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Problem 1. Find a 1-parameter family of solutions of each of the differential equations 1-16 below. Be careful to justify all steps used in obtaining a solution and to indicate intervals for which the differential equation and the solution are valid. Also try to discover particular solutions which are not members of the family of solutions.

1. $y\prime = y$.
2. $xdy - ydx = 0$.
3. $\frac{dr}{d\theta} = -\sin\theta$.
4. $(y^2+1)dx - (x^2+1)dy = 0$.
5. $\frac{dr}{d\theta}\cot\theta - r = 2$.
6. $yx^2dy - y^3dx = 2x^2dy$.
7. $(y^2-1)dx - (2y+xy)dy = 0$.
8. $x\log xdy + \sqrt{1+y^2}dx = 0$.
9. $\exp^{x+1}\tan ydx + \cos ydyd = 0$.
10. $x\cos ydx + x^2\sin ydy = a^2\sin ydy$.
11. $\frac{dr}{d\theta} = r\tan\theta$.
12. $(x-1)\cos ydy = 2x\sin ydx$.
13. $y\prime = y\log y\cot x$.
14. $xdy + (1+y^2)\arctan ydx = 0$.
15. $dy + x(y+1)dx = 0$.
16. $e^{y^2}(x^2+2x+1)dx + (xy+y)dy = 0$.

Solution: 1.

$$\frac{dy}{y} = dx$$
$$\int\frac{dy}{y} = \int dx$$
$$\log y = x + c$$
$$y = c\exp^x$$

2.

$$\frac{dy}{y} - \frac{dx}{x} = 0$$
$$\int\frac{dy}{y} = \int\frac{dx}{x}$$
$$\log y = \log x + c$$
$$y = cx$$

3.

$$dr = -\sin\theta d\theta$$
$$r = cos\theta + c$$

4.

$$\frac{dx}{x^2+1}dx - \frac{dy}{y^2+1} = 0$$
$$\arctan x = \arctan y + c$$

5.

$$\cot\theta dr + (2-r)d\theta = 0$$
$$\frac{1}{2-r}dr + \tan\theta d\theta = 0$$
$$-\ln(2-r) + \ln\sec\theta = c$$
$$r = c\sec\theta - 2$$

6.

$$
\begin{aligned}
(y-2)x^2dy - y^3dx &= 0 \\
\frac{y-2}{y^3}dy - \frac{dx}{x^2} &= 0 \\
(cx+1)y^2 &= x(y-1)
\end{aligned}
$$

7.

$$
\begin{aligned}
\frac{dx}{2+x} - \frac{y}{y^2-1}dy &= 0 \\
\frac{1}{2}\log(y^2-1) &= \log(2+x)+c \\
\sqrt{y^2-1} &= c(x+2)
\end{aligned}
$$

8.

$$
\begin{aligned}
\frac{dy}{\sqrt{1+y^2}} + \frac{dx}{x\log x} &= 0 \\
\sinh^{-1} y &= \log\log x + c
\end{aligned}
$$

9.

$$
\begin{aligned}
e^{x+1}dx + \cot y \cos y dy &= 0 \\
e^{x+1} + \cos y + \log\sin\frac{y}{2} - \log\cos\frac{y}{2} &= 0
\end{aligned}
$$

10.

$$
\begin{aligned}
x\cos y dx + (x^2-a^2)\sin y dy &= 0 \\
\frac{x}{x^2-a^2} + \tan y &= 0 \\
\frac{1}{2}\log(x^2-a^2) + c &= \log\cos y \\
\cos^2 y &= c(a^2-x^2)
\end{aligned}
$$

11.

$$
\begin{aligned}
\frac{dr}{r} &= \frac{\sin\theta}{\cos\theta}d\theta \\
\log r &= -\log|\cos\theta| + c \\
r\cos\theta &= c
\end{aligned}
$$

12.

$$
\begin{aligned}
\cot y dy &= \frac{2x}{x-1}dx \\
\log\sin y dy &= 2x + 2\log(x-1) + c \\
\sin y &= (x-1)^2e^{2x+c}
\end{aligned}
$$

13.

$$\begin{aligned}
\frac{dy}{y\log y} &= \frac{\cos x}{\sin x}dx \\
\log|\log y| &= \log|\sin x| + c \\
\log y &= c\sin x
\end{aligned}$$

14.

$$\begin{aligned}
\frac{dy}{(1+y^2)\arctan y} + \frac{dx}{x} = 0 \\
\log\arctan y &= -\log x + c \\
y &= \tan\frac{c}{x}
\end{aligned}$$

15.

$$\begin{aligned}
\frac{dy}{y+1} + xdx = 0 \\
\log y + 1 + \frac{1}{2}x^2 = c \\
y &= ce^{-\frac{1}{2}x^2} - 1
\end{aligned}$$

16.

$$\begin{aligned}
(x+1)dx + \frac{y}{e^{y^2}}dy = 0 \\
\frac{1}{2}x^2 + x - \frac{1}{2}e^{-y^2} + c = 0
\end{aligned}$$

□

Problem 2. Find a particular solution satisfying the initial condition, of each of the following differential equations 17-21. The initial condition is indicated alongside each equation.
17. $\frac{dy}{dx} + y = 0, y(1) = 1$.
18. $\sin x\cos 2ydx + \cos x\sin 2ydy = 0, y(0) = \frac{\pi}{2}$.
19. $(1-x)dy = x(y+1)dx, y(0) = 0$.
20. $ydy + xdx = 3xy^2dx, y(2) = 1$.
21. $dy = e^{x+y}dx, y(0) = 0$.
22. Define the differential of a function of three independent variables; of n independent variables.

Solution: 17.

$$\begin{aligned}
\frac{dy}{y} &= -dx \\
\log y &= -x + c \\
y &= ce^{-x}
\end{aligned}$$

From the initial condition we can obtain $c = e$, thus $y = e^{1-x}$.
18.

$$\tan x dx + \tan 2y dy = 0$$
$$\log|cosx| + \frac{1}{2}\log|cos2y| = c$$
$$\cos 2y \cos^2 x = c$$

From the initial condition we can obtain $c = 1$, thus $\cos 2y \cos^2 x = 1$.
19.

$$\frac{x}{1-x}dx = \frac{dy}{y+1}$$
$$-x - \log(1-x) = \log(y+1) + c$$

From the initial condition we can obtain $c = 0$, thus $e^{-x} = (y+1)(1-x)$.
20.

$$\frac{y}{1-3y^2}dy + xdx = 0$$
$$-\frac{1}{6}\log(1-3y^2) + \frac{1}{2}x^2 = c$$

From the initial condition we can obtain $c = 2 - \frac{1}{6-\log 2}$, thus $3y^2 = 1 + 2e^{3x^2-12}$.
21.

$$\frac{dy}{e^y} = e^x dx$$
$$-e^{-y} = e^x + c$$

From the initial condition we can obtain $c = -2$, thus $2 = e^{-y} + e^x$.

$\square$