Lecture 11: Duality Theory

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Announcement

- Homework 4 due date changed to 10/24
- Midterm exam changed to Wednesday, 10/31, in class (location TBD)

Recap: Dual Problems

In the last lecture, we introduced the dual problems of LPs. For example, for the LP in the standard form

$$\begin{array}{ll} \text{minimize}_{\mathbf{x}} & \mathbf{c}^T\mathbf{x} \\ \text{subject to} & A\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq 0 \end{array}$$

The dual problem is:

$$\begin{array}{ll} \text{maximize}_{\mathbf{y}} & \mathbf{b}^T\mathbf{y} \\ \text{subject to} & A^T\mathbf{y} \leq \mathbf{c} \end{array}$$

We showed by some arguments that the optimal values of these two optimization problems should be the same.

Write the Dual

Primal	minimize	maximize	Dual
constraints	$\geq b_i$	≥ 0	variables
	$\leq b_i$	≤ 0	
	$= b_i$	free	
variables	≥ 0	$\leq c_j$	constraints
	≤ 0	$\geq c_j$	
	free	$= c_j$	

1. Each primal constraint is associated with a dual variable; each primal variable is associated with a dual constraint.
2. Equality constraints correspond to free variables, vice versa.
3. Usual (unusual) constraints correspond to usual (unusual) constraints.

Today we are going to show some important relations between primal and dual problems.

Invariance of Transformations

Theorem

If we transform a linear program to an equivalent one (such as by replacing free variables, adding slack variables, etc), then the dual of the two problems will be equivalent.

Theorem

If we transform the primal to its dual, then transform the dual to its dual, then we will obtain a problem equivalent to the primal problem, that is, the dual of dual is the primal.

Duality Theorems

Remember that by the way we construct the dual problem, we argued that they should have the same optimal value.

- In the following, we formally prove it.
- Since we have the invariant of transformation property, we only need to discuss the standard form. All the following theorems also apply to general cases as well.

Weak Duality Theorem

Primal		**Dual**	
min	$\mathbf{c}^T\mathbf{x}$	max	$\mathbf{b}^T\mathbf{y}$
s.t.	$A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq 0$	s.t.	$A^T\mathbf{y} \leq \mathbf{c}$

Theorem (Weak Duality Theorem)

If $\mathbf{x}$ *is feasible to the primal and* $\mathbf{y}$ *is feasible to the dual, then*

$$\mathbf{b}^T\mathbf{y} \leq \mathbf{c}^T\mathbf{x}$$

If the primal is a minimization and dual is a maximization, then

- Any dual feasible solution will give a lower bound on the primal optimal value
- Any primal feasible solution will give an upper bound on the dual optimal value
- The optimal value of primal is larger than that of dual

Proof

Assume $\mathbf{x}$ is feasible to the primal problem and $\mathbf{y}$ is feasible to the dual problem. Then we have

$$\mathbf{b}^T\mathbf{y} = (A\mathbf{x})^T\mathbf{y} = \mathbf{x}^T(A^T\mathbf{y}) \leq \mathbf{c}^T\mathbf{x}$$

The last inequality is because that $\mathbf{x} \geq 0$ and $A^T\mathbf{y} \leq \mathbf{c}$. $\square$

Corollary

- *If the primal problem is unbounded (i.e., the optimal value is* $-\infty$*), then the dual problem must be infeasible*
- *If the dual problem is unbounded (i.e., the optimal value is* ∞*), then the primal problem must be infeasible*

Another Corollary

Corollary

Let $\mathbf{x}$ *and* $\mathbf{y}$ *be feasible solutions to the primal and dual problems respectively. If* $\mathbf{c}^T\mathbf{x} = \mathbf{b}^T\mathbf{y}$*, then* $\mathbf{x}$ *and* $\mathbf{y}$ *must be optimal solutions to the primal and dual, respectively.*

Optimality conditions for LP: If $\mathbf{x}$, $\mathbf{y}$ satisfy:

1. $\mathbf{x}$ is primal feasible
2. $\mathbf{y}$ is dual feasible
3. The objective values are the same, i.e., $\mathbf{c}^T\mathbf{x} = \mathbf{b}^T\mathbf{y}$

Then $\mathbf{x}$ and $\mathbf{y}$ are optimal solutions to the primal and dual problems respectively.

The reverse is also true (see the next theorem)

Strong Duality Theorem

Theorem (Strong Duality Theorem)

If a linear program has a finite optimal solution, so does its dual, and the optimal values of the primal and dual are equal

- We present a constructive proof. That is, for a given primal optimal solution, we construct a dual optimal solution and show that their objective values are equal
- We use simplex method in our proof
- In the proof, we will see that the simplex method actually also finds the dual optimal solution when it finishes

Proof

We prove by using the simplex method. Without loss of generality, we assume the primal problem is in the standard form.

If the primal problem has an optimal solution $\mathbf{x}^*$, then it must be associated with some optimal basis B such that $\mathbf{x}_B = A_B^{-1}\mathbf{b}$ ($\mathbf{x}_B$ is the basic part of $\mathbf{x}^*$). Also, when the simplex method terminates, the reduced costs

$$\mathbf{c}^T - \mathbf{c}_B^T A_B^{-1} A \geq 0 \tag{1}$$

Now we define $\mathbf{y}^T = \mathbf{c}_B^T A_B^{-1}$. By (1), $A^T\mathbf{y} \leq \mathbf{c}$, i.e., $\mathbf{y}$ is feasible to the dual problem. In addition

$$\mathbf{b}^T\mathbf{y} = \mathbf{c}_B^T A_B^{-1}\mathbf{b} = \mathbf{c}_B^T\mathbf{x}_B = \mathbf{c}^T\mathbf{x}^*$$

Therefore by the weak duality theorem, $\mathbf{y}$ must be optimal to the dual problem and the theorem holds. $\square$

Remark

From the proof, we see that the dual optimal solution actually comes as a by-product when we use the simplex algorithm. The term $\mathbf{c}_B^T A_B^{-1}$ will be the dual optimal solution (if primal has a finite optimal solution). Therefore, when we solve the primal problem, the dual is also solved.

This is not a coincidence. Nearly all LP algorithms (simplex method, interior point method or ellipsoid method) solve both primal and dual problems simultaneously.

Discussions Continued

Based on the strong duality theorem, we know that $(\mathbf{x}, \mathbf{y})$ is optimal to the primal and dual respectively if and only if

- $\mathbf{x}$ is primal feasible
- $\mathbf{y}$ is dual feasible
- They achieve the same objective value

Therefore solving LP is in fact equivalent as solving the following linear system:

- $A\mathbf{x} = \mathbf{b}$, $\mathbf{x} \geq 0$
- $A^T\mathbf{y} \leq \mathbf{c}$
- $\mathbf{b}^T\mathbf{y} = \mathbf{c}^T\mathbf{x}$

Later, we will show how this perspective helps solving LPs

Question

What of the following states are possible for an LP and its dual?

	Finite Optimum	Unbounded	Infeasible
Finite Optimum	?	?	?
Unbounded	?	?	?
Infeasible	?	?	?

An Example of Primal/Dual Both Infeasible

Primal:

$$\begin{array}{rll} \text{minimize} & x_1 + 2x_2 & \\ \text{subject to} & x_1 + x_2 & = 1 \\ & 2x_1 + 2x_2 & = 3 \end{array}$$

Dual:

$$\begin{array}{rll} \text{maximize} & y_1 + 3y_2 & \\ \text{subject to} & y_1 + 2y_2 & = 1 \\ & y_1 + 2y_2 & = 2 \end{array}$$

Answer

The only possible cases for LPs are:

	Finite Optimum	Unbounded	Infeasible
Finite Optimum	$\surd$		
Unbounded			$\surd$
Infeasible		$\surd$	$\surd$

- If both the primal and dual are feasible, then both must have optimal solutions. (This might be one way to quickly determine if an LP is bounded.) And by the strong duality theorem, their optimal values must be the same.

We have studied the relation between primal and dual optimal values. Next we study the relation between primal and dual optimal solutions.

Complementarity Conditions

Primal		**Dual**	
min	$\mathbf{c}^T\mathbf{x}$	max	$\mathbf{b}^T\mathbf{y}$
s.t.	$A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq 0$	s.t.	$A^T\mathbf{y} \leq \mathbf{c}$

Theorem

Let $\mathbf{x}$ and $\mathbf{y}$ be feasible solutions to the primal and dual problems respectively. Then $\mathbf{x}$ and $\mathbf{y}$ are optimal solutions if and only if

$$\begin{aligned} x_i > 0 &\implies A_i^T\mathbf{y} = c_i \\ A_i^T\mathbf{y} < c_i &\implies x_i = 0 \end{aligned}$$

Or in other words,

$$x_i \cdot (c_i - A_i^T\mathbf{y}) = 0, \quad \forall i.$$

Example

Primal:

$$\begin{array}{rrrrcl} \text{minimize} & 13x_1 & +10x_2 & +6x_3 & & \\ \text{s.t.} & 5x_1 & +x_2 & +3x_3 & = & 8 \\ & 3x_1 & +x_2 & & = & 3 \\ & x_1, & x_2, & x_3 & \geq & 0 \end{array}$$

Optimal solution $(1, 0, 1)$

Dual problem:

$$\begin{array}{rrrcl} \text{maximize} & 8y_1 & +3y_2 & & \\ & 5y_1 & +3y_2 & \leq & 13 \\ & y_1 & +y_2 & \leq & 10 \\ & 3y_1 & & \leq & 6 \end{array}$$

Optimal solution $(2, 1)$

Verify the complementarity conditions

$$x_1 \cdot (13 - 5y_1 - 3y_2) = 0, \;\; x_2 \cdot (10 - y_1 - y_2) = 0, \;\; x_3 \cdot (6 - 3y_1) = 0$$

Proof

By the strong duality theorem, if $\mathbf{x}$ and $\mathbf{y}$ are optimal solutions for the primal and dual respectively, we must have

$$\mathbf{c}^T\mathbf{x} = \mathbf{b}^T\mathbf{y}$$

Thus we have

$$0 = \mathbf{c}^T\mathbf{x} - \mathbf{b}^T\mathbf{y} = \mathbf{c}^T\mathbf{x} - \mathbf{y}^T A\mathbf{x} = \sum_{i=1}^{n}(c_i - A_i^T\mathbf{y}) \cdot x_i \qquad (2)$$

Since $\mathbf{x}$ and $\mathbf{y}$ are both feasible, for each i, $c_i - A_i^T\mathbf{y} \geq 0$ and $x_i \geq 0$. Therefore, in order for (1) to hold, we must have

$$(c_i - A_i^T\mathbf{y}) \cdot x_i = 0, \quad \forall i$$

The other direction follows from the same arguments $\square$

Complementarity in Another Form

Sometimes, we write the dual problem (equivalently) as

Primal		**Dual**	
min	$\mathbf{c}^T\mathbf{x}$	max	$\mathbf{b}^T\mathbf{y}$
s.t.	$A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq 0$	s.t.	$A^T\mathbf{y} + \mathbf{s} = \mathbf{c}, \quad \mathbf{s} \geq 0$

We call **s** the *dual slack* variables. Then the complementarity conditions can be written as

$$x_i \cdot s_i = 0 \quad \forall i$$

We sometimes call the complementarity conditions the *complementary slackness condition*.

General Complementarity Conditions

Consider the primal-dual pair:

Primal			**Dual**		
minimize	$\mathbf{c}^T\mathbf{x}$		maximize	$\mathbf{b}^T\mathbf{y}$	
subject to	$\mathbf{a}_i^T\mathbf{x} \geq b_i,$	$i \in M_1,$	subject to	$y_i \geq 0,$	$i \in M_1$
	$\mathbf{a}_i^T\mathbf{x} \leq b_i,$	$i \in M_2,$		$y_i \leq 0,$	$i \in M_2$
	$\mathbf{a}_i^T\mathbf{x} = b_i,$	$i \in M_3,$		y_i free,	$i \in M_3$
	$x_j \geq 0,$	$j \in N_1,$		$A_j^T\mathbf{y} \leq c_j,$	$j \in N_1$
	$x_j \leq 0,$	$j \in N_2,$		$A_j^T\mathbf{y} \geq c_j,$	$j \in N_2$
	x_j free,	$j \in N_3,$		$A_j^T\mathbf{y} = c_j,$	$j \in N_3$

Theorem

Let $\mathbf{x}$ *and* $\mathbf{y}$ *are feasible solutions to the primal and dual problems respectively. Then* $\mathbf{x}$ *and* $\mathbf{y}$ *are optimal if and only if*

$$y_i \cdot (\mathbf{a}_i^T\mathbf{x} - b_i) = 0, \quad \forall i; \qquad x_j \cdot (A_j^T\mathbf{y} - c_j) = 0, \quad \forall j.$$

Optimality Conditions for LP in Another Form

Remember we had the optimality conditions for LPs:

1. $\mathbf{x}$ is primal feasible
2. $\mathbf{y}$ is dual feasible
3. $\mathbf{c}^T\mathbf{x} = \mathbf{b}^T\mathbf{y}$

Now with the complementarity conditions, we can write an equivalent set of conditions:

1. $\mathbf{x}$ is primal feasible
2. $\mathbf{y}$ is dual feasible
3. All complementarity conditions are satisfied

Solve Problems Using Complementarity Conditions

If we have obtained an optimal solution to the primal problem, then we may be able to quickly find a dual optimal solution.

If the primal problem is of the standard form and $\mathbf{x}$ is the optimal basic solution with basis B. And assume $x_i > 0$ for $i \in B$. Then by complementarity conditions, we know that the optimal solution $\mathbf{y}$ to the dual problem must satisfy:

$$A_i^T \mathbf{y} = c_i \quad i \in B$$

i.e., $\mathbf{y} = (A_B^{-1})^T \mathbf{c}_B$.

Example

Primal:

$$
\begin{array}{rrrrcl}
\text{minimize} & 13x_1 & +10x_2 & +6x_3 & & \\
\text{s.t.} & 5x_1 & +x_2 & +3x_3 & = & 8 \\
 & 3x_1 & +x_2 & & = & 3 \\
 & x_1, & x_2, & x_3 & \geq & 0
\end{array}
$$

Optimal solution $(1, 0, 1)$

Dual problem:

$$
\begin{array}{rrrcl}
\text{maximize} & 8y_1 & +3y_2 & & \\
 & 5y_1 & +3y_2 & \leq & 13 \\
 & y_1 & +y_2 & \leq & 10 \\
 & 3y_1 & & \leq & 6
\end{array}
$$

Because x_1 and x_3 are positive, we must have that the first and third constraints are tight for the dual optimal solution. Thus the optimal solution to the dual must be $(2, 1)$.

Finding Dual Optimal Solution from Simplex Tableau

If the primal optimal solution is obtained from using the simplex tableau, and the initial problem was constructed from adding slack variables, then one can find the optimal dual solution $(A_B^{-1})^T \mathbf{c}_B$ from the simplex tableau when it finishes.

When the initial tableau is constructed from adding slack variables (thus it is naturally a canonical form), we can write the initial tableau as follows:

$\mathbf{c}$	$\mathbf{0}_m$	$\mathbf{0}$
A	$\mathbf{I}_m$	$\mathbf{b}$

Continue..

$\mathbf{c}$	$\mathbf{0}_m$	0
A	$\mathbf{I}_m$	$\mathbf{b}$

Suppose after some iterations, it reaches an optimal solution with basis B. Then the tableau becomes:

$\mathbf{c} - \mathbf{c}_B^T A_B^{-1} A$	$-\mathbf{c}_B^T A_B^{-1}$	$-\mathbf{c}_B^T A_B^{-1}\mathbf{b}$
$A_B^{-1} A$	A_B^{-1}	$A_B^{-1}\mathbf{b}$

Therefore, the opposite to the reduced cost corresponding to the original identity matrix part is the optimal dual solution.

Example

Consider the production planning problem

$$
\begin{array}{rrrrrrl}
\text{minimize} & -x_1 & -2x_2 & & & & \\
\text{subject to} & x_1 & & +s_1 & & & = 100 \\
 & & 2x_2 & & +s_2 & & = 200 \\
 & x_1 & +x_2 & & & +s_3 & = 150 \\
 & x_1, & x_2, & s_1, & s_2, & s_3 & \geq 0
\end{array}
$$

Dual:

$$
\begin{array}{rrrrl}
\text{maximize} & 100p_1 & +200p_2 & +150p_3 & \\
\text{subject to} & p_1 & & +p_3 & \leq -1 \\
 & & 2p_2 & +p_3 & \leq -2 \\
 & p_1, & p_2, & p_3 & \leq 0
\end{array}
$$

Simplex Tableau

The initial tableau for the primal:

B	-1	-2	0	0	0	0
3	1	0	1	0	0	100
4	0	2	0	1	0	200
5	1	1	0	0	1	150

Final tableau:

B	0	0	0	1/2	1	250
1	1	0	0	-1/2	1	50
3	0	0	1	1/2	-1	50
2	0	1	0	1/2	0	100

What is the dual optimal solution?

- $(p_1, p_2, p_3) = (0, -1/2, -1)$, with objective value -250

Discussions

Caveat: If the problem is not derived from adding slack variables (therefore the initial top row is not the actual $\mathbf{c}$), then this method may not give the right answer.

- In that case, one can just compute $(A_B^{-1})^T\mathbf{c}_B$.