Assignment 2 MAT2040 WU, Chenhao 117010285

1. $AB = \begin{bmatrix}1&0&a&b\\0&1&c&d\end{bmatrix}\begin{bmatrix}-a&-b\\-c&-d\\1&0\\0&1\end{bmatrix} = \begin{bmatrix}0&0\\0&0\end{bmatrix}$

$BA = \begin{bmatrix}-a&-b\\-c&-d\\1&0\\0&1\end{bmatrix}\begin{bmatrix}1&0&a&b\\0&1&c&d\end{bmatrix} = \begin{bmatrix}-a&-b&-a^2-bc&-ab-bd\\-c&-d&-ca-dc&-bc-d^2\\1&0&a&b\\0&1&c&d\end{bmatrix}$

2.(a). Matrix is Invertible $\Leftrightarrow \det\left(\begin{bmatrix}1&\lambda&0\\1&1&1\\0&0&1\end{bmatrix}\right) \neq 0. \Rightarrow 1-\lambda \neq 0 \Rightarrow \lambda \neq 1$

(b) $\left[\begin{array}{ccc|ccc}1&\lambda&0&1&0&0\\1&1&1&0&1&0\\0&0&1&0&0&1\end{array}\right] \xrightarrow{R_2\to R_2-R_1} \left[\begin{array}{ccc|ccc}1&\lambda&0&1&0&0\\0&1-\lambda&1&-1&1&0\\0&0&1&0&0&1\end{array}\right] \xrightarrow{R_2\to R_2-R_3} \left[\begin{array}{ccc|ccc}1&\lambda&0&1&0&0\\0&1-\lambda&0&-1&1&-1\\0&0&1&0&0&1\end{array}\right]$

$\xrightarrow{R_2\to \frac{1}{1-\lambda}R_2} \left[\begin{array}{ccc|ccc}1&\lambda&0&1&0&0\\0&1&0&\frac{1}{\lambda-1}&\frac{1}{1-\lambda}&\frac{1}{1-\lambda}\\0&0&1&0&0&1\end{array}\right] \xrightarrow{R_1\to R_1-\lambda R_2} \left[\begin{array}{ccc|ccc}1&0&0&\frac{1}{1-\lambda}&\frac{\lambda}{\lambda-1}&\frac{\lambda}{\lambda-1}\\0&1&0&\frac{1}{\lambda-1}&\frac{1}{1-\lambda}&\frac{1}{1-\lambda}\\0&0&1&0&0&1\end{array}\right]$

Hence, $M^{-1} = \begin{bmatrix}\frac{1}{1-\lambda}&\frac{\lambda}{\lambda-1}&\frac{\lambda}{\lambda-1}\\\frac{1}{\lambda-1}&\frac{1}{1-\lambda}&\frac{1}{1-\lambda}\\0&0&1\end{bmatrix}$

3.(a). $\left[\begin{array}{ccc|ccc}1&2&2&1&0&0\\3&7&9&0&1&0\\-1&-4&-7&0&0&1\end{array}\right] \xrightarrow[R_3\to R_3+R_1]{R_2\to R_2-3R_1} \left[\begin{array}{ccc|ccc}1&2&2&1&0&0\\0&1&3&-3&1&0\\0&-2&-5&1&0&1\end{array}\right] \xrightarrow{R_3\to R_3+2R_2} \left[\begin{array}{ccc|ccc}1&2&2&1&0&0\\0&1&3&-3&1&0\\0&0&1&-5&2&1\end{array}\right]$

$\xrightarrow{R_2\to R_2-3R_3} \left[\begin{array}{ccc|ccc}1&2&2&1&0&0\\0&1&0&12&-5&-3\\0&0&1&-5&2&1\end{array}\right] \xrightarrow{R_1\to R_1-2R_3} \left[\begin{array}{ccc|ccc}1&2&0&11&-4&-2\\0&1&0&12&-5&-3\\0&0&1&-5&2&1\end{array}\right] \xrightarrow{R_1\to R_1-2R_2}$

$\left[\begin{array}{ccc|ccc}1&0&0&-13&6&4\\0&1&0&12&-5&-3\\0&0&1&-5&2&1\end{array}\right] \Rightarrow A^{-1} = \begin{bmatrix}-13&6&4\\12&-5&-3\\-5&2&1\end{bmatrix}$

(b). $\left[\begin{array}{ccc|ccc}1&1&1&1&0&0\\3&5&4&0&1&0\\3&6&5&0&0&1\end{array}\right] \xrightarrow[R_3\to R_3-3R_1]{R_2\to R_2-3R_1} \left[\begin{array}{ccc|ccc}1&1&1&1&0&0\\0&2&1&-3&1&0\\0&3&2&-3&0&1\end{array}\right] \xrightarrow{R_2\to \frac{1}{2}R_2} \left[\begin{array}{ccc|ccc}1&1&1&1&0&0\\0&1&\frac{1}{2}&-\frac{3}{2}&\frac{1}{2}&0\\0&3&2&-3&0&1\end{array}\right] \xrightarrow{R_3\to R_3-3R_2}$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} & -\frac{3}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{3}{2} & -\frac{3}{2} & 1 \end{array}\right] \xrightarrow{R_2 \to R_2 - R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & 2 & -1 \\ 0 & 0 & \frac{1}{2} & \frac{3}{2} & -\frac{3}{2} & 1 \end{array}\right] \xrightarrow{R_3 \to 2R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & 2 & -1 \\ 0 & 0 & 1 & 3 & -3 & 2 \end{array}\right] \xrightarrow{R_1 \to R_1 - R_3}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & -2 & 3 & -2 \\ 0 & 1 & 0 & -3 & 2 & -1 \\ 0 & 0 & 1 & 3 & -3 & 2 \end{array}\right] \xrightarrow{R_1 \to R_1 - R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & -3 & 2 & -1 \\ 0 & 0 & 1 & 3 & -3 & 2 \end{array}\right]$$

4. (a). $A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$A^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

$A^4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(b). ~~From~~ For $k=2n$, $n \in \mathbb{Z}^+$, $A^k = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

For $k=2n+1$, $n \in \mathbb{Z}^+$, $A^k = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

5. (a) $(A^2-B^2)^T = (A^2)^T - (B^2)^T = A^TA^T - B^TB^T$.

As given pre-requirements, $A^T = A$., Hence $\Rightarrow (A^2-B^2)^T = A^TA^T - B^TB^T$

$= AA - BB$

$= A^2 - B^2$

$\Rightarrow A^2 - B^2$ is symmetric

(b) $[(A+B)(A-B)]^T = (A-B)^T(A+B)^T = (A^T-B^T)(A^T+B^T) = (A-B)(A+B)$

$(A-B)(A+B) \neq (A+B)(A-B) \Rightarrow (A+B)(A-B)$ is not symmetric

(c) $(ABA)^T = A^T(AB)^T = A^TB^TA^T = ABA \Rightarrow ABA$ is symmetric

(d) $(ABAB)^T = B^T(ABA)^T = B^T(BA)^TA^T = B^TA^TB^TA^T = BABA$

$BABA \neq ABAB \Rightarrow ABAB$ is not symmetric

6. $C^3 = (B^{-1}AB)(B^{-1}AB)(B^{-1}AB) = B^{-1}A(BB^{-1})A(BB^{-1})AB$

$= B^{-1}AIAIAB = B^{-1}(AI)(AI)AB$

$= B^{-1}AAAB = B^{-1}A^3B$. QED.

7. (a). True. Reason: a matrix with all entries are zero, can never obtain a trival solution from $A\vec{x} = \vec{0}$

(b). False ~~True~~. Reason: $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ is not invertible

(c). True. Reason, Let suppose $A(E_1E_2\cdots E_k) = I$, & $A^{-1} = E_1E_2\cdots E_k$.
We can always find a inverse operation for E_i, and therefore.
$A^{-1}(A^{-1})^{-1} = E_1E_2\cdots E_kE_k^{-1}\cdots E_2^{-1}E_1^{-1} = I$, $(A^{-1})^{-1} = E_k^{-1}\cdots E_2^{-1}E_1^{-1}$

(d) True. Reason,. Assume that the inverse of A^T is $(A^T)^{-1}$, and we can start with $A^T(A^T)^{-1} = I \Rightarrow [A^T(A^T)^{-1}]^T = I^T \Rightarrow [(A^T)^{-1}]^T(A^T)^T = I \Rightarrow [(A^T)^{-1}]^T A = I$
$\Rightarrow [(A^{-1})^T]^T A = I \Rightarrow A^{-1}A = I$. $\Rightarrow$ ~~$A^{-1}AA^{-1} = A$~~ we've found an inverse of A

(e). True. Reason: A is invertible $\Leftrightarrow$ A is row equivalent to I
B is row-eq. to A $\Leftrightarrow$ B is row-eq. to I $\Leftrightarrow$ B is invertible
Similarly, A is not invertible $\Leftrightarrow$ A is not row-eq. to I
B is row-eq. to A $\Leftrightarrow$ B is not row-eq. to I $\Leftrightarrow$ B is not invertible

8. Suppose that $BE_1E_2\cdots E_k = I \Rightarrow E_1E_2\cdots E_kB = I$

$\Rightarrow E_1E_2\cdots E_kA = E_1E_2\cdots E_kBC \Rightarrow E_1\cdots E_kA = C$

9. (a): Interpret the elementary matrixes to row operations as following:

$$E_1\cdots E_8 I = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&-2&1\end{bmatrix} \xrightarrow{E_7} \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&-2&1\\0&0&1&0\end{bmatrix} \xrightarrow{E_6} \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&-2&1\\0&5&1&0\end{bmatrix} \xrightarrow{E_5} \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&-1&-2&1\\0&5&1&0\end{bmatrix} \xrightarrow{E_4}$$

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&-1&-2&1\\2&5&1&0\end{bmatrix} \xrightarrow{E_3} \begin{bmatrix}1&0&0&0\\0&1&0&0\\-3&-1&-2&1\\2&5&1&0\end{bmatrix} \xrightarrow{E_2} \begin{bmatrix}1&0&0&0\\1&1&0&0\\-3&-1&-2&1\\2&5&1&0\end{bmatrix} \xrightarrow{E_1} \begin{bmatrix}2&0&0&0\\1&1&0&0\\-3&-1&-2&1\\2&5&1&0\end{bmatrix}$$

(b). The inverse of $E_1\cdots E_8$ is $E_8^{-1}E_7^{-1}\cdots E_1^{-1}$

$$\Rightarrow E_8^{-1}E_7^{-1}\cdots E_1^{-1} I = \begin{bmatrix}\frac{1}{2}&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix} \xrightarrow{E_2^{-1}} \begin{bmatrix}\frac{1}{2}&0&0&0\\-\frac{1}{2}&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix} \xrightarrow{E_3^{-1}} \begin{bmatrix}\frac{1}{2}&0&0&0\\-\frac{1}{2}&1&0&0\\\frac{3}{2}&0&1&0\\0&0&0&1\end{bmatrix} \xrightarrow{\cdots} \begin{bmatrix}\frac{1}{2}&0&0&0\\-\frac{1}{2}&1&0&0\\\frac{3}{2}&-5&0&1\\4&-9&1&2\end{bmatrix}$$

10. $$A=\begin{bmatrix}a&a&a&a\\a&b&b&b\\a&b&c&c\\a&b&c&d\end{bmatrix} \xrightarrow[R_4\to R_4-R_1]{R_2\to R_2-R_1,\ R_3\to R_3-R_1} \begin{bmatrix}a&a&a&a\\0&b-a&b-a&b-a\\0&b-a&c-a&c-a\\0&b-a&c-a&d-a\end{bmatrix} \xrightarrow[R_4\to R_4-R_2]{R_3\to R_3-R_2} \begin{bmatrix}a&a&a&a\\0&b-a&b-a&b-a\\0&0&c-b&c-b\\0&0&c-b&d-b\end{bmatrix} \xrightarrow{R_4\to R_4-R_3} \begin{bmatrix}a&a&a&a\\0&b-a&b-a&b-a\\0&0&c-b&c-b\\0&0&0&d-c\end{bmatrix}$$

A is non-singular $\Leftrightarrow a\neq 0,\ a\neq b,\ b\neq c,\ c\neq d.$

$$L=\begin{bmatrix}1&0&0&0\\1&1&0&0\\1&1&1&0\\1&1&1&1\end{bmatrix},\quad U=\begin{bmatrix}a&a&a&a\\0&b-a&b-a&b-a\\0&0&c-b&c-b\\0&0&0&d-c\end{bmatrix}$$

11. $$\left[\begin{array}{cccc|c}3&3&1&-4&5\\3&5&-1&-3&5\\-9&-3&-4&16&-5\\15&13&-8&-21&-5\end{array}\right] = \begin{bmatrix}1&0&0&0\\1&1&0&0\\-3&-3&1&0\\5&-1&-3&1\end{bmatrix}\begin{bmatrix}3&3&1&-4\\0&2&-2&1\\0&0&5&1\\0&0&0&-1\end{bmatrix}$$

$$\begin{bmatrix}3&3&1&-4\\0&2&-2&1\\0&0&5&1\\0&0&0&-1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}1&0&0&0\\-1&1&0&0\\6&-3&1&0\\12&-8&3&1\end{bmatrix}\begin{bmatrix}5\\5\\-5\\-5\end{bmatrix} \Rightarrow \begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}-1\\2\\2\\0\end{bmatrix}$$