Exercises 8

These used to be called “Fundamental Questions”. Do not hand the solutions to these exercises in; they are just to make sure you can have some practice with the current material.

1. Find $\det A$ using the Cofactor Formula.

   (a) $A = \begin{bmatrix} 1 & 5 & 2 \\ -1 & -1 & -2 \\ 2 & 1 & -3 \end{bmatrix}$

   (b) $A = \begin{bmatrix} 4 & -1 & 7 & 1 \\ 2 & 0 & -1 & -3 \\ 2 & 0 & 2 & 0 \\ 1 & 0 & 5 & 0 \end{bmatrix}$

2. Let $A = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 0 \\ 2 & 0 & -3 \end{bmatrix}$

   (a) Find $\det A$.

   (b) Find $\operatorname{adj} A$.

   (c) Find A^{-1}.

   (d) Find the solution to $A\mathbf{x} = \begin{bmatrix} -12 \\ 0 \\ 3 \end{bmatrix}$.

   (e) Verify your solution, by also calculating it using Cramer’s Rule.

3. Consider $A - \lambda I = \begin{bmatrix} 4-\lambda & 2 \\ 1 & 3-\lambda \end{bmatrix}$. For which values of λ is $A - \lambda I$ not invertible?

4. If all entries of A are integers, and $\det A = 1$ or -1, prove that all entries of A^{-1} are integers.

5. For invertible $n \times n$ matrices A and B, show that the following statements are true:

   (a) $\operatorname{adj} I = I$.

   (b) $\operatorname{adj}(A^T) = (\operatorname{adj} A)^T$.

   (c) $\operatorname{adj}(tA) = t^{n-1}(\operatorname{adj} A)$.

   (d) $\operatorname{adj}(AB) = (\operatorname{adj} A)(\operatorname{adj} B)$.

6. Given a permutation σ of $1, 2, 3, \ldots, n$, we can define the permutation matrix $P_\sigma = \begin{bmatrix} \mathbf{e}_{\sigma(1)}, & \mathbf{e}_{\sigma(2)}, & \cdots, & \mathbf{e}_{\sigma(n)} \end{bmatrix}$ where $\mathbf{e}_i$ are the standard basis vectors of $\mathbb{R}^n$ for $i = 1, 2, \ldots, n$. (In lecture we used the transpose of this matrix.)

(a) Write down P_σ for $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix}$ (this is shorthand notation for $\sigma(1) = 3, \sigma(2) = 2, \sigma(3) = 4, \sigma(4) = 1$).

(b) How many pairs of rows are outof order in P_σ in (a) (where the order of the rows in I is the correct order)?

(c) What is $\det P_\sigma$ for P_σ in (a)?

(d) Given a permutation σ of $1, 2, 3, \ldots, n$, define a new permutation π of $1, 2, 3, \ldots, n$ as follows: for $i = 1, 2, \ldots, n$ we define $\pi(i) = j$ for the j so that $\sigma(j) = i$. (Note that π is well defined, because there is exactly one such j.)

What is π when $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix}$ (as in (a))?

(e) What is $\sigma(\pi(1))$? What is $\sigma(\pi(2))$?

(For a general permutation σ of $1, 2, \ldots, n$, and π defined as in (d), what is $\sigma(\pi(i))$ for $i = 1, 2, \ldots, n$?)

(f) Write down P_π for the π you found in (d).

(g) Find the product $P_\sigma P_\pi$ of the matrices in (a) and (f).

(h) What is P_σ^{-1}? What is P_π^{-1}?

This is not a coincidence! π as defined above is known as the inverse permutation of σ. You can write down the matrix product of P_σ and P_π for general σ and π (in terms of its columns), and if you think about it a little bit, you will see that the definition of π is exactly so that this is equal to the identity.

7. Finish the (first) proof of Cramer's Rule, by plugging in the definition of adj A in

$$\mathbf{x} = A^{-1}\mathbf{b} = \frac{1}{\det A}(\operatorname{adj} A)\mathbf{b},$$

and doing matrix algebra until you reach the conclusion of Cramer's Rule.

8. If you know all 16 cofactors of a 4×4 invertible matrix A, how would you find A?