Lecture 14: Sensitivity Analysis

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Announcements

- Homework 4 due today
- Homework 5 due next Wednesday (Oct 31), 12pm (noon)
- No office hour today, office hour on Friday (10/26), 1-3pm

Midterm Exam Information

- Time: Oct 31, Wednesday, 10:30am - 12:00pm.
- Location: Li Wen Building, first floor.
- One (double-sided) piece of note is allowed (on A4 paper), no book/calculator/electronic device or anything else.
- Sample midterm and its solution are posted.
- Review will be given on Friday (10/26).

Recap: Sensitivity Analysis

Let

$$\begin{aligned} V = \min_{\mathbf{x}} \quad & \mathbf{c}^T\mathbf{x} \\ \text{s.t.} \quad & A\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq 0 \end{aligned}$$

Then $\nabla V(\mathbf{b}) = \mathbf{y}^*$ if the dual problem has a unique optimal solution $\mathbf{y}^*$ and $\nabla V(\mathbf{c}) = \mathbf{x}^*$ if the primal problem has a unique optimal solution $\mathbf{x}^*$.

- If we change b_i by a small amount Δb_i, then the change of objective value will be $\Delta b_i y_i^*$
- If we change c_i by a small amount Δc_i, then the change of objective value will be $\Delta c_i x_i^*$
- The results hold for general LPs (not only standard form)

Sensitivity Analysis

The above analysis only holds for small changes

- Small change means that the optimal basis does not change
- If the optimal basis changes, then the change of optimal value and solution may be quite different, usually requires much more efforts

It is useful to know what range of change is a “small change” so that the local sensitivity analysis will work

Global Sensitivity

Now we study what will happen if

1. $\mathbf{b}$ changes to $\mathbf{b} + \Delta\mathbf{b}$
2. $\mathbf{c}$ changes to $\mathbf{c} + \Delta\mathbf{c}$

Recall the simplex tableau:

$\mathbf{c}^T - \mathbf{c}_B^T A_B^{-1} A$	$-\mathbf{c}_B^T {A_B}^{-1}\mathbf{b}$
$A_B^{-1} A$	$A_B^{-1}\mathbf{b}$

At optimal, the reduced costs $\mathbf{c}^T - \mathbf{c}_B^T A_B^{-1} A \geq 0$. And $A_B^{-1}\mathbf{b}$ and $(A_B^{-1})^T \mathbf{c}_B$ are the basic part of the optimal primal solution and the optimal dual solution, respectively.

Change on $\mathbf{b}$

Suppose $\mathbf{b}$ becomes $\tilde{\mathbf{b}} = \mathbf{b} + \Delta\mathbf{b}$. Now the basic solution corresponding to the original optimal basis is

$$\tilde{\mathbf{x}}_B = A_B^{-1}(\mathbf{b} + \Delta\mathbf{b}) = \mathbf{x}^* + A_B^{-1}\Delta\mathbf{b}$$

Note that the reduced cost $\mathbf{c}^T - \mathbf{c}_B^T A_B^{-1} A$ doesn't depend on $\mathbf{b}$.

- If $\tilde{\mathbf{x}}_B \geq 0$, then B is still the optimal basis, and the new optimal solution is $(\tilde{\mathbf{x}}_B, 0)$ with the new optimal value

$$V(\tilde{\mathbf{b}}) = V^* + \mathbf{c}_B^T A_B^{-1}\Delta\mathbf{b} = V^* + (\mathbf{y}^*)^T\Delta\mathbf{b}$$

  where $\mathbf{y}^*$ is the optimal dual solution (this explains the local theorem).
- If the original basis is still optimal, then the local sensitivity analysis holds.

Change on $\mathbf{b}$

Now we study when the change only occurs to one component of $\mathbf{b}$, what ranges of changes qualify for a *small* change (i.e., the local sensitivity analysis holds).

Assume $\Delta\mathbf{b} = \lambda\mathbf{e}_i$ ($\mathbf{e}_i$ is a vector with 1 at position i). Then we need to have

$$\mathbf{x}^* + \lambda A_B^{-1}\mathbf{e}_i \geq 0$$

in order that the optimal basis remains the same. We can then find the range of λ by solving these inequalities.

Example

Consider the production example:

$$
\begin{array}{rrrrl}
\text{maximize} & x_1 & +2x_2 & & \\
\text{subject to} & x_1 & & \leq & 100 \\
 & & 2x_2 & \leq & 200 \\
 & x_1 & +x_2 & \leq & 150 \\
 & x_1, & x_2 & \geq & 0
\end{array}
$$

At optimal, the basis is $\{1, 2, 3\}$, and the optimal solution is $(50, 100, 50, 0, 0)$

- How much we can change the 3rd right hand side coefficient (150) such that the optimal basis remains the same?

Example Continued

The final simplex tableau is

B	0	0	0	1/2	1	250
1	1	0	0	-1/2	1	50
3	0	0	1	1/2	-1	50
2	0	1	0	1/2	0	100

Thus $A_B^{-1} = \begin{pmatrix} 0 & -0.5 & 1 \\ 0 & 0.5 & 0 \\ 1 & 0.5 & -1 \end{pmatrix}$. If $\mathbf{b}$ changes to $\mathbf{b} + \lambda \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, then

$$\tilde{\mathbf{x}}_B = \mathbf{x}_B^* + \lambda A_B^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 50 \\ 100 \\ 50 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

In order for this to be positive, we need $-50 \leq \lambda \leq 50$.

Changes in $\mathbf{c}$

Now suppose $\mathbf{c}$ changes to $\tilde{\mathbf{c}} = \mathbf{c} + \Delta\mathbf{c}$. In order for the basic solution to be still optimal, we need to guarantee that the reduced costs (we only need to consider the non-basic part since the basic part must still be 0):

$$\tilde{\mathbf{c}}_N^T - \tilde{\mathbf{c}}_B^T A_B^{-1} A_N \geq 0$$

Note that this basis still provides a basic feasible solution since the feasibility doesn't depend on $\mathbf{c}$.

Next we assume $\Delta\mathbf{c} = \lambda \mathbf{e}_j$. We discuss two cases: $j \in B$ and $j \in N$. We study how to find the ranges of λ such that the original basis is still optimal (and thus one can apply the local sensitivity analysis)

Case 1: $j \in B$

In this case, the reduced costs are

$$
\begin{aligned}
& \mathbf{c}_N^T - (\mathbf{c}_B^T + \lambda \mathbf{e}_j^T) A_B^{-1} A_N \\
= \; & \mathbf{c}_N^T - \mathbf{c}_B^T A_B^{-1} A_N - \lambda \mathbf{e}_j^T A_B^{-1} A_N
\end{aligned}
$$

Note that $\mathbf{c}_N^T - \mathbf{c}_B^T A_B^{-1} A_N$ is the reduced costs for the original problem. We denote it by $\mathbf{r}_N^T$. Therefore, in order to maintain the optimality of the current basis, we need to have

$$
\mathbf{r}_N^T - \lambda \mathbf{e}_j^T A_B^{-1} A_N \geq 0 \tag{1}
$$

- We can solve the range of λ from (1).
- This is a set of inequalities

Case 2: $j \in N$

In this case, the reduced costs are:

$$\mathbf{c}_N^T + \lambda \mathbf{e}_j^T - \mathbf{c}_B^T A_B^{-1} A_N = \mathbf{r}_N^T + \lambda \mathbf{e}_j^T$$

Therefore, in order to maintain the optimality of the current basis, we need to have

$$\mathbf{r}_N + \lambda \mathbf{e}_j \geq 0 \tag{2}$$

- We can solve the range of λ from (2).

Example

Consider the same production example:

$$\begin{array}{lrrl} \text{maximize} & x_1 & +2x_2 & \\ \text{subject to} & x_1 & & \leq 100 \\ & & 2x_2 & \leq 200 \\ & x_1 & +x_2 & \leq 150 \\ & x_1, & x_2 & \geq 0 \end{array}$$

The final simplex tableau is

B	0	0	0	1/2	1	250
1	1	0	0	-1/2	1	50
3	0	0	1	1/2	-1	50
2	0	1	0	1/2	0	100

How much we can change the first objective coefficient so that we can use the local sensitivity analysis?

Example Continued

We have

$$A_B^{-1} = \begin{pmatrix} 0 & -0.5 & 1 \\ 1 & 0.5 & -1 \\ 0 & 0.5 & 0 \end{pmatrix}; \quad A_N = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}; \quad \mathbf{r}_N = \begin{pmatrix} 0.5 \\ 1 \end{pmatrix}$$

Assume we change the profit 1 from 1 to $1+\lambda$ (i.e., $-1-\lambda$ in the standard form). Then we need

$$\mathbf{r}_N - \lambda A_N^T (A_B^{-1})^T \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.5 \\ 1 \end{pmatrix} - \lambda \begin{pmatrix} 0.5 \\ -1 \end{pmatrix} \geq 0$$

Therefore, $-1 \leq \lambda \leq 1$

- It means that when the profit of the first product is between 0 and 2, we can use the local sensitivity theorem to compute the optimal value

What if the Change is Outside the Range

If the change of $\mathbf{c}$ is so much that the reduced cost of the current solution contains negative number, then

- We can continue with the simplex tableau until it reaches optimal solution.

If the change of $\mathbf{b}$ is so much that the solution corresponding to the original optimal basis B is no longer feasible

- Then we may need to solve the problem from the start
- However, it can be viewed as that the objective coefficients of the dual problem changed. Then one can use the method that deals with changes in objective coefficients

Changes to A

If the change is for a number in a non-basic column, say A_j, then the original optimal solution is still feasible, the only change is to the reduced cost of jth variable.

- Recompute $\bar{c}_j$. If it is still nonnegative, then the original optimal solution stays optimal. Otherwise, update the tableau for the jth column as well as the reduced cost and continue from there.

If the change is for a number in a basic column, then nearly all the numbers in the tableau will change. In general, there is not a simple way to deal with it.

Other Changes

Adding a variable (the rest are kept the same):

- The original BFS is still a BFS, the reduced cost is unchanged
- Only need to check the reduced cost corresponding to the new variable. If it is non-negative, then the original optimal solution is still optimal; otherwise continue the simplex method from there

Adding a constraint:

- If the original optimal solution satisfies the constraint, then it is still optimal
- If not, then the best way to deal with it is to think it as adding a dual variable, then use the simplex tableau for the dual problem to continue calculations