Assignment 5

Hand-in Evaluation Deadline: 5:00 pm, November 25th
In-class Evaluation: L1: 2:40 pm - 2:50 pm, November 29th
L2: 9:40 am - 9:50 am, November 29th

The material in lectures may differ between {L1, L2} on the one hand and {L3, L4} on the other, and therefore the homework assignment will differ for {L1, L2} and {L3, L4}.

It is therefore **not** advisable to go to lecture L3 or L4 for the in-class homework evaluation if you attend L1 or L2!

1. Let A and B be similar matrices. Show that $\det A = \det B$.

2. Let $A = \begin{bmatrix} 3 & 1 & -2 \\ -1 & 0 & 0 \\ 5 & 0 & 6 \end{bmatrix}$

   (a) Find $\det A$.

   (b) Compute $\det A^4$

   (c) Find $\operatorname{adj} A$.

   (d) Find A^{-1}.

   (e) Find the solution to $A\mathbf{x} = \begin{bmatrix} 16 \\ -2 \\ -8 \end{bmatrix}$.

   (f) Verify your solution, by also calculating it using Cramer's Rule.

3. (a) Consider the transformation $T(A) = A^T$ that maps a matrix in $\mathbb{R}^{n\times n}$ to its transpose. Is T a linear transformation? Make sure to show convincing reasoning.

   (b) Let B be an $m \times n$ matrix. Consider the transformation $T(A) = BA$ that maps a matrix A in $\mathbb{R}^{n\times n}$ to BA. Is T a linear transformation? Make sure to show convincing reasoning.

   (c) Consider the transformation $T(A) = A^{-1}$ that maps a matrix in $\mathbb{R}^{n\times n}$ to its inverse. Is T a linear transformation? Make sure to show convincing reasoning.

   (d) Consider the transformation $T(\mathbf{x}) = 2x_1\mathbf{w}_1 + x_2\mathbf{w}_2 - 3x_3\mathbf{w}_3$ from $\mathbb{R}^3$ to a vector space W, where $\mathbf{w}_1, \mathbf{w}_2$ and $\mathbf{w}_3 \in W$. Is T a linear transformation? Make sure to show convincing reasoning.

   (e) Consider the transformation $T(\mathbf{x}) = 2x_1\mathbf{w}_1 + x_2\mathbf{w}_2 - 3\mathbf{w}_3$ from $\mathbb{R}^2$ to a vector space W, where $\mathbf{w}_1, \mathbf{w}_2$ and $\mathbf{w}_3 \in W$. Is T a linear transformation? Make sure to show convincing reasoning.

4. Any counter clockwise rotation in $\mathbb{R}^2$ around the origin is a linear transformation. Suppose $\mathbf{x} \mapsto L(\mathbf{x})$ corresponds to the linear transformation that rotates points in $\mathbb{R}^2$ an α degree

angle around the origin (counter clockwise). Find the matrix A so that $L(\mathbf{x}) = A\mathbf{x}$, clearly showing how you derived it (i.e., do not just copy it from a textbook — derive the matrix yourself).

5. Let $T : P_2 \to P_2$ be defined by $T(a_0 + a_1 x + a_2 x^2) = 2a_0 + (3a_0 + 2a_2)x - 4a_1 x^2$. T is a linear mapping (you do not have to show this).

   Find the matrix representation of T relative to the basis $\mathcal{B} = \{1, x, x^2\}$.

6. Define $T : P_2 \to \mathbb{R}^3$ by $T(\mathbf{p}) = \begin{bmatrix} \mathbf{p}(-1) \\ \mathbf{p}(0) \\ \mathbf{p}(1) \end{bmatrix}$.

   (a) Find $T(\mathbf{p})$ for $\mathbf{p}(x) = x - 5$.

   (b) Show that T is a linear transformation.

   (c) Find the matrix for T relative to the basis $\{1, x, x^2\}$ for P_2 and the standard basis for $\mathbb{R}^3$.

7. Find a matrix representation of the transformation in 3(d). Clearly state how this matrix is representing the transformation.

8. Consider the linear transformation $T(\mathbf{x}) = A\mathbf{x}$ where $A = \begin{bmatrix} 4 & -6 & -6 \\ 3 & -5 & -3 \\ -3 & 15 & 13 \end{bmatrix}$.

   Give the $\mathcal{B}$-matrix for T, for $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$.

   Also express A in terms of this $\mathcal{B}$-matrix.

9. (a) Suppose T is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$. Show that $T(\mathbf{0}) = \mathbf{0}$.

   (b) Suppose T is a linear transformation from a vector space V to a vector space W. Show that $T(\mathbf{0}) = \mathbf{0}$.

10. (a) Suppose L_1 is linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$, and L_2 is linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^k$.

    Show that $T(\mathbf{x}) = L_2(L_1(\mathbf{x}))$ is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^k$.

    (b) Suppose L_1 is linear transformation from V to U, and L_2 is linear transformation from U to W.

    Show that $T(\mathbf{x}) = L_2(L_1(\mathbf{x}))$ is a linear transformation from V to W.