Lecture 7: The Simplex Method

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Announcement

- Homework 2 due today
- Homework 3 due on Oct 10th
- Lectures this week: Today, Friday (9/28) and Sunday (9/30)
- Tutorials only on Thursday and Friday
- Today's office hour: 1:30pm - 3:30pm

Recap

In the previous lectures, we showed an important fact about linear optimization

- If there is an optimal solution, then there is an optimal basic feasible solution
- Therefore, when we solve LP, we only need to search among the BFS

We introduced the simplex method, which is a *smart* way to search among the BFS

- It starts from a BFS
- Then it searches among its neighbors (differ by only one index among the basic indices)
- Either one can find a better neighbor or one knows that the current solution is already optimal

Simplex Method

When at a BFS, the simplex method considers its neighbors by choosing one entering basis.

- When the incoming index is chosen, the (jth feasible) direction is also set:

$$\mathbf{d} = [-A_B^{-1}A_j; 0; ..; 1; ..; 0]$$

- The change to the objective value when going in jth feasible direction (in a unit step size) is given by the reduced cost $\bar{c}_j$

$$\bar{c}_j = c_j - \mathbf{c}_B^T A_B^{-1} A_j$$

- When all the reduced costs are non-negative, the current BFS is optimal

Change of Basis

What if some $\bar{c}_j < 0$?

- Then it means that by bringing jth variable (non-basic) into the basis, we can decrease the objective value. Thus we want to go in that direction

Assume $\mathbf{d}$ is the jth basic direction with $\bar{c}_j < 0$ (there could be multiple such j, we now consider any one of them, we will discuss how to choose among such j's later). We know that going in this direction can reduce the objective value. But how much can we go?

- We need to make sure that $\mathbf{x} + \theta \mathbf{d} \geq 0$.
- We also want to go as far as possible
- Therefore, we choose

$$\theta^* = \max\{\theta \geq 0 | \mathbf{x} + \theta \mathbf{d} \geq 0\}$$

Stepsize

Now we know that we can go as much as

$$\theta^* = \max\{\theta \geq 0 | \mathbf{x} + \theta \mathbf{d} \geq 0\}$$

without violating any constraints. There are two cases for θ^*

- If $\mathbf{d} \geq 0$, then $\theta^* = \infty$. In this case, one can go unlimitedly far without making the solution infeasible, while keeping the objective decreasing. Therefore, the original LP is unbounded
- If $d_i < 0$ for some i, then we can solve:

$$\theta^* = \min_{\{i | d_i < 0\}} \left(-\frac{x_i}{d_i} \right)$$

Since $d_i \geq 0$ for $i \in N$. We can also write it as:

$$\theta^* = \min_{\{i \in B | d_i < 0\}} \left(-\frac{x_i}{d_i} \right)$$

Moving to a New Basis

Assume θ^* is finite (otherwise the problem is unbounded), then we can move to another feasible solution

$$\mathbf{y} = \mathbf{x} + \theta^* \mathbf{d}$$

Assuming that $B(\ell) \in \{B(1), ..., B(m)\}$ is the index such that $\theta^* = -\frac{x_{B(\ell)}}{d_{B(\ell)}}$ (it is possible for multiple indices to achieve this, we will discuss this later). Then we must have

$$y_{B(\ell)} = x_{B(\ell)} + \theta^* d_{B(\ell)} = 0$$

Thus, the basic variable $x_{B(\ell)}$ has become zero, whereas the non-basic variable x_j becomes positive (equal to θ^*). Meaning the basis has changed to

$$B(1), ..., B(\ell - 1), j, B(\ell + 1), ..., B(m)$$

An Iteration of the Simplex Method

We start from a BFS $\mathbf{x}$ (with corresponding basis B)

1. We first compute the *reduced costs* $\bar{\mathbf{c}}$

$$\bar{c}_j = c_j - \mathbf{c}_B^T A_B^{-1} A_j$$

   - If no reduced costs is negative, then $\mathbf{x}$ is already optimal
   - Otherwise choose some j such that $\bar{c}_j < 0$
2. Compute the jth basic direction $\mathbf{d} = [-A_B^{-1}A_j; 0; ...; 1; ...; 0]$
   - If $\mathbf{d} \geq 0$, then the problem is unbounded, i.e., the optimal value is $-\infty$.
   - Otherwise, compute $\theta^* = \min_{i \in B, d_i < 0}\{-\frac{x_i}{d_i}\}$
3. Let $\mathbf{y} = \mathbf{x} + \theta^* \mathbf{d}$. Then $\mathbf{y}$ is the new BFS with index j replacing $B(\ell)$ in the basis, where $B(\ell)$ is the index attaining the minimum in θ^*. Objective value is changed by $\theta^* \mathbf{c}^T \mathbf{d} = \theta^* \bar{c}_j$.
4. Repeat these procedures

Degeneracy

In most of the cases, the objective value will strictly decrease after one simplex method iteration. However, it is possible that the objective stays the same.

Since the change of the objective value (if one chooses to have x_j enter the basis) is $\theta^* \bar{c}_j$ and we know that $\bar{c}_j < 0$. This can only happen if $\theta^* = 0$.

Recall that

$$\theta^* = \min_{\{i \in B | d_i < 0\}} \left(-\frac{x_i}{d_i} \right)$$

If for i's such that $d_i < 0$, there exists $x_i = 0$, then $\theta^* = 0$. This is the case when there are 0s in the BFS $\mathbf{x}$.

Degeneracy

Definition (Degeneracy)

We call a basic feasible solution $\mathbf{x}$ degenerate if some of the basic variables are 0.

- Degeneracy could happen. As an algorithm, we need to consider what consequences it may have

An example:

$$\begin{array}{cccccc} x_1 & -x_2 & & = & 0 \\ x_1 & +x_2 & +2x_3 & = & 2 \\ x_1 & ,x_2 & ,x_3 & \geq & 0 \end{array}$$

If we choose the basic indices to be $\{2,3\}$, then the corresponding basic solution is $(0,0,1)$. It is therefore degenerate.

- The is equivalent to that the number of non-zeros at a basic solution is strictly less than m

Degeneracy

Assume degeneracy happens at some point:

- Given a BFS $\mathbf{x}$ with negative reduced cost $\bar{c}_j < 0$ and $\theta^* = 0$. By definition, there must be some x_i in the basis that $x_i = 0$.

We can still view that we changed the basis from i (i leaving the basis) to j (j entering the basis) and proceed to the next iteration.

- Although the solution (and the objective value) does not change, the basis changed. Therefore, the reduced costs vector will change in the next iteration (issue seems resolved)

However, we need to guarantee that there won't be any cycle, i.e., we will not visit the same BFS more than once

- This can only happen together with degeneracy, since otherwise the objective value will strictly decrease

Example of Cycling

If not dealt properly, cycle can happen. Consider the following LP:

$$A = \begin{pmatrix} -2 & -9 & 1 & 9 & 1 & 0 \\ 1/3 & 1 & -1/3 & -2 & 0 & 1 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\mathbf{c} = (-2, -3, 1, 12, 0, 0)$$

If we set $B = \{5, 6\}$ initially, then the sequence shown below leads to a cycle (objective value doesn't change, and there is always an index with negative reduced cost):

Step #	1	2	3	4	5	6
Exiting	x_6	x_5	x_2	x_1	x_4	x_3
Entering	x_2	x_1	x_4	x_3	x_6	x_5
Basis	$(5,2)$	$(1,2)$	$(1,4)$	$(3,4)$	$(3,6)$	$(5,6)$

We will show that cycle can be avoided by designing how to choose incoming/outgoing basis when there are multiple choices.

Pivoting Rules: Choose the Entering Basis

In the description of the algorithm, we say that at each feasible solution, we can choose *any* j with negative reduced cost to enter the basis in the next iteration. Sometimes, there are more than one j with $\bar{c}_j < 0$. In this case, we need to make some rules to choose the entering basis.

Here are several possible rules:

1. *Smallest subscript rule*: choose the smallest index j such that $\bar{c}_j < 0$
2. *Most negative rule*: choose the smallest $\bar{c}_j$
3. *Most decrement rule*: choose j with the smallest $\theta^* \bar{c}_j$

Pivoting Rules: Choose the Exiting Basis

Recall that

$$\theta^* = \min_{\{i \in B | d_i < 0\}} \left(-\frac{x_i}{d_i} \right)$$

And we choose the index that attains this minimum to leave the basis. It is possible that there are two or more indices that attain the minimum (tie). Then we also need a rule to decide the outgoing basis.

- The most commonly used rule is the *smallest index rule*

When this tie happens, the next BFS will be degenerate

Bland's Rule

Theorem (Bland's Rule)

If we use both the smallest index rule for choosing the entering basis and the exiting basis, then no cycle will occur in the simplex algorithm.

- The proof is given in the book

Using the Bland's rule when applying the simplex method, we can guarantee to stop within a finite number of iterations at an optimal solution.

Finding an Initial BFS

In our previous discussion, we assumed that we start with a certain BFS

- This can be done easily if the standard form is derived by adding slacks to each constraint. (Why?)

However, in general, it is not necessarily easy to get an initial BFS from the standard form. For example,

$$
\begin{array}{lrrrrcr}
\text{minimize} & x_1 & +x_2 & +x_3 & & & \\
\text{subject to} & x_1 & +2x_2 & +3x_3 & & = & 3 \\
 & & -4x_2 & -9x_3 & & = & -5 \\
 & & & +3x_3 & +x_4 & = & 1 \\
 & x_1 & , x_2 & , x_3 & , x_4 & \geq & 0
\end{array}
$$

Finding an Initial BFS

- One could test different basis B, to see if $A_B^{-1}\mathbf{b} \geq 0$.
- However, this may take a long time.
- In fact, in terms of computational complexity (which we will define later), finding one BFS is as hard as finding the optimal solution!

We will discuss an initialization method next — two-phase method.

Two-Phase Simplex Method

In the two-phase simplex method, we first solve an auxiliary problem ($\mathbf{e}$ means an all-one vector).

$$\begin{array}{rc} \text{minimize}_{\mathbf{x},\mathbf{y}} & \mathbf{e}^T\mathbf{y} \\ \text{subject to} & A\mathbf{x} + \mathbf{y} = \mathbf{b} \\ & \mathbf{x}, \mathbf{y} \geq 0 \end{array}$$

Without loss of generality, we assume $\mathbf{b} \geq 0$ (otherwise, we pre-multiply that row by -1).

There is a trivial BFS to the auxiliary problem: $(\mathbf{x} = 0, \mathbf{y} = \mathbf{b} \geq 0)$ so one can apply the Simplex method to solve it.

Theorem

The original problem is feasible if and only if the optimal value of the auxiliary problem is 0.

Proof of the Theorem. First, if the original problem is feasible with a feasible solution $\mathbf{x}_0$. Then $\mathbf{x} = \mathbf{x}_0$, $\mathbf{y} = 0$ is a feasible solution to the auxiliary problem with objective value 0. Note that the optimal value of the auxiliary problem cannot be less than 0. Therefore the optimal value of the auxiliary problem must be 0. ("only if" part proved)

Second, if the optimal value of the auxiliary problem is 0. Say $(\mathbf{x}^*, \mathbf{y}^*)$ is the optimal solution. Then it must be that $\mathbf{y}^* = 0$. Then $\mathbf{x}^*$ is a feasible solution to the original problem. $\square$

Two-Phase Simplex Method

By this theorem, we can solve the auxiliary problem by the Simplex method, and

1. If the optimal value is not 0, then we can claim that the original problem is infeasible;
2. If the optimal value is 0 with solution $(\mathbf{x}^*, \mathbf{0})$. Then we know that $\mathbf{x}^*$ must be a BFS for the auxiliary problem. Then it must be a BFS for the original problem as well. And we can start from there to initialize the simplex method.

If $\mathbf{x}^*$ is degenerate (has less then m positive entries) and still contains basic indices in the auxiliary variables, then one can pick any other columns in the nonbasic part in $\mathbf{x}$ to make it a BFS for the original problem.

Procedure of the Two-Phase Method

Phase I:

1. Construct the auxiliary problem such that $\mathbf{b} \geq 0$
2. Solve the auxiliary problem using the Simplex method
   - If we reach an optimal solution with optimal value greater than 0, then the original problem is infeasible
3. If the optimal value is 0 with optimal solution $\mathbf{x}^*$, then we enter phase II

Phase II: Solve the original problem starting from the BFS $\mathbf{x}^*$

- If $\mathbf{x}^*$ is degenerate, then we need to supplement some indices to make it a BFS for the original problem

Now we have completed the simplex method and showed that it is a valid algorithm for any linear optimization problem

- In the next lecture, we will discuss how to implement these steps efficiently in practice — simplex tableau.