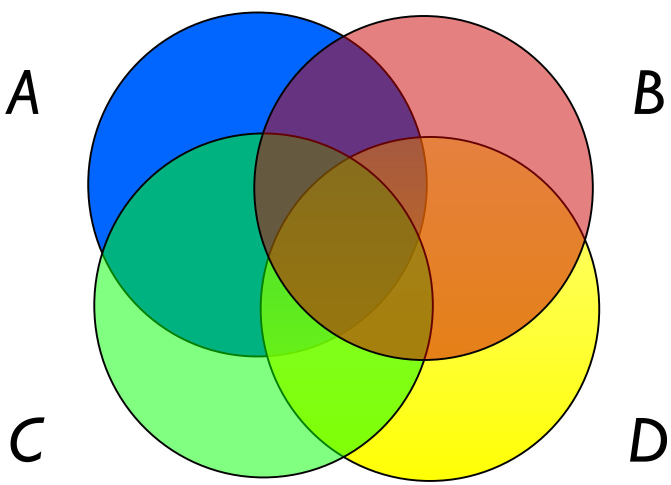
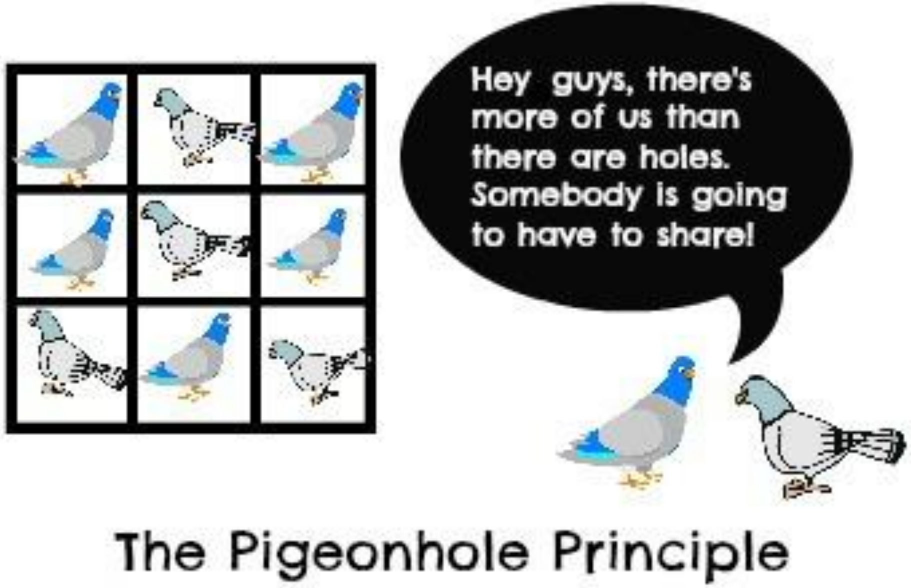
Combinatorial Proofs and its Principles





Plan

Combinatorics is a typical technique in discrete mathematics. This technique is proved very useful in counting and enjoys wide range applications from evolutionary biology to computer science, etc.

- Binomial coefficients, combinatorial proof
- Inclusion-exclusion principle
- Pigeonhole principle

Binomial Theorem

$$\boxed{(1+x)^n = c_0 + c_1 x + c_2 x^2 + \ldots + c_n x^n}$$

We can compute the coefficients c_i by counting arguments.

e.g. $$(1+x)^3 = (1+x)(1+x)(1+x)$$

$$= 1\cdot 1\cdot 1 + 1\cdot 1\cdot x + 1\cdot x\cdot 1 + 1\cdot x\cdot x$$

$$+ x\cdot 1\cdot 1 + x\cdot 1\cdot x + x\cdot x\cdot 1 + x\cdot x\cdot x$$

(expand by taking either 1 or x from each factor and multiply)

$$= 1 + 3x + 3x^2 + x^3$$

(group the terms with the same power and add)

So in this case, $c_0 = 1$, $c_1 = 3$, $c_2 = 3$, $c_3 = 1$.

Binomial Theorem

$$\boxed{(1+x)^n = c_0 + c_1 x + c_2 x^2 + \ldots + c_n x^n}$$

We can compute the coefficients c_i by counting arguments.

$$(1+x)^n = \underbrace{(1+x)(1+x)(1+x)\cdots\cdots(1+x)}_{\text{n factors}}$$

Each term corresponds to selecting 1 or x from each of the n factors.

So the coefficient c_k corresponds to the number of ways for choosing k positions of x from n factors.

Therefore, $c_k = \binom{n}{k}$

These are called the **binomial coefficients**.

Binomial Theorem

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots + \binom{n}{n}x^n$$

$$
\begin{array}{ll}
(1+X)^0 = & 1 \\
(1+X)^1 = & 1 + 1X \\
(1+X)^2 = & 1 + 2X + 1X^2 \\
(1+X)^3 = & 1 + 3X + 3X^2 + 1X^3 \\
(1+X)^4 = & 1 + 4X + 6X^2 + 4X^3 + 1X^4
\end{array}
$$

We see that the coefficients are the sums of two coefficients in the upper level.

This is called the Pascal's formula and we will prove it soon.

Binomial Coefficients

In general we have the following identity:

$$\boxed{(y+x)^n = \binom{n}{0} y^n + \binom{n}{1} x y^{n-1} + \ldots + \binom{n}{k} x^k y^{n-k} + \ldots + \binom{n}{n} x^n}$$

because if we choose k x's then there will be n-k y's.

Corollary: When x = 1, y = 1, it implies that

$$2^n = \binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{k} + \ldots + \binom{n}{n}$$

That is, the sum of the binomial coefficients is equal to 2^n.

Binomial Coefficients

In general we have the following identity:

$$\boxed{(y+x)^n = \binom{n}{0}y^n + \binom{n}{1}xy^{n-1} + \ldots + \binom{n}{k}x^k y^{n-k} + \ldots + \binom{n}{n}x^n}$$

Corollary: When x = -1, y = 1, it implies that

$$0 = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \binom{n}{4} + \ldots + (-1)^n\binom{n}{n}$$

$\Rightarrow$ The sum of "odd" binomial coefficients $=$ the sum of "even" binomial coefficients

Proving Identities

$$\boxed{\binom{n}{k} = \binom{n}{n-k}}$$

One can often prove identities of binomial coefficients by a counting argument.

Direct proof: $$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \binom{n}{n-k}$$

Combinatorial proof:

Number of ways to choose k items from n items	=	number of ways to choose n-k items from n items

Finding a Combinatorial Proof

A **combinatorial proof** is an argument that establishes algebraic facts by counting principles.

Many such proofs follow the same basic outline:

1. Define a set S.

2. Show that $|S| = n$ by counting one way.

3. Show that $|S| = m$ by counting another way.

4. Conclude that $n = m$.

Double counting

Proving Identities

Pascal's Formula

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

Direct proof:

$$\binom{n}{k-1} + \binom{n}{k} = \frac{n!}{(k-1)!(n-k+1)!} + \frac{n!}{k!(n-k)!}$$

$$= \frac{n!k + n!(n-k+1)}{k!(n-k+1)!}$$

$$= \frac{n!(n+1)}{k!(n-k+1)!}$$

$$= \frac{(n+1)!}{k!(n-k+1)!} = \binom{n+1}{k}$$

Proving Identities

Pascal's Formula

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

Combinatorial proof:

LHS = number of ways to choose k elements from n+1 elements

For RHS, fix an element x in the n+1 elements.

1) If the k elements contain x, then we need to choose k-1 elements from the remaining n elements, so $\binom{n}{k-1}$.
2) If the k elements do not contain x, then we need to choose k elements from the remaining n elements, so $\binom{n}{k}$.

Hence, we complete the proof.

Combinatorial Proof

$$\boxed{\binom{n}{0}^2 + \binom{n}{1}^2 + \ldots + \binom{n}{n}^2 = \binom{2n}{n}}$$

Consider 2n balls, half red, half blue.

RHS = number of ways to choose n balls from 2n balls.

On the other hand, to choose n balls, we can

- choose 0 red ball and n blue balls, so $\binom{n}{0}\binom{n}{n} = \binom{n}{0}^2$
- choose 1 red ball and n-1 blue balls, so $\binom{n}{1}\binom{n}{n-1} = \binom{n}{1}^2$
- …
- choose i red balls and n-i blue balls, so $\binom{n}{i}\binom{n}{n-i} = \binom{n}{i}^2$
- …
- choose n red balls and 0 blue ball, so $\binom{n}{n}\binom{n}{0} = \binom{n}{n}^2$

Therefore, LHS = RHS.

Another Combinatorial Proof

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \ldots + \binom{n}{n}^2 = \binom{2n}{n}$$

This can be also proved by calculating a coefficient in two different ways.

Consider the identity $(1+x)^n(1+x)^n = (1+x)^{2n}$

1. For LHS, we have

$$(1+x)^n(1+x)^n = (\binom{n}{0}+\binom{n}{1}x+\ldots+\binom{n}{n}x^n)(\binom{n}{0}+\binom{n}{1}x+\ldots+\binom{n}{n}x^n)$$

So the coefficient of x^n is $\binom{n}{0}^2 + \binom{n}{1}^2 + \ldots + \binom{n}{n}^2$

2. For RHS, the coefficient of x^n is $\binom{2n}{n}$

Exercises

Prove that

$$3^n = 1+2n+4\binom{n}{2}+8\binom{n}{3}+\ldots+2^k\binom{n}{k}+\ldots+2^n\binom{n}{n}$$

Give a combinatorial proof of the following identity.

$$\binom{n}{0}\binom{2n}{n}+\binom{n}{1}\binom{2n}{n-1}+\ldots+\binom{n}{k}\binom{2n}{n-k}+\ldots+\binom{n}{n}\binom{2n}{0}=\binom{3n}{n}$$

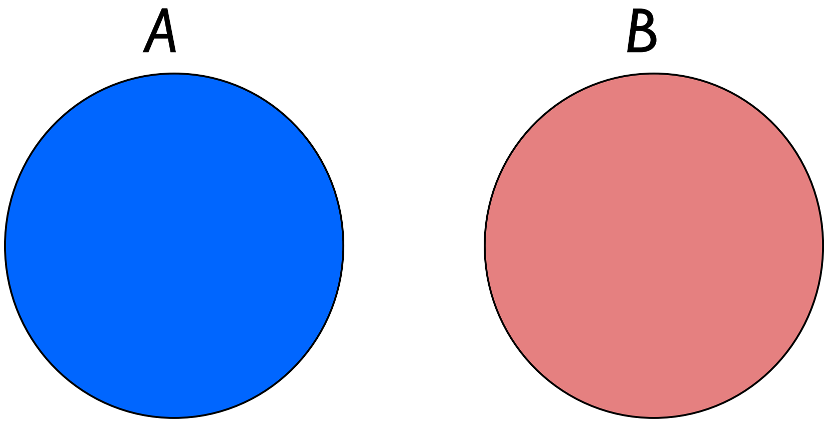
Plan

- Binomial coefficients, combinatorial proof
- Inclusion-exclusion principle
- Pigeonhole principle

Sum Rule

If sets A and B are disjoint, then

$$|A \cup B| = |A| + |B|$$

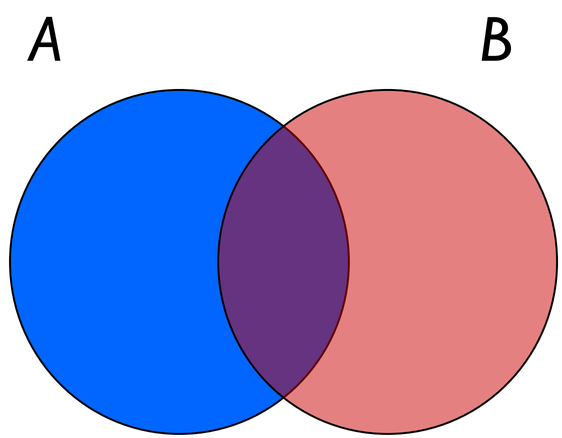


What if A and B are not disjoint?

Inclusion-Exclusion (2 sets)

For two arbitrary sets A and B

$$|A \cup B| = |A| + |B| - |A \cap B|$$



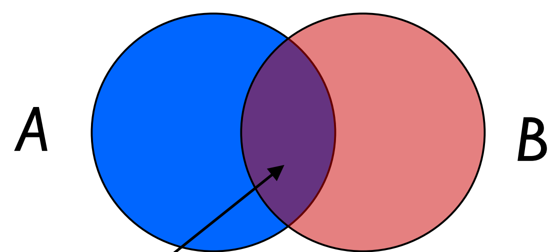
Inclusion-Exclusion (2 sets)

Let S be the set of integers from 1 through 1000 that are multiples of 3 or multiples of 5.

Let A = {integers from 1 to 1000 that are multiples of 3}.

Let B = {integers from 1 to 1000 that are multiples of 5}.

It is clear that S is the union of A and B, but notice that A and B are not disjoint.



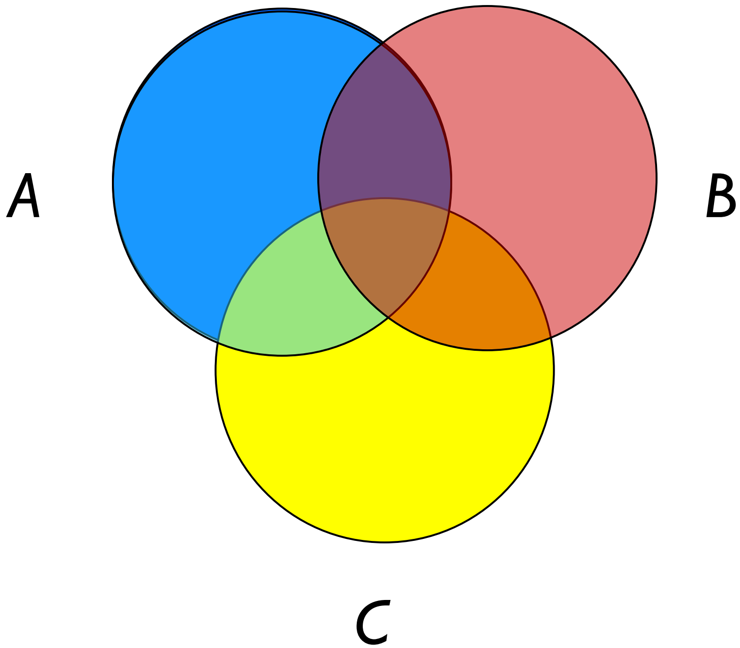
$|A| = \lfloor 1000/3 \rfloor = 333$ $|B| = \lfloor 1000/5 \rfloor = 200$

$A \cap B$ is the set of integers that are multiples of 15, and so $|A \cap B| = \lfloor 1000/15 \rfloor = 66$

So, by the inclusion-exclusion principle, we have $|S| = |A| + |B| - |A \cap B| = 467$.

Inclusion-Exclusion (3 sets)

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$



Inclusion-Exclusion (3 sets)

From a total of 50 students:

$|A| \longrightarrow$ 30 know Java

$|B| \longrightarrow$ 18 know C++

$|C| \longrightarrow$ 26 know C#

$|A \cap B| \longrightarrow$ 9 know both Java and C++

$|A \cap C| \longrightarrow$ 16 know both Java and C#

$|B \cap C| \longrightarrow$ 8 know both C++ and C#

$|A \cup B \cup C| \longrightarrow$ 47 know at least one language.

How many know all?

$|A \cap B \cap C|$

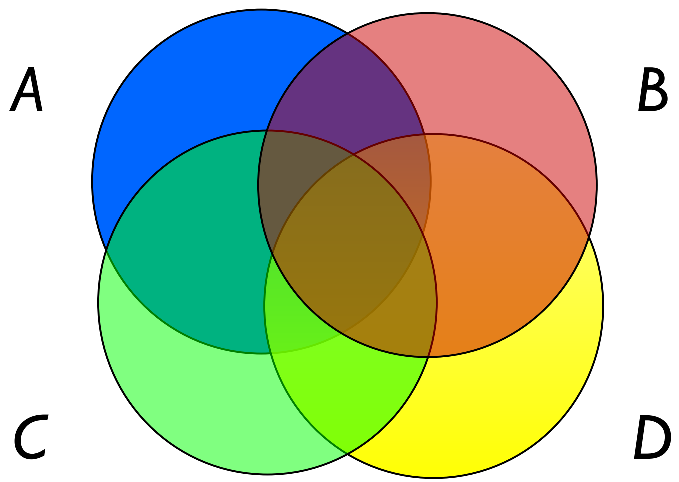
$$\boxed{|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|}$$

$$47 = 30 + 18 + 26 - 9 - 16 - 8 + |A \cap B \cap C|$$

$$|A \cap B \cap C| = 6$$

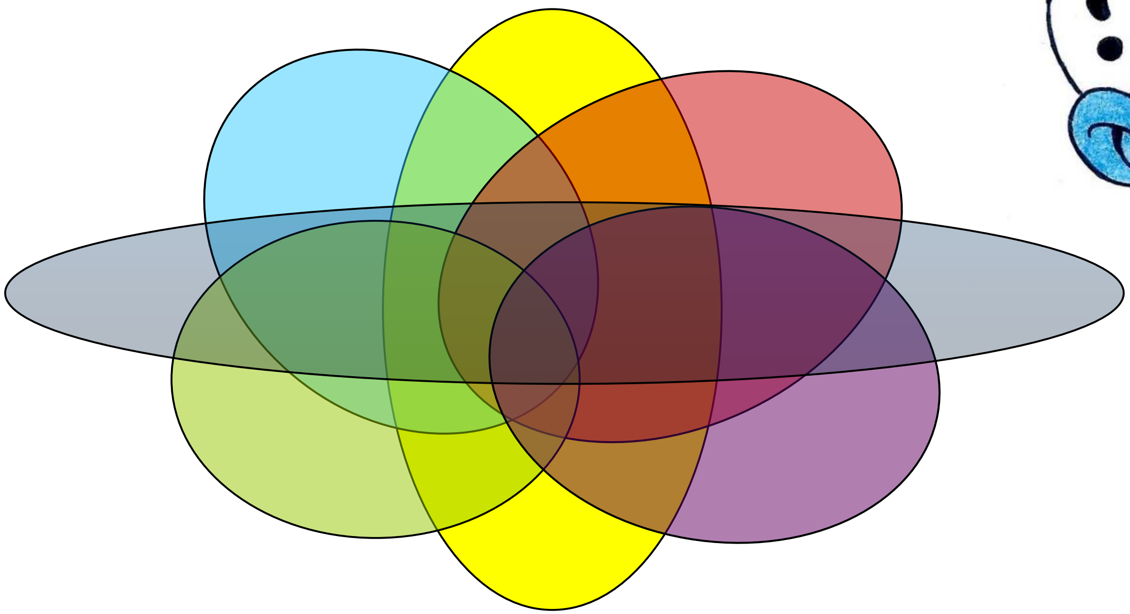
Inclusion-Exclusion (4 sets)

$$
\begin{aligned}
|A \cup B \cup C \cup D| = &|A| + |B| + |C| + |D| \\
&- |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D| \\
&+ |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D| \\
&- |A \cap B \cap C \cap D|
\end{aligned}
$$



Inclusion-Exclusion (n sets)

What is the inclusion-exclusion formula for the union of n sets?



Inclusion-Exclusion (n sets)

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i| - \sum_{1 \le i \le j \le n} |A_i \cap A_j| + \sum_{1 \le i \le j \le k \le n} |A_i \cap A_j \cap A_k| + \cdots + (-1)^{n+1} |A_1 \cap \cdots \cap A_n|$$

$|A_1 \cup A_2 \cup \ldots \cup A_n| =$

sum of sizes of all single sets
- sum of all 2-set intersection sizes
+ sum of all 3-set intersection sizes
- sum of all 4-set intersection sizes
…
+ $(-1)^{n+1} \times$ sum of all n-set intersection sizes

Inclusion-Exclusion (n sets)

Proof of the inclusion-exclusion formula:

Consider an element x in $A_1 \cap A_2 \cap \ldots \cap A_k$. How many times it is counted in RHS?

- Single sets: $|A_1|, |A_2|, \ldots, |A_k|$ $\quad \Longrightarrow (-1)^{1+1}\binom{k}{1}$
- 2-sets: $-|A_1 \cap A_2|, -|A_1 \cap A_3|, \ldots, -|A_{k-1} \cap A_k|$ $\quad \Longrightarrow (-1)^{2+1}\binom{k}{2}$
- 3-sets: $|A_1 \cap A_2 \cap A_3|, \ldots, |A_{k-2} \cap A_{k-1} \cap A_k|$ $\quad \Longrightarrow (-1)^{3+1}\binom{k}{3}$

$\vdots \qquad\qquad \vdots$

- k-sets: $(-1)^{k+1}|A_1 \cap A_2 \cap \ldots \cap A_k|$ $\quad \Longrightarrow (-1)^{k+1}\binom{k}{k}$

From RHS we see that x is counted

$$\binom{k}{1} - \binom{k}{2} + \binom{k}{3} - \binom{k}{4} + \ldots + (-1)^{k+1}\binom{k}{k} = \binom{k}{0} = 1$$

see slide 7

Example

Suppose a, b, c are integers such that $0 \leq a \leq 3, \quad 0 \leq b \leq 4, \quad 0 \leq c \leq 6.$

How many solutions does the following equation have?

$$a + b + c = 11$$

- The universal set U = { (a,b,c) | a+b+c=11 }, let N = |U|.
- P_1 = { solutions with a > 3 }
- P_2 = { solutions with b > 4 }
- P_3 = { solutions with c > 6 }
- We need to calculate $|\overline{P_1} \cap \overline{P_2} \cap \overline{P_3}|$ = N - $|P_1 \cup P_2 \cup P_3|$.

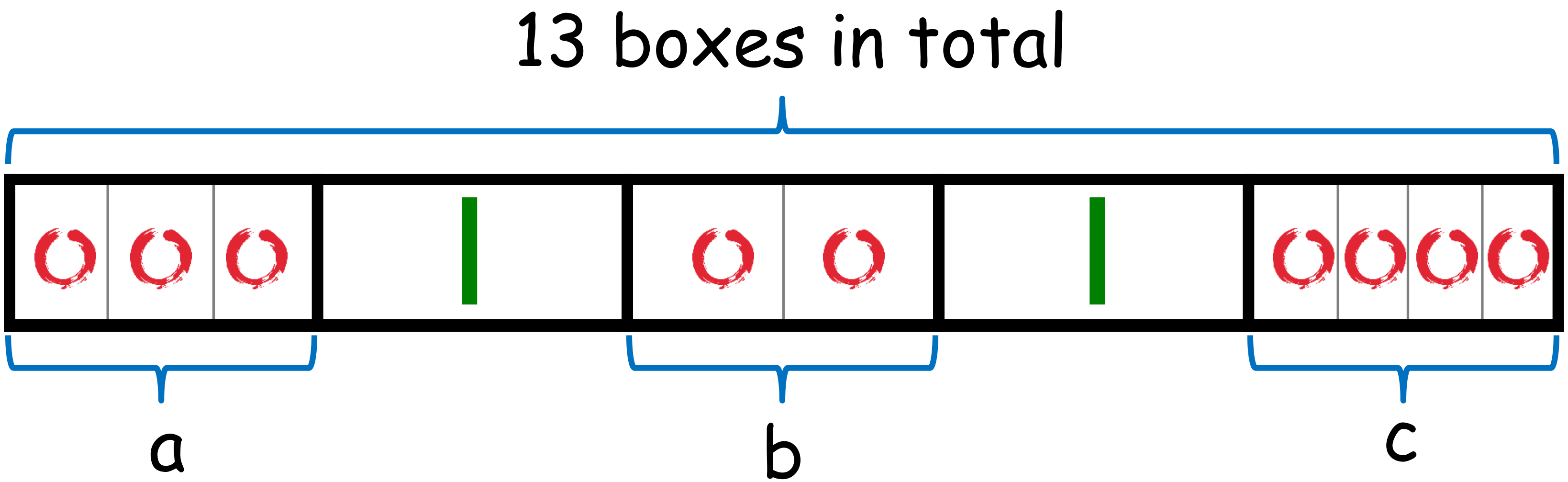
By the inclusion-exclusion formula we have

$$|\overline{P_1} \cap \overline{P_2} \cap \overline{P_3}| = N - |P_1| - |P_2| - |P_3| + |P_1 \cap P_2| + |P_1 \cap P_3| + |P_2 \cap P_3| - |P_1 \cap P_2 \cap P_3|$$

Example

The universal set U = { (a,b,c) | a+b+c=11 }, let N = |U|.

How to count N?



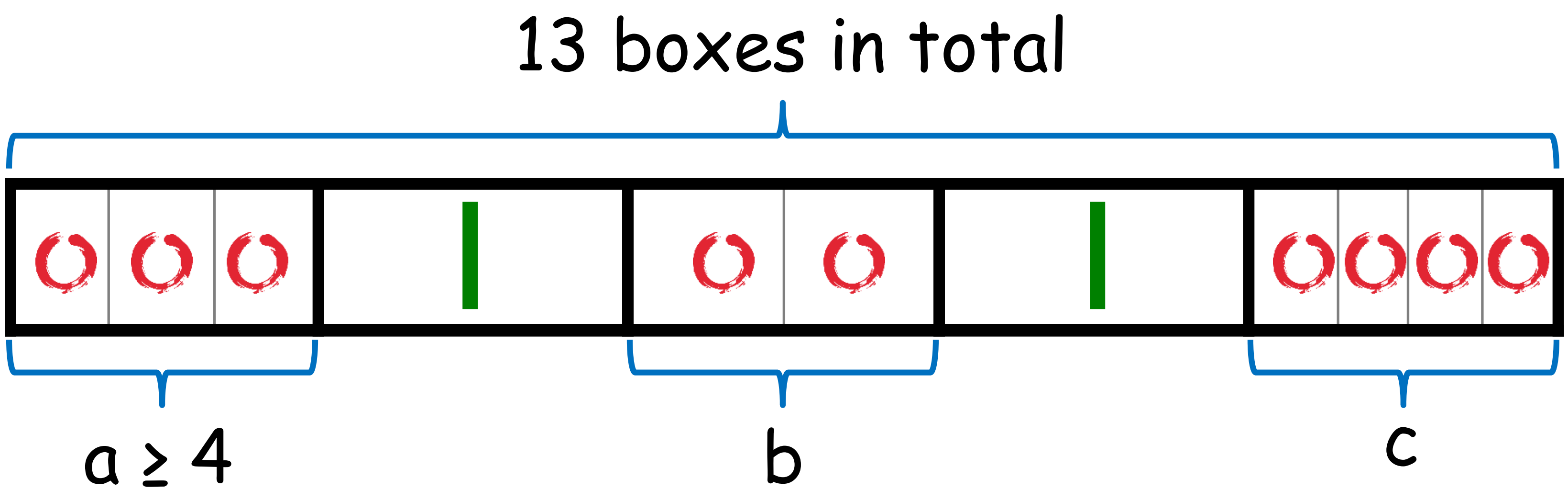
are placed in the boxes above, and separated by .

So N = $\binom{11+3-1}{11}$ = 78.

Example

P$_1$ = { solutions with a > 3 } = { solutions with a ≥ 4 }

How to count |P$_1$|?



There are 7 (= 11 - 4) circles we need to choose,
so $|P_1| = \binom{7+3-1}{7} = 36$.

Example

So we have

- $N = |U| = |\{ (a,b,c) \mid a+b+c=11 \}| = \binom{11+3-1}{11} = 78$
- $|P_1| = |\{ \text{solutions with } a >= 4 \}| = \binom{7+3-1}{7} = 36$
- $|P_2| = |\{ \text{solutions with } b >= 5 \}| = \binom{6+3-1}{6} = 28$
- $|P_3| = |\{ \text{solutions with } c >= 7 \}| = \binom{4+3-1}{4} = 15$
- $|P_1 \cap P_2| = |\{ \text{solutions with } a >= 4,\ b >= 5 \}| = \binom{2+3-1}{2} = 6$
- $|P_1 \cap P_3| = |\{ \text{solutions with } a >= 4,\ c >= 7 \}| = \binom{0+3-1}{0} = 1$
- $|P_2 \cap P_3| = |\{ \text{solutions with } b >= 5,\ c >= 7 \}| = 0$
- $|P_1 \cap P_2 \cap P_3| = |\{ \text{solutions with } a >= 4,\ b >= 5,\ c >= 7 \}| = 0$

The number of solutions is

$$|\overline{P_1} \cap \overline{P_2} \cap \overline{P_3}| = 78 - 36 - 28 - 15 + 6 + 1 + 0 - 0 = 6$$

Christmas Presents

In a Christmas party, everyone brings his/her present.

There are n people and so there are totally n presents.

Suppose the host collects and shuffles all the presents.

Now everyone picks a random present.

What is the probability that no one picks his/her own present?

- The universal set U = {people-present matching} $\Rightarrow$ N = |U| = n!
- P_i = {people i picks his/her own present}.
- We need to calculate $|\overline{P_1} \cap \overline{P_2} \cap \ldots \cap \overline{P_n}|$ = N - $|P_1 \cup P_2 \cup \cdots \cup P_n|$.

By the inclusion-exclusion formula we have

$$|\overline{P_1} \cap \overline{P_2} \cap \ldots \cap \overline{P_n}| = N - \sum_{i=1}^{n} |P_i| + \sum_{1 \le i < j \le n} |P_i \cap P_j| - \sum_{1 \le i < j < k \le n} |P_i \cap P_j \cap P_k| + \cdots + (-1)^n |P_1 \cap \cdots \cap P_n|$$

Christmas Presents

$$\left|\overline{P_1} \cap \overline{P_2} \cap \ldots \cap \overline{P_n}\right| = N - \sum_{i=1}^{n} |P_i| + \sum_{1 \le i < j \le n} |P_i \cap P_j| - \sum_{1 \le i < j < k \le n} |P_i \cap P_j \cap P_k| + \cdots + (-1)^n |P_1 \cap \cdots \cap P_n|$$

- What is $|P_i|$?
  - $|P_i| = (n-1)!$ as there are n-1 remaining.
  - There are $\binom{n}{1}$ of $|P_i|$
- What is $|P_i \cap P_j|$?
  - $|P_i \cap P_j| = (n-2)!$ as there are n-2 remaining.
  - There are $\binom{n}{2}$ of $|P_i \cap P_j|$

…

So $\left|\overline{P_1} \cap \overline{P_2} \cap \ldots \cap \overline{P_n}\right| = N + \sum_{i=1}^{n} (-1)^i \binom{n}{i} (n-i)!$

Christmas Presents

$$\left|\overline{P_1}\cap\overline{P_2}\cap\ldots\cap\overline{P_n}\right| = N + \sum_{i=1}^{n}(-1)^i\binom{n}{i}(n-i)!$$

$$= n! + n!\sum_{i=1}^{n}(-1)^i\frac{1}{i!}$$

(Recall N = n!)

$$= n!\sum_{i=0}^{n}(-1)^i\frac{1}{i!}$$

Thus, the probability hat no one picks his/her own present is

$$p = \left|\overline{P_1}\cap\overline{P_2}\cap\ldots\cap\overline{P_n}\right|/N = \sum_{i=0}^{n}\frac{(-1)^i}{i!}$$

Christmas Presents

The probability hat no one picks his/her own present is

$$p = |\overline{P_1} \cap \overline{P_2} \cap \ldots \cap \overline{P_n}|/N = \sum_{i=0}^{n} \frac{(-1)^i}{i!}$$

Recall the Taylor series

$$e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^k$$

So

$$p \rightarrow e^{-1} = 1/e \approx 0.3679 \qquad (\text{as } n \rightarrow \infty)$$

Euler's totient function

Given a number n, how many numbers from 1 to n are relatively prime to n?

This number is denoted by $\varphi(n)$, and φ is called Euler's totient function.

Let $n = p_1^{c_1} p_2^{c_2} \cdots p_r^{c_r}$

- The universal set U = { 1, …, n } ⇒ N = |U| = n
- P_i = {the number that is divisible by p_i}
- We need to calculate $|\overline{P_1} \cap \overline{P_2} \cap \ldots \cap \overline{P_r}|$ = N - $|P_1 \cup P_2 \cup \cdots \cup P_r|$.

Using the inclusion-exclusion formula we can show that the answer is

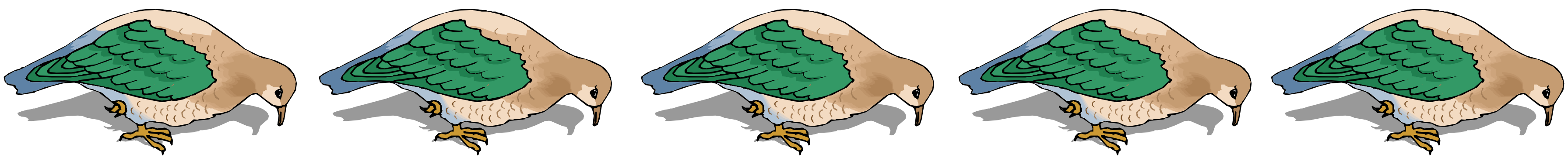
$n(1-1/p_1)(1-1/p_2)\ldots(1-1/p_r)$

Plan

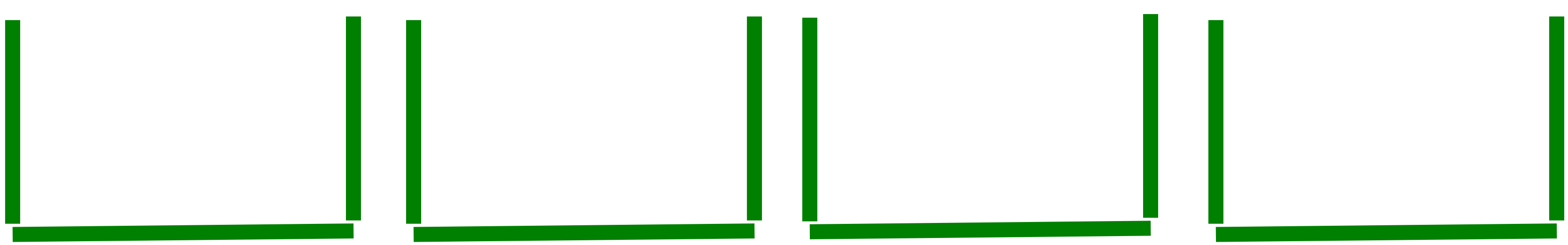
- Binomial coefficients, combinatorial proof
- Inclusion-exclusion principle
- Pigeonhole principle

Pigeonhole Principle

If more pigeons

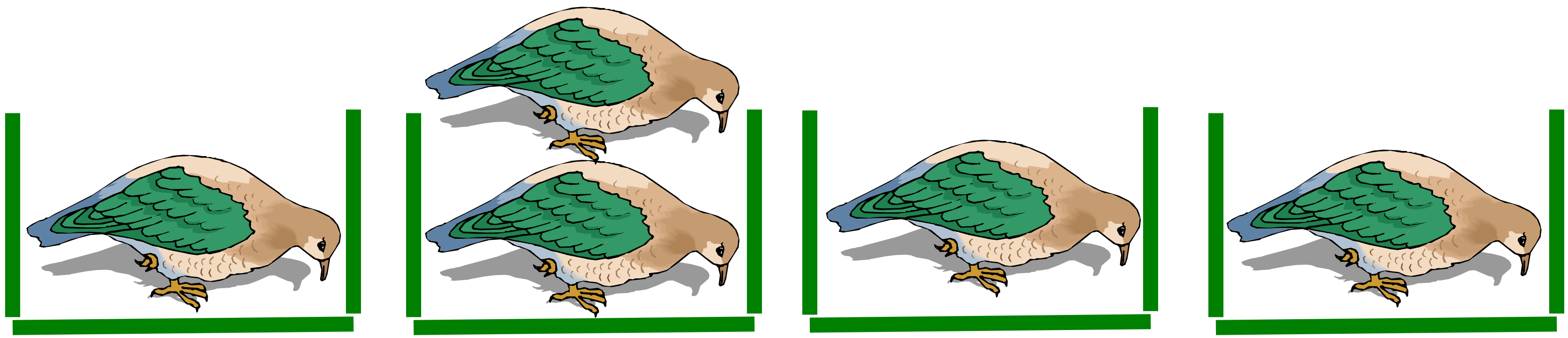


than pigeonholes,



Pigeonhole Principle

then some hole must have at least two pigeons!



Pigeonhole principle

A function from a larger set to a smaller set cannot be injective.

(There must be at least two elements in the domain that are mapped to the same element in the range.)

Picking Pairs

Question: Let $A = \{1,2,3,4,5,6,7,8\}$

If five distinct integers are selected from A, must a pair of integers have a sum of 9?

Consider the pairs $\{1,8\}$, $\{2,7\}$, $\{3,6\}$, $\{4,5\}$.
These are all the pairs whose sum is equal to 9.

These pairs cover each number exactly once, so our problem is to pick 5 numbers from these 4 pairs.

By the pigeonhole principle, at least one such pair will have to be picked.

Handshaking

Question: In a party of n people, is it always true that there are two people shaking hands with equal number of people?

Everyone can shake hand with 0 to n-1 people, and there are n people, and so it does not seem that it must be the case, but think about it carefully:

Case 1: if there is a person who does not shake hand with others, then any person can shake hands with at most n-2 people. So everyone (n people) shakes hand with 0 to n-2 people (n-1 numbers), by the pigeonhole principle the answer is "yes".

Case 2: if everyone shakes hand with at least one person, then everyone (n people) shakes hand with 1 to n-1 people (n-1 numbers), hence the answer is also "yes" by the pigeonhole principle.

Birthday Problem

In a group of 367 people, there must be two people having the same birthday.

Suppose $n \le 365$, what is the probability that in a random set of n people, some pair of them will have the same birthday?

We can think of it as picking n random numbers from 1 to 365 without repetition.

There are 365^n ways of picking n numbers from 1 to 365.

There are $365 \cdot 364 \cdot 363 \cdot \ldots \cdot (365-n+1)$ ways of picking n numbers from 1 to 365 without repetition.

So the probability that no pairs have the same birthday is equal to $365 \cdot 364 \cdot 363 \cdot \ldots \cdot (365-n+1) / 365^n$

This is smaller than 50% for 23 people, smaller than 1% for 57 people.

Subset Sum

20480135385502964448038
4894459918669156762409920
1082662032430379651370981
1178480894769706178994993
1253127351683239693851327
1301505129234077811069011
1311567111143866433882194
1470029452721203587686214
1578271047286257499433886
1638243921852176243192354
1763580219131985963102365
1826227795601842231029694
1843971862675102037201420
2396951193722134526177237
2781394568268599801096354
2796605196713610405408019
2931016394761975263190347
2933458058294405155197296
3075514410490975920315348
3111474985252793452860017
3145621587936120118438701
3148901255628881103198549
3157693105325111284321993
3171004832173501394113017
3208234421597368647019265
3437254656355157864869113
3574883393058653923711365
3644909946040480189969149
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3870332127437971355322815
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4167283461025702348124920
4235996831123777788211249
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4815379351865384279613427
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9913237476341764299813987

Question. Given the 90 25-digit numbers above, can we find two different subsets that give the same sum?

Subset Sum

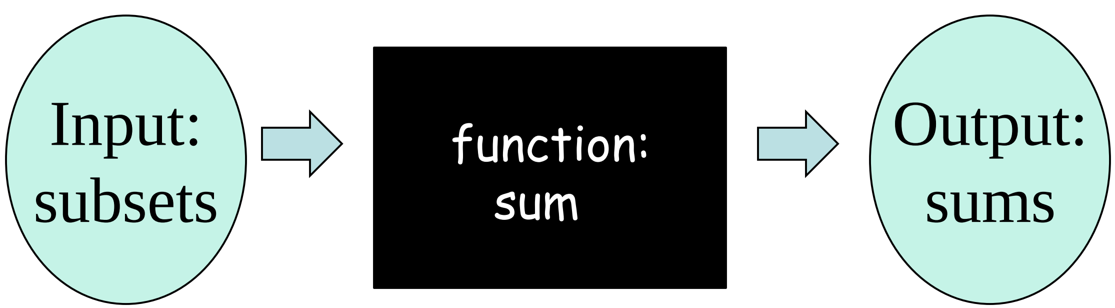
How to solve this problem?

How about the opposite? Can all the sums be different?

We can count the following sets.

- X = { all possible different subsets of the 90 numbers set }
- Y = { all possible different sums that these subsets may yield }

If $|X| > |Y|$, then by the pigeonhole principle, there are more inputs than outputs and thus it is not possible for all subsets to have different sums.



Subset Sum

Let A be the set of the 90 numbers, each with at most 25 digits.

So the total sum of the 90 numbers is at most 90×10^{25}.

Let X be the set of all subsets of the 90 numbers. **(pigeons)**

Let Y be the set of integers from 0 to 90×10^{25}. **(pigeonholes)**

Let $f: X \to Y$ be a function mapping each subset of A into its sum.

If we could show that $|X| > |Y|$, then by the pigeonhole principle, the function f must map two elements in X into a single element in Y.

This means that there are two subsets having the same sum.

Subset Sum

Let A be the set of the 90 numbers, each with at most 25 digits.
So the total sum of the 90 numbers is at most 90×10^{25}.

Let X be the set of all subsets of the 90 numbers. **(pigeons)**

Let Y be the set of integers from 0 to 90×10^{25}. **(pigeonholes)**

$$|X| = |pow(A)| = 2^{90} \geq 1.237 \times 10^{27}$$

$$|Y| \leq 90 \times 10^{25} \leq 0.901 \times 10^{27}$$

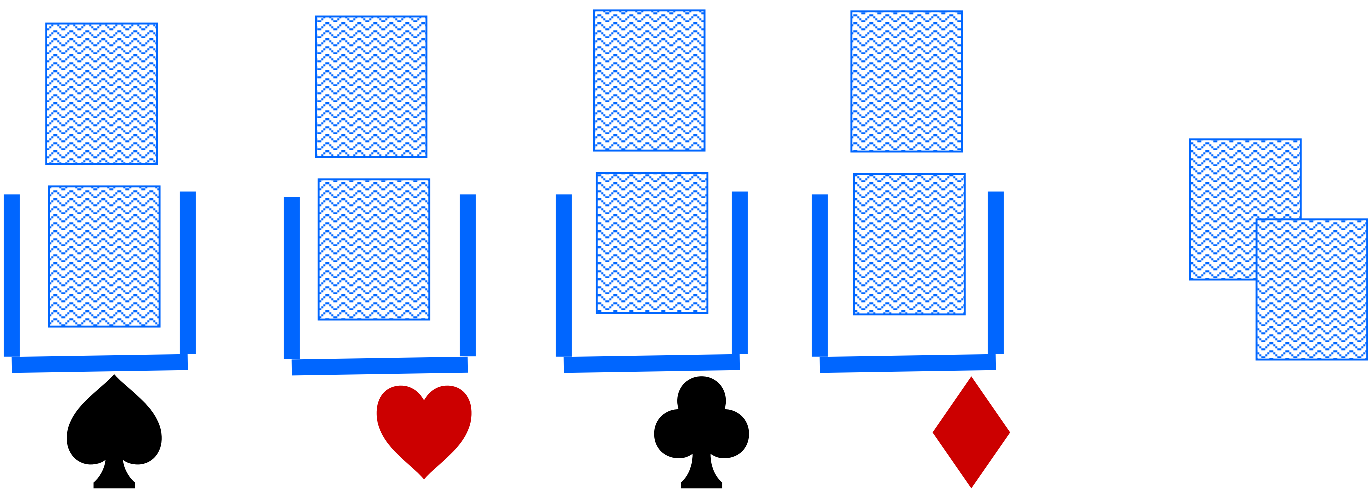
So, $|X| > |Y|$.

By the pigeonhole principle, there are two different subsets with the same sum.

Generalized Pigeonhole Principle

Generalized Pigeonhole Principle

If n pigeons and h holes,
then some hole has at least $\left\lceil \frac{n}{h} \right\rceil$ pigeons.



Cannot have < 3 cards in every hole.

Club vs Strangers

Let's agree that given any two people, either they have met or not.

If every people in a group has met, then we'll call the group a club.

If every people in a group has not met, then we'll call a group of strangers.

Theorem. Every collection of 6 people includes a club of 3 people, or a group of 3 strangers.

Let x be one of the six people.

By the (generalized) pigeonhole principle, we have the following claim.

Claim. For the remaining 5 people, either 3 of them have met x, or 3 of them have not met x.

Club vs Strangers

Theorem. Every collection of 6 people includes a club of 3 people,
or a group of 3 strangers.

Claim. For the remaining 5 people, either 3 of them have met x,
or 3 of them have not met x.

Case 1: "3 people have met x"

Case 1.1: No pair in these 3 people has met each other.
Then there is a group of 3 strangers.

Case 1.2: Some pair in these 3 people has met each other.
Then that pair, together with x, form a club of 3 people.

Club vs Strangers

Theorem. Every collection of 6 people includes a club of 3 people, or a group of 3 strangers.

Claim. For the remaining 5 people, either 3 of them have met x, or 3 of them have not met x.

Case 2: "3 people have not met x"

Case 2.1: Every pair in these 3 people has met each other.
Then there is a club of 3 people.

Case 2.2: Some pair in these 3 people has not met each other.
Then that pair, together with x, form a group of 3 strangers.

Club vs Strangers

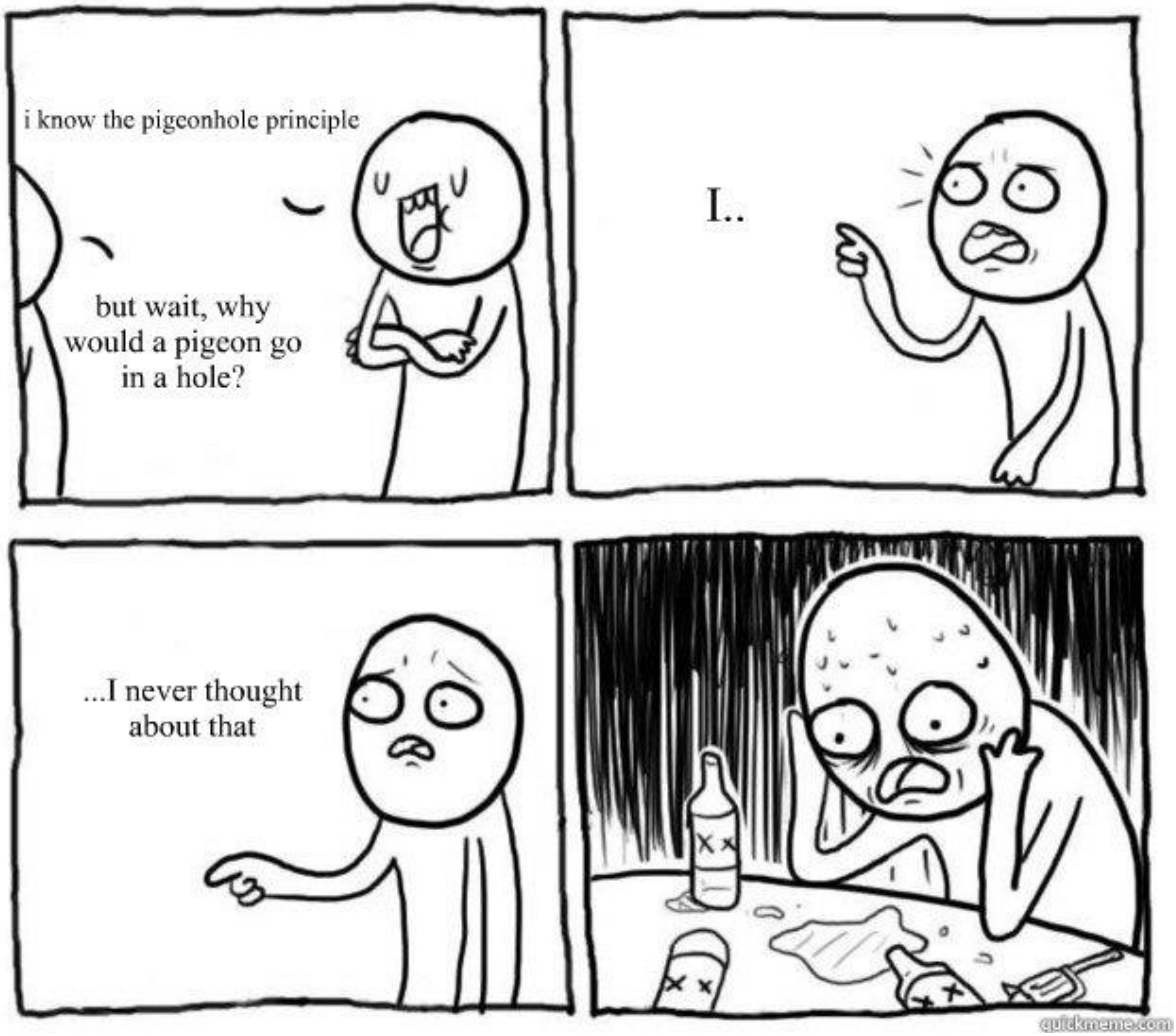
Theorem. Every collection of 6 people includes a club of 3 people, or a group of 3 strangers.

Theorem. For every k, if there are enough people, then there exists either a club of k people, or a group of k strangers.

A large enough structure cannot be totally disorder.

This is a basic result of Ramsey theory.

More Questions about Pigeonhole..



Quick Summary

We prove the binomial theorem and study combinatorial proofs of identities.

We also learn the inclusion-exclusion principle and see some applications.
You should be able to apply the inclusion-exclusion formula to solve some simple problems.

Finally we learn the pigeonhole principle and some of its applications.
As pointed out by Ramsey theory, “a large enough structure cannot be totally disorder.” This idea is very powerful in many complicated problems.