Lecture 12: Interpret the Dual Problems

Zizhuo Wang

Institute of Data and Decision Analytics (iDDA)
Chinese University of Hong Kong, Shenzhen

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Announcement

- Homework 4 due next Wednesday (10/24)

Recap: Duality

What have we learned so far about LP duality?

- Given any linear program, we can construct its dual
- Duality theorems (suppose the primal problem is a minimization problem)

Theorem (Weak Duality Theorem)

If $\mathbf{x}$ is a feasible solution to the primal problem and $\mathbf{y}$ is a feasible solution to the dual problem, then

$$\mathbf{b}^T\mathbf{y} \leq \mathbf{c}^T\mathbf{x}$$

- Primal (dual) feasible solution can be used to construct bounds for the objective value of the dual (primal) problem
- If one problem is unbounded, the other one must be infeasible
- If $\mathbf{x}$ is primal feasible, $\mathbf{y}$ is dual feasible, and $\mathbf{c}^T\mathbf{x} = \mathbf{b}^T\mathbf{y}$, then they are both optimal — Optimality Conditions

Theorem (Strong Duality Theorem)

If a linear program has an optimal solution, so does its dual, and the optimal value of the primal and dual problems are equal

- In the proof of the strong duality theorem, we showed that the simplex method can actually find the dual optimal solution when it finishes
- We can also find the dual optimal solution in the final simplex tableau in certain circumstances

All possible states of a pair of primal/dual linear programs

	Finite Optimum	Unbounded	Infeasible
Finite Optimum	$\surd$		
Unbounded			$\surd$
Infeasible		$\surd$	$\surd$

Interpret the Dual Problem

In the following, we will look at some examples and provide interpretations to the dual problems, and see how the duality theory can help us solve problems

- The production planning problem
- The multi-firm alliance problem
- The transportation problem
- The alternative systems problem
- The maximum flow problem

Production Planning Problem

Recall the production planning problem:

$$\begin{array}{rrrl} \text{maximize} & x_1 & +2x_2 & \\ \text{subject to} & x_1 & & \leq 100 \\ & & 2x_2 & \leq 200 \\ & x_1 & +x_2 & \leq 150 \\ & x_1, & x_2 & \geq 0 \end{array}$$

The objective corresponds to profits; the constraints are resource constraints.

Dual of the Production Planning Problem

We write down its dual and associate the three constraints with dual variables p_1, p_2 and p_3.

$$\begin{array}{rrrrl} \text{minimize} & 100p_1 & +200p_2 & +150p_3 & \\ \text{subject to} & p_1 & & +p_3 & \geq 1 \\ & & 2p_2 & +p_3 & \geq 2 \\ & p_1, & p_2, & p_3 & \geq 0 \end{array}$$

What is the meaning of this optimization problem?

- Suppose someone wants to buy all resources of the firm at unit prices p_1, p_2 and p_3. What prices he should offer?
- Since one unit of resource 1 and resource 3 can produce one unit of product 1 which worths \$1. Therefore, if $p_1 + p_3 < 1$, the firm would rather produce by itself (than agree to sell).
- Similarly, it wouldn't sell if $2p_2 + p_3 < 2$.
- Other than that, the buyer wants to buy the resources for the lowest price, and the dual problem finds that price.

Discussions

In this example, we interpret the dual variables to be the fair price of each resource

- This is a very common type of interpretation for dual variables
- In the next lecture we will see why this makes sense
- Below we look at an extension of the production planning problem and see how the dual problem can provide help for solving an important question

Multi-Firm Alliance Problem

Suppose there is a collection of firms $1, 2, ..., m$, each making the same set of products.

- Firm i has resource $\mathbf{b}_i$ and has consumption matrix A and profit $\mathbf{c}$ for each product (A and $\mathbf{c}$ are the same across all firms)

Therefore, each firm is trying to solve a production problem:

$$\begin{aligned} \text{maximize}_{\mathbf{x}} \quad & \mathbf{c}^T\mathbf{x} \\ \text{s.t.} \quad & A\mathbf{x} \le \mathbf{b}_i \\ & \mathbf{x} \ge 0 \end{aligned}$$

Denote the optimal profit for firm i by V^i

Multi-Firm Alliance Problem

Now suppose the firms can form an *alliance* so that they can *pool* their resources and produce together.

Assume a subset of firms S form an alliance, then to maximize the profit of the alliance S, we solve the following LP:

$$\begin{array}{rl} \text{maximize}_{\mathbf{x}} & \mathbf{c}^T\mathbf{x} \\ \text{s.t.} & A\mathbf{x} \le \sum_{i\in S} \mathbf{b}_i \\ & \mathbf{x} \ge 0 \end{array}$$

The optimal profit for the alliance is denoted by V^S.

We call the alliance involving all firms the *grand alliance*. In a grand alliance, all resources are pooled together. We denote the optimal profit for the grand alliance by V^*.

Multi-Firm Alliance Problem

One important question before forming an alliance is:

- How to allocate the profit of the alliance to each of its members so that each member/suballiance has the incentive to stay in that alliance?

Now let's consider the grand alliance. Our question is whether the firms have incentive to form a grand alliance and what allocation rule we should use?

What constraints do we have?

1. The total allocation equals the total profit
2. The allocated profits to any suballiance should be greater than what the suballiance could get if they separate from the grand alliance (no subset of players would want to leave the alliance)

Multi-Firm Alliance Problem

In order to make the grand alliance stable, an allocation $(z_1, ..., z_m)$ must satisfy

$$\sum_{i=1}^{m} z_i = V^*$$
$$\sum_{i \in S} z_i \geq V^S, \quad \forall S \subseteq \{1, ..., m\}$$

We call such z_i's the *core* of the grand alliance.

- It is not trivial to find such z_i's or even prove its existence
- It involves checking 2^m inequalities
- However, we can use duality to find such a solution easily

Multi-Firm Alliance Problem

The solution: consider the dual of the grand alliance production problem:

$$\begin{array}{rl} \text{minimize}_{\mathbf{y}} & \left(\sum_{i=1}^{m} \mathbf{b}_i\right)^T \mathbf{y} \\ \text{s.t.} & A^T\mathbf{y} \geq \mathbf{c} \\ & \mathbf{y} \geq 0 \end{array}$$

Denote the optimal solution by $\mathbf{y}^*$

Theorem

Define $z_i = \mathbf{b}_i^T \mathbf{y}^*$. Then $(z_1, ..., z_m)$ is in the core of the grand alliance.

- Basically, $\mathbf{y}^*$ can be viewed as the prices for the resource and we just allocate according to the value of resource of each firm

Proof

First, by the strong duality theorem, we know that $\sum_{i=1}^m z_i = V^*$. Now we only need to show that for any subset S,

$$\sum_{i \in S} z_i \geq V^S.$$

Consider the production problem for alliance S and its dual:

$$\begin{array}{lll} \textbf{Primal} & \max_{\mathbf{x}} & \mathbf{c}^T\mathbf{x} \\ & \text{s.t.} & A\mathbf{x} \leq \sum_{i \in S} \mathbf{b}_i \\ & & \mathbf{x} \geq 0 \end{array} \Bigg\| \begin{array}{lll} \textbf{Dual} & \min_{\mathbf{y}} & \left(\sum_{i \in S} \mathbf{b}_i\right)^T \mathbf{y} \\ & \text{s.t.} & A^T\mathbf{y} \geq \mathbf{c} \\ & & \mathbf{y} \geq 0 \end{array}$$

Note that $\mathbf{y}^*$ is still a feasible solution to the dual. Therefore, by the weak duality theorem,

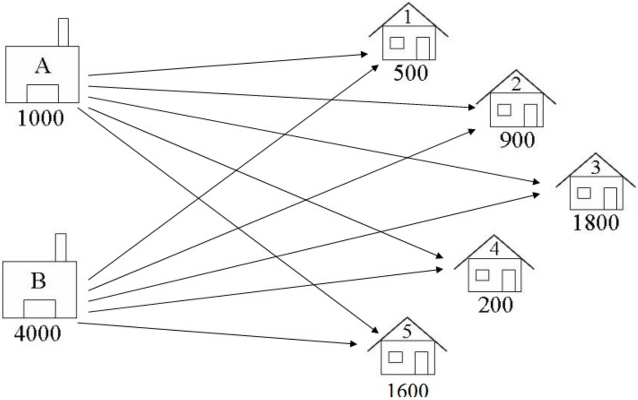
$$\sum_{i \in S} z_i = (\sum_{i \in S} \mathbf{b}_i)^T \mathbf{y}^* \geq V^S$$

Transportation Problem

Consider a transportation problem:

- We have m warehouses, each contains a_i amount of a certain product
- We have n stores, each needs b_j amount of that product
- It costs c_{ij} to transport one unit of product from warehouse i to store j
- Objective: Minimize the total transportation cost while satisfying all stores' demands.
- We assume that $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$ (total supply equals total demand)

Illustration



LP Formulation

Let x_{ij} denote the amount of products to be transported from warehouse i to store j.

Then the LP formulation is:

$$
\begin{array}{ll}
\text{minimize} & \displaystyle\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} \\
\text{subject to} & \displaystyle\sum_{j=1}^{n} x_{ij} = a_i, \quad i = 1, \dots, m \\
& \displaystyle\sum_{i=1}^{m} x_{ij} = b_j, \quad j = 1, \dots, n \\
& x_{ij} \geq 0, \quad \forall i, j
\end{array}
$$

The Dual Problem

The dual of the transportation problem can be written as follows:

$$\begin{array}{ll} \text{maximize} & \sum_{i=1}^{m} a_i u_i + \sum_{j=1}^{n} b_j v_j \\ \text{subject to} & u_i + v_j \leq c_{ij}, \quad \forall i, j \end{array}$$

What is the meaning of this optimization problem?

- We do a little trick, we use $-u_i$ instead of u_i
- This is equivalent since u_is are free variables

Dual of Transportation Problem

$$\begin{array}{ll} \text{maximize} & \sum_{j=1}^{n} b_j v_j - \sum_{i=1}^{m} a_i u_i \\ \text{subject to} & v_j - u_i \le c_{ij}, \quad \forall i, j \end{array}$$

Assume a logistic firm buys the product from warehouse i at price u_i and sells them at store j for price v_j. What prices will make the firm competitive and earn the most money?

- The profit: $\sum_{j=1}^{n} b_j v_j - \sum_{i=1}^{m} a_i u_i$
- Constraints? We must have $v_j - u_i \le c_{ij}$. If $v_j - u_i > c_{ij}$, then no one will hire this firm to ship between i and j
- Strong duality tells us that the optimal values are the same

Alternative Systems

Given a set of linear inequalities:

$$A^T\mathbf{y} \le \mathbf{c}$$

An important question is: whether the system has a solution?

- It is easy to verify that it has a solution, one only needs to find a solution (we call it a *certificate*)
- To disprove the existence, can we also have such a certificate?

The answer: Yes.

- If we can find a vector $\mathbf{x}$ satisfying

  $$A\mathbf{x} = 0, \quad \mathbf{x} \ge 0, \quad \mathbf{c}^T\mathbf{x} < 0$$

  Then there must be no solution to the system $A^T\mathbf{y} \le \mathbf{c}$

Alternative Systems

Proof. Consider the pair of primal-dual linear program:

Primal		**Dual**	
min	$\mathbf{c}^T\mathbf{x}$	max	0
s.t.	$A\mathbf{x} = 0, \mathbf{x} \geq 0$	s.t.	$A^T\mathbf{y} \leq \mathbf{c}$

If there exists $\mathbf{x}$ such that

$$A\mathbf{x} = 0, \quad \mathbf{x} \geq 0, \quad \mathbf{c}^T\mathbf{x} < 0$$

Then by scaling this $\mathbf{x}$, we can make $\mathbf{c}^T\mathbf{x}$ arbitrarily negative, therefore, the primal problem must be unbounded.

However, for an LP, if the primal problem is unbounded, then the dual problem must be infeasible. Thus such an $\mathbf{x}$ is a *certificate* that the linear system $A^T\mathbf{y} \leq \mathbf{c}$ is infeasible. $\square$

Alternative Systems

One can construct many more pairs of such alternative systems.

- It is hard to directly prove something is not feasible
- LP duality provides an alternative approach, transforming the problem to proving something is feasible