Lecture 22: Eigenvalues and Eigenvectors

MAT2040 Linear Algebra

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But first let's recall what we already know about this decomposition and how we can interpret it.

Recall:

Example 18.2

Consider $T(\mathbf{x}) = A\mathbf{x}$ where $A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$.

What is the $\mathcal{B}$-matrix for T for $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\}$?

Express A in terms of the $\mathcal{B}$-matrix for T.

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Express A in terms of the $\mathcal{B}$-matrix for T.

$$\boxed{\begin{gathered} A = P \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix} P^{-1}. \\ \text{where } P = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}. \end{gathered}}$$

This decomposition is useful if we think about matrix powers:

$$\begin{aligned}
A^k &= \overbrace{(PDP^{-1})(PDP^{-1})(P\cdots P^{-1})(PDP^{-1})}^{k\text{ times}} \\
&= PD(P^{-1}P)D(P^{-1}P)\cdots(P^{-1}P)DP^{-1} \\
&= PD^kP^{-1} \\
&= P\begin{bmatrix} 5^k & 0 \\ 0 & 3^k \end{bmatrix} P^{-1}.
\end{aligned}$$

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(It also turns out to be very useful for differential equations, where you will often get a *matrix exponential*. The matrix exponential of a diagonal matrix is, again, easy to compute.)

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(The image of any vector that is not a multiple of $\mathbf{b}_1$ and $\mathbf{b}_2$ will **not** just be scaled: if $\mathbf{v} = t_1\mathbf{b}_1 + t_2\mathbf{b}_2$ $(t_1 \neq 0,\ t_2 \neq 0)$ then

$$A\mathbf{v} = t_1 A\mathbf{b}_1 + t_2 A\mathbf{b}_2 = 5t_1\mathbf{b}_1 + 3t_2\mathbf{b}_2$$

(by linearity of $A\mathbf{x}$, or by matrix algebra).)

The interpretation on the previous slide gives us an idea of how to find this decomposition.

Alexander Yakovlev / Fotolia

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If these vectors, say $\mathcal{B}$, form a basis for $\mathbb{R}^n$, we have a **diagonal** $\mathcal{B}$-matrix for the transformation $A\mathbf{x}$.

(We will prove a precise statement later.)

Definition 22.1

Let A be an $n \times n$ (square!) matrix.
A scalar λ is called an **eigenvalue** of A if there exists a nonzero vector $\mathbf{x}$ so that $A\mathbf{x} = \lambda\mathbf{x}$. The vector $\mathbf{x}$ is called an **eigenvector** corresponding to eigenvalue λ.

Example 22.2

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Is $\mathbf{u} = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$ an eigenvector of A?

What about $\mathbf{v} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$?

Example 22.2 (continued)

Let $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$.

Show that 7 is an eigenvalue of A and find (all) eigenvectors corresponding to eigenvalue 7.

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Rewriting gives

$$(A - \lambda I)\mathbf{x} = \mathbf{0}.$$

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We want λ so that $A - \lambda I$ is not invertible!
(Recall that all matrices are square.)

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(Recall that all matrices are square.)

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This is a polynomial equation where λ is the variable!

Conclusion:

The eigenvalues of A are exactly the solutions to $\det(A - \lambda I) = 0$ (where λ is the variable).

Example 22.3

Find the eigenvalues of $A = \begin{bmatrix} 3 & 2 \\ 3 & -2 \end{bmatrix}$.

The polynomial

$$p(\lambda) = \det(A - \lambda I)$$

is called the **characteristic polynomial** for matrix A, and

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Note that λ is a variable in both the characteristic polynomial and the characteristic equation.

Summarizing:

- Finding eigenvalues of A is equivalent to finding roots of the characteristic equation for matrix A.

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- Finding eigenvalues of A is equivalent to finding roots of the characteristic equation for matrix A.

- Finding all eigenvectors belonging to a particular eigenvalue $\bar{\lambda}$ is equivalent to finding $\mathrm{Null}(A - \bar{\lambda} I)$
(except that $\mathbf{0}$ is never an eigenvector).

  $\mathrm{Null}(A - \bar{\lambda} I)$ is known as the **eigenspace** belonging to eigenvalue $\bar{\lambda}$.

Theorem 22.4

If A and B are $n \times n$ matrices that are similar, then A and B have the same characteristic polynomial.