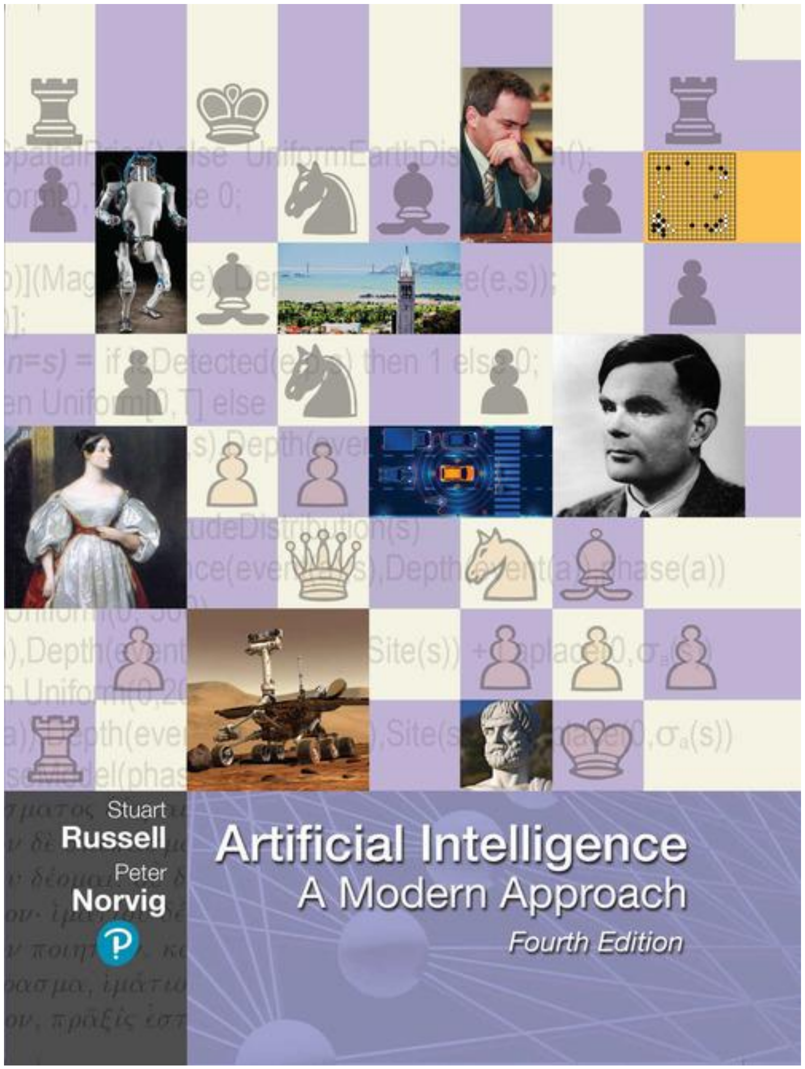
Artificial Intelligence Fundamentals

2023-2024



Probability, too, if regarded as something endowed with some kind of objective existence, is no less a misleading misconception, an illusory attempt to exteriorize or materialize our true probabilistic beliefs.

- Bruno de Finetti

AIMA Chapter 12

Quantifying Uncertainty

Outline

- Acting Under Uncertainty
- Basic Probability Notation
- Inference Using Full Joint Distributions
- Independence
- Bayes' Rule and Its Use
- Naive Bayes Models
- The Wumpus World Revisited

Acting Under Uncertainty

Real world problems contain **uncertainties due to:**

- partial observability,
- nondeterminism, or
- adversaries.

Example of dental diagnosis using propositional logic:

$$Toothache \Rightarrow Cavity.$$

However **inaccurate**, not all patients with toothaches have cavities

$$Toothache \Rightarrow Cavity \lor GumProblem \lor Abscess\ldots$$

In order to make the rule true, we have to add an almost **unlimited list** of possible problems.

The only way to fix the rule is to make it **logically exhaustive**

Acting Under Uncertainty

An agent strives to choose the right thing to do—the rational decision—depends on both the **relative importance** of various goals and the **likelihood** that, and degree to which, they will be achieved.

Large domains such as *medical diagnosis* fail to three main reasons:

- **Laziness:** It is too much work to list the complete set of antecedents or consequents needed to ensure an exceptionless rule
- **Theoretical ignorance:** Medical science has no complete theory for the domain
- **Practical ignorance:** Even if we know all the rules, we might be uncertain about a particular patient because not all the necessary tests have been or can be run.

An agent only has a **degree of belief** in the relevant sentences.

Further Reading: Bruno De Finetti, "Sul Significato Soggettivo della Probabilità", 1930

Acting Under Uncertainty

Probability theory

- tool to deal with degrees of belief of relevant sentences.
- summarizes the uncertainty that comes from our *laziness* and *ignorance*

Uncertainty and rational decisions

- An AI agent requires **preference** among **different possible outcomes** of various plans
- **Utility Theory**: the quality of the outcome being useful
  - Every state has a degree of usefulness/utility
  - Higher utility is preferred
- **Decision Theory:** Preferences (Utility Theory) combined with probabilities
  - *Decision theory = probability theory + utility theory.*
  - agent is **rational** *if and only if* it chooses the action that **yields the highest expected utility**, averaged over all the **possible outcomes** of the action.
  - principle of *maximum expected utility* (MEU).

Acting Under Uncertainty

Function of a **decision-theoretic agent** that selects rational actions:

function DT-AGENT(*percept*) **returns** an *action*
persistent: *belief state*, probabilistic beliefs about the current state of the world
        action, the agent's action
update *belief state* based on *action* and *percept*
calculate outcome probabilities for actions,
        given action descriptions and current *belief state*
select *action* with highest expected utility
        given probabilities of outcomes and utility information
return *action*

Basic Probability Notation

For our agent to represent and use probabilistic information, we need a *formal language*.

Sample space: the set of all possible worlds

- The possible worlds are *mutually exclusive* and *exhaustive*

A fully specified probability model associates a numerical probability $P(\omega)$ with each possible world.

The **basic axioms** of probability theory say that every possible world has a probability between 0 and 1 and that the **total probability** of the set of possible worlds is 1:

$$0 \leq P(\omega) \leq 1 \text{ for every } \omega \text{ and } \sum_{\omega \in \Omega} P(\omega) = 1 .$$

Unconditional or prior probability: degrees of belief in propositions in the absence of any other information

Other basic axioms

1

$$\begin{aligned} P(\neg a) &= \textstyle\sum_{\omega \in \neg a} P(\omega) && \text{by Equation (12.2)} \\ &= \textstyle\sum_{\omega \in \neg a} P(\omega) + \sum_{\omega \in a} P(\omega) - \sum_{\omega \in a} P(\omega) \\ &= \textstyle\sum_{\omega \in \Omega} P(\omega) - \sum_{\omega \in a} P(\omega) && \text{grouping the first two terms} \\ &= 1 - P(a) && \text{by (12.1) and (12.2).} \end{aligned}$$

2

$$P(a \vee b) = P(a) + P(b) - P(a \wedge b)\,.$$

Basic Probability Notation

Conditional or posterior probability: given evidence that has happened, degree of belief of new event

- Make use of unconditional probabilities

Probability of a given b:

$$P(a|b) = \frac{P(a \wedge b)}{P(b)}$$

Can also written as:

$$P(a \wedge b) = P(a \mid b)\, P(b)\ .$$

- Example of rolling fair dice, rolling doubles when the first dice is 5

$$P(doubles|Die_1 = 5) = \frac{P(doubles \wedge Die_1 = 5)}{P(Die_1 = 5)}\ .$$

Basic Probability Notation

Factored representation: possible world is represented by a set of variable/value pairs.

- Variables in probability theory are called **random variables**, and their names begin with an uppercase letter. ($Total$ and Die_1)

Sometimes we will want to talk about the probabilities of all the possible values of a random variable. We could write:

$$P(Weather = sun) = 0.6$$
$$P(Weather = rain) = 0.1$$
$$P(Weather = cloud) = 0.29$$
$$P(Weather = snow) = 0.01\,,$$

Abbreviation of this will be:

$$\boldsymbol{P}(Weather) = (0.6, 0.1, 0.29, 0.01),$$

P statement defines a **probability distribution** for the random variable *Weather*

Inference Using Full Joint Distributions

Start with the joint distribution:

	toothache		*toothache*	
	catch	*¬catch*	*catch*	*¬catch*
cavity	**.108**	**.012**	**.072**	**.008**
¬cavity	**.016**	**.064**	**.144**	**.576**

For any proposition φ, sum the atomic events where it is true:

$P(\varphi) = \Sigma_{\omega:\omega \models \varphi} P(\omega)$

marginalization

Inference Using Full Joint Distributions

Start with the joint distribution:

	toothache		*¬toothache*	
	catch	*¬catch*	*catch*	*¬catch*
cavity	**.108**	**.012**	**.072**	**.008**
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For any proposition φ, sum the atomic events where it is true:

$$P(\varphi) = \Sigma_{\omega:\omega \models \varphi} P(\omega)$$

$$P(toothache) = 0.108 + 0.012 + 0.016 + 0.064 = 0.2$$

Inference Using Full Joint Distributions

Start with the joint distribution:

	toothache		*¬toothache*	
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cavity	**.108**	**.012**	**.072**	**.008**
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For any proposition φ, sum the atomic events where it is true:

$$P(\varphi) = \Sigma_{\omega:\omega \models \varphi} P(\omega)$$

$$P(cavity \vee toothache) = 0.108 + 0.012 + 0.072 + 0.008 + 0.016 + 0.064 = 0.28$$

Inference Using Full Joint Distributions

Start with the joint distribution:

	toothache		*¬toothache*	
	catch	*¬catch*	*catch*	*¬catch*
cavity	**.108**	**.012**	**.072**	**.008**
¬cavity	**.016**	**.064**	**.144**	**.576**

Can also compute conditional probabilities:

$$P(\neg cavity \mid toothache) = \frac{P(\neg cavity \wedge toothache)}{P(toothache)}$$

$$= \frac{0.016 + 0.064}{0.108 + 0.012 + 0.016 + 0.064} = 0.4$$

Normalization

	toothache		*¬toothache*	
	catch	*¬catch*	*catch*	*¬catch*
cavity	**.108**	**.012**	**.072**	**.008**
¬cavity	**.016**	**.064**	**.144**	**.576**

Denominator can be viewed as a normalization constant α

$$
\begin{aligned}
\mathbf{P}(Cavity \mid toothache) &= \alpha\, \mathbf{P}(Cavity, toothache) \\
&= \alpha\, [\mathbf{P}(Cavity, toothache, catch) + \mathbf{P}(Cavity, toothache, \neg catch)] \\
&= \alpha\, [\langle 0.108, 0.016 \rangle + \langle 0.012, 0.064 \rangle] \\
&= \alpha\, \langle 0.12, 0.08 \rangle = \langle 0.6, 0.4 \rangle
\end{aligned}
$$

General idea: compute distribution on query variable by fixing evidence variables and summing over hidden variables

Inference Using Full Joint Distributions

Let X be all the variables. Typically, we want
the posterior joint distribution of the query variables Y
given specific values e for the evidence variables E

Let the hidden variables be $\mathrm{H} = \mathrm{X} - \mathrm{Y} - \mathrm{E}$

Then the required summation of joint entries is done by summing out the hidden variables:

$$\mathrm{P}(\mathrm{Y} \,|\, \mathrm{E} = \mathrm{e}) = \alpha \mathrm{P}(\mathrm{Y}, \mathrm{E} = \mathrm{e}) = \alpha \Sigma_{\mathrm{h}} \mathrm{P}(\mathrm{Y}, \mathrm{E} = \mathrm{e}, \mathrm{H} = \mathrm{h})$$

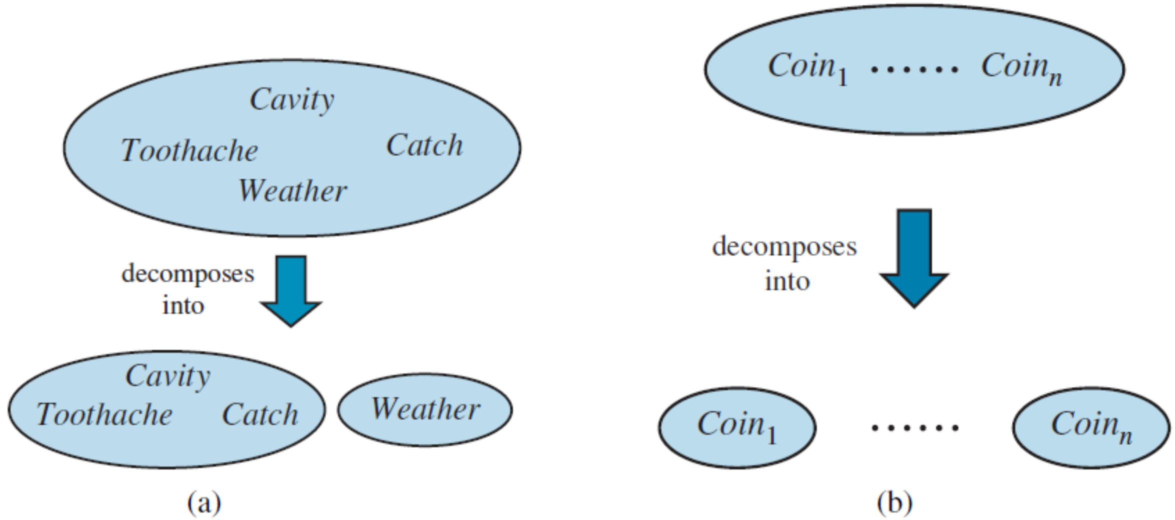
The terms in the summation are joint entries because Y, E, and H together exhaust the set of random variables

Obvious problems (n variables, d domain space, e.g., 2 if boolean):

1) Worst-case time complexity $O(d^n)$ where d is the largest arity
2) Space complexity $O(d^n)$ to store the joint distribution
3) How to *find the numbers* for $O(d^n)$ entries? (n=100, d=2 means 10^30 probabilities)

Independence

Two examples of factoring a large joint distribution into smaller distributions, using absolute independence. (a) **Weather** and **dental** problems are independent. (b) **Coin flips** are independent.



$$P(a \mid b) = P(a) \quad \text{or} \quad P(b \mid a) = P(b) \quad \text{or} \quad P(a \wedge b) = P(a)P(b)\ .$$

probability of a cloudy day + toothache:

$$P(toothache, catch, cavity, cloud) = P(cloud \mid toothache, catch\ cavity)\, P(toothache, catch, cavity)\ .$$

$$P(cloud \mid toothache, catch, cavity) = P(cloud)\ .$$

$$P(toothache, catch, cavity, cloud) = P(cloud)P(toothache, catch, cavity)$$

Pearson

Bayes' Rule and Its Use

Bayes’ rule is derived from the product rule

$P(a \wedge b) = P(a \mid b)P(b)$ and $P(a \wedge b) = P(b \mid a)P(a)$.

Equating the two right-hand sides and dividing by $P(a)$, we get

$$P(b|a) = \frac{P(a|b)P(b)}{P(a)} .$$

Often, we perceive as *evidence the effect of some unknown cause* and we would like to determine that cause. In that case, Bayes’ rule becomes

$$P(cause|effect) = \frac{P(effect\,|cause)P(cause)}{P(effect)}$$

The conditional probability $P(effect|cause)$ quantifies the relationship in the **causal** direction, whereas $P(cause|effect)$ describes the **diagnostic** direction.

Bayes' Rule and Its Use

For example, a doctor knows that the disease meningitis causes a patient to have a stiff neck, say, 70% of the time. The doctor also knows some unconditional facts: the prior probability that any patient has meningitis is 1/50000, and the prior probability that any patient has a stiff neck is 1%. Letting s be the proposition that the patient has a stiff neck and m be the proposition that the patient has meningitis, we have

$$P(s|m) = 0.7$$
$$P(m) = 1/50000$$
$$P(s) = 0.01$$

$$P(m|s) = \frac{P(s|m)P(m)}{P(s)} = \frac{0.7 \times 1/50000}{0.01} = 0.0014$$

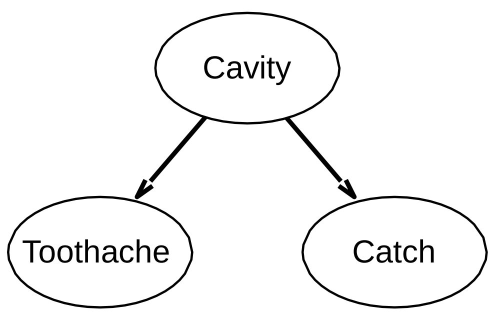
That is, we expect only 0.14% of patients with a stiff neck to have meningitis. Notice that even though a stiff neck is quite strongly indicated by meningitis (with probability 0.7), the probability of meningitis in patients with stiff necks remains small. This is because the prior probability of stiff necks (from any cause) is much higher than the prior for meningitis.

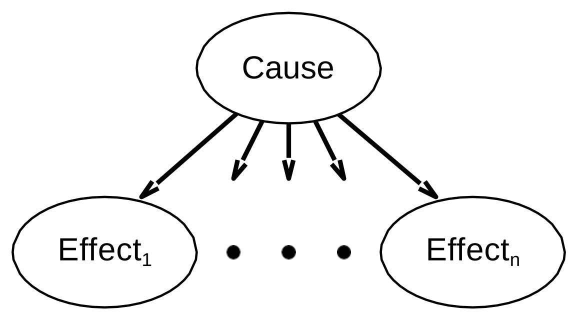
Bayes' Rule and conditional independence

$$\mathbf{P}(Cavity \mid toothache \wedge catch)$$
$$= \alpha \mathbf{P}(toothache \wedge catch \mid Cavity)\mathbf{P}(Cavity)$$
$$= \alpha \mathbf{P}(toothache \mid Cavity)\mathbf{P}(catch \mid Cavity)\mathbf{P}(Cavity)$$

This is an example of a naive Bayes model:

$$\mathbf{P}(Cause, Effect_1, \ldots, Effect_n) = \mathbf{P}(Cause)\,\Pi_i \mathbf{P}(Effect_i \mid Cause)$$





Total number of parameters is linear in n

Naïve Bayes Models

The full joint distribution can be written as

$$\mathbf{P}(Cause, Effect_1, \ldots, Effect_n) = \mathbf{P}(Cause) \prod_i \mathbf{P}(Effect_i \mid Cause)$$

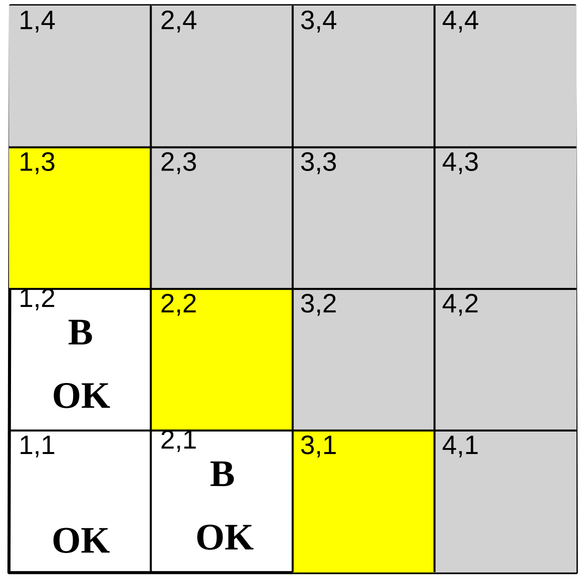
Such a probability distribution is called a naive Bayes model—"naive" because it is often used (as a **simplifying assumption**) in cases where the "effect" variables are not strictly independent given the cause variable (*still useful in practice*).

Call the observed effects **E**=**e**, while the remaining effect variables **Y** are unobserved

$$\begin{aligned}
\mathbf{P}(Cause \mid \mathbf{e}) &= \alpha \sum_{\mathbf{y}} \mathbf{P}(Cause)\mathbf{P}(\mathbf{y} \mid Cause)\left(\prod_j \mathbf{P}(e_j \mid Cause)\right) \\
&= \alpha\, \mathbf{P}(Cause)\left(\prod_j \mathbf{P}(e_j \mid Cause)\right)\sum_{\mathbf{y}} \mathbf{P}(\mathbf{y} \mid Cause) \\
&= \alpha\, \mathbf{P}(Cause) \prod_j \mathbf{P}(e_j \mid Cause)
\end{aligned}$$

because the summation over y is 1

The Wumpus World Revisited



$P_{ij} = true$ iff $[i, j]$ contains a pit

$B_{ij} = true$ iff $[i, j]$ is breezy

Include only $B_{1,1}, B_{1,2}, B_{2,1}$ in the probability model

Specifying the probability model

The full joint distribution is $\mathbf{P}(P_{1,1},\ldots,P_{4,4},B_{1,1},B_{1,2},B_{2,1})$

Apply product rule: $\mathbf{P}(B_{1,1},B_{1,2},B_{2,1} \mid P_{1,1},\ldots,P_{4,4})\mathbf{P}(P_{1,1},\ldots,P_{4,4})$

(Do it this way to get $P(Effect|Cause)$.)

First term: 1 if pits are adjacent to breezes, 0 otherwise

Second term (prior): pits are placed randomly, probability 0.2 per square

$$\mathbf{P}(P_{1,1},\ldots,P_{4,4}) = \Pi_{i,j=1,1}^{4,4}\mathbf{P}(P_{i,j}) = 0.2^{n} \times 0.8^{16-n}$$

for n pits.

Observations and query

We know the following facts:

$b = \neg b_{1,1} \wedge b_{1,2} \wedge b_{2,1}$

$known = \neg p_{1,1} \wedge \neg p_{1,2} \wedge \neg p_{2,1}$

Query is $\mathbf{P}(P_{1,3} | known, b)$

Define $Unknown$ = P_{ij}s other than $P_{1,3}$ and $Known$

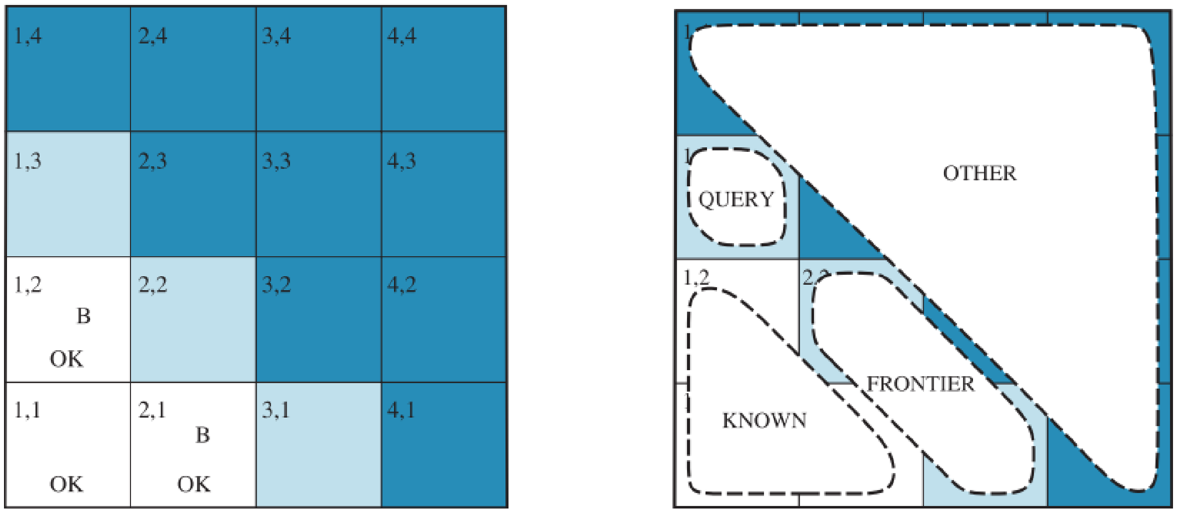
For *inference by enumeration*, we have

$$\mathbf{P}(P_{1,3} | known, b) = \alpha \Sigma_{unknown} \mathbf{P}(P_{1,3}, unknown, known, b)$$

Grows exponentially with number of squares!

Using conditional independence

Basic insight: observations are **conditionally independent** of other *hidden squares* given neighbouring hidden squares



(a) (b)

Define $Unknown = Fringe \cup Other$

$\mathbf{P}(b|P_{1,3}, Known, Unknown) = \mathbf{P}(b|P_{1,3}, Known, Fringe)$

Manipulate query into a form where we can use this!

Using conditional independence contd.

$$
\begin{aligned}
&\mathbf{P}(P_{1,3} \mid known, b) \\
&= \alpha \sum_{unknown} \mathbf{P}(P_{1,3}, known, b, unknown) \qquad \text{(from Equation (12.23))} \\
&= \alpha \sum_{unknown} \mathbf{P}(b \mid P_{1,3}, known, unknown)\mathbf{P}(P_{1,3}, known, unknown) \quad \text{(product rule)} \\
&= \alpha \sum_{frontier} \sum_{other} \mathbf{P}(b \mid known, P_{1,3}, frontier, other)\mathbf{P}(P_{1,3}, known, frontier, other) \\
&= \alpha \sum_{frontier} \sum_{other} \mathbf{P}(b \mid known, P_{1,3}, frontier)\mathbf{P}(P_{1,3}, known, frontier, other),
\end{aligned}
$$

By **independence**, the term on the right can be factored, and then the terms can be reordered:

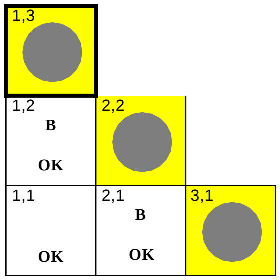
$$
\begin{aligned}
&\mathbf{P}(P_{1,3} \mid known, b) \\
&= \alpha \sum_{frontier} \mathbf{P}(b \mid known, P_{1,3}, frontier) \sum_{other} \mathbf{P}(P_{1,3})P(known)P(frontier)P(other) \\
&= \alpha P(known)\mathbf{P}(P_{1,3}) \sum_{frontier} \mathbf{P}(b \mid known, P_{1,3}, frontier)P(frontier) \sum_{other} P(other) \\
&= \alpha' \mathbf{P}(P_{1,3}) \sum_{frontier} \mathbf{P}(b \mid known, P_{1,3}, frontier)P(frontier),
\end{aligned}
$$

Absorbed in α’

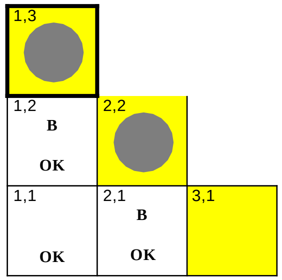
Sums to 1

are 1 when the breeze observations are consistent with the other variables and 0 otherwise

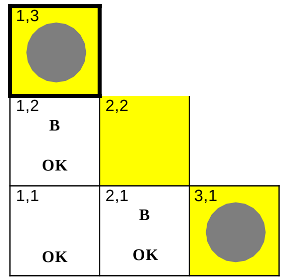
Using conditional independence contd.



0.2 x 0.2 = 0.04



0.2 x 0.8 = 0.16



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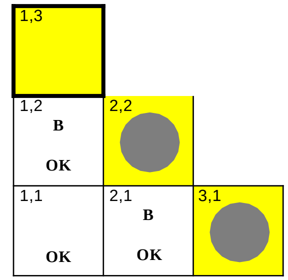
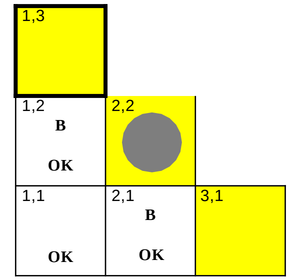


0.2 x 0.2 = 0.04



0.2 x 0.8 = 0.16

$$\alpha' \mathbf{P}(P_{1,3}) \sum \mathbf{P}(b \mid known, P_{1,3}, frontier) P(frontier)$$

$$\begin{aligned} \mathbf{P}(P_{1,3} \mid known, b) &= \alpha\, \langle 0.2(0.04 + 0.16 + 0.16),\ 0.8(0.04 + 0.16) \rangle \\ &\approx \langle 0.31, 0.69 \rangle \end{aligned}$$

$$\mathbf{P}(P_{2,2} \mid known, b) \approx \langle 0.86, 0.14 \rangle$$

AIMA-python - “Probability.ipynb”

6536 lines (6536 sloc) | 397 KB

Raw Blame

Probability

This IPy notebook acts as supporting material for topics covered in **Chapter 13 Quantifying Uncertainty**, **Chapter 14 Probabilistic Reasoning**, **Chapter 15 Probabilistic Reasoning over Time**, **Chapter 16 Making Simple Decisions** and parts of **Chapter 25 Robotics** of the book* Artificial Intelligence: A Modern Approach*. This notebook makes use of the implementations in probability.py module. Let us import everything from the probability module. It might be helpful to view the source of some of our implementations. Please refer to the Introductory IPy file for more details on how to do so.

In [1]:
```
from probability import *
from utils import print_table
from notebook import psource, pseudocode, heatmap
```

CONTENTS

- Probability Distribution
  - Joint probability distribution
  - Inference using full joint distributions

- Bayesian Networks - BayesNode - BayesNet - Exact Inference in Bayesian Networks - Enumeration - Variable elimination - Approximate Inference in Bayesian Networks - Prior sample - Rejection sampling - Likelihood weighting - Gibbs sampling
- Hidden Markov Models - Inference in Hidden Markov Models - Forward-backward - Fixed lag smoothing - Particle filtering

- Monte Carlo Localization - Decision Theoretic Agent - Information Gathering Agent

[aima-python/probability.ipynb at master · aimacode/aima-python (github.com)](https://github.com/aimacode/aima-python)

Summary

- **Probabilities** express the agent's inability to reach a definite decision regarding the truth of a sentence.
- **Decision theory** combines the agent's beliefs and desires, defining the best action as the one that maximizes expected utility.
- Basic probability statements include **prior or unconditional probabilities** and **posterior or conditional probabilities** over simple and complex propositions.
- The axioms of probability constrain the probabilities of logically related propositions.
- The **full joint probability distribution** specifies the probability of each complete assignment of values to random variables
- **Absolute independence** between subsets of random variables allows the full joint distribution to be factored into smaller joint distributions, greatly reducing its complexity.
- **Bayes' rule** allows unknown probabilities to be computed from known conditional probabilities, usually in the causal direction.
- **Conditional independence** brought about by direct causal relationships in the domain allows the full joint distribution to be factored into smaller, conditional distributions.

In the next lecture...

- Representing Knowledge in an Uncertain Domain
- Semantics of Bayesian Networks
- Exact Inference in Bayesian Networks
- Approximate Inference for Bayesian Networks
- Causal Networks