Assignment 3

Hand-in Evaluation Deadline: 5:00 pm, 28th October
In-class Evaluation: L1: 2:40 pm - 2:50 pm, 1th November
L2: 9:40 am - 9:50 am, 1th November
L3: 2:40 pm - 2:50 pm, 31th October
L4: 4:40 pm - 4:50 pm, 1th November

1. Determine whether each of the following set of vectors is linearly independent or linearly dependent. If it is linearly dependent, write a nontrivial relation of dependence:
(a)
$$\mathbf{u_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{u_2} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}, \mathbf{u_3} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}.$$
(b)
$$\mathbf{v_1} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \mathbf{v_2} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \end{bmatrix}, \mathbf{v_3} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v_4} = \begin{bmatrix} 0 \\ 0 \\ 3 \\ 8 \end{bmatrix}.$$

2. Find a maximal linearly independent subset of the following set:
$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ -3 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ -6 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} -4 \\ 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -5 \\ 9 \\ -4 \\ 7 \end{bmatrix} \right\}.$$

3. Determine whether the following sets are subspaces of $\mathbb{R}^{2\times 2}$. Show your reasoning.
(a) The set of all 2×2 diagonal matrices.
(b) The set of all 2×2 triangular matrices.
(c) The set of all 2×2 lower triangular matrices.
(d) The set of all 2×2 matrices A such that $a_{12} = 1$.
(e) The set of all 2×2 matrices B such that $b_{11} = 0$.
(f) The set of all symmetric 2×2 matrices.
(g) The set of all singular 2×2 matrices.

4. (a) Write the solution set of $A\mathbf{x} = \mathbf{b}$ in parametric vector form.

$$A = \begin{bmatrix} 3 & 4 & 4 \\ -3 & -2 & 0 \\ 6 & 2 & -4 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 2 \\ 2 \\ -8 \end{bmatrix}$$

(b) For $A\mathbf{x} = \mathbf{b}'$ the following is a solution, Give the full solution set of $A\mathbf{x} = \mathbf{b}'$ in parametric vector form. Explain why you could write this down without doing any work.

$$A = \begin{bmatrix} 3 & 4 & 4 \\ -3 & -2 & 0 \\ 6 & 2 & -4 \end{bmatrix}, \mathbf{b}' = \begin{bmatrix} 7 \\ -1 \\ -4 \end{bmatrix},$$

and

$$\mathbf{x} = \begin{bmatrix} -5/3 \\ 3 \\ 0 \end{bmatrix}$$

5. Let $\mathbf{b_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{b_2} = \begin{bmatrix} -3 \\ 4 \\ 0 \end{bmatrix}, \mathbf{b_3} = \begin{bmatrix} 3 \\ -6 \\ 3 \end{bmatrix}$.

(a) Show that the set $\mathcal{B} = \{\mathbf{b_1}, \mathbf{b_2}, \mathbf{b_3}\}$ is a basis of $\mathbb{R}^3$.

(b) Given a vector $\mathbf{x} = \begin{bmatrix} -8 \\ 2 \\ 3 \end{bmatrix}$ in $\mathcal{R}^3$. Find the coordinate vector of $\mathbf{x}$ with respect to the basis $\mathcal{B}$.

(c) Given the coordinate of $\mathbf{y}$ with respect to the basis $\mathcal{B}$ is $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. What is the vector $\mathbf{y}$ in $\mathcal{R}^3$?

6. Determine the dimension of each of the following vector spaces:

(a) $\text{span}\left\{ \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} -3 \\ 3 \\ 6 \end{bmatrix} \right\}$.

(b) $\text{span}\{(x-2)(x+2), x^2(x^4-2), x^6-8\}$.

7. Let V be a vector space of dimension $n > 0$, show that

(a) Any set of n linearly independent vectors in V forms a basis.

(b) Any set of n vectors that span V forms a basis.

8. Let

$$V = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathcal{R}^4 | x_1 + x_2 + x_3 + x_4 = 0 \right\}.$$

Show that

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right\}$$

is a basis for V. What is $\dim(V)$?

9. Show that if U and V are subspaces of $\mathbb{R}^n$ and $U \cap V = \{\mathbf{0}\}$, then

$$\dim(U + V) = \dim(U) + \dim(V)$$

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10. Given vectors $\mathbf{u_1}, \cdots, \mathbf{u_p}$ and $\mathbf{w}$ in a vector space V with basis B, show that $\mathbf{w}$ is a linear combination of $\mathbf{u_1}, \cdots, \mathbf{u_p}$ if and only if $[\mathbf{w}]_B$ is a linear combination of the coordinate vectors $[\mathbf{u_1}]_B, [\mathbf{u_2}]_B, ..., [\mathbf{u_p}]_B$.

11. Let $H = \text{Span}\{\mathbf{v_1}, \mathbf{v_2}\}$ and $K = \text{Span}\{\mathbf{v_3}, \mathbf{v_4}\}$ where $\mathbf{v_1} = [1, -1, -3]^T, \mathbf{v_2} = [8, -9, 6]^T, \mathbf{v_3} = [-3, -1, 8]^T, \mathbf{v_4} = [3, -5, 4]^T$. Geometrically, H and K are planes in $\mathbb{R}^3$ through the origin, and they intersect in a line through origin. Find a nonzero vector $\mathbf{w}$ that generates that line (i.e., $\text{Span}\{\mathbf{w}\}$ is that line).(Hint: $\mathbf{w}$ can be written as a linear combination of $\mathbf{v_1}$ and $\mathbf{v_2}$, and as a linear combination of $\mathbf{v_3}$ and $\mathbf{v_4}$. To build $\mathbf{w}$, solve the equation $c_1\mathbf{v_1} + c_2\mathbf{v_2} = c_3\mathbf{v_3} + c_4\mathbf{v_4}$ for the unknown c_1, c_2, c_3, c_4.)

12. (a) Suppose $T = \{\mathbf{u_1}, \cdots, \mathbf{u_n}\}, S = \{\mathbf{v_1}, \cdots, \mathbf{v_m}\}$. Let $\mathbf{u_i} \in \text{Span}(S)$ for all $i = 1, \cdots, n$, show that $\text{Span}(T) \subseteq \text{Span}(S)$.

    (b) Let $T = \{\mathbf{u_1}, \cdots, \mathbf{u_n}\}$ and $S = \{\mathbf{u_1} + \mathbf{3u_2}, \mathbf{u_2}, \cdots, \mathbf{u_n}\}$, show that Span($T$)=Span($S$).