CIE6020 Assignment 4

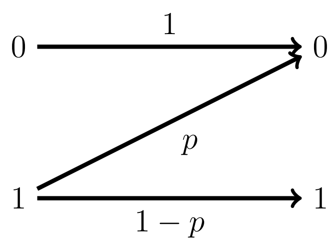
Due: 23:55, 10 April 2019

1. *Fano's inequality.* Consider a random variable X over $\{1, 2, \ldots, m\}$ with $\Pr(X = i) = p_i, i = 1, 2, ..., m$, where $p_1 \geq p_2 \geq \cdots \geq p_m$. The minimal probability of error predictor when there is no information about the instance of X is $\hat{X} = 1$, the most probable value of X, with resulting probability of error $P_e = 1 - p_1$. Maximize $H(X)$ subject to the constraint $1 - p_1 = P_e$ to find a bound on P_e in terms of $H(X)$. This is Fano's inequality in the absence of conditioning.

2. *Z-channel.* The Z-channel is a binary input and binary output channel with the transition probabilities $W(y|x)$ given by

$$W = \begin{bmatrix} 1 & 0 \\ p & 1-p \end{bmatrix}.$$

See an illustration in the figure below.



Find the capacity of the Z-channel and the maximizing input probability distribution.

3. Consider the discrete memoryless channel $Y = X + Z$ (mod 11), where Z is uniformly distributed on $\{1, 2, 3\}$. Assume Z is independent of X. Find the capacity of this channel and the maximizing input probability distribution.

4. Consider a channel with the input and output alphabet $\{0, 1\}$. The ith input X_i and the ith output Y_i, $i = 1, 2, \ldots$ are related by

$$Y_i = X_i + U_i$$

where the addition is modulo 2 and U_i has distribution $\Pr\{U_i = 1\} = 1 - \Pr\{U_i = 0\} = q$. Here U_j and $(X_i, i = 1, \ldots)$ are independent.

   (a) When $U_i, i = 1, 2, \ldots$ and $(X_j, j = 1, \ldots)$ are independent, show the channel is a memoryless binary symmetric channel and give its capacity.

(Hint: show that for any integer $n > 0$,

$$\Pr\{Y_i = y_i, i = 1, \dots, n | X_i = x_i, i = 1, \dots, n\} = \prod_{i=1}^{n} \Pr\{Y_i = y_i | X_i = x_i\},$$

i.e., the channel is memoryless.)

(b) When $U_i = U_{i+1}, i = 1, 3, 5, \dots$, and $U_i, i = 1, 3, 5, \dots$ and $(X_j, j = 1, \dots)$ are independent, show that the channel is not memoryless.

(Hint: calculate $\Pr\{Y_1 = y_1, Y_2 = y_2 | X_1 = x_1, X_2 = x_2\}$ and show that it is not the same as $\Pr\{Y_1 = y_1 | X_1 = x_1\} \Pr\{Y_2 = y_2 | X_2 = x_2\}$.)

(c) Under the condition of (b), the channel can be equivalent to a DMC by combining two consecutive uses of the channel. Give the transition matrix of this DMC, and calculate its capacity.

(d) Assume you are given a set of capacity achieving codes for the memoryless binary symmetric channel under the condition of (a). Using these codes, construct a capacity achieving code for the channel under the condition of (b).

5. Consider a stochastic process $U_1, U_2, \dots$ with $U_i \in \mathcal{U}$, a finite set, such that the entropy rate H exists and $-\frac{1}{n} \log p(U_1, U_2, \dots, U_n) \to H$ in probability. For any integer $n > 0$ and real number $\epsilon > 0$, find a subset $A_\epsilon^{(n)} \subset \mathcal{U}^n$ such that $|A_\epsilon^{(n)}| \leq 2^{n(H+\epsilon)}$ and $\Pr\{(U_1, \dots, U_n) \in A_\epsilon^{(n)}\} > 1 - \epsilon$ when n is sufficiently large.