Lecture 14: More Vector Spaces of Matrices

MAT2040 Linear Algebra

Definition 14.1

The **row space** of a matrix A is the set of all possible linear combinations of the rows of A (written as column vectors). It is denoted by $\operatorname{Row} A$.

In other words,

$$\operatorname{Row} A = \operatorname{Span}\{\vec{\mathbf{a}}_1^T, \vec{\mathbf{a}}_2^T, \ldots, \vec{\mathbf{a}}_m^T\}.$$

Fact 14.2

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Fact 14.3

$Row\, A = Col\, A^T$ for any matrix A.

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Suppose we want to find a basis for both $\operatorname{Col} A$ and $\operatorname{Row} A$. Do we really need to find a row echelon form for both A and A^T to identify these bases? No!

Theorem 14.4

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So to find a basis for both $\mathrm{Col}\, A$ and $\mathrm{Row}\, A$ we only have to find the row echelon form of A (not of A and A^T both).

Also, we immediately see the following.

Corollary 14.5

$\dim(Col\, A) = \dim(Row\, A)$ *for any matrix A.*

Example 14.6

Find Col A and Row A for $A = \begin{bmatrix} 2 & 5 & -2 & -3 & 1 \\ 4 & 10 & -8 & -16 & -6 \\ 6 & 15 & -8 & -8 & 9 \\ 2 & 5 & 0 & -1 & 0 \end{bmatrix}$

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This may be helpful: A is row equivalent to $\begin{bmatrix} 2 & 5 & -2 & -3 & 1 \\ 0 & 0 & 2 & 2 & -1 \\ 0 & 0 & 0 & 3 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

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We just saw that $\operatorname{rank}(A)$ therefore also equals the dimension of $\operatorname{Row} A$.

Example 14.8 (continuation of previous example)

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What is $\operatorname{rank}(A)$? Also, what was $\dim(\operatorname{Null} A)$ again? What about $\operatorname{rank}(A) + \dim(\operatorname{Null} A)$?

Side note: When considering generalizations of matrices (where the entries are not real or complex numbers), it can be the case that the corresponding notion of $\operatorname{Col} A$ and $\operatorname{Row} A$ do **not** have the same dimension.

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Therefore these dimensions have a separate names — **column rank** and **row rank**. In this course the entries in a matrix are only real or complex numbers, and that means column rank and row rank have the same value. We will thus use "rank" for this quantities.

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- a basis for $\operatorname{Row} A = \{\vec{\mathbf{b}}_i^T : \vec{\mathbf{b}}_i \text{ is not an all zero row}\}$.

Summarizing, a basis for $\operatorname{Col} A$ and $\operatorname{Row} A$ can be found from a row echelon form of matrix A (say matrix B):

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- a basis for $\operatorname{Row} A = \{\vec{\mathbf{b}}_i^T : \vec{\mathbf{b}}_i \text{ is not an all zero row}\}$.

Note that the first $k = \operatorname{rank}(A)$ rows of A are not necessarily a basis for $\operatorname{Row} A$! (Why?)

Recall:

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($\dim(\text{Null}\,A)$ is sometimes called "nullity".)

Special case 1: Matrices with rank 0?

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And therefore every column of A is a linear combination of the vector $\mathbf{u}$ (because $\mathbf{u}$ spans $\operatorname{Col} A$).

We see that

$$A = \begin{bmatrix} v_1\mathbf{u}, & v_2\mathbf{u}, & \ldots, & v_n\mathbf{u} \end{bmatrix}$$

for some $v_1, v_2, \ldots v_n \in \mathbb{R}$.

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Again, to avoid getting $A = \mathbf{O}$ (the all zero matrix that has rank 0), we do need $\mathbf{v} \neq \mathbf{0}$.

We just proved one implication of the following fact (and the other implication is easy).

Fact 14.9

The $m \times n$ matrix A has rank 1 if and only if there exist nonzero vectors $\mathbf{u} \in \mathbb{R}^m$ and $\mathbf{v} \in \mathbb{R}^n$ so that

$$A = \mathbf{u}\mathbf{v}^T.$$

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($\mathbf{u}\mathbf{v}^T$ is sometimes referred to as the "outer product of $\mathbf{u}$ and $\mathbf{v}$".)

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Suppose A is an $m \times n$ matrix and it has full rank.
What is $\operatorname{rank}(A)$?

Definition 14.10

The **left null space** of a matrix A is $\operatorname{Null} A^T$.

Example 14.11

Show that the left null space of the $m \times n$ matrix A is the set $\{\mathbf{y} \in \mathbb{R}^m : \mathbf{y}^T A = \mathbf{0}^T\}$.

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The null space of A can similarly be described as exactly the vectors corresponding to the weights of a linear dependence relations of the columns of A.

Theorem 14.12 (Invertible Matrix Theorem)

Given a $n \times n$ matrix A (square!). The following are equivalent:

1. *A is invertible*
2. *the only solution to $A\mathbf{x} = \mathbf{0}$ is the trivial solution*
3. *A is row equivalent to I*
4. *A is a product of elementary matrices*
5. *the columns of A are linearly independent*
6. *$A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b} \in \mathbb{R}^m$*
7. *the columns of A span $\mathbb{R}^n$*
8. $Null\, A = \{\mathbf{0}\}$
9. $Col\, A = \mathbb{R}^n$
10. *the columns of A form a basis of $\mathbb{R}^n$*
11. $\dim(Col\, A) = n$
12. $\dim(Null\, A) = 0$
13. $rank(A) = n$.