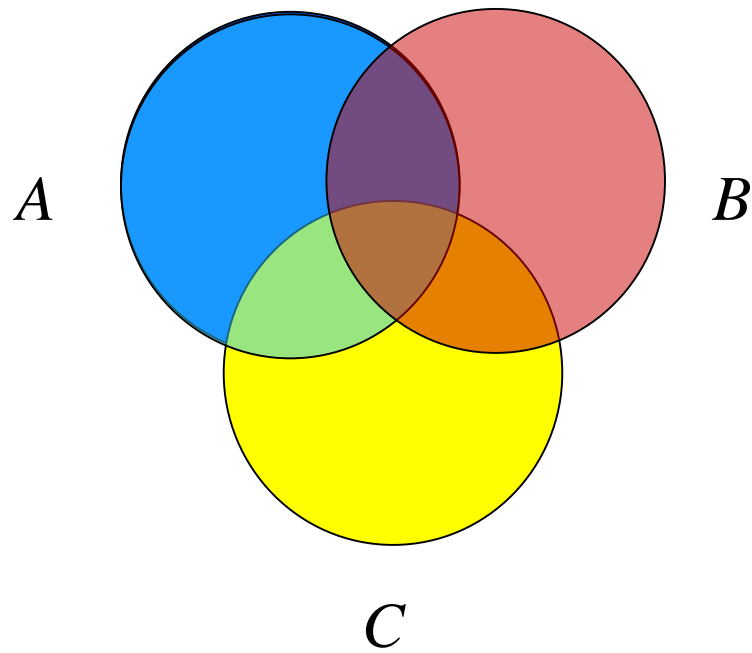


# Sets



# This Lecture

We will introduce some basic set theory in this session.

- Basic Definitions
- Operations on Sets
- Set Identities
- Russell's Paradox

# Defining Sets

**Definition:** A set is an **unordered** collection of **distinct** objects.

The objects in a set are called the **elements** or **members** of the set  $S$ , and we say  $S$  **contains** its elements.

e.g.  $S = \{2, 3, 5, 7\} = \{3, 5, 7, 2\}$

$S = \{\{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}\}$

$\text{pow}(\{a,b\}) = \{\emptyset, \{a\}, \{b\}, \{a,b\}\}$  (power set of  $\{a,b\}$ )

Which of the following are sets?

- $S = \{2, 3, 5, 3, 7\}$  NO
- $S = \{\{a\}, a\}$  YES

# Classical Sets

The following are some well-known examples of sets.

$\mathbb{Z}$ : the set of all integers

$\mathbb{Z}^+$ : the set of all positive integers

$\mathbb{Z}^-$ : the set of all negative integers

$\mathbb{N}$ : the set of all nonnegative integers

$\mathbb{R}$ : the set of all real numbers

$\mathbb{Q}$ : the set of all rational numbers

$\mathbb{C}$ : the set of all complex numbers

# Defining Sets by Properties

It is inconvenient, and sometimes impossible, to define a set by listing all its elements.

Alternatively, we use the notation  $\{x \in A \mid P(x)\}$  to define the set as the **set of elements**,  $x$ , in  $A$  **such that**  $x$  satisfies property  $P$ .

e.g.  $S = \{x \in \mathbb{R} \mid -2 < x < 5\}$

$$S = \{x \mid x \text{ is a prime and } x < 70,000,000\}$$

**Definition:** The **size** of a set  $S$ , denoted by  $|S|$ , is defined as the number of elements contained in  $S$ .

# Membership

The most basic question in set theory is whether an element is in a set.

$x \in A$     $x$  is an **element** of  $A$        $x \notin A$     $x$  is not an **element** of  $A$   
 $x$  is **in**  $A$                                        $x$  is not **in**  $A$

## Definition:

- $A \subseteq B$  ( $A$  is a **subset** of  $B$ )  $\longleftrightarrow$  For any  $x \in A$  we have  $x \in B$ .
- $A = B$  ( $A$  is **equal** to  $B$ )  $\longleftrightarrow$   $A \subseteq B$  and  $B \subseteq A$ .
- $A \subset B$  ( $A$  is a **proper subset** of  $B$ )  $\longleftrightarrow$   $A \subseteq B$  and  $A \neq B$ .

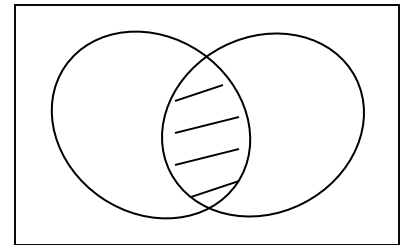
# This Lecture

- Basic Definitions
- Operations on Sets
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- Russell's Paradox

# Basic Operations on Sets

Let  $A, B$  be two subsets of a *universal* set  $U$   
(depending on the context  $U$  could be  $\mathbb{R}$ ,  $\mathbb{Z}$ , or other sets).

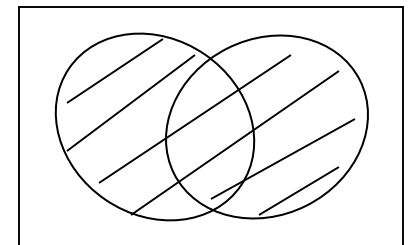
**intersection:**  $A \cap B = \{x \in U \mid x \in A \text{ and } x \in B\}$



**Defintion:** Two sets are said to be **disjoint** if their intersection is an empty set.

e.g. Let  $A$  be the set of odd numbers, and  $B$  be the set of even numbers.  
Then  $A$  and  $B$  are disjoint.

**union:**  $A \cup B = \{x \in U \mid x \in A \text{ or } x \in B\}$



**Fact:**  $|A \cup B| = |A| + |B| - |A \cap B|$



# Basic Operations on Sets

- Definition

## Unions and Intersections of an Indexed Collection of Sets

Given sets  $A_0, A_1, A_2, \dots$  that are subsets of a universal set  $U$  and given a nonnegative integer  $n$ ,

$$\bigcup_{i=0}^n A_i = \{x \in U \mid x \in A_i \text{ for at least one } i = 0, 1, 2, \dots, n\}$$

$$\bigcup_{i=0}^{\infty} A_i = \{x \in U \mid x \in A_i \text{ for at least one nonnegative integer } i\}$$

$$\bigcap_{i=0}^n A_i = \{x \in U \mid x \in A_i \text{ for all } i = 0, 1, 2, \dots, n\}$$

$$\bigcap_{i=0}^{\infty} A_i = \{x \in U \mid x \in A_i \text{ for all nonnegative integers } i\}.$$

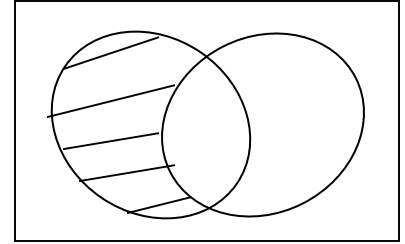
# Basic Operations on Sets

For each positive integer  $i$ , let  $A_i = \left\{x \in \mathbf{R} \mid -\frac{1}{i} < x < \frac{1}{i}\right\} = A_i = \left(-\frac{1}{i}, \frac{1}{i}\right)$ .

- a. Find  $A_1 \cup A_2 \cup A_3$  and  $A_1 \cap A_2 \cap A_3$ .      b. Find  $\bigcup_{i=1}^{\infty} A_i$  and  $\bigcap_{i=1}^{\infty} A_i$ .

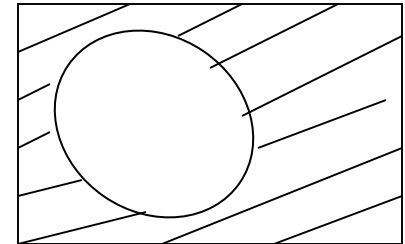
# Basic Operations on Sets

**difference:**  $A - B = \{x \in U \mid x \in A \text{ and } x \notin B\}$



**Fact:**  $|A - B| = |A| - |A \cap B|$

**complement:**  $\overline{A} = A^c = \{x \in U \mid x \notin A\}$



e.g. Let  $U = \mathbb{Z}$  and  $A$  be the set of odd numbers.

Then  $\overline{A}$  is the set of even numbers.

**Fact:** If  $A \subseteq B$ , then  $\overline{B} \subseteq \overline{A}$

# Examples

$$A = \{1, 3, 6, 8, 10\} \quad B = \{2, 4, 6, 7, 10\}$$

$$A \cap B = \{6, 10\}, \quad A \cup B = \{1, 2, 3, 4, 6, 7, 8, 10\} \quad A - B = \{1, 3, 8\}$$

$$\text{Let } U = \{x \in \mathbb{Z} \mid 1 \leq x \leq 100\}.$$

$$A = \{x \in U \mid x \text{ is divisible by } 3\}, \quad B = \{x \in U \mid x \text{ is divisible by } 5\}$$

$$A \cap B = \{x \in U \mid x \text{ is divisible by } 15\}$$

$$A \cup B = \{x \in U \mid x \text{ is divisible by } 3 \text{ or is divisible by } 5 \text{ (or both)}\}$$

$$A - B = \{x \in U \mid x \text{ is divisible by } 3 \text{ but is not divisible by } 5\}$$

**Exercise:** compute  $|A|$ ,  $|B|$ ,  $|A \cap B|$ ,  $|A \cup B|$ ,  $|A - B|$ .

# Partitions of Sets

Two sets are **disjoint** if their intersection is empty.

A collection of nonempty sets  $\{A_1, A_2, \dots, A_n\}$  is a **partition** of a set  $A$  if and only if

$$A = A_1 \cup A_2 \cup \dots \cup A_n$$

$A_1, A_2, \dots, A_n$  are **mutually disjoint** (or pairwise disjoint).

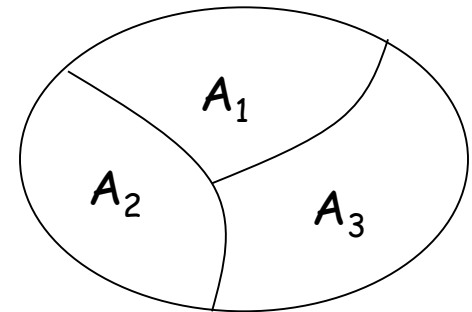
e.g. Let  $A$  be the set of integers.

$$A_1 = \{x \in A \mid x = 3k+1 \text{ for some integer } k\}$$

$$A_2 = \{x \in A \mid x = 3k+2 \text{ for some integer } k\}$$

$$A_3 = \{x \in A \mid x = 3k \text{ for some integer } k\}$$

Then  $\{A_1, A_2, A_3\}$  is a partition of  $A$



# Partitions of Sets

e.g.  $A = \{x \in \mathbb{Z} \mid x \text{ is divisible by } 6\}$ .

$A_1 = \{x \in \mathbb{Z} \mid x \text{ is divisible by } 2\}$ .

$A_2 = \{x \in \mathbb{Z} \mid x \text{ is divisible by } 3\}$ .

Then  $\{A_1, A_2\}$  is not a partition of  $A$ , because

- $A_1 \cap A_2 \neq \emptyset$
- $A \subset A_1 \cup A_2$

e.g.  $A = \mathbb{Z}$ .

$A_1 = \{x \in \mathbb{Z} \mid x < 0\}$ .

$A_2 = \{x \in \mathbb{Z} \mid x > 0\}$ .

Then  $\{A_1, A_2\}$  is not a partition of  $A$ , because

$A \not\supset A_1 \cup A_2$  as 0 is contained in  $A$ .

# Cartesian Products

**Definition:** Given two sets  $A$  and  $B$ , the **Cartesian product**  $A \times B$  is the set of all ordered pairs  $(a,b)$ , where  $a$  is in  $A$  and  $b$  is in  $B$ . That is,

$$A \times B = \{(a,b) \mid a \in A, b \in B\}$$

**Ordered pairs** means the ordering is important, e.g.  $(1,2) \neq (2,1)$

e.g. Let  $A$  be the set of letters, i.e.  $\{a,b,c,\dots,x,y,z\}$ .

Let  $B$  be the set of digits, i.e.  $\{0,1,\dots,9\}$ .

$A \times A$  is just the set of strings with two letters.

$B \times B$  is just the set of strings with two digits.

$A \times B$  is the set of strings where the first character is a letter and the second character is a digit.

# Cartesian Products

The definition can be generalized to any number of sets, e.g.

$$A \times B \times C = \{(a, b, c) \mid a \in A \text{ and } b \in B \text{ and } c \in C\}$$

Using the above examples,  $A \times A \times A$  is the set of strings with three letters.

e.g. the set of the vectors in  $\mathbb{R}^3$  is the set  $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$ .

**Fact:** If  $|A| = n$  and  $|B| = m$ , then  $|A \times B| = nm$ .

**Fact:** If  $|A| = n$  and  $|B| = m$  and  $|C| = l$ , then  $|A \times B \times C| = nml$ .

**Fact:**  $|A_1 \times A_2 \times \dots \times A_k| = |A_1| \times |A_2| \times \dots \times |A_k|$ .



## Exercises

1. Let  $A$  be the set of prime numbers, and let  $B$  be the set of even numbers. What is  $A \cap B$  and  $|A \cap B|$ ?
2. Is  $|A \cup B| > |A| > |A \cap B|$  always true?
3. Let  $A$  be the set of all  $n$ -bit binary strings,  $A_i$  be the set of all  $n$ -bit binary strings with  $i$  ones. Is  $(A_1, A_2, \dots, A_i, \dots, A_n)$  a partition of  $A$ ?

# This Lecture

- Basic Definitions
- Operations on Sets
- Set Identities
- Russell's Paradox

# Set Identities

Let  $A, B, C$  be subsets of a universal set  $U$ .

Commutative Law: (a)  $A \cup B = B \cup A$  and (b)  $A \cap B = B \cap A$

Associative Law: (a)  $(A \cup B) \cup C = A \cup (B \cup C)$   
(b)  $(A \cap B) \cap C = A \cap (B \cap C)$

Distributive Law: (a)  $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$   
(b)  $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Identity Law: (a)  $A \cup \emptyset = A$  and (b)  $A \cap U = A$

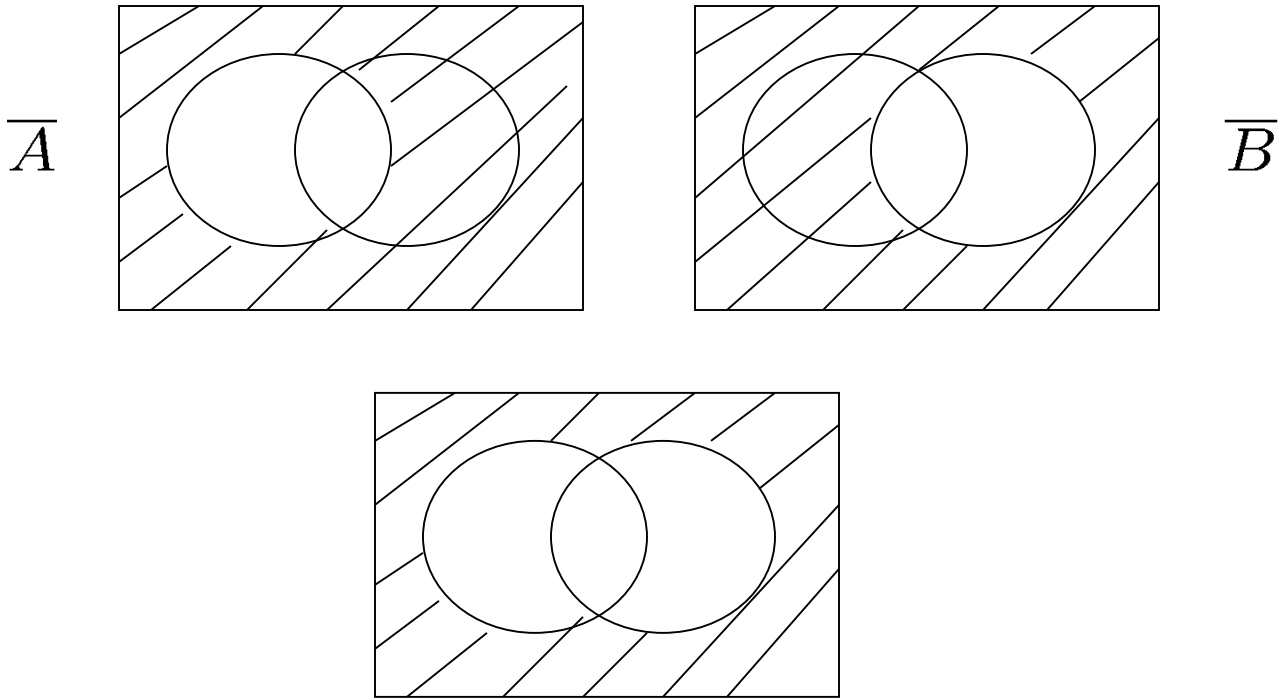
Complement Law: (a)  $A \cup A^c = U$  and (b)  $A \cap A^c = \emptyset$

De Morgan's Law: (a)  $(A \cup B)^c = A^c \cap B^c$  and (b)  $(A \cap B)^c = A^c \cup B^c$

Set difference Law:  $A - B = A \cap B^c$

# Venn Diagram

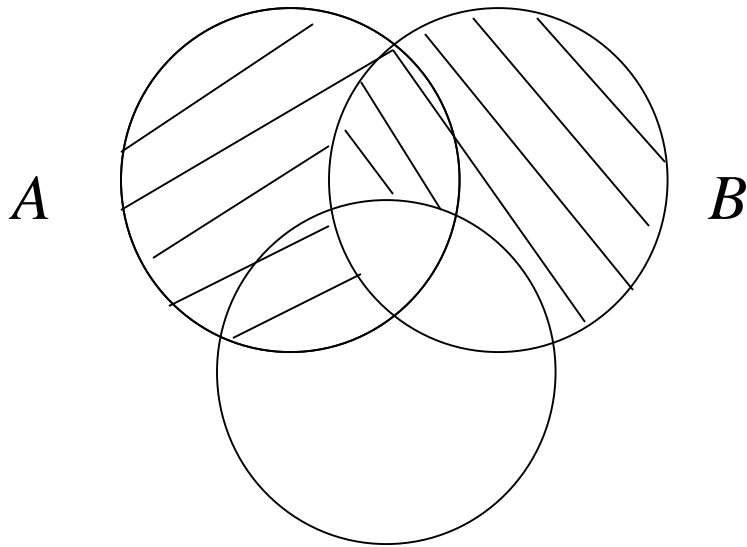
De Morgan's Law:  $\overline{A \cup B} = \overline{A} \cap \overline{B}$



$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

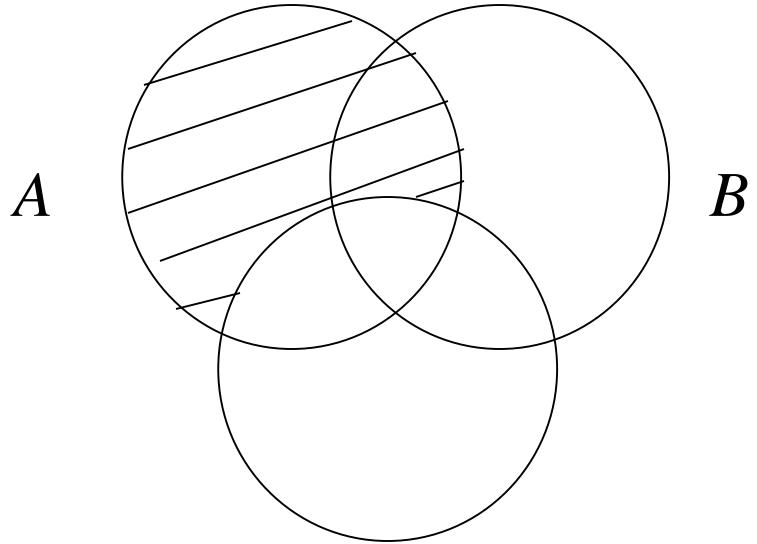
## Disproof

$$(A - B) \cup (B - C) = A - C?$$



$C$

L.H.S

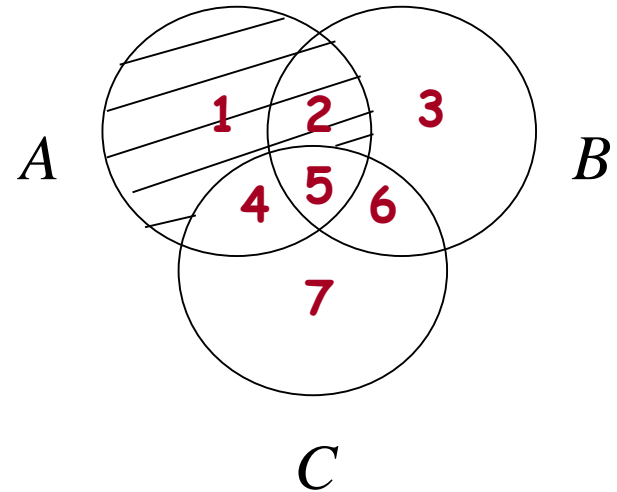
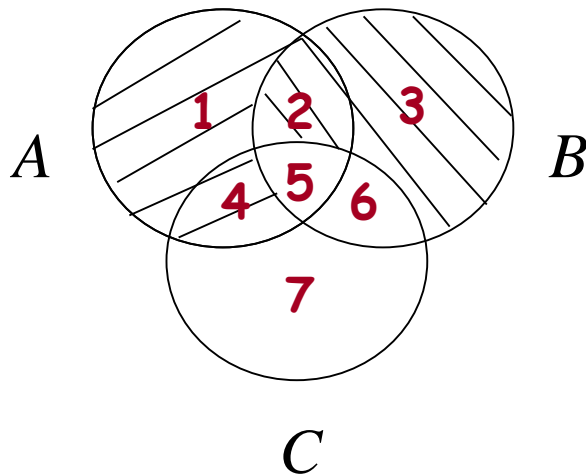


$C$

R.H.S

## Disproof

$$(A - B) \cup (B - C) = A - C?$$



We can easily construct a **counterexample** to the equality, by putting a number in each region in the figure.

Let  $A = \{1, 2, 4, 5\}$ ,  $B = \{2, 3, 5, 6\}$ ,  $C = \{4, 5, 6, 7\}$ .

Then we see that L.H.S =  $\{1, 2, 3, 4\}$  and R.H.S =  $\{1, 2\}$ .

## Algebraic Proof

$$\overline{((A \cup C) \cap (B \cup C))} = (\overline{A} \cup \overline{B}) \cap \overline{C}?$$

$$\overline{((A \cup C) \cap (B \cup C))}$$

$$= \overline{(A \cup C)} \cup \overline{(B \cup C)} \quad \text{by DeMorgan's law}$$

$$= (\overline{A} \cap \overline{C}) \cup \overline{(B \cup C)} \quad \text{by DeMorgan's law}$$

$$= (\overline{A} \cap \overline{C}) \cup (\overline{B} \cap \overline{C}) \quad \text{by DeMorgan's law}$$

$$= (\overline{A} \cup \overline{B}) \cap \overline{C} \quad \text{by Distributive law}$$

# Proof by Definition

How to prove  $(A \cap B) \times C = (A \times C) \cap (B \times C)$ ?

1.  $LHS \subseteq RHS$ .

Since  $(A \cap B) \times C \subseteq A \times C$ ,  $(A \cap B) \times C \subseteq B \times C$

2.  $RHS \subseteq LHS$ .

$(x,y) \in (A \times C) \cap (B \times C) \Rightarrow (x,y) \in A \times C$  and  $(x,y) \in B \times C$

So  $x \in A$  and  $x \in B \Rightarrow x \in A \cap B$

Hence  $(x,y) \in (A \cap B) \times C$ .

Therefore, we complete the proof.



## Exercises

$$A - (A \cap B) = A - B?$$

$$(A \cup B) - C = (A - C) \cup (B - C)?$$

$$\overline{(A \cup B \cup C)} = \overline{A} \cap \overline{B} \cap \overline{C}?$$

# This Lecture

- Basic Definitions
- Operations on Sets
- Set Identities
- Russell's Paradox

# Russell's Paradox

$$\text{Let } W ::= \{S \in \text{Sets} \mid S \notin S\}$$

In words,  $W$  is the set that contains all the sets that don't contain themselves.

Is  $W$  in  $W$ ?

If  $W$  is in  $W$ , then  $W$  contains itself.

But the  $W$  property implies that  $W$  is not in  $W$ .

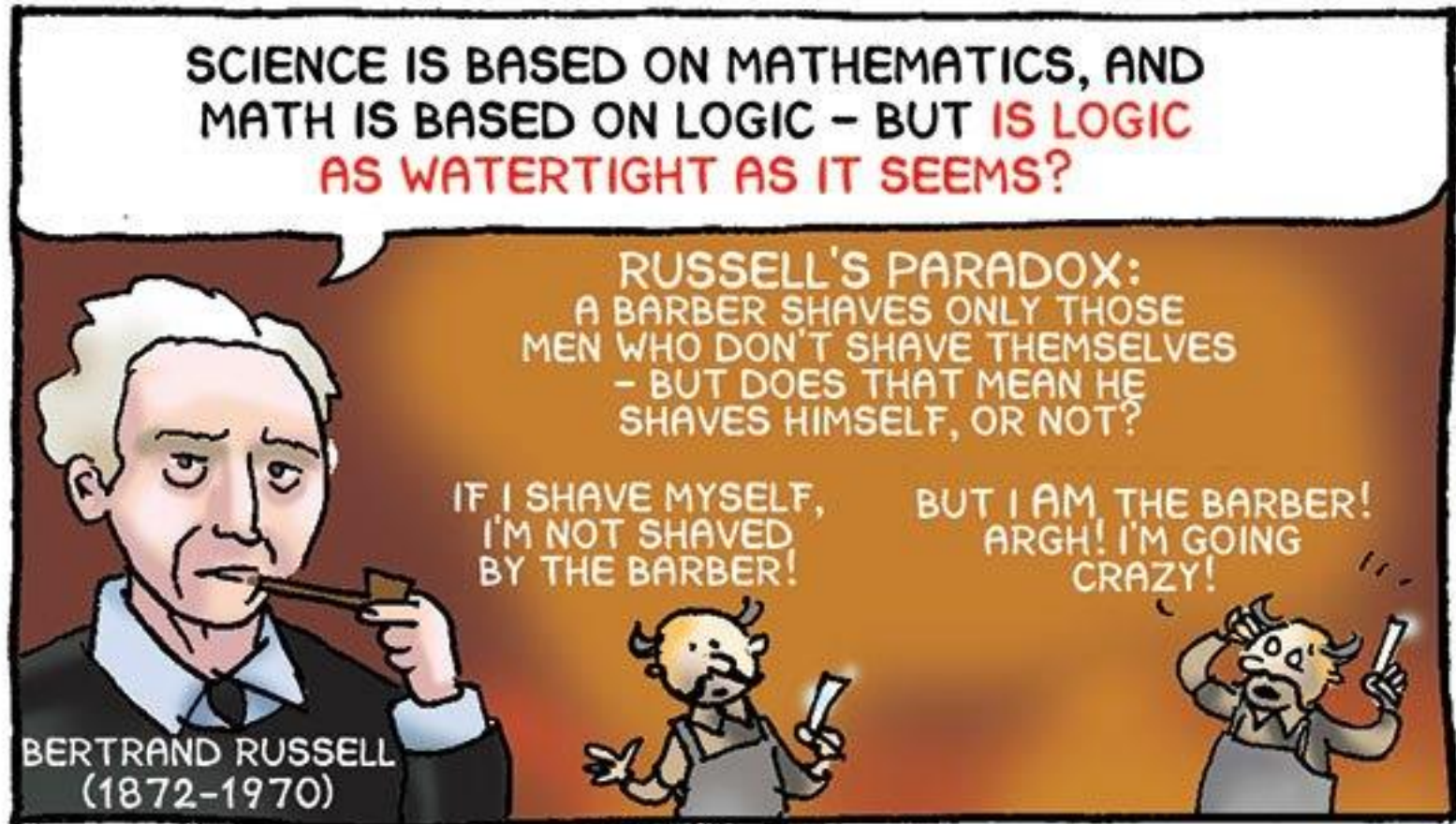
So  $W$  is not in  $W$ .

If  $W$  is not in  $W$ , then it satisfies the  $W$  property.

So  $W$  is in  $W$ .

What's wrong???

# Barber's Paradox



## Solution to Russell's Paradox

A man either shaves himself or not shaves himself.

A barber either shaves himself or not shaves himself.

Perhaps such a barber does not exist?

Actually this is the way out of the paradox.

Going back to the Russell's paradox,

we conclude that  $W$  cannot be a set,

because every set either contains itself or not,

but neither case can happen for  $W$ .

This paradox tells us that not everything we define is a set.

Later on mathematicians define sets more carefully,

e.g. using the sets that we already know.

# Halting Problem (Optional)

Now we mention one of the most famous problems in computer science.

**The halting problem:** Can we write a program which detects infinite loop?

We want a program  $H$  that given any program  $P$  and input  $I$ :

$H(P,I)$  returns "halt" if  $P$  will terminate given input  $I$ ;

$H(P,I)$  returns "loop forever" if  $P$  will not terminate given input  $I$ .

And  $H$  itself must terminate in finite time.

**The halting problem:** Does such a program  $H$  exist?

**NO!**

The reasoning used in solving the halting problem is very similar to that of Russell's paradox.

# Halting Problem (Optional)

We want a program  $H$  that given any program  $P$  and input  $I$ :

- $H(P, I)$  returns "halt" if  $P$  will terminate given input  $I$ ;
- $H(P, I)$  returns "loop forever" if  $P$  will not terminate given input  $I$ .
- $H$  itself must terminate in finite time.

Program  $P$  consists of characters, and hence it is an input.

Construct a program  $\text{Test}(P)$  such that

- $\text{Test}(P)$  loops forever if  $H(P, P)$  halts;
  - $\text{Test}(P)$  halts if  $H(P, P)$  loops forever.
- 
- If  $\text{Test}(\text{Test})$  loops forever, then so does  $H(\text{Test}, \text{Test})$ , hence  $\text{Test}(\text{Test})$  halts. **A contradiction!**
  - If  $\text{Test}(\text{Test})$  halts, then so does  $H(\text{Test}, \text{Test})$ , hence  $\text{Test}(\text{Test})$  loops forever. **A contradiction!**

**The halting problem:** Does such a program  $H$  exist?

**NO!**

# Summary

Recall what we have covered so far.

- Basic Definitions (defining sets, membership, subsets, size)
- Operations on Sets (intersection, union, difference, complement, partition, power set, Cartesian product)
- Set Identities (Distributive law, DeMorgan's law, checking set identities - proof & disproof, algebraic)