Lecture 17: Linear Transformations

MAT2040 Linear Algebra

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We will now commence the study of transformations like $x \mapsto A\mathbf{x}$. In particular we will generalize the notion of transformations, and think about transformations from *one vector space to another vector space*.

Definition 17.1

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Example: mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$: $\mathbf{x} \mapsto A\mathbf{x}$ (where A is an $m \times n$ matrix).

For the mapping $\mathbf{x} \mapsto A\mathbf{x}$ (from $\mathbb{R}^n$ to $\mathbb{R}^m$) we defined the following concepts:

- $\boxed{}$ set of all possible values of $\mathbb{R}^m$ this mapping can achieve,
- $\boxed{}$ set of all possible value of $\mathbb{R}^n$ that are mapped to $\mathbf{0}$.

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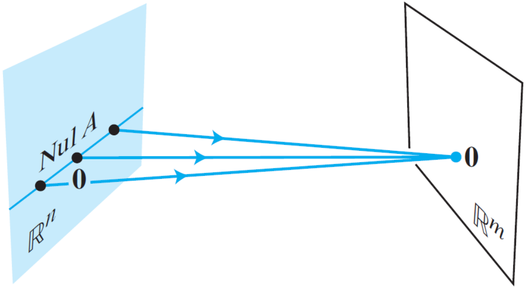

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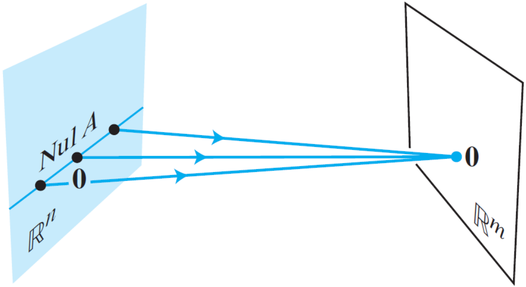

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We want to have words to refer to these concepts for general mappings from V to W as well!

Definition 17.2

For the mapping $\mathbf{v} \mapsto L(v)$ (from V to W) we define:

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- $\boxed{\textbf{the range of } L =}$ set of all possible values of W this mapping can achieve,
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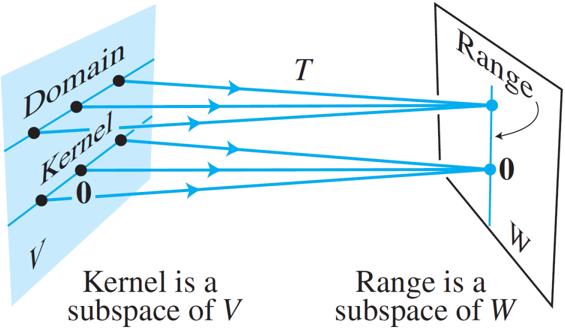

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Example 17.3

Consider the mapping $f(x)$ from $\mathbb{R}$ to $\mathbb{R}$ given by $x \mapsto \max\{x, 0\}$ (the absolute value of x).

What is the range of $f(x)$? What is the kernel of $f(x)$?

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Consider $L(\mathbf{x}) = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} \mathbf{x}$.

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For instance the mapping $x \mapsto x^2$ (from $\mathbb{R}$ to $\mathbb{R}$) cannot be expressed as a matrix multiplication.

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Definition 17.4

A mapping $L(\cdot)$ from $\mathbb{R}^n$ to $\mathbb{R}^m$ is called **a linear transformation**, if

$$L(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha L(\mathbf{x}) + \beta L(\mathbf{y})$$

for every $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$.

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In other words: every linear combination of vectors in $\mathbb{R}^n$ is mapped to the linear combination of the image of these vectors with the same weights.

Lemma 17.5

$L(\mathbf{x})$ from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a linear transformation, if and only if $L(\mathbf{x}) = A\mathbf{x}$ for some $m \times n$ matrix A.

Example 17.6 (Linear Transformation)

$$\mathbf{x} \mapsto A\mathbf{x} \text{ where } A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

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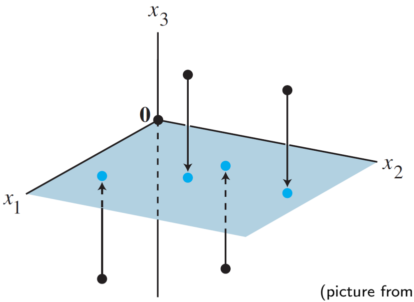

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Projection Transformation

Example 17.7 (Linear Transformation)

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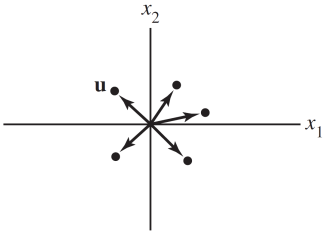

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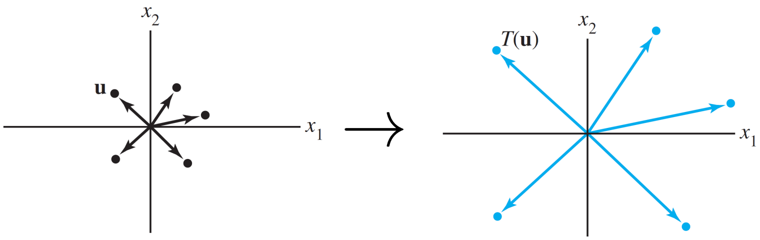

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Dilation Transformation

Example 17.8 (Linear Transformation)

$T(\mathbf{x}) = A\mathbf{x}$ where $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

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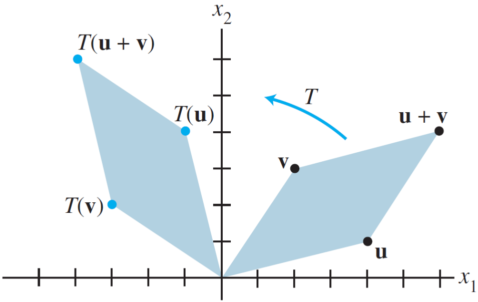

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Rotation Transformation

Example 17.9 (Linear Transformation)

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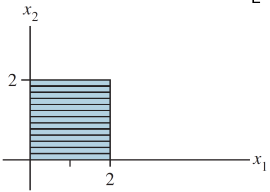

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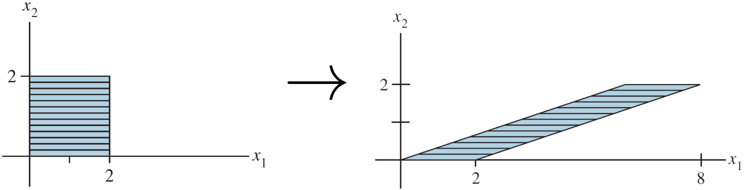

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Shear Transformation

Key takeaway: If we have

- a linear transformation L from $\mathbb{R}^n$ to $\mathbb{R}^m$,
- a basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ for $\mathbb{R}^n$, and
- we know $L(\mathbf{b}_1), L(\mathbf{b}_2), \ldots, L(\mathbf{b}_n)$,

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then we know $L(\mathbf{x})$ for any $\mathbf{x} \in \mathbb{R}^n$: we know that we can write

$$\mathbf{x} = t_1\mathbf{b}_1 + t_2\mathbf{b}_2 + \cdots + t_n\mathbf{b}_n$$

for some $t_1, t_2, \ldots, t_n \in \mathbb{R}$ (because $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n$ is a basis for $\mathbb{R}^n$),

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Note that $[\mathbf{x}]_{\mathcal{B}} = (t_1, t_2, \ldots, t_n)^T$!

Key takeaway: If we have

- a linear transformation L from $\cancel{\mathbb{R}^n}$ a vector space V to $\cancel{\mathbb{R}^m}$ a vector space W,
- a basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ for $\cancel{\mathbb{R}^n}$ V, and
- we know $L(\mathbf{b}_1), L(\mathbf{b}_2), \ldots, L(\mathbf{b}_n)$,

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Note that $[\mathbf{x}]_{\mathcal{B}} = (t_1, t_2, \ldots, t_n)^T$!

Definition 17.8 (for general vector spaces)

A mapping $L(\cdot)$ from V to W is called **a linear transformation**, if

$$L(\alpha \mathbf{x} + \beta \mathbf{y}) = \alpha L(\mathbf{x}) + \beta L(\mathbf{y})$$

for every $\mathbf{x}, \mathbf{y} \in V$ and $\alpha, \beta \in \mathbb{R}$.

Example 17.10

Consider the vector space P_k (polynomials of degree at most k).

Is *differentiation* a linear transformation?

Example 17.11

Let V be a vector space with a basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$.

Is $L(\mathbf{x}) = [\mathbf{x}]_{\mathcal{B}}$ a linear transformation?

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Recall:

Lemma 12.6

Let V be a vector space with a basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$, and let $\mathbf{x}, \mathbf{y} \in V$. For any $\alpha, \beta \in \mathbb{R}$

$$[\alpha\mathbf{x} + \beta\mathbf{y}]_{\mathcal{B}} = \alpha[\mathbf{x}]_{\mathcal{B}} + \beta[\mathbf{y}]_{\mathcal{B}}.$$

Theorem 17.12

Let V be a vector space with basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ and W be a vector space with basis $\mathcal{C} = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m\}$.
If $T(\mathbf{x})$ from V to W is a linear transformation then there exists an $m \times n$ matrix A so that

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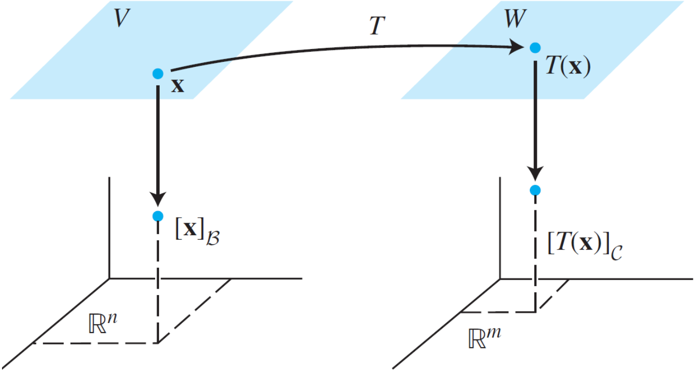
If $T(\mathbf{x})$ from V to W is a linear transformation then there exists an $m \times n$ matrix A so that

$$[T(\mathbf{x})]_{\mathcal{C}} = A[\mathbf{x}]_{\mathcal{B}}.$$

In fact,

$$A = \left[[T(\mathbf{v}_1)]_{\mathcal{C}}, \quad [T(\mathbf{v}_2)]_{\mathcal{C}}, \quad \ldots, [T(\mathbf{v}_n)]_{\mathcal{C}}\right].$$

A is called the **matrix for T relative to the bases $\mathcal{B}$ and $\mathcal{C}$**.



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The last theorem is saying that every linear transformation T from a vector space V to a vector space W can be represented (and thus studied) by a matrix!

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The trick is the following:

- determine a basis $\mathcal{B}$ for V, and $\mathcal{C}$ for W
- for every $\mathbf{v}_i \in \mathcal{B}$, determine $T(\mathbf{v}_i)$
- the **coordinates** of $T(\mathbf{v}_i)$ with respect to basis $\mathcal{C}$ are the i-th column vector of A.

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- for every $\mathbf{v}_i \in \mathcal{B}$, determine $T(\mathbf{v}_i)$
- the **coordinates** of $T(\mathbf{v}_i)$ with respect to basis $\mathcal{C}$ are the i-th column vector of A.

Linearity makes sure (the $\mathcal{B}$-coordinates) of all other vectors in V are mapped to the correct $\mathcal{C}$-coordinates in W.

Example 17.13

Integration is a linear transformation from P_k to P_{k+1}.

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Choose a basis for P_k and P_{k+1}, and find the matrix for integration relative to your chosen bases.