IMECC-UNICAMP
Atividade 1 - Métodos computacionais
Professor: Dr. Guilherme Ludwig
Vitor Macedo Rocha RA: 216394

1 Distribuições condicionais completas e implementação Gibbs

Considere $\mathbf{p} = [p_1, ..., p_{14}]^\top$, $\mathbf{x} = [n_1, ..., n_{14}, m_1, ..., m_{14}]^\top$ e $\mathbb{B} = \{0, 1, 2, ..., N\}$, temos que a verosimilhança e as distribuições a priori são dadas por

$$\mathcal{L}(N, \mathbf{p}|\mathbf{x}) \propto \frac{N!}{(N-r)!} \prod_{i=1}^{14} p_i^{n_i}(1-p_i)^{N-n_i} \mathbb{I}(N \in \mathbb{N})\mathbb{I}(p_i \in [0,1])\mathbb{I}(r \in \mathbb{B}) \tag{1}$$

$$\pi(N) = \frac{e^{-\lambda}\lambda^N}{N!}\mathbb{I}(N \in \mathbb{N}) \tag{2}$$

$$\pi(p_i) = \mathbb{I}(p_i \in [0,1]), i = 1, 2, ..., 14. \tag{3}$$

A seguir vamos derivar a distribuição condicional de $N|\mathbf{p}, \mathbf{x}$. Note que $\mathbb{I}(N \in \mathbb{N})\mathbb{I}(p_i \in [0,1])\mathbb{I}(r \in \mathbb{B}) = \mathbb{I}(N \in r, r+1, ...)\mathbb{I}(p_i \in [0,1])$, pois $\mathbb{I}(r \in \mathbb{B}) = \mathbb{I}(N \in \{r, r+1, ...\})$.

$$\begin{aligned}
\pi(N|\mathbf{p}, \mathbf{x}) &\propto \mathcal{L}(N, \mathbf{p}|\mathbf{x})\pi(N) \\
&= \frac{N!}{(N-r)!} \prod_{i=1}^{14} p_i^{n_i}(1-p_i)^{N-n_i} \mathbb{I}(N \in \{r, r+1, ...\})\mathbb{I}(p_i \in [0,1]) \frac{e^{-\lambda}\lambda^N}{N!}\mathbb{I}(N \in \mathbb{N}) \\
&= \frac{e^{-\lambda}\lambda^N}{(N-r)!} \prod_{i=1}^{14} p_i^{n_i}(1-p_i)^{N}(1-p_i)^{-n_i} \mathbb{I}(N \in \{r, r+1, ...\})\mathbb{I}(p_i \in [0,1]) \\
&= \frac{e^{-\lambda}\lambda^N}{(N-r)!} \prod_{i=1}^{14} [p_i^{n_i}] \prod_{i=1}^{14} \left[(1-p_i)^{N}\right] \prod_{i=1}^{14} \left[(1-p_i)^{-n_i}\right] \mathbb{I}(N \in \{r, r+1, ...\})\mathbb{I}(p_i \in [0,1]) \\
&\propto \frac{e^{-\lambda}\lambda^N}{(N-r)!} \left[\prod_{i=1}^{14}(1-p_i)\right]^N \mathbb{I}(N \in \{r, r+1, ...\})\mathbb{I}(p_i \in [0,1]) \\
&= \frac{e^{-\lambda}\left[\lambda \prod_{i=1}^{14}(1-p_i)\right]^N}{(N-r)!} \mathbb{I}(N \in \{r, r+1, ...\})\mathbb{I}(p_i \in [0,1]) \frac{e^{-\lambda \prod_{i=1}^{14}(1-p_i)}}{e^{-\lambda \prod_{i=1}^{14}(1-p_i)}} \\
&\propto \frac{\exp\{-\lambda \prod_{i=1}^{14}(1-p_i)\}\left[\lambda \prod_{i=1}^{14}(1-p_i)\right]^N}{(N-r)!} \mathbb{I}(N \in \{r, r+1, ...\})\mathbb{I}(p_i \in [0,1]) \\
&\propto \frac{\exp\{-\lambda \prod_{i=1}^{14}(1-p_i)\}\left[\lambda \prod_{i=1}^{14}(1-p_i)\right]^{N-r}}{(N-r)!} \mathbb{I}(N \in \{r, r+1, ...\})\mathbb{I}(p_i \in [0,1]) \\
&= \frac{\exp\{-\lambda \prod_{i=1}^{14}(1-p_i)\}\left[\lambda \prod_{i=1}^{14}(1-p_i)\right]^{N-r}}{(N-r)!} \mathbb{I}(N - r \in \mathbb{N})\mathbb{I}(p_i \in [0,1])
\end{aligned}$$

Portanto

$$\pi(N|\mathbf{p}, \mathbf{x}) \propto \frac{\exp\{-\lambda \prod_{i=1}^{14}(1-p_i)\}\left[\lambda \prod_{i=1}^{14}(1-p_i)\right]^{N-r}}{(N-r)!} \mathbb{I}(N - r \in \mathbb{N})\mathbb{I}(p_i \in [0,1]) \tag{4}$$

Temos que $(N-r)|(\mathbf{p}, \mathbf{x}) \sim Poisson(\lambda \prod_{i=1}^{14}(1-p_i))$, identificado pelo *kernel* apresentado em 4.

2 Implementação ***Hamiltonian*** Monte Carlo

3 Implementação *Hamiltonian* Monte Carlo no STAN