Lecture 15: Determinants

MAT2040 Linear Algebra

Before studying $A\mathbf{x}$ through the lens of mappings (transformations) — in other words studying $\mathbf{x} \mapsto A\mathbf{x}$ — we will first study a quantity that is known as *the determinant* of A, denoted by $\det A$ or

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}.$$

The determinant is only defined for square matrices! For the next two or three lectures all the matrices will be square $n \times n$ matrices.

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Motivation is the volume of the geometric object the edges of which come from the rows of A (parallelepiped).

(Note that "volume" is length in 1 dimension, "area" in 2 dimensions, and some people prefer "hypervolume" for dimensions 4 and larger.)

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Note that the volume of a geometric object is zero if the object is "flat" in the space it lives in.

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For a square matrix this is equivalent to saying that A is not invertible!

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Our starting point for defining the determinant will be the *geometric properties* of the determinant.

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   (b) by changing row $\vec{\mathbf{a}}_i$ to $\vec{\mathbf{b}}_i$ — call the new matrix B, we have
   $$\det B = \det A + \det A'$$
   where A' is the matrix A where row i is replaced by $\vec{\mathbf{b}}_i - \vec{\mathbf{a}}_i$.

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Let's see what other properties we can find so that we can find a formula or way of computing the determinant of a matrix.

Property 4

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Property 7

If A is triangular, then $\det A =$ *product of diagonal entries.*

So we almost have a method to calculate $\det A$!

If we can transform A into triangular form using only the elementary row operation that subtracts multiples of one row from another, the determinant does not change (property 6).

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Let's do an example before we try to find out what happens to the determinant when rows are exchanged.

Example 15.2

Find $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ where $a \neq 0$, and check geometrically that this equals the area of the parallelogram with sides $(0,0) \to (a,b)$ and $(0,0) \to (c,d)$.

Property 8

If B is the matrix A with rows i and j exchanged then

$$\det B =$$

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$$\det B = -\det A.$$

We can conclude that the following is an algorithm to compute $\det A$:

1. Use elementary row operations except the scaling operation, to transform A into row echelon form. Keep track of the number of row interchanges, say r.
2. $\det A = (-1)^r$ product of the diagonal entries of row echelon form.

Example 15.3

Find $\begin{vmatrix} 5 & 9 & 17 \\ 1 & 2 & 0 \\ -5 & -11 & 3 \end{vmatrix}$.

Note that our intuition of getting a volume was not quite correct: we got a volume that could also be negative.

The absolute value of the determinant of A, however, exactly equals the volume of the parallelepiped defined by the rows of A.

You may be (should be) wondering whether the determinant is well defined: is the number of row interchanges to get the rows in a certain order always odd or always even (no matter which ones and which order you choose)?

Suppose you have a matrix, and you want to move row 1 to row $\pi(1)$, row 2 to row $\pi(2)$, etc., row n to row $\pi(n)$, using row interchanges, where π is a *permutation*.

Fact 15.4

The number of row interchanges you have to do has the same parity as the number of pairs that are out of order in π.

Corollary 15.5

Let P be a permutation matrix.

$$\det P = (-1)^{\#\ \textit{of pairs of rows that are out of order compared to } I}.$$

Example 15.6

What is $\begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$?

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Definition 15.7

The **sign of a permutation** π of size n is the determinant of the $n \times n$ permutation matrix that corresponds to π:

$$\mathrm{sign}(\pi) = \begin{vmatrix} e_{\pi(1)}^T \\ e_{\pi(2)}^T \\ e_{\pi(3)}^T \\ \vdots \\ e_{\pi(n)}^T \end{vmatrix}.$$

Property 9

If A has a column of all zeros then

$$\det A = 0.$$

We will now focus on getting a formula for $\det A$ (as well as derive some more properties).

Let's split up every row into rows that just contain one nonzero entry.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & d \end{vmatrix}$$ by property 3(a)

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$$= 0 + ad + (-1)\begin{vmatrix} c & 0 \\ 0 & b \end{vmatrix} + 0 \qquad \text{by properties 9, 7, 8 and 9, respectively}$$

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&= 0 + ad + (-1)\begin{vmatrix} c & 0 \\ 0 & b \end{vmatrix} + 0 && \begin{array}{l}\text{by properties 9, 7,} \\ \text{8 and 9, respectively}\end{array} \\
&= ad - bc && \text{by property 7.}
\end{aligned}
$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

We can do the same for $n \times n$ matrices — we get $\boxed{}$ matrices, but most of the matrices we get have determinant 0 (because they have an all zero column).

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How many matrices will there be that do not have an all zero column?

Formula (Leibniz formula)

$$\det A = \sum_{\pi \text{ is a permutation of } 1,2,\ldots,n} sign(\pi) a_{1\pi(1)} a_{2\pi(2)} \cdots a_{n\pi(n)}.$$

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What if we group terms — for instance starting with all terms involving a_{11}?

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$$= a_{11} \det M_{11}$$

where M_{11} is the matrix A with row 1 and column 1 deleted.