CIE6020 Assignment 1

Due: 23:55, 25 Jan 2019

1. If the base of the logarithm is b, we denote the entropy as $H_b(X)$. Show that $H_b(X) = (\log_b a)H_a(X)$.

2. *Coin flips.* A fair coin is flipped until the first head occurs. Let X denote the number of flips required.

   (a) Find the entropy $H(X)$ in bits. The following expressions may be useful:

   $$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}, \qquad \sum_{n=0}^{\infty} nr^n = \frac{r}{(1-r)^2}.$$

   (b) A random variable X is drawn according to this distribution. Find an "efficient" sequence of yes-no questions of the form, "Is X contained in the set S?" Compare $H(X)$ to the expected number of questions required to determine X.

3. *Entropy of functions.* Let X be a random variable taking on a finite number of values. What is the (general) inequality relationship of $H(X)$ and $H(Y)$ if

   (a) $Y = 2^X$?

   (b) $Y = \cos(X)$?

4. What is the minimum value of $H(p_1, \ldots, p_n) = H(\mathbf{p})$ as $\mathbf{p}$ ranges over the set of n-dimensional probability vectors? Find all $\mathbf{p}$'s that achieve this minimum.

5. Let X be a discrete random variable. Show that the entropy of a function of X is less than or equal to the entropy of X, i.e., $H(g(X)) \leq H(X)$. (Hint: apply chain rule on $H(X, g(X))$.)

6. Let $p(x, y)$ be given by

$Y \backslash X$	0	1
0	$\frac{1}{3}$	$\frac{1}{3}$
1	0	$\frac{1}{3}$

Find by definition: (a) $H(X)$, $H(Y)$. (b) $H(X|Y)$, $H(Y|X)$. (c) $H(X,Y)$. (d) $I(X;Y)$. Check that $H(X)+H(Y|X)=H(Y)+H(X|Y)$, and $H(X)-H(X|Y)=H(Y)-H(Y|X)$. Draw a Venn diagram (information diagram) for the quantities in parts (a) through (d).

7. *Chain rule for conditional entropy.* Show that

$$H(X_1, X_2, ..., X_n|Y) = \sum_{i=1}^{n} H(X_i|X_1, ..., X_{i-1}, Y).$$

8. *Entropy of a sum.* Let X and Y be random variables that take on values $x_1, x_2, ..., x_r$ and $y_1, y_2, ..., y_s$, respectively. Let $Z = X + Y$.

   (a) Show that $H(Z|X) = H(Y|X)$. Argue that if X, Y are independent, then $H(Y) \leq H(Z)$ and $H(X) \leq H(Z)$. Thus, the addition of *independent* random variables adds uncertainty.

   (b) Give an example of (necessarily dependent) random variables in which $H(X) > H(Z)$ and $H(Y) > H(Z)$.

   (c) Under what conditions does $H(Z) = H(X) + H(Y)$?