

**Problem 1.** Compute the pagerank vector  $\pi^*$  of the graph in Figure, for  $\theta = 0.1, 0.3, 0.5, 0.85$ . What do you observe?

*Solution:* Initially we characterize the graph in matrix  $H$

$$H = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1/3 & 0 & 1/3 & 0 & 1/3 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Notice that there is a dangling node in row 5, thus we define the vector  $\mathbf{w} = (0, 0, 0, 0, 1)^T$  represents the dangling node and obtain the second matrix  $\hat{H}$  by

$$\hat{H} = H + \frac{1}{N} \mathbf{w} \mathbf{1}^T$$
$$\hat{H} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1/3 & 0 & 1/3 & 0 & 1/3 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \end{pmatrix}$$

To obtain the matrix  $G$  we substitute substitute corresponding entries by formula

$$G = \theta \hat{H} + (1 - \theta) \frac{1}{N} \mathbf{1} \mathbf{1}^T$$

(1). When  $\theta = 0.1$

$$G = \begin{pmatrix} 0.18 & 0.28 & 0.18 & 0.18 & 0.18 \\ 0.18 & 0.28 & 0.18 & 0.18 & 0.18 \\ 0.213 & 0.18 & 0.213 & 0.18 & 0.213 \\ 0.18 & 0.18 & 0.23 & 0.18 & 0.23 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \end{pmatrix}$$

Apply iterative approach to compute the ranking vector, we obtain

$$\pi^* = (0.1906 \quad 0.2256 \quad 0.1998 \quad 0.1840 \quad 0.1998)^T$$

(2). When  $\theta = 0.3$

$$G = \begin{pmatrix} 0.14 & 0.44 & 0.14 & 0.14 & 0.14 \\ 0.14 & 0.44 & 0.14 & 0.14 & 0.14 \\ 0.24 & 0.14 & 0.24 & 0.14 & 0.24 \\ 0.14 & 0.14 & 0.29 & 0.14 & 0.29 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \end{pmatrix}$$

Apply iterative approach to compute the ranking vector, we obtain

$$\pi^* = (0.1710 \quad 0.2899 \quad 0.1937 \quad 0.1516 \quad 0.1937)^T$$

(3). When  $\theta = 0.5$

$$G = \begin{pmatrix} 0.1 & 0.6 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.6 & 0.1 & 0.1 & 0.1 \\ 0.2667 & 0.1 & 0.2667 & 0.1 & 0.2667 \\ 0.1 & 0.1 & 0.35 & 0.1 & 0.35 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \end{pmatrix}$$

Apply iterative approach to compute the ranking vector, we obtain

$$\pi^* = (0.1471 \quad 0.3824 \quad 0.1765 \quad 0.1177 \quad 0.1765)^T$$

(4). When  $\theta = 0.85$

$$G = \begin{pmatrix} 0.03 & 0.88 & 0.03 & 0.03 & 0.03 \\ 0.03 & 0.88 & 0.03 & 0.03 & 0.03 \\ 0.3133 & 0.03 & 0.3133 & 0.03 & 0.3133 \\ 0.03 & 0.03 & 0.455 & 0.03 & 0.455 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \end{pmatrix}$$

Apply iterative approach to compute the ranking vector, we obtain

$$\pi^* = (0.0708 \quad 0.7030 \quad 0.0900 \quad 0.0453 \quad 0.0900)^T$$

After calculating the result based on formula and corresponding  $\theta$ , one can find that when  $\theta$  is a small scalar, the distinction is not significant among webpages with more incoming links and webpages with less incoming links. And when  $\theta$  increases, the numbers of incoming links have a much significant influence on the aggregate importance score, and therefore the distinction is more significant.  $\square$

**Problem 2.** (a) Solve for  $\mathbf{b}$  in the following least squares problem, by hand or programming in any language:

$$\text{minimize}_{\mathbf{b}} \quad \|\mathbf{A}\mathbf{b} - \mathbf{c}\|_2^2$$

where

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 0 \\ 0 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{c} = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 3 \end{pmatrix}$$

(b) Solve the above least squares problem again with regularization. Vary the regularization parameter  $\lambda$  for  $\lambda = 0, 0.2, 0.4, \dots, 5.0$ , and plot both  $\|\mathbf{A}\mathbf{b} - \mathbf{c}\|_2^2$  and  $\|\mathbf{b}\|_2^2$  against  $\lambda$ .

*Solution:* (a) The objective function can be transformed by following

$$\begin{aligned}\|\mathbf{A}\mathbf{b} - \mathbf{c}\|_2^2 &= (\mathbf{A}\mathbf{b} - \mathbf{c})^T(\mathbf{A}\mathbf{b} - \mathbf{c}) \\ &= \mathbf{b}^T \mathbf{A}^T \mathbf{A} \mathbf{b} - 2\mathbf{b}^T \mathbf{A}^T \mathbf{c} + \mathbf{c}^T \mathbf{c}\end{aligned}$$

Take derivative with respect to  $\mathbf{b}$  and set

$$\begin{aligned}2(\mathbf{A}^T \mathbf{A})\mathbf{b} - 2\mathbf{A}^T \mathbf{c} &= 0 \\ (\mathbf{A}^T \mathbf{A})\mathbf{b} &= \mathbf{A}^T \mathbf{c} \\ \begin{pmatrix} 6 & 3 & 4 \\ 3 & 6 & 3 \\ 4 & 3 & 6 \end{pmatrix} \mathbf{b} &= \begin{pmatrix} 9 \\ 6 \\ 8 \end{pmatrix}\end{aligned}$$

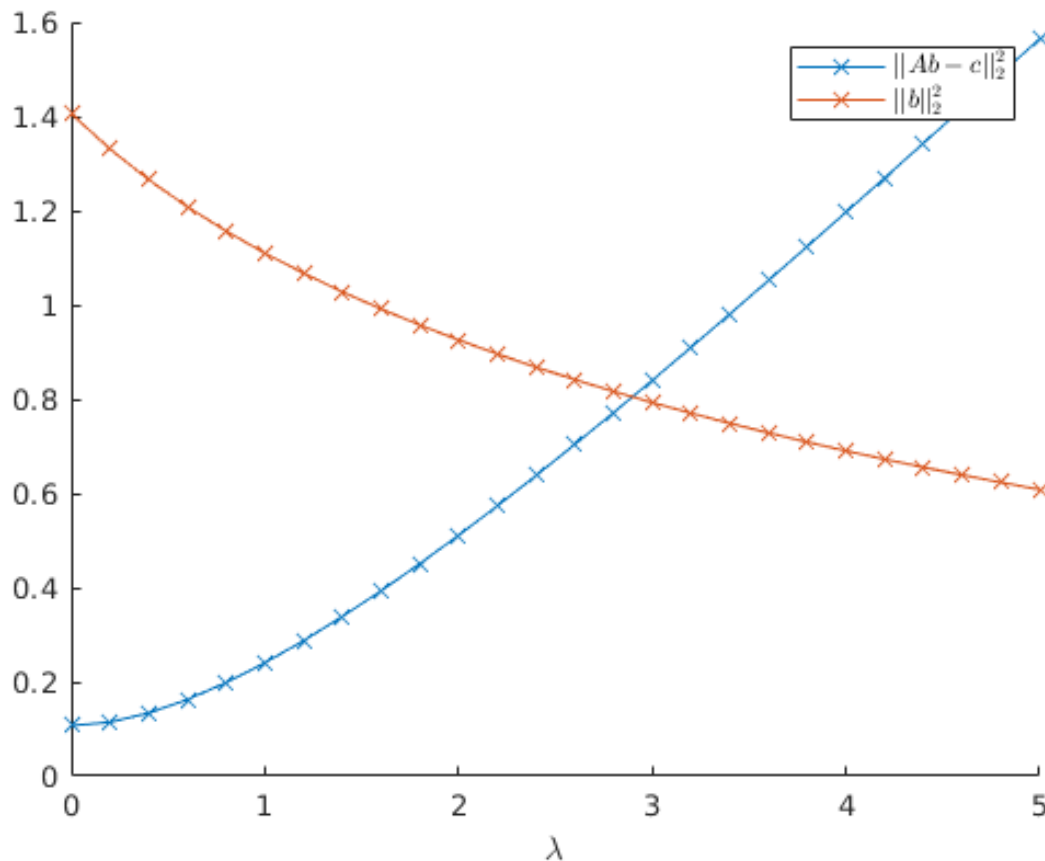
By solving the linear system we can obtain

$$\mathbf{b} = \left(\frac{29}{28} \quad \frac{3}{14} \quad \frac{15}{28}\right)^T$$

(b). Take the derivative of  $\|\mathbf{A}\mathbf{b} - \mathbf{c}\|_2^2 + \lambda\|\mathbf{b}\|_2^2$  respect to  $\mathbf{b}$  and set it to zero

$$\begin{aligned}2\mathbf{A}^T(\mathbf{A}\mathbf{b} - \mathbf{c}) + 2\lambda\mathbf{b} &= 0 \\ (\mathbf{A}^T \mathbf{A} + \lambda)\mathbf{b} &= \mathbf{A}^T \mathbf{c}\end{aligned}$$

By solving the linear system and plot using computer software we obtain



**Problem 3.** Consider 3 people making dependent estimates of a number, with the following expectations of errors and correlations of errors:

$$\begin{aligned}E[\epsilon_1^2] &= 1773 \\E[\epsilon_2^2] &= 645 \\E[\epsilon_3^2] &= 1796 \\E[\epsilon_1\epsilon_2] &= 1057 \\E[\epsilon_1\epsilon_3] &= 970 \\E[\epsilon_2\epsilon_3] &= 708\end{aligned}$$

Compute the average of errors and the error of the average in this case.

*Solution:* The average of errors can be given by

$$\begin{aligned}E_{AE} &= \frac{1}{N} \sum_{i=1}^N E_x[\epsilon_i^2(x)] \\&= \frac{1}{3}(1773 + 645 + 1796) \\&= \frac{4214}{3}\end{aligned}$$

The error of the average can be given by

$$\begin{aligned}E_{EA} &= \frac{1}{N^2} E_x[(\sum_{i=1}^N \epsilon_i(x))^2] \\&= \frac{1}{9}(1773 + 645 + 1796 + 2 \times 1057 + 2 \times 970 + 2 \times 708) \\&= 1076\end{aligned}$$

□