Lecture 15: Introduction to Interior Point Method

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A New Perspective of Solving LPs

Now we have studied some duality theories of linear optimization. Let's review the simplex method from some new perspective.

We start with the optimality conditions of LP (consider the standard form):

1. (Primal Feasible) $A\mathbf{x} = \mathbf{b}$, $\mathbf{x} \geq 0$
2. (Dual Feasible) $A^T\mathbf{y} \leq \mathbf{c}$
3. (Complementarity) $x_i \cdot s_i = x_i \cdot (c_i - A_i^T\mathbf{y}) = 0$ for each i

They are necessary and sufficient conditions for the optimal solution of the primal and dual problems.

Review of the Simplex Method

In the simplex method, we search among basic feasible solutions.

- Therefore, we always keep primal feasibility

For any basis B, define $\mathbf{y}^T = \mathbf{c}_B^T A_B^{-1}$. Note that $\mathbf{c}^T - \mathbf{y}^T A$ is the reduced cost. And we have for basic variables, the reduced costs are zero. Therefore,

$$x_i \cdot (c_i - A_i^T \mathbf{y}) = 0 \quad \forall i$$

- Therefore, we always keep complementarity conditions

Review of the Simplex Method

During the simplex method, the reduced costs may be negative

- That is, the dual solution $\mathbf{y}^T = \mathbf{c}_B^T A_B^{-1}$ may not always satisfy $A^T\mathbf{y} \le \mathbf{c}$
- It stops whenever the reduced costs are non-negative
- We seek $\mathbf{y}$ that satisfies $A^T\mathbf{y} \le \mathbf{c}$, i.e., dual feasibility

Proposition

During each iteration, the simplex method maintains primal feasibility and complementarity conditions, and seeks solution that is dual feasible.

What if we choose two other conditions to maintain?

- It will result in dual-simplex method and interior point method

Dual Simplex Method

One can view the dual simplex method to be a simplex method applied to the dual problem of an LP.

- It maintains dual feasibility
- It maintains complementarity conditions
- However, it doesn't need to be primal feasible during the process. It seeks for primal feasibility

The tableau can be viewed as rotated from the primal one.

There are cases where using a dual simplex method is more convenient, for example

- If there is a dual BFS available (but no primal BFS available)
- We have mentioned one such scenario in the discussion of sensitivity analysis (when $\mathbf{b}$ is changed by a large amount or a constraint is added)

Interior Point Method

Remember the optimality condition

1. (Primal Feasible) $A\mathbf{x} = \mathbf{b}$, $\mathbf{x} \geq 0$
2. (Dual Feasible) $A^T\mathbf{y} \leq \mathbf{c}$
3. (Complementarity) $x_i \cdot s_i = x_i \cdot (c_i - A_i^T\mathbf{y}) = 0$ for each i

Interior point method maintains both primal feasibility and dual feasibility during its iterations and seeks for a pair of solutions that satisfy the complementarity conditions.

High-Level Idea

We want to find **x**, **y** and **s** such that

$$\begin{aligned} A\mathbf{x} &= \mathbf{b}, & \mathbf{x} &\geq 0 \\ A^T\mathbf{y} + \mathbf{s} &= \mathbf{c}, & \mathbf{s} &\geq 0 \\ x_i \cdot s_i &= 0, & &\forall i \end{aligned}$$

This is a set of nonlinear equations. It is not very easy to find a solution.

High-Level Idea Continued

We consider a relaxed version of the problem.

$$
\begin{aligned}
A\mathbf{x} = \mathbf{b}, &\quad \mathbf{x} \geq 0 \\
A^T\mathbf{y} + \mathbf{s} = \mathbf{c}, &\quad \mathbf{s} \geq 0 \\
x_i \cdot s_i = \mu, &\quad \forall i
\end{aligned}
$$

We call $\mu > 0$ the complementarity gap.

Idea: If we have found a solution for a certain μ, then we might be able to find a solution for a smaller μ. Then we keep decreasing μ and finally we can get the solution to the LP

- The essential steps in the interior point method is to show that this is possible to do — the method is similar to Newton's method, which we will discuss in the second half of the semester.

Why Called Interior Point Method?

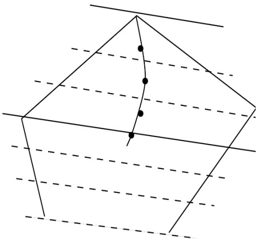

Figure: Central path in an interior point method

- In the simplex method, we only search among the extreme points (all on the boundaries of the polytope)
- In the interior point method, we search in the interior of the feasible region
- Until we reach optimal solution, we keep $\mathbf{x} > 0$ and $\mathbf{s} > 0$
- The optimal solution output by interior point method may not be a BFS (if the solution is unique, then it must be a BFS)

Interior Point Method

Start with any feasible solution $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{s}$ satisfying

$$\begin{aligned} A\mathbf{x} = \mathbf{b}, &\quad \mathbf{x} > 0 \\ A^T\mathbf{y} + \mathbf{s} = \mathbf{c}, &\quad \mathbf{s} > 0 \\ x_i \cdot s_i = \mu_i, &\quad \forall i \end{aligned}$$

Now we want to find the next solution $\bar{\mathbf{x}} = \mathbf{x} + \mathbf{dx}$, $\bar{\mathbf{y}} = \mathbf{y} + \mathbf{dy}$ and $\bar{\mathbf{s}} = \mathbf{s} + \mathbf{ds}$ such that:

$$\begin{aligned} A(\mathbf{x} + \mathbf{dx}) = \mathbf{b}, &\quad \bar{\mathbf{x}} > 0 \\ A^T(\mathbf{y} + \mathbf{dy}) + (\mathbf{s} + \mathbf{ds}) = \mathbf{c}, &\quad \bar{\mathbf{s}} > 0 \\ (x_i + dx_i) \cdot (s_i + ds_i) = \eta\mu_i, &\quad \forall i \end{aligned}$$

where η is a constant less than 1.

$$\begin{aligned} A(\mathbf{x}+\mathbf{dx}) &= \mathbf{b}, \quad \bar{\mathbf{x}} > 0 \\ A^T(\mathbf{y}+\mathbf{dy}) + (\mathbf{s}+\mathbf{ds}) &= \mathbf{c}, \quad \bar{\mathbf{s}} > 0 \\ (x_i + dx_i)\cdot(s_i + ds_i) &= \eta\mu_i, \qquad \forall i \end{aligned}$$

This is a system of nonlinear equations. If we can solve it, we can continue the procedure until it reaches a solution with $\mu_i = 0$.

Important observation

- Everything is linear except the $dx_i \cdot ds_i$ term in the last equation
- If we keep our step size small, that term will be negligible (second-order quantity)

Interior Point Method

We solve dx_i, ds_i that satisfies:

$$\begin{aligned} A\mathbf{dx} &= 0 \\ A^T\mathbf{dy} + \mathbf{ds} &= 0 \\ s_i dx_i + x_i ds_i &= \eta\mu_i - \mu_i \quad \forall i \end{aligned}$$

There are $2n + m$ unknowns, and $2n + m$ equations, so there exists a unique solution

Interior Point Method

We then choose the next solution to be

$$\begin{aligned} \mathbf{x}^+ &= \mathbf{x} + \alpha \mathbf{dx} \\ \mathbf{y}^+ &= \mathbf{y} + \alpha \mathbf{dy} \\ \mathbf{s}^+ &= \mathbf{s} + \alpha \mathbf{ds} \end{aligned}$$

By choosing α sufficiently small, we can guarantee that

- $\mathbf{x}^+$ is primal feasible
- $(\mathbf{y}^+, \mathbf{s}^+)$ is dual feasible
- Each $x_i^+ \cdot s_i^+$ will be smaller than μ_i

Then we keep shrinking μ_i, we will get a sequence of solutions with smaller and smaller complementarity gaps

Interior Point Method

We can also prove that when μ is small enough, if we can get a solution that satisfies:

$$\begin{aligned} A\mathbf{x} &= \mathbf{b}, \quad \mathbf{x} \geq 0 \\ A^T\mathbf{y} + \mathbf{s} &= \mathbf{c}, \quad \mathbf{s} \geq 0 \\ x_i \cdot s_i &\leq \mu, \quad\ \ \forall i, \end{aligned}$$

then we can identify an exact optimal solution.

- We can prove that this algorithm will converge to an optimal solution after $O(\sqrt{n})$ iterations.
- $O(\sqrt{n})$ means a number between $C_1\sqrt{n}$ and $C_2\sqrt{n}$ where C_1 and C_2 are constants that do not depend on n.

Initialization of Interior Point Method

In the simplex method, we need an initial BFS to start.

- In the interior point method, we need to have an initial interior point to start.

We need $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{s}$ that satisfy:

$$\begin{aligned} A\mathbf{x} = \mathbf{b}, &\quad \mathbf{x} > 0 \\ A^T\mathbf{y} + \mathbf{s} = \mathbf{c}, &\quad \mathbf{s} > 0 \end{aligned}$$

Like in the simplex method, it is not trivial.

Initialization of Interior Point Method

The current method is to consider an augmented problem called *homogeneous self-dual problem* (HSDP):

$$
\begin{array}{lrrrrcl}
\text{minimize}_{\mathbf{x},\mathbf{y},\tau,\theta} & & & & (n+1)\theta & & \\
\text{s.t.} & & A\mathbf{x} & -\mathbf{b}\tau & +\bar{\mathbf{b}}\theta & = & 0 \\
 & -A^T\mathbf{y} & & +\mathbf{c}\tau & -\bar{\mathbf{c}}\theta & \geq & 0 \\
 & \mathbf{b}^T\mathbf{y} & -\mathbf{c}^T\mathbf{x} & & +\bar{z}\theta & \geq & 0 \\
 & -\bar{\mathbf{b}}^T\mathbf{y} & +\bar{\mathbf{c}}^T\mathbf{x} & -\bar{z}\tau & & = & -(n+1) \\
 & \mathbf{y}\text{ free,} & \mathbf{x} \geq 0, & \tau \geq 0 & \theta\text{ free} & &
\end{array}
$$

where $\bar{\mathbf{b}} = \mathbf{b} - A\mathbf{e}$, $\bar{\mathbf{c}} = \mathbf{c} - \mathbf{e}$, $\bar{z} = \mathbf{c}^T\mathbf{e} + 1$.

- This problem is self-dual, i.e., the dual is equivalent to itself.
- This problem has a natural initial interior point ($\mathbf{y} = 0$, $\mathbf{x} = \mathbf{e}$, $\tau = 1$, $\theta = 1$)
- Solving this problem will give the optimal solution to the original problem (if exist)

Interior Point Method

Recall that we need $O(\sqrt{n})$ iterations for the interior point method to converge.

Furthermore, in each iteration, it solves a set of linear equations with size $2n + m$, which has complexity $O(n^3)$.

Therefore, interior point method is a polynomial-time algorithm with overall complexity roughly $O(n^{3.5})$

- There are several variants of the interior point method. The one we introduced is called the *primal-dual* type of interior point method.
- The main ideas of those variants are similar, i.e., going through the *interior* of the feasible region and seeking for complementarity

Practical Performance

On average, the speed of the simplex method is comparable with the speed of the interior point method despite their difference in theoretical complexity

- For some problems, simplex method can finish very fast (within a few iterations); for other problems, simplex method needs some extra time (worst-case exponential)
- In contrast, the running time of interior point method is quite stable, it doesn't vary much from problem to problem (given fixed size)

One Property of Interior Point Method

Theorem

The interior point method will always find the optimal solution with the maximum possible number of non-zeros.

Consider a simple case:

$$\begin{aligned} \text{minimize}_{\mathbf{x}} \quad & x_1 + x_2 + x_3 \\ \text{s.t.} \quad & x_1 + x_2 + x_3 = 1 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

Simplex method will give a BFS as optimal (one of $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$), while the interior point method will output $(1/3, 1/3, 1/3)$.

Therefore..

In the case of multiple optimal solutions..

- If we want a high-rank solution (with the maximum possible non-zeros), then choose the interior point method
- If we want a low-rank solution (with small number of non-zeros), then choose the simplex method

In particular, the highest-rank solution is unique for linear programs, but the lowest-rank solution may not be unique.

- The optimal solution output by the simplex method may depend on the initial solution (as well as the pivoting rules).

High-Rank v.s. Low-Rank

Either could be desirable in practice.

High-rank:

- In the multi-firm alliance problem, we want a unique allocation
- Location problem (e.g., building the fountain), it will give the solution that lies in the center of all solutions.

Low-rank

- In portfolio problems, we want to minimize the number of stocks chosen (reduce the transaction cost).
- In graph problems, we want to have fewer nodes/edges chosen.
- In other cases, we prefer integer solutions over fractional solutions (given that their objective values are equal). This usually corresponds to low-rank solutions.

Software Choices

Both simplex method and interior point method are used by major commercial software

- In MATLAB, we can specify which method we want to use in the function called *linprog* (the interior point method is the default)
- Same in CPLEX
- CVX uses interior point method
- Excel uses simplex method