Lecture 16: Determinants (continued)

MAT2040 Linear Algebra

Warning!

Note: My lectures (L1 and L2) may from now on present material quite differently from the way it is presented L3 and L4. So for the next in-class homework evaluation you need to be in L1 or L2.

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The material here is based on Gilbert Strang's excellent textbook *Linear Algebra and its Applications, Fourth Edition*, Chapter 4.

We can conclude that the following is an algorithm to compute $\det A$:

1. Use elementary row operations except the scaling operation, to transform A into row echelon form. Keep track of the number of row interchanges, say r (actually parity suffices).
2. $\det A = (-1)^r$ product of the diagonal entries of row echelon form.

Example 16.1

Find $\begin{vmatrix} 5 & 9 & 17 \\ 1 & 2 & 0 \\ -5 & -11 & 3 \end{vmatrix}$.

Note that our intuition of getting a volume was not quite correct: we got a volume that could also be negative.

The absolute value of the determinant of A, however, exactly equals the volume of the parallelepiped defined by the rows of A.

You may be (should be) wondering whether the determinant is well defined: is the number of row interchanges to get the rows in a certain order always odd or always even (no matter which ones and which order you choose)?

Suppose you have a matrix, and you want to move row 1 to row $\pi(1)$, row 2 to row $\pi(2)$, etc., row n to row $\pi(n)$, using row interchanges, where π is a *permutation*.

Fact 16.2

The number of row interchanges you have to do has the same parity as the number of pairs that are out of order in π.

Corollary 16.3

Let P be a permutation matrix.

$$\det P = (-1)^{\#\textit{ of pairs of rows that are out of order compared to } I}.$$

Example 16.4

What is $\begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$?

How many pairs of rows are out of order?

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How many pairs of rows are out of order?

Example 16.5

What is $\begin{vmatrix} 0 & a & 0 & 0 \\ b & 0 & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{vmatrix}$?

Lemma 16.6

Let E be an elementary matrix.

$$\det E = \begin{cases} 1 & \textit{if } E \textit{ corresponds to an el. row op. } R_i \to R_i + \beta R_j \\ \alpha & \textit{if } E \textit{ corresponds to an el. row op. } R_i \to \alpha R_i \\ -1 & \textit{if } E \textit{ corresponds to an el. row op. } R_i \leftrightarrow R_j. \end{cases}$$

Lemma 16.7

Let E be an elementary $n \times n$ matrix, and A be an $n \times n$ matrix.

$$\det EA = (\det E)(\det A).$$

Lemma 16.8

For any two $n \times n$ matrices A, B,

$$\det AB = (\det A)(\det B).$$

Example 16.9

Give an expression in terms of $\det A$:

$$\det A^2 = \det AA = ?$$

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Give an expression in terms of $\det A$:

$\det A^2 = \det AA = ?$

$\det A^{-1} = ?$

We will now work towards two formulas for the determinant.

Property 9

If A has a column of all zeros then

$$\det A = 0.$$

Let's split up every row of A into rows that just contain one nonzero entry.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & d \end{vmatrix}$$

by property 3(a)

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & d \end{vmatrix} \qquad \text{by property 3(a)}$$

$$= \begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix} + \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix} + \begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix} \qquad \text{by property 3(a)}$$

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$$= \begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix} + \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix} + \begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix}$$ by property 3(a)

$$= 0 + ad + (-1) \begin{vmatrix} c & 0 \\ 0 & b \end{vmatrix} + 0$$ by properties 9, 7, 8 and 9, respectively

$$
\begin{aligned}
\begin{vmatrix} a & b \\ c & d \end{vmatrix} &= \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & d \end{vmatrix} && \text{by property 3(a)} \\
&= \begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix} + \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix} + \begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix} && \text{by property 3(a)} \\
&= 0 + ad + (-1) \begin{vmatrix} c & 0 \\ 0 & b \end{vmatrix} + 0 && \text{by properties 9, 7, 8 and 9, respectively} \\
&= ad - bc && \text{by property 7.}
\end{aligned}
$$

Doing the same for 3×3 matrix would give how many matrices? How many of those do not have an all zero column? (A matrix with an all zero column has determinant zero, and we can ignore it in our calculations.)

$$\begin{aligned}
&\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\
&\quad = \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & a_{32} & 0 \end{vmatrix} \\
&\quad + \begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & 0 \\ 0 & 0 & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ 0 & 0 & a_{23} \\ a_{31} & 0 & 0 \end{vmatrix} \\
&\quad + \begin{vmatrix} 0 & 0 & a_{13} \\ a_{21} & 0 & 0 \\ 0 & a_{32} & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & 0 & 0 \end{vmatrix}
\end{aligned}$$

$$
\begin{aligned}
\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}
&= a_{11}a_{22}a_{33}\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} + a_{11}a_{23}a_{32}\begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} \\
&+ a_{12}a_{21}a_{33}\begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} + a_{12}a_{23}a_{31}\begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} \\
&+ a_{13}a_{21}a_{32}\begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} + a_{13}a_{22}a_{31}\begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix}
\end{aligned}
$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= \sum_{\substack{\pi \text{ is a permutation} \\ \text{of } 1,2,3}} a_{1\pi(1)} a_{2\pi(2)} a_{3\pi(3)} \begin{vmatrix} e^T_{\pi(1)} \\ e^T_{\pi(2)} \\ e^T_{\pi(3)} \end{vmatrix}$$

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$$= \sum_{\substack{\pi \text{ is a permutation} \\ \text{of } 1,2,3}} a_{1\pi(1)} a_{2\pi(2)} a_{3\pi(3)} \begin{vmatrix} e_{\pi(1)}^T \\ e_{\pi(2)}^T \\ e_{\pi(3)}^T \end{vmatrix}$$

$$= \sum_{\substack{\pi \text{ is a permutation} \\ \text{of } 1,2,3}} \operatorname{sign}(\pi) a_{1\pi(1)} a_{2\pi(2)} a_{3\pi(3)}.$$

Definition 16.10

The **sign of a permutation** π of size n is the determinant of the $n \times n$ permutation matrix that corresponds to π:

$$\operatorname{sign}(\pi) = \begin{vmatrix} e_{\pi(1)}^T \\ e_{\pi(2)}^T \\ e_{\pi(3)}^T \\ \vdots \\ e_{\pi(n)}^T \end{vmatrix}.$$

We can do the same for $n \times n$ matrices — we get $\boxed{}$ matrices, but most of the matrices we get have determinant 0 (because they have an all zero column).

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How many matrices will there be that do not have an all zero column?

Formula (Leibniz formula)

$$\det A = \sum_{\substack{\pi \text{ is a permutation} \\ \text{of } 1,2,\ldots,n}} \operatorname{sign}(\pi) a_{1\pi(1)} a_{2\pi(2)} \cdots a_{n\pi(n)}.$$

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Argh! So many terms!

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Lemma 16.11

$\det A^T = \det A$ *for any* $n \times n$ *matrix* A*.*

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Lemma 16.11

$\det A^T = \det A$ *for any* $n \times n$ *matrix* A*.*

Also means you can do column operations, or switch to transpose when calculating the determinant!

Example 16.12

Sometimes we can actually use Leibniz formula to calculate a determinant — when the matrix has a lot of zero entries.

Let's find $\begin{vmatrix} 0 & 2 & 0 \\ -5 & 3 & 0 \\ -1 & -2 & 3 \end{vmatrix}$.

Formula (Leibniz formula)

$$\det A = \sum_{\substack{\pi \text{ is a permutation} \\ \text{of } 1,2,\ldots,n}} \operatorname{sign}(\pi) a_{1\pi(1)} a_{2\pi(2)} \cdots a_{n\pi(n)}.$$

What if we group terms — for instance starting with all terms involving a_{11}?

There is a term that includes a_{11} for every permutation where $\pi(1) = 1$.

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The sum of all terms that include a_{11} in Leibniz Formula can be written as

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$$= a_{11} \det M_{11}$$

where M_{11} is the matrix A with row 1 and column 1 deleted.

Similarly, there is a term that includes a_{21} for every permutation where $\pi(2) = 1$.

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The sum of all terms that include a_{21} in Leibniz Formula can be written as

$$a_{21} \sum_{\substack{\pi \text{ is a permutation} \\ \text{of } 1,2,\ldots,n \\ \text{where } \pi(2)=1}} \text{sign}(\pi) a_{1\pi(1)} a_{3\pi(3)} a_{4\pi(4)} \cdots a_{n\pi(n)}$$

Similarly, there is a term that includes a_{21} for every permutation where $\pi(2) = 1$.

The sum of all terms that include a_{21} in Leibniz Formula can be written as

$$a_{21} \sum_{\substack{\pi \text{ is a permutation} \\ \text{of } 1,2,\ldots,n \\ \text{where } \pi(2)=1}} \operatorname{sign}(\pi) a_{1\pi(1)} a_{3\pi(3)} a_{4\pi(4)} \cdots a_{n\pi(n)}$$

$$= -a_{21} \det M_{21}$$

where M_{21} is the matrix A with row 2 and column 1 deleted.

Similarly, there is a term that includes a_{21} for every permutation where $\pi(2) = 1$.

The sum of all terms that include a_{21} in Leibniz Formula can be written as

$$a_{21} \sum_{\substack{\pi \text{ is a permutation} \\ \text{of } 1,2,\ldots,n \\ \text{where } \pi(2)=1}} \operatorname{sign}(\pi) a_{1\pi(1)} a_{3\pi(3)} a_{4\pi(4)} \cdots a_{n\pi(n)}$$
$$= -a_{21} \det M_{21}$$

where M_{21} is the matrix A with row 2 and column 1 deleted.

The minus sign comes from the fact that when calculating $\det M_{21}$ row 2 was ignored in the ordering, and row 2 is actually out of order with (only) row one for all these terms.

Formula (Cofactor Formula for Expansion using 1st Column)

$$\det A = \sum_{i=1}^{n} a_{i1}(-1)^{i+1} \det M_{i1},$$

where M_{i1} is the matrix A with row i and column 1 deleted.

Formula (Cofactor Formula for Expansion using 1st Column)

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where M_{i1} is the matrix A with row i and column 1 deleted.

The term "$(-1)^{i+j} \det M_{ij}$" is called the (i,j)-**cofactor of** A, sometimes denoted by C_{ij} (the textbook uses A_{ij}).

($j = 1$ here.)

Formula (Cofactor Formula for Expansion using 1st Row)

$$\det A = \sum_{j=1}^{n} a_{1j}(-1)^{1+j} \det M_{1j},$$

where M_{1j} is the matrix A with row 1 and column j deleted.

Formula (Cofactor Formula for Expansion using 1st Row)

$$\det A = \sum_{j=1}^{n} a_{1j}(-1)^{1+j} \det M_{1j},$$

where M_{1j} is the matrix A with row 1 and column j deleted.

Formula (Cofactor Formula for Expansion using ith Row)

$$\det A = \sum_{j=1}^{n} a_{ij}(-1)^{i+j} \det M_{ij},$$

where M_{ij} is the matrix A with row i and column j deleted.

Formula (Cofactor Formula for Expansion using 1st Row)

$$\det A = \sum_{j=1}^{n} a_{1j}(-1)^{1+j} \det M_{1j},$$

where M_{1j} is the matrix A with row 1 and column j deleted.

Formula (Cofactor Formula for Expansion using ith Row)

$$\det A = \sum_{j=1}^{n} a_{ij}(-1)^{i+j} \det M_{ij},$$

where M_{ij} is the matrix A with row i and column j deleted.

And similar formula for expansion using jth column.

Example 16.13

Calculate $\begin{vmatrix} 3 & 2 & -1 \\ 2 & 2 & 1 \\ 0 & 0 & 1 \end{vmatrix}$ using the cofactor formula for expansion using the 1st row.

Example 16.13

Calculate $\begin{vmatrix} 3 & 2 & -1 \\ 2 & 2 & 1 \\ 0 & 0 & 1 \end{vmatrix}$ using the cofactor formula for expansion using the 1st row.

Was this the fastest way to compute this determinant?

Example 16.14

Calculate $\begin{vmatrix} 5 & 2 & -1 & 6 \\ -5 & 3 & 0 & 8 \\ 2 & 0 & 0 & 0 \\ 17 & 3 & 0 & 0 \end{vmatrix}$ using the cofactor formula.

Lemma 16.15

Let A be an $n \times n$ matrix, and let C_{ij} be the (i,j)-cofactor of A.

$$a_{i1}C_{k1} + a_{i2}C_{k2} + \cdots + a_{in}C_{kn} = \begin{cases} \det A & \text{if } i = k \\ 0 & \text{if } i \neq k. \end{cases}$$

Definition 16.16

Let A be an $n \times n$ matrix. The **adjoint of** A is the matrix

$$\begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix},$$

which is denoted by "adj A".

(Note that the row and columns order is different than usual!)

Example 16.17

Suppose $A = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

Write down $\operatorname{adj} A$.

Lemma 16.18

Let A be an invertible matrix.

$$A^{-1} = \frac{1}{\det A} \operatorname{adj} A.$$

Example 16.17 (continued)

Suppose $A = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

Find A^{-1} using $\operatorname{adj} A$.

Theorem 16.19 (Cramer's Rule (first published by Maclaurin))

Let A be an invertible $n \times n$ matrix, and $b \in \mathbb{R}^n$. The (unique) solution to $A\mathbf{x} = \mathbf{b}$ is

$$x_i = \frac{\det A_i}{\det A}$$

where A_i is equal to the matrix A except that column i is replaced by $\mathbf{b}$ (i.e., $A_i = \begin{bmatrix} \mathbf{a}_1, & \mathbf{a}_2, & \dots, & \mathbf{a}_{i-1}, & \mathbf{b}, & \mathbf{a}_{i+1}, & \dots, & \mathbf{a}_n \end{bmatrix}$).

Example 16.17 (continued)

Suppose $A = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

Solve $A\mathbf{x} = \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix}$ using Cramer's Rule.

Summarizing:

- Defined $\det A$ for square matrices based on 3 geometric properties
- Found algorithm to calculate $\det A$ using elementary row operations
- Found some properties of $\det A$:
  - $\det A^T = \det A$
  - $\det AB = (\det A)(\det B)$ (in particular $\det A^{-1} = \dfrac{1}{\det A}$)

- Found 2 formulas for $\det A$

Formula (Leibniz formula)

$$\det A = \sum_{\substack{\pi \text{ is a permutation} \\ \text{of } 1,2,\ldots,n}} \operatorname{sign}(\pi) a_{1\pi(1)} a_{2\pi(2)} \cdots a_{n\pi(n)}.$$

Formula (Cofactor Formula for Expansion using ith Row)

$$\det A = \sum_{j=1}^{n} a_{ij}(-1)^{i+j} \det M_{ij},$$

where M_{ij} is the matrix A with row i and column j deleted.

- Both of these formulas are in general not practical to calculate $\det A$ — they are of theoretical interest
- We showed that

$$A^{-1} = \frac{1}{\det A} \operatorname{adj} A, \text{where}$$

$$\operatorname{adj} A = \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}.$$

Again, this is not usually practical for calculating A^{-1}, but it gives a theoretical handle on what the entries in A^{-1} look like.

- In particular, we have Cramer's Rule.

Let A be an invertible $n \times n$ matrix, and $b \in \mathbb{R}^n$. The (unique) solution to $A\mathbf{x} = \mathbf{b}$ is

$$x_i = \frac{\det A_i}{\det A}$$

where A_i is equal to the matrix A except that column i is replaced by $\mathbf{b}$.

Again, usually not practical for calculating the solution, but it gives a theoretical handle on how the solution changes when $\mathbf{b}$ changes.