Lecture 1: Introduction to Optimization

Zizhuo Wang

Institute of Data and Decision Analytics (iDDA)
Chinese University of Hong Kong, Shenzhen

Sep 5, 2018

Course Logistics

Meeting times:

- Wed/Fri 10:30am-11:50am
- Cheng Dao Building 103

My office hours (Daoyuan Building 501):

- Wednesday 2:00pm-4:00pm
- or by appointments

Teaching assistants:

- Tong Li: `216019016@link.cuhk.edu.cn` - first half
- Hao Liang: `217019008@link.cuhk.edu.cn` - second half
- Office hours: Tuesday 3:00pm-5:00pm
- Location: Cheng Dao Building 319 (Tong Li) and Dao Yuan Building 225 (Hao Liang)

Logistics

Reference books

- Mainly follow the lecture notes
- Introduction to Linear Optimization, by D. Bertsimas and J. Tsitsiklis
- Convex Optimization, by S. Boyd and L. Vandenberghe (Free access online)

Homework

- There will be 10 assignments
- The first one will be due on Wednesday, Sep 19th (by 12pm)
- Submit electronically

Exams

- Midterm: Oct 26th (Friday), in class; Final: TBD

Logistics

Grades:

- Midterm: 40%
- Final: 40%
- Homework: 20%

Student conduct codes apply to all homework and exams

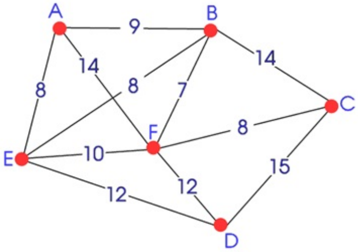
Course website

- Blackboard: https://bb.cuhk.edu.cn
- All course materials will be posted on Blackboard

What is Optimization?

Example 1. (Maximum Area Problem) You have 80 meters of fencing and want to enclose a rectangle yard as large as possible (in area). How should you do it?

Example 2. (Traveling Salesman Problem) A salesman needs to visit a number of places in a day. How should he schedule his trip so that the total distance is shortest (or the total cost is smallest)?



Example 3. (Production Problem) A firm produces two types of alloy products. In order to produce them, they need to use steel, iron and copper as the resource. The production requirement and profit for each type of alloy are given as follows:

	Steel	Iron	Copper	Profit
Alloy 1	1	0	1	\$1
Alloy 2	0	2	1	\$2
Resources	100	200	150	

- How much of each type of products should the firm produce in order to maximize its profit?

Summary

What are the common components of optimization problems?

- Decision
- Objective
- Constraints

Optimization concerns choosing a *decision* (or decisions) to *optimize* certain *objectives* while subject to certain *constraints*

- Optimize could mean *maximize* or *minimize* depending on the problem context.

Back to Our Examples

Maximum Area Problem

- Decision: length and width of the yard; Objective: maximize the area; Constraints: the total length of the fence.

Traveling Salesman Problem

- Decision: the order of visits; Objective: minimize the total distance; Constraints: has to visit each place once.

Production Problem:

- Decision: production plan; Objective: maximize profit; Constraints: resource.

Optimization Problems

Mathematically, an optimization problem is usually represented as:

$$\begin{aligned} \text{maximize/minimize}_x \quad & f(x) \\ \text{subject to} \quad & x \in \Omega \end{aligned}$$

- x: Decision
- $f(\cdot)$: Objective
- Ω: Constraints

Optimization

It is ubiquitous in our life

- Economics, finance, science, engineering, business, government, sports, marketing

Nothing at all takes place in the Universe in which some rule of maximum or minimum does not appear. – L. Euler, 1707-1783

- The earliest study was in 300-100BC when Euclid studied geometry / Heron studied the reflection principle of light
- It has mainly been part of mathematics (main contributors: Newton, Langrange, Euler, etc.)
- Modern optimization starts in 20th century, getting much attention during the World War II.
- Today the use of computer has enabled new methods and perspectives for optimization

Content of the Course

Modeling techniques: How to translate a practical problem into an optimization problem, especially a *good* optimization problem?

- Examples will be shown today and throughout the semester
- Systematic methods will be taught

Optimization algorithms: How to solve optimization problems efficiently?

- Several fundamental algorithms (simplex, gradient, Newton, interior point methods) will be discussed

Implementation: How to solve optimization problems using available tools?

- Computer software will be used: MATLAB

My Expectation for You

- Being active: both in class and off class
- Some basic background about calculus and linear algebra
- A little computer skill

Today's Agenda

- Introduce basic terminologies and classifications of optimization problems

Terminologies

Recall an optimization problem can be expressed as:

$$\begin{array}{ll} \text{minimize}_x & f(x) \\ \text{subject to} & x \in \Omega \end{array}$$

We consider a slightly more restrictive form:

$$\begin{array}{lll} \text{minimize}_x & f(x) & \\ \text{subject to} & g_i(x) \leq 0, & \forall i = 1, ..., s \\ & h_j(x) = 0, & \forall j = 1, ..., t \end{array}$$

Terminologies

$$
\begin{aligned}
\text{minimize}_x \quad & f(x) \\
\text{subject to} \quad & g_i(x) \leq 0, \quad \forall i = 1, ..., s \\
& h_j(x) = 0, \quad \forall j = 1, ..., t
\end{aligned}
$$

- x: Decision variables
- $f(x)$: Objective function
- $g(x), h(x)$: Inequality/equality constraints
- Feasible solution: A decision that satisfies all constraints
- Feasible set (region): The set of feasible solutions
- Optimal solution: The (feasible) decision variable that attains at least as good objective value as any other feasible solution
- Optimal value: The objective value of any optimal solution

Properties of Optimization Problems

In optimization world, we usually don't allow strict inequality constraints.

- Easy to analyze and without loss of generality

Facts about optimization problems:

- An optimization problem may not always have a feasible solution.
- Even if it is feasible, it may not have an optimal solution.
- The optimal value could be unbounded.
- Even if an optimization problem is feasible, finite and can attain its optimal value, the optimal solution may not be unique (however, the optimal value must be unique).

Possible States for Optimization Problems

By classifying the outcomes, an optimization problem may have the following states:

1. Infeasible
2. Feasible, optimal value finite but not attainable
3. Feasible, optimal value finite and attainable
4. Feasible, but optimal value is unbounded

Classifications

$$\begin{aligned} \text{minimize}_x \quad & f(x) \\ \text{subject to} \quad & g_i(x) \le 0, \quad \forall i = 1, ..., s \\ & h_j(x) = 0, \quad \forall j = 1, ..., t \end{aligned}$$

- Unconstrained optimization: If $s = t = 0$. Otherwise constrained optimization.
- Linear optimization (LP): Constraints and objective are linear
- Nonlinear optimization (NLP): Either some of the constraints or the objective function is nonlinear
- Integer/Discrete optimization (IP): If some of the decision variables have to be integers
- Continuous optimization: All the decision variables can take continuous values

Classifications

- Constrained vs Unconstrained
- Linear vs Nonlinear
- Continuous vs Discrete

By default, when we talk about an optimization problem, we assume it is continuous, unless we explicitly say that it is *discrete*

Remarks

The above classifications are based on what an optimization problem appears.

- Sometimes, an NLP can be equivalently transformed to an LP
- Sometimes, an IP can be equivalently transformed to a continuous optimization problem

Linear optimization is the most well-studied and the easiest optimization problem.

- Nonlinear optimization and integer optimization could be significantly harder than LP.
- Therefore, in many cases, people strive to find LP formulations for problems