CIE 6020 Assignment 2

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1. Let X, Y, Z be three random variables with a joint probability mass function $p(x, y, z)$. The relative entropy between the joint distribution and the product of the marginal is

$$D(p(x,y,z)||p(x)p(y)p(z)) = E[\log \frac{p(x,y,z)}{p(x)p(y)p(z)}]$$

Expand this in terms of entropies. When is this quantity zero?

Answer

$$\begin{aligned}
E[log\frac{p(x,y,z)}{p(x)p(y)p(z)}] &= \sum_{z\in\mathcal{Z}} p(z) \sum_{y\in\mathcal{Y}} p(y \mid z) \sum_{x\in\mathcal{X}} p(x \mid y, z) \log \frac{p(x,y,z)}{p(x)p(y)p(z)} \\
&= \sum_{z\in\mathcal{Z}} p(z) \sum_{y\in\mathcal{Y}} p(y \mid z) \sum_{x\in\mathcal{X}} p(x \mid y, z) [\log p(x,y,z) - \log p(x)p(y)p(z)] \\
&=
\end{aligned}$$

2. Let the random variable X have three possible outcomes a, b, c. Consider two distributions on this random variable:

symbol	$p(x)$	$p(y)$
a	$\frac{1}{2}$	$\frac{1}{3}$
b	$\frac{1}{4}$	$\frac{1}{3}$
c	$\frac{1}{4}$	$\frac{1}{3}$

Calculate $H(p)$, $H(q)$, $D(p||q)$ and $D(q||p)$. Verify that in this case, $D(p||q) \neq D(q||p)$

Answer

$$\begin{aligned}
H(p) &= -(\frac{1}{2}\log\frac{1}{2} + \frac{1}{4}\log\frac{1}{4} + \frac{1}{4}\log\frac{1}{4}) = \frac{3}{2} \\
H(q) &= -(\frac{1}{3}\log\frac{1}{3} + \frac{1}{3}\log\frac{1}{3} + \frac{1}{3}\log\frac{1}{3}) = \log 3 \\
D(p||q) &= \frac{1}{2}\log\frac{\frac{1}{2}}{\frac{1}{3}} + \frac{1}{4}\log\frac{\frac{1}{4}}{\frac{1}{3}} + \frac{1}{4}\log\frac{\frac{1}{4}}{\frac{1}{3}} = \log 3 - \frac{3}{2} \\
D(q||p) &= \frac{1}{3}\log\frac{\frac{1}{3}}{\frac{1}{2}} + \frac{1}{3}\log\frac{\frac{1}{3}}{\frac{1}{4}} + \frac{1}{3}\log\frac{\frac{1}{3}}{\frac{1}{4}} = -\log 3 + \frac{5}{3}
\end{aligned}$$

3. Show that $lnx \geq 1 - \frac{1}{x}$ for $x > 0$, where the equality holds when $x = 1$.

Proof

Let $f(x) = \ln x + \frac{1}{x} - 1$, and $f(1) = 0$.

The derivative of $f(x)$ is $\frac{d}{dx}f(x) = \frac{1}{x} - \frac{1}{x^2}$.

(1). When

4. *Conditioning reduces entropy.* Show that $H(Y|X) \leq H(Y)$ with equality iff X and Y are independent.

5. Show that $I(X;Y|Z) \geq 0$ with equality iff $X \to Z \to Y$.

6. *Data processing.* Let $X_1 \to X_2 \to X_3 \to ... \to X_n$ form a Markov chain, i.e.,

$$p(x_1, x_2, ..., x_n) = p(x_1)p(x_2|x_1)...p(x_n|x_{n-1})$$

Reduce $I(X_1; X_2, ..., X_n)$ to its simplest form.

7. Let X and Y be two random variables and let Z be independent of (X, Y). Show that $I(X;Y) \geq I(X; g(Y, Z))$ for any function g.

8. *Bottleneck.* Suppose that a (non-stationary) Markov chain starts in one of n states, necks down to $k < n$ states, and then fans back to $m > k$ states. Thus $X_1 \rightarrow X_2 \rightarrow X_3$, that is, $p(x_1, x_2, x_3) = p(x_1)p(x_2|x_1)p(x_3|x_2)$, for all $x_1 \in \{1, 2, ..., n\}$, $x_2 \in \{1, 2, ..., k\}$, $x_3 \in \{1, 2, ..., m\}$