CSC3001: Discrete Mathematics

Final Exam (Spring 2018)

Instructions:

1. This exam is 120 minute long.
2. The total mark in this exam paper is 120, and the maximum you can get is 100. **You should try to attempt as many questions as possible.**
3. This exam has 16 pages, consisting of 8 questions. **Write down your full working in this exam paper**.
4. You should check the number of questions in the first 30 minutes. **Instructor is not responsible for any missing questions when exam ends.**
5. No calculator is allowed.
6. This exam is in closed book format. No books, dictionaries or blank papers to be brought in except one page of A4 size paper note which you can write anything on both sides. Any cheating will be given **ZERO** mark.
7. Please note that all the graphs in this exam paper are **SIMPLE GRAPHS**.

Student Number: ____________________ **Name:** _______________

1. (*13 points*) Suppose there are jugs A, B with volume 198ml and 252ml respectively. Is it possible to obtain exactly 90ml water in one of the jugs by pouring water between jugs? If yes, describe how you do it; if not, explain the reason.

2. (*10 points*) Let G be a graph with 1001 vertices. Suppose that the degree of each vertex is either 99 or 100. Prove that there are either at least 501 vertices of degree 100 or at least 502 vertices of degree 99.

3. (*25 points*) Let $x, y, z, n \in \mathbb{N}$. Count the number of solutions for the following equation:

$$x + y + z = 3n$$

where $x < n - 1, y < 2n - 1, z < 2n$.

4. (*15 points*) Given the following program, what is the minimum number of registers needed to store values of the variables throughout the computation? Deduce your claim.

Input:	a
Step 1.	$b = a - 2$
Step 2.	$c = b \cdot \frac{1}{4}$
Step 3.	$d = c + 1$
Step 4.	$e = 3^a$
Step 5.	$f = b \cdot e$
Step 6.	$g = e^2 - f$
Step 7.	$h = b - d$
Output:	f, g, h

5. (*14 points*) Check by definition which of the following arguments are valid. (**Note:** full mark will be given ONLY if you check by definition.)

(i)
$$\begin{array}{l} p \wedge q \to \neg r \\ p \to \neg q \\ \neg q \to p \\ \hline \therefore\ \neg r \end{array}$$

(ii)
$$\begin{array}{l} p \to q \\ q \to \neg p \\ \hline \therefore\ p \leftrightarrow q \end{array}$$

6. (*12 points*) Let $n \geq 2$ be an integer. Give a combinatorial proof for the following identity.

$$\binom{3n}{2} = 3\binom{n}{2} + 3n^2$$

7. (*11 points*) For the stable matching problem with equal number of boys and girls, prove that there is always a girl who does not receive any proposals until the last day of the marrying procedure.

8. (*20 points*) Let G be a k-regular graph with n vertices such that

- every pair of adjacent vertices has λ common neighbors; and
- every pair of non-adjacent vertices has μ common neighbors.

for some $k, n \in \mathbb{Z}^+$ and $\lambda, \mu \in \mathbb{N}$.

(i) Prove that $(n - k - 1)\mu = k(k - \lambda - 1)$. [**11 marks**]

(ii) Suppose $k \geq 2$. When will G be disconnected? What is the relation between λ and n when G is disconnected? [**9 marks**]