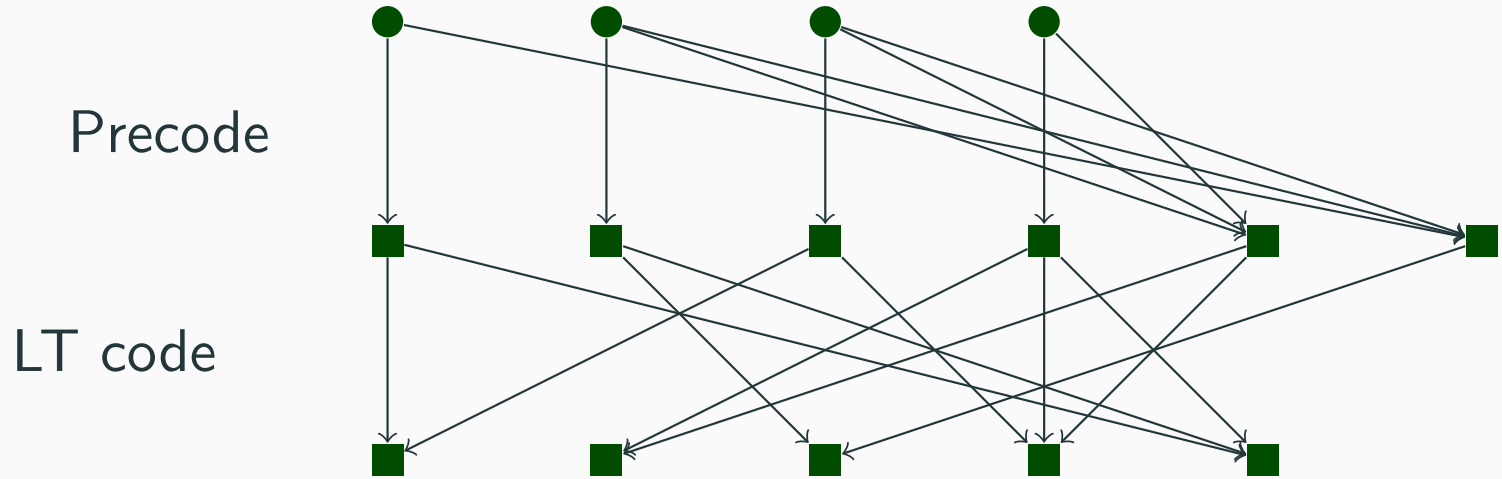
$$\rho_1 = 1/k$$
$$\rho_i = \frac{1}{i(i-1)}, \forall i = 2, 3, \ldots, k.$$

- Robust Soliton distribution:

Raptor codes

Raptor codes [Shokrollahi 2000]

- The trick: precode
- Encoding/decoding complexity $O(1)$ per packet



Degree distribution of Raptor codes

- BP decoding recovers at least $1 - \eta$ fraction of the (intermediate) input packets.
- The maximum degree $D \leq 1/\eta$. So $\mathbb{E}[\Psi] = O(1)$.
- The gap $\frac{n-k}{k}$ can be any positve value but is not vanishing for a fixed degree distribution when $k \to \infty$.

Performance analysis

- Asymptotic analysis: performance when $k \to \infty$.
  - Tree analysis [LMS98]
  - Differential equation approach (see [Wor99])
- Finite-length analysis: performance when k is relative small.
  - Iterative formula for the distribution of the decoder status [KLS04]

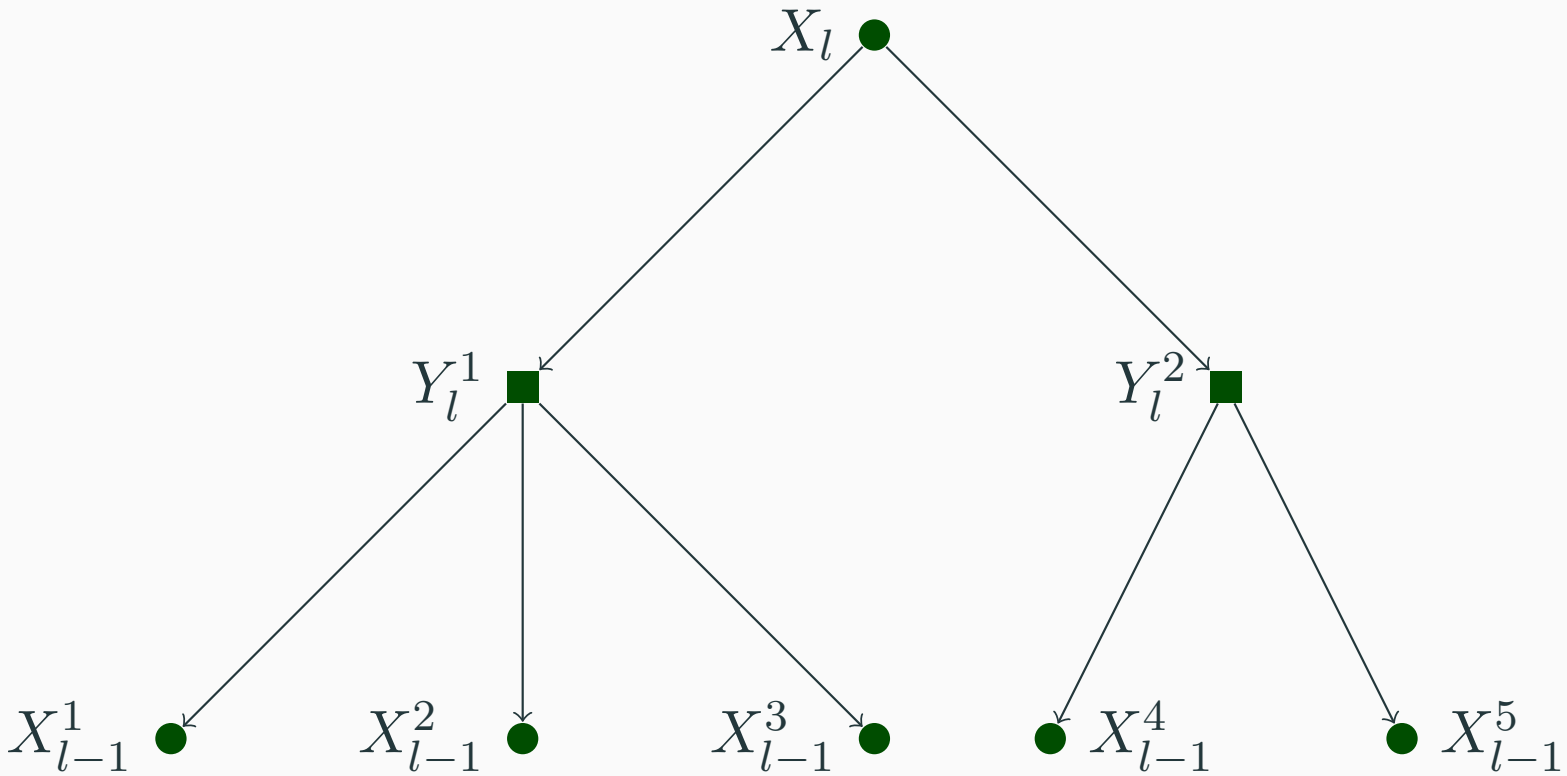
[LMS98] M. Luby, M. Mitzenmacher, and M. A. Shokrollahi, "Analysis of Random Processes via And-Or Tree Evaluation", in Proc. *SODA*, 1998, pp. 364–373.

[Wor99] N. C. Wormald, "The differential equation method for random graph processes and greedy algorithms," Karonsky and Proemel, eds., Lectures on Approximation and Randomized Algorithms PWN, Warsaw, pp. 73–155, 1999.

[KLS04] R. Karp, M. Luby, and A. Shokrollahi, "Finite length analysis of LT- codes," in Proc. IEEE ISIT'04, 2004.

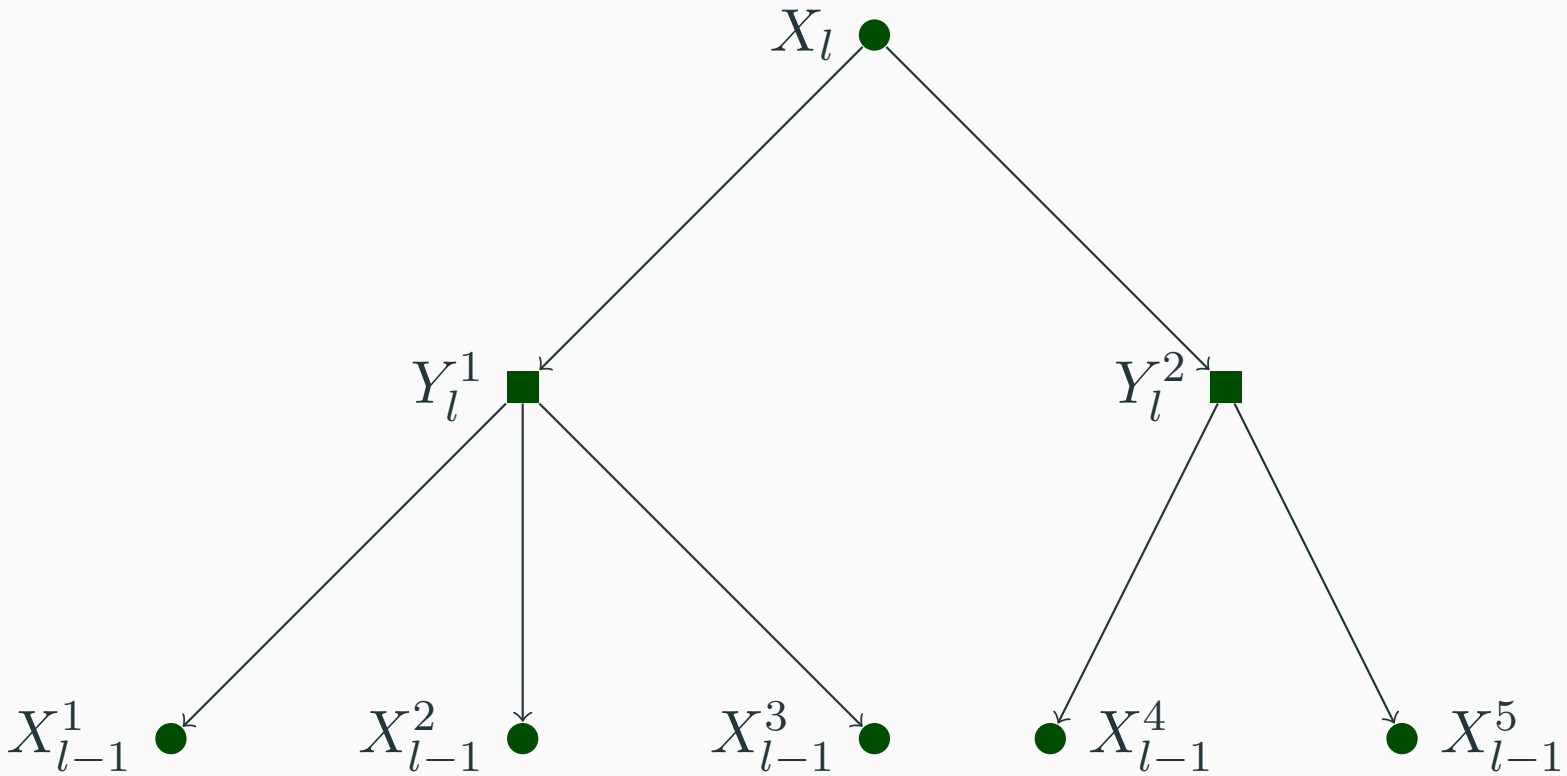
Tree based analysis

And-or tree



And-or tree

$$X_l = Y_l^1 \vee Y_l^2,\ Y_l^1 = X_{l-1}^1 \wedge X_{l-1}^2 \wedge X_{l-1}^3,\ Y_l^2 = X_{l-1}^4 \wedge X_{l-1}^5.$$



And-or tree

- Random variables in the same level are independent.
- Let $x_l = \Pr\{X_l^i = 1\}$, $y_l = \Pr\{Y_l^i = 1\}$.
- Let α_i (β_i) be the probability that an OR (AND) node has i children.

Lemma

$$x_l = 1 - \alpha(1 - \beta(x_{l-1})).$$

Proof.

$$\begin{aligned} x_l &= 1 - \sum_i \alpha_i (1 - y_l)^i \\ &= 1 - \alpha(1 - y_l) \end{aligned} \qquad \begin{aligned} y_{l+1} &= \sum_i \beta_i x_l^i \\ &= \beta(x_l). \end{aligned}$$

□

Variable node degree distribution

- The variable node degree distribution converges to a Poisson distribution

$$\Lambda_k = \frac{(a/C)^k e^{-a/C}}{k!},$$

where $a = \sum_i i\Psi_i$ and $C = K/n$.

Computation graph

- Computation graph T_l: Choose an edge (v, w) and then T_l is the subgraph induced by v and all neighbors of v within distance $2l$ after deleting the edge (v, w).
- A variable node has $i-1$ children with probability $\lambda_i = i\Lambda_i / \sum_i i\Lambda_i$, and a check node has $i-1$ children with probability $\psi_i = i\Psi_i / \sum_i i\Psi_i$.
- The performance of T_l converges to the and-or tree of level $2l+1$ generated using $\{\lambda_i\}$ and $\{\psi_i\}$, when $k \to \infty$.

Tree analysis of LT codes

- Let x_i be the probability that the variable nodes in the $2i$th level is decodable. Using the and-or-tree Lemma, $x_{l+1} = 1 - \lambda(1 - \psi(x_l))$, where

$$\begin{aligned} \psi(z) &= \sum_i \psi_i z^{i-1} = \Psi'(z)/\Psi'(1) = \Psi'(z)/a \\ \lambda(z) &= \sum_i \lambda_i z^{i-1} = \Lambda'(z)/\Lambda'(1) = \exp((z-1)a/C). \end{aligned}$$

Sufficient condition

- To guarantee the success of decoding with high probability, we can require

$$x < 1 - \exp(-\frac{1}{C}\Psi'(x)), \quad \text{for } x \in [0, 1-\eta],$$

which implies

$$\Psi'(x) + C\ln(1-x) > 0, \quad \text{for } x \in [0, 1-\eta].$$

- Let $D = \lfloor 1/(1-\eta) \rfloor - 1$. For any $C < 1$, let

$$\Psi(x) = C\left((1/C-1)x + \sum_{i=2}^{D-1} \frac{x^i}{(i-1)i} + \frac{x^D}{D-1}\right).$$

Differential equation approach

Differential equation: The general method

- Compute the expected changes in random variables of the process per unit time at time t and, regarding the variables as continuous
- Write down the differential equations suggested by the expected changes
- Use large deviation theorems to show that with high probability the solution of the differential equations is close to the values of the variables.

Differential equation: Raptor codes

- Let R_d be the number of edges of degree d. Then

$$E[R_d(t+1) - R_d(t)|R(t)] = (R_{d+1}(t) - R_d(t))\frac{d}{k-t} \quad d > 1$$

$$E[R_1(t+1) - R_1(t)|R(t)] = (R_2(t) - R_1(t))\frac{1}{k-t} - 1 + O(1/k)$$

- The differential equation approach guides us to consider the following system of differential equations

$$\frac{d\rho_d(\tau)}{d\tau} = [\rho_{d+1}(\tau) - \rho_d(\tau)]\frac{d}{C-\tau} \quad d > 1$$

$$\frac{d\rho_1(\tau)}{d\tau} = [\rho_2(\tau) - \rho_1(\tau)]\frac{1}{C-\tau} - 1,$$

  where $\rho_d(0) = d\Psi_d$ and $C = k/n$.

- $R_d(t) \to n\rho_d(t/n)$ (in probability) uniformly for all t before decoding stops.

Solve the system of differential equations

- Since we hope to decode the decoding stops after $t > (1-\eta)K$, we want

$$\rho_1(\tau) > 0, \quad t \in [0, (1-\eta)C].$$

- Solving the system of differential equations, we get

$$\rho_1(\tau) = (1-\tau/C)(\Psi'(\tau/C) + C\ln(1-\tau/C)).$$

- Therefore,

$$\Psi'(x) + C\ln(1-x) > 0, \quad x \in [0, 1-\eta],$$

which is the same requirement as we have obtained using tree analysis.

Finite-length analysis

Finite-length analysis

- Define R^t as number of decodable input symbols at time t
- $P^t_{\text{stop}} = \Pr\{R^t = 0, R^\tau > 0, \tau < t\}$.

Finite-length analysis

- Define R^t as number of decodable input symbols at time t
- $P^t_{\text{stop}} = \Pr\{R^t = 0, R^\tau > 0, \tau < t\}$.
- Define C^t as the number of undecodable received symbols at time t.
- Let $\Lambda^t_{c,r} = \Pr\{C^t = c, R^t = r, R^\tau > 0, \tau < t\}$.

Initial status ($t = 0$)

$$\begin{aligned}
\Lambda^0_{c,r} &= \Pr\{R^0 = r | C^0 = c\} \Pr\{C^0 = c\} \\
&= \Pr\{R^0 = r | C^0 = c\} \mathsf{Bi}(n, c, 1 - \Psi_1) \\
&= \Pr\{|A_{n-c}| = r\} \mathsf{Bi}(n, c, 1 - \Psi_1)
\end{aligned}$$

where $A_{n-c} = \{X_1, X_2, \ldots, X_{n-c}\}$ with X_i be i.i.d. and uniformly distributed in $\{1, 2, \ldots, k\}$.

Recursive formula

$$
\begin{aligned}
&\Lambda^t_{c,r|n} \\
&= \Pr\left\{C^t = c, R^t = r, R^\tau > 0, \tau < t\right\} \\
&= \sum_{c',r'>0} \Pr\left\{C^t = c, C^{t-1} = c', R^t = r, R^{t-1} = r', R^\tau > 0, \tau < t\right\} \\
&= \sum_{c',r'>0} \Pr\left\{C^t = c, R^t = r, |C^{t-1} = c', R^{t-1} = r', R^\tau > 0, \tau < t-1\right\} \Lambda^{t-1}_{c',r'|n} \\
&= \sum_{c',r'>0} \underbrace{\Pr\left\{R^t = r | C^t = c, C^{t-1} = c', R^{t-1} = r', E_{t-1}\right\}}_{(a)} \times \\
&\quad \times \underbrace{\Pr\left\{C^t = c | C^{t-1} = c', R^{t-1} = r', E_{t-1}\right\}}_{(b)} \Lambda^{t-1}_{c',r'|n}.
\end{aligned}
$$

- (a) can be calculated similar to $\Pr\{R^0 = r | C^0 = c\}$.
- (b) can be calculated similar to $\Pr\{C^0 = c\}$.