CIE6020/MAT3350 HW Assignment 2

Due: 23:55, 18 Feb 2019

1. Let X, Y, Z be three random variables with a joint probability mass function $p(x, y, z)$. The relative entropy between the joint distribution and the product of the marginals is

$$D(p(x, y, z)||p(x)p(y)p(z)) = \mathbb{E}\left[\log \frac{p(x, y, z)}{p(x)p(y)p(z)}\right].$$

   Expand this in terms of entropies. When is this quantity zero?

2. Let the random variable X have three possible outcomes $\{a, b, c\}$. Consider two distributions on this random variable:

symbol	$p(x)$	$q(x)$
a	$\frac{1}{2}$	$\frac{1}{3}$
b	$\frac{1}{4}$	$\frac{1}{3}$
c	$\frac{1}{4}$	$\frac{1}{3}$

   Calculate $H(p)$, $H(q)$, $D(p||q)$ and $D(q||p)$. Verify that in this case, $D(p||q) \neq D(q||p)$.

3. Show that $\ln x \geq 1 - \frac{1}{x}$ for $x > 0$, where the equality holds when $x = 1$.

4. *Conditioning reduces entropy.* Show that $H(Y|X) \leq H(Y)$ with equality iff X and Y are independent.

5. Show that $I(X; Y|Z) \geq 0$ with equality iff $X \to Z \to Y$.

6. *Data processing.* Let $X_1 \to X_2 \to X_3 \to \cdots \to X_n$ form a Markov chain, i.e.,

$$p(x_1, x_2, ..., x_n) = p(x_1)p(x_2|x_1) \cdots p(x_n|x_{n-1}).$$

   Reduce $I(X_1; X_2, ..., X_n)$ to its simplest form.

7. Let X and Y be two random variables and let Z be independent of (X, Y). Show that $I(X; Y) \geq I(X; g(Y, Z))$ for any function g.

8. *Bottleneck.* Suppose that a (nonstationary) Markov chain starts in one of n states, necks down to $k < n$ states, and then fans back to $m > k$ states. Thus, $X_1 \to X_2 \to X_3$, that is, $p(x_1, x_2, x_3) = p(x_1)p(x_2|x_1)p(x_3|x_2)$, for all $x_1 \in \{1, 2, ..., n\}, x_2 \in \{1, 2, ..., k\}, x_3 \in \{1, 2, ..., m\}$.

(a) Show that the dependence of X_1 and X_3 is limited by the bottleneck by proving that $I(X_1; X_3) \leq \log k$.

(b) Evaluate $I(X_1; X_3)$ for $k = 1$, and conclude that no dependence can survive such a bottleneck.