IMECC-UNICAMP
Atividade 1 - Métodos computacionais
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1 Distribuições condicionais completas e implementação Gibbs

Considere $\mathbf{p} = [p_1, ..., p_{14}]^\top$, $\mathbf{x} = [n_1, ..., n_{14}, m_1, ..., m_{14}]^\top$ e $\mathbb{B} = \{0, 1, 2, ..., N\}$, temos que a verosimilhança e as distribuições a priori são dadas por

$$\mathcal{L}(N, \mathbf{p}|\mathbf{x}) \propto \frac{N!}{(N-r)!} \prod_{i=1}^{14} p_i^{n_i}(1-p_i)^{N-n_i} \mathbb{I}(N \in \mathbb{N})\mathbb{I}(p_i \in [0,1])\mathbb{I}(r \in \mathbb{B}) \tag{1}$$

$$\pi(N) = \frac{e^{-\lambda}\lambda^N}{N!}\mathbb{I}(N \in \mathbb{N}) \tag{2}$$

$$\pi(p_i) = \mathbb{I}(p_i \in [0,1]), i = 1, 2, ..., 14. \tag{3}$$

A seguir vamos derivar a distribuição condicional completa de $N|\mathbf{p}, \mathbf{x}$. Note que $\mathbb{I}(N \in \mathbb{N})\mathbb{I}(p_i \in [0,1])\mathbb{I}(r \in \mathbb{B}) = \mathbb{I}(N \in r, r+1, ...)\mathbb{I}(p_i \in [0,1])$, pois $\mathbb{I}(r \in \mathbb{B}) = \mathbb{I}(N \in \{r, r+1, ...\})$.

$$\begin{aligned}
\pi(N|\mathbf{p}, \mathbf{x}) &\propto \mathcal{L}(N, \mathbf{p}|\mathbf{x})\pi(N) \\
&= \frac{N!}{(N-r)!} \prod_{i=1}^{14} p_i^{n_i}(1-p_i)^{N-n_i} \mathbb{I}(N \in \{r, r+1, ...\})\mathbb{I}(p_i \in [0,1]) \frac{e^{-\lambda}\lambda^N}{N!}\mathbb{I}(N \in \mathbb{N}) \\
&= \frac{e^{-\lambda}\lambda^N}{(N-r)!} \prod_{i=1}^{14} p_i^{n_i}(1-p_i)^{N}(1-p_i)^{-n_i} \mathbb{I}(N \in \{r, r+1, ...\})\mathbb{I}(p_i \in [0,1]) \\
&= \frac{e^{-\lambda}\lambda^N}{(N-r)!} \prod_{i=1}^{14} \left[p_i^{n_i}\right] \prod_{i=1}^{14} \left[(1-p_i)^{N}\right] \prod_{i=1}^{14} \left[(1-p_i)^{-n_i}\right] \mathbb{I}(N \in \{r, r+1, ...\})\mathbb{I}(p_i \in [0,1]) \\
&\propto \frac{e^{-\lambda}\lambda^N}{(N-r)!} \left[\prod_{i=1}^{14}(1-p_i)\right]^N \mathbb{I}(N \in \{r, r+1, ...\})\mathbb{I}(p_i \in [0,1]) \\
&= \frac{e^{-\lambda}\left[\lambda\prod_{i=1}^{14}(1-p_i)\right]^N}{(N-r)!} \mathbb{I}(N \in \{r, r+1, ...\})\mathbb{I}(p_i \in [0,1]) \frac{e^{-\lambda\prod_{i=1}^{14}(1-p_i)}}{e^{-\lambda\prod_{i=1}^{14}(1-p_i)}} \\
&\propto \frac{\exp\{-\lambda\prod_{i=1}^{14}(1-p_i)\}\left[\lambda\prod_{i=1}^{14}(1-p_i)\right]^N}{(N-r)!} \mathbb{I}(N \in \{r, r+1, ...\})\mathbb{I}(p_i \in [0,1]) \\
&\propto \frac{\exp\{-\lambda\prod_{i=1}^{14}(1-p_i)\}\left[\lambda\prod_{i=1}^{14}(1-p_i)\right]^{N-r}}{(N-r)!} \mathbb{I}(N \in \{r, r+1, ...\})\mathbb{I}(p_i \in [0,1]) \\
&= \frac{\exp\{-\lambda\prod_{i=1}^{14}(1-p_i)\}\left[\lambda\prod_{i=1}^{14}(1-p_i)\right]^{N-r}}{(N-r)!} \mathbb{I}(N - r \in \mathbb{N})\mathbb{I}(p_i \in [0,1])
\end{aligned}$$

Portanto

$$\pi(N|\mathbf{p}, \mathbf{x}) \propto \frac{\exp\{-\lambda\prod_{i=1}^{14}(1-p_i)\}\left[\lambda\prod_{i=1}^{14}(1-p_i)\right]^{N-r}}{(N-r)!} \mathbb{I}(N - r \in \mathbb{N})\mathbb{I}(p_i \in [0,1]) \tag{4}$$

Temos que $(N-r)|(\mathbf{p}, \mathbf{x}) \sim Poisson(\lambda\prod_{i=1}^{14}(1-p_i))$, identificado pelo *kernel* apresentado em 4.

Agora vamos derivar a distribuição condicional completa de $p_i|N, \mathbf{x}, i = 1, 2, ..., 14$. Considere $\mathbf{p_{(i)}} = (p_1, p_2, ..., p_{i-1}, p_{i+1}, ..., p_{14}$, isto é, o vetor de parâmetros $\mathbf{p}$ sem o i-ésimo elemento.

$$
\begin{aligned}
\pi(p_i|N, \mathbf{p_{(i)}}, \mathbf{x}) &\propto \mathcal{L}(N, \mathbf{p}|\mathbf{x})\pi(\mathbf{p_{(i)}}) \\
&= \frac{N!}{(N-r)!}\prod_{l=1}^{14} p_i^{n_l}(1-p_l)^{N-n_l}\mathbb{I}(N \in \{r, r+1, ...\})\mathbb{I}(p_l \in [0,1]) \left[\prod_{j \neq i} \mathbb{I}(p_j \in [0,1])\right] \\
&= \frac{N!}{(N-r)!}\prod_{l=1}^{14} p_l^{n_l}(1-p_l)^{N-n_l}\mathbb{I}(N \in \{r, r+1, ...\})\mathbb{I}(p_l \in [0,1]) \\
&\propto p_i^{n_i}(1-p_i)^{N-n_i}\mathbb{I}(p_i \in [0,1]) \\
&= p_i^{(n_i-1)+1}(1-p_i)^{(N-n_i-1)+1}\mathbb{I}(p_i \in [0,1])
\end{aligned}
$$

Podemos então identificar que $p_i|N, \mathbf{x} \sim \text{Beta}(n_i + 1, N - n_i + 1), i = 1, 2, ..., 14$ pelo kernel que apresentado acima. Chegamos a conclusão que

$$(N-r)|(\mathbf{p}, \mathbf{x}) \sim Poisson(\lambda \prod_{i=1}^{14}(1-p_i)) \tag{5}$$

$$p_i|N, \mathbf{x} \sim Beta(n_i + 1, N - n_i + 1), i = 1, 2, ..., 14 \tag{6}$$

1.1 Implementação Gibbs

2 Implementação *Hamiltonian* Monte Carlo

3 Implementação *Hamiltonian* Monte Carlo no STAN