MAT2040: Linear Algebra

Final Exam (2017-18, Term2)

Instructions:

1. This exam consists of 9 questions (3 pages). This exam is 2.5 hour long, and worth 100 points.
2. This exam is in closed book format. No books, calculators, dictionaries or blank papers to be brought except one page of A4 size paper which you can write anything on both sides. Any cheating will be given **ZERO** mark. **Please show your steps.**

Student Number: ____________________ **Name:** _______________

1. (*10 points*) Consider the linear system:

$$\begin{aligned} x_1 - x_2 &= a_1 \\ x_2 - x_3 &= a_2 \\ x_3 - x_4 &= a_3 \\ x_4 - x_5 &= a_4 \\ x_5 - x_1 &= a_5 \end{aligned}$$

where a_1, a_2, a_3, a_4, a_5 are given real numbers.

(i) Show that the above linear system is consistent if and only if $a_1+a_2+a_3+a_4+a_5 = 0$. **[6 marks]**

(ii) Find the solution in vector form for the above linear system when $a_1 + a_2 + a_3 + a_4 + a_5 = 0$. **[4 marks]**

2. (*8 points*) Find the matrix B such that the following equation is valid,

$$AB = 2A - B$$

where A, B are 3×3 real matrices and $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 3 \\ 3 & 4 & 5 \end{bmatrix}$.

3. (*8 points*) Given the matrix

$$A = \begin{bmatrix} a & 2 & 2 & 2 \\ 2 & a & 2 & 2 \\ 2 & 2 & a & 2 \\ 2 & 2 & 2 & a \end{bmatrix}$$

where a is a real number.

(i) Compute the determinant for the above matrix A. **[6 marks]**

(ii) Find the condition that the matrix A is singular. **[2 marks]**

4. (*12 points*) Let $M_2(\mathcal{R})$ be the vector space of 2×2 real matrices, the mapping L from $M_2(\mathcal{R})$ to $M_2(\mathcal{R})$ is defined as follows:

$$L(A) = A^T + 2A, \quad \text{for} \quad A \in M_2(\mathcal{R})$$

where A^T is the transpose of A.

(i) Show that the mapping L is a linear transformation. **[4 marks]**

(ii) Find the kernel of L and the range of L. **[6 marks]**

(iii) Determine whether L is injective and surjective. **[2 marks]**

5. (*16 points*) Given the linear system $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

(i) Show that the linear system is inconsistent. **[2 marks]**

(ii) Find the least square solution of the linear system. **[6 marks]**

(iii) Find the projection matrix and projection vector corresponding to the least square solution in (ii). **[6 marks]**

(ii) Find the distance between $\mathbf{b}$ and column space $C(A)$. **[2 marks]**

6. (*14 points*) Let

$$A = \begin{bmatrix} 1 & -1 & 4 \\ 1 & 2 & -2 \\ 1 & 2 & 2 \\ 1 & -1 & 0 \end{bmatrix} = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3]$$

where $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ are three column vectors of A.

(i) Show that $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ are linearly independent. **[2 marks]**

(ii) Using Gram-Schmidt process for $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ to obtain three orthonormal vectors $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$. **[8 marks]**

(ii) Let $A = QR$ be the QR factorization of A, find Q and R. **[4 marks]**

7. (*12 points*) Let

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

(i) Find all the eigenvalues of A and show that A is positive definite. **[6 marks]**

(ii) Find the diagonal matrix Λ and an orthogonal matrix U, such that

$$U^T A U = \Lambda$$

[6 marks]

8. (*10 points*) Let $A \in M_n(\mathcal{R})$, and $A^3 + 2A^2 = 3A$.

(i) Find all possible eigenvalues of A. **[6 marks]**

(ii) If $A^2 + 2A \neq 3I_n$, prove that $r(A) < n$, where $r(A)$ is the rank of A. **[4 marks]**

9. (*10 points*)

(i) Let $A \in M_n(\mathcal{R})$ be a real symmetric positive definite matrix, prove that there is another real symmetric positive definite matrix $B \in M_n(\mathcal{R})$ such that $A = B^2$. **[6 marks]**

(ii) Let $A, B \in M_n(\mathcal{R})$, prove that $\det(I_n - AB) = \det(I_n - BA)$. **[4 marks]**