Trabalho III

1. Considere o Hamiltoniano $H_t = -\hbar\omega_t S_z$ com $S_z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, em que ω_t é uma função escalar do tempo tal que $\Omega_t = \int_0^t \omega_{t'} dt'$. Responda as seguintes questões.

a) Verifique que $[H_t, H_{t'}] = 0$, $\forall t, t'$.

R: Para tempos diferentes, t e t', verifiquemos:

$$[H_t, H_{t'}] = H_t H_{t'} - H_{t'} H_t$$

$$\Rightarrow \hbar^2 \omega_t \omega_{t'} S_z^2 - \hbar^2 \omega_{t'} \omega_t S_z^2 = \hbar^2 \omega_t \omega_{t'} S_z^2 - \hbar^2 \omega_t \omega_{t'} S_z^2 = \mathbb{O}\ \forall t, t'$$

c.q.d.

b) Qual a expressão geral (forma exponencial) para o operador de evolução temporal que podemos utilizar neste caso?

R: Considerando que $[H_t, H_{t'}] = 0$, devemos ter o operador de evolução temporal escrito na forma

$$U_t = \exp\left(-\frac{i}{\hbar}\int_0^t H_{t'} dt'\right)$$

Assim, dado $H_{t'} = -\hbar\omega_{t'} S_z$, temos:

$$U_t = \exp\left(-\frac{i}{\hbar}\int_0^t (-\hbar\omega_{t'} S_z) dt'\right) = \exp\left(iS_z \int_0^t \omega_{t'} dt'\right) = \exp(iS_z\Omega_t)$$

$$\therefore \boxed{U_t = \exp(iS_z\Omega_t)}$$

c) Qual é a matriz que representa U_t na base de autovetores de H_t?

R: Inicialmente, vamos obter os autovetores de H_t. Pra isso, cabe notar que H_t é proporcional a S_z, isto é, $H_t = -\hbar\omega_t S_z$, garantido pelo fato de $-\hbar\omega_t$ ser um fator escalar.

Além disso, é fácil ver que os autovetores de S_z são também autovetores de H_t. Pra mostrar isso, consideremos que $|s_j\rangle$ seja um autovetor de S_z com um autovalor λ_j correspondente:

$$S_z|s_j\rangle = \lambda_j|s_j\rangle$$

Consideremos, agora, a atuação de H_t em $|s_j\rangle$:

$$H_t|s_j\rangle = (-\hbar\omega_t S_z)|s_j\rangle = (-\hbar\omega_t)\, S_z|s_j\rangle = (-\hbar\omega_t)\lambda_j|s_j\rangle$$

Donde se verifica que, de fato, os autovetores $|s_j\rangle$ de S_z são também autovetores de H_t, com autovalores $-\hbar\omega_t\lambda_j$.

Da Álgebra Linear, também sabemos que dois operadores compartilham a base de autovetores se, e somente se, comutarem:

$$[H_t, S_z] = H_t S_z - S_z H_t = -\hbar\omega_t S_z^2 - S_z(-\hbar\omega_t S_z) = -\hbar\omega_t S_z^2 + \hbar\omega_t S_z^2 = \mathbb{O}$$

Portanto, S_z e H_t, de fato, compartilham a mesma base de autovetores, como já se havia determinado.

Segue a determinação dos autovetores.

- Autovalores:

$$\det(S_z - \lambda_j \mathbb{I}) = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda_j & 0 & 0 \\ 0 & -1-\lambda_j & 0 \\ 0 & 0 & 0-\lambda_j \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(-1-\lambda)(-\lambda_j) = 0$$

$$\lambda_j \in \{-1, 0, 1\}$$

- Autovetores:

$$\lambda = -1$$

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} x_1 = 0 \\ y_1 \text{ qualquer} \\ z_1 = 0 \end{cases}$$

$$\therefore \boxed{|s_1\rangle = (0, y_1, 0) = |2\rangle}$$

onde $y_1 = 1$.

$$\lambda = 0$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} x_2 = 0 \\ y_2 = 0 \\ z_2 \text{ qualquer} \end{cases}$$

$$\therefore \boxed{|s_2\rangle = (0, 0, z_2) = |3\rangle}$$

onde $z_2 = 1$.

$$\lambda = 1$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} x_3 \text{ qualquer} \\ y_3 = 0 \\ z_3 = 0 \end{cases}$$

$$\therefore \boxed{|s_3\rangle = (x_3, 0, 0) = |1\rangle}$$

onde $x_3 = 1$.

Como temos a base canônica $\{|1\rangle, |2\rangle, |3\rangle\}$ para S_z e S_z é diagonal, temos:

$$U_t = \exp\left(i\Omega_t \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} e^{i\Omega_t} & 0 & 0 \\ 0 & e^{-i\Omega_t} & 0 \\ 0 & 0 & e^{0} \end{pmatrix}$$

$$\therefore \boxed{U_t = \begin{pmatrix} e^{i\Omega_t} & 0 & 0 \\ 0 & e^{-i\Omega_t} & 0 \\ 0 & 0 & 1 \end{pmatrix}}$$

d) Qual é a representação produto externo de U_t nessa base? Calcule o estado evoluído $|\psi_t\rangle$ desse sistema sob a ação de H_t se o estado inicial for $|\psi_0\rangle = \frac{1}{\sqrt{3}}(|1\rangle + |2\rangle + |3\rangle)$, com $|1\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $|2\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $|3\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

R: Para um espaço simples, podemos escrever um operador linear $A: \mathbb{C}^n \to \mathbb{C}^n$ em uma base $|\beta_j\rangle \in \mathbb{C}^n$, da seguinte forma:

$$A = \mathbb{I}_{\mathbb{C}^n} A \mathbb{I}_{\mathbb{C}^n} = \sum_{j=1}^{n} |\beta_j\rangle\langle\beta_j| \, A \sum_{k=1}^{n} |\beta_k\rangle\langle\beta_k| = \sum_{j,k=1}^{n} \langle\beta_j|A|\beta_k\rangle |\beta_j\rangle\langle\beta_k|$$

Para o operador U_t, temos:

$$U_t = \sum_{i,j=1}^{3} \langle i|U_t|j\rangle |i\rangle\langle j|$$

Aplicando U_t em cada vetor da base:

$$U_t|1\rangle = \begin{pmatrix} e^{i\Omega_t} & 0 & 0 \\ 0 & e^{-i\Omega_t} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} e^{i\Omega_t} \\ 0 \\ 0 \end{pmatrix} = e^{i\Omega_t}|1\rangle$$

$$U_t|2\rangle = \begin{pmatrix} e^{i\Omega_t} & 0 & 0 \\ 0 & e^{-i\Omega_t} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ e^{-i\Omega_t} \\ 0 \end{pmatrix} = e^{-i\Omega_t}|2\rangle$$

$$U_t|3\rangle = \begin{pmatrix} e^{i\Omega_t} & 0 & 0 \\ 0 & e^{-i\Omega_t} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = |3\rangle$$

Assim:

$$\begin{aligned} U_t &= \langle 1|U_t|1\rangle|1\rangle\langle 1| + \langle 2|U_t|1\rangle|2\rangle\langle 1| + \langle 3|U_t|1\rangle|3\rangle\langle 1| + \langle 1|U_t|2\rangle|1\rangle\langle 2| + \langle 2|U_t|2\rangle|2\rangle\langle 2| + \langle 3|U_t|2\rangle|3\rangle\langle 2| \\ &\quad + \langle 1|U_t|3\rangle|1\rangle\langle 3| + \langle 2|U_t|3\rangle|2\rangle\langle 3| + \langle 3|U_t|3\rangle|3\rangle\langle 3| \end{aligned}$$

$$\begin{aligned} &= \langle 1|e^{i\Omega_t}|1\rangle|1\rangle\langle 1| + \langle 2|e^{i\Omega_t}|1\rangle|2\rangle\langle 1| + \langle 3|e^{i\Omega_t}|1\rangle|3\rangle\langle 1| + \langle 1|e^{-i\Omega_t}|2\rangle|1\rangle\langle 2| + \langle 2|e^{-i\Omega_t}|2\rangle|2\rangle\langle 2| \\ &\quad + \langle 3|e^{-i\Omega_t}|2\rangle|3\rangle\langle 2| + \langle 1||3\rangle|1\rangle\langle 3| + \langle 2||3\rangle|2\rangle\langle 3| + \langle 3||3\rangle|3\rangle\langle 3| \\ &= e^{i\Omega_t}|1\rangle\langle 1| + 0|2\rangle\langle 1| + 0|3\rangle\langle 1| + 0|1\rangle\langle 2| + e^{-i\Omega_t}|2\rangle\langle 2| + 0|3\rangle\langle 2| + 0|1\rangle\langle 3| + 0|2\rangle\langle 3| + 1|3\rangle\langle 3| \\ &= \boxed{e^{i\Omega_t}|1\rangle\langle 1| + e^{-i\Omega_t}|2\rangle\langle 2| + |3\rangle\langle 3|} \end{aligned}$$

O estado evoluído fica:

$$|\psi_t\rangle = U_t|\psi_0\rangle = \left(e^{i\Omega_t}|1\rangle\langle 1| + e^{-i\Omega_t}|2\rangle\langle 2| + |3\rangle\langle 3|\right)\cdot\frac{1}{\sqrt{3}}(|1\rangle + |2\rangle + |3\rangle) = \boxed{\frac{1}{\sqrt{3}}\left(e^{i\Omega_t}|1\rangle + e^{-i\Omega_t}|2\rangle + |3\rangle\right)}$$

e) Para o sistema preparado no estado $|\psi_t\rangle$, quais são as probabilidades para medidas do seguinte observável $S_x = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$?

R: Cálculo dos autovalores de S_x:

$$\det\left(S_x - \lambda_j \mathbb{I}\right) = 0$$

$$\Rightarrow \begin{vmatrix} -\lambda_j & 0 & 1 \\ 0 & -\lambda_j & 0 \\ 1 & 0 & -\lambda_j \end{vmatrix} = 0$$

$$\Rightarrow \lambda_j - \lambda_j^3 = 0$$

$$\Rightarrow \lambda_j \in \{-1, 0, 1\}$$

Os autovetores são:

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} x_1 + z_1 = 0 \\ y_1 = 0 \end{cases} \Rightarrow (x_1, 0, -x_1)$$

Para $x_1 = 1$, e normalizando:

$$\boxed{|\phi_1\rangle = \frac{1}{\sqrt{2}}(|1\rangle - |3\rangle)}$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} z_2 = 0 \\ y_2 \text{ qualquer} \Rightarrow (0, y_2, 0) \\ x_2 = 0 \end{cases}$$

Para $y_2 = 0$ e normalizando:

$$\boxed{|\phi_2\rangle = |2\rangle}$$

$$\begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} -x_3 + z_3 = 0 \\ -y_3 = 0 \quad \Rightarrow (x_3, 0, x_3) \\ x_3 - z_3 = 0 \end{cases}$$

Para $x_3 = 1$ e normalizando:

$$\boxed{|\phi_3\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |3\rangle)}$$

Cálculo dos produtos internos entre o estado $|\psi_t\rangle$ e o os autovetores de S_x:

$$\langle\phi_1|\psi_t\rangle = \left(\frac{1}{\sqrt{2}}(|1\rangle - |3\rangle)\right)^\dagger \left(\frac{1}{\sqrt{3}}\left(e^{i\Omega_t}|1\rangle + e^{-i\Omega_t}|2\rangle + |3\rangle\right)\right) = \frac{1}{\sqrt{6}}\left(e^{i\Omega_t} - 1\right)$$

$$\langle\phi_2|\psi_t\rangle = \langle 2|\left(\frac{1}{\sqrt{3}}\left(e^{i\Omega_t}|1\rangle + e^{-i\Omega_t}|2\rangle + |3\rangle\right)\right) = \frac{e^{-i\Omega_t}}{\sqrt{3}}$$

$$\langle\phi_3|\psi_t\rangle = \left(\frac{1}{\sqrt{2}}(|1\rangle + |3\rangle)\right)^\dagger \left(\frac{1}{\sqrt{3}}\left(e^{i\Omega_t}|1\rangle + e^{-i\Omega_t}|2\rangle + |3\rangle\right)\right) = \frac{1}{\sqrt{6}}\left(e^{i\Omega_t} + 1\right)$$

Probabilidades:

$$\Pr(\lambda = -1) = |\langle\phi_1|\psi_t\rangle|^2 = \left|\frac{1}{\sqrt{6}}\left(e^{i\Omega_t} - 1\right)\right|^2 =$$

$$= \frac{1}{6}\left(e^{i\Omega_t} - 1\right)\left(e^{-i\Omega_t} - 1\right) =$$

$$= \frac{1}{6}\left(e^{-i\Omega_t}e^{i\Omega_t} - e^{-\Omega_t} - e^{i\Omega_t} + 1\right) =$$

$$= \frac{1}{6}\left(2 - (\cos(\Omega_t) - \sin(\Omega_t) + \cos(\Omega_t) + \sin(\Omega_t))\right)$$

$$= \frac{1}{6}(2 - 2\cos(\Omega_t)) = \boxed{\frac{1}{3}(1 - \cos(\Omega_t))}$$

$$\Pr(\lambda = 0) = |\langle\phi_2|\psi_t\rangle|^2 = \left|\frac{e^{-i\Omega_t}}{\sqrt{3}}\right|^2 = \boxed{\frac{1}{3}}$$

$$\Pr(\lambda = +1) = |\langle\phi_3|\psi_t\rangle|^2 = \left|\frac{1}{\sqrt{6}}\left(e^{i\Omega_t} + 1\right)\right|^2 = \boxed{\frac{1}{3}(1 + \cos(\Omega_t))}$$

Fazendo

$$\Pr(-1) + \Pr(0) + \Pr(1) = \frac{1}{3}(1 - \cos(\Omega_t)) + \frac{1}{3} + \frac{1}{3}(1 + \cos(\Omega_t)) = 1$$

como esperado.

f) Calcule o valor médio de S_x em função do tempo.

R: O valor médio é dado por:

$$\langle S_x \rangle = \langle \psi_t | S_x | \psi_t \rangle$$

Calculando:

$$S_x|\psi_t\rangle =$$

$$(|3\rangle\langle 1| + |1\rangle\langle 3|)\left(\frac{1}{\sqrt{3}}\left(e^{i\Omega_t}|1\rangle + e^{-i\Omega_t}|2\rangle + |3\rangle\right)\right) =$$

$$= \frac{1}{\sqrt{3}} e^{i\Omega_t}|3\rangle + \frac{1}{\sqrt{3}}|1\rangle = \frac{1}{\sqrt{3}}\left(|1\rangle + e^{i\Omega_t}|3\rangle\right)$$

Logo:

$$\langle \psi_t | S_x | \psi_t \rangle = \frac{1}{\sqrt{3}}\left(e^{-i\Omega_t}\langle 1| + e^{i\Omega_t}\langle 2| + \langle 3|\right)\frac{1}{\sqrt{3}}\left(|1\rangle + e^{i\Omega_t}|3\rangle\right) = \frac{1}{3}\left(e^{-i\Omega_t} + e^{i\Omega_t}\right) = \boxed{\frac{2}{3}\cos(\Omega_t)}$$

2. Considere um sistema quântica com hamiltoniano da forma $H_t = H_c + V_t$, com $H_c = \begin{bmatrix} e_1 & 0 & 0 \\ 0 & e_2 & 0 \\ 0 & 0 & e_3 \end{bmatrix}$ e $V_t = \begin{bmatrix} 0 & 0 & \gamma e^{i\omega t} \\ 0 & 0 & 0 \\ \gamma e^{-i\omega t} & 0 & 0 \end{bmatrix}$. Determine as probabilidades de transição do estado inicial $|1\rangle$ para os estados $|2\rangle$ e $|3\rangle$ ao longo do tempo.

R: Podemos escrever o estado do sistema como uma combinação linear da base:

$$|\psi(t)\rangle = c_1(t)|1\rangle + c_2(t)|2\rangle + c_3(t)|3\rangle$$

Para o hamiltoniano total do sistema:

$$i\hbar \begin{bmatrix} \partial_t c_1(t) \\ \partial_t c_2(t) \\ \partial_t c_3(t) \end{bmatrix} = \begin{bmatrix} e_1 & 0 & \gamma e^{i\omega t} \\ 0 & e_2 & 0 \\ \gamma e^{-i\omega t} & 0 & e_3 \end{bmatrix} \begin{bmatrix} c_1(t) \\ c_2(t) \\ c_3(t) \end{bmatrix}$$

Para simplificar a notação, vamos omitir a dependência no tempo dos coeficientes:

$$\Rightarrow \begin{cases} i\hbar\partial_t c_1 = e_1 c_1 + \gamma e^{i\omega t} c_3 \ (I) \\ i\hbar\partial_t c_2 = e_2 c_2 \ (II) \\ i\hbar\partial_t c_3 = e_3 c_3 + \gamma e^{-i\omega t} c_1 \ (III) \end{cases}$$

Derivando no tempo as equações (I) e (III), temos:

$$\Rightarrow \begin{cases} i\hbar\partial_{tt} c_1 = e_1 \partial_t c_1 + \gamma\left(i\omega e^{i\omega t} c_3 + e^{i\omega t}\partial_t c_3\right) (IV) \\ i\hbar\partial_{tt} c_3 = e_3 \partial_t c_3 + \gamma\left(-i\omega e^{-i\omega t} c_1 + e^{-i\omega t}\partial_t c_1\right) (V) \end{cases}$$

Das equações (I) e (III):

$$\partial_t c_1 = -\frac{i}{\hbar}\left(e_1 c_1 + \gamma e^{i\omega t} c_3\right)$$

$$\partial_t c_3 = -\frac{i}{\hbar}\left(e_3 c_3 + \gamma e^{-i\omega t} c_1\right)$$

Substituindo nas equações (V) e (IV), respectivamente:

$$\begin{cases} i\hbar\partial_{tt}c_1 = e_1\partial_t c_1 + \gamma e^{i\omega t}\left(c_3\left(\frac{e_3}{i\hbar} + i\omega\right) + c_1\left(\frac{\gamma e^{-i\omega t}}{i\hbar}\right)\right) \\ i\hbar\partial_{tt}c_3 = e_3\partial_t c_3 + \gamma e^{-i\omega t}\left(c_1\left(\frac{e_1}{i\hbar} - i\omega\right) + c_3\left(\frac{\gamma e^{i\omega t}}{i\hbar}\right)\right) \end{cases}$$

$$\Rightarrow \begin{cases} i\hbar\partial_{tt}c_1 = e_1\partial_t c_1 + c_3\gamma e^{i\omega t}\left(i\omega - \frac{ie_3}{\hbar}\right) - \frac{i\gamma^2}{\hbar}c_1 \\ i\hbar\partial_{tt}c_3 = e_3\partial_t c_3 + c_1\gamma e^{-i\omega t}\left(-i\omega - \frac{ie_1}{\hbar}\right) - \frac{i\gamma^2}{\hbar}c_3 \end{cases}$$

Novamente, das equações (I) e (III), temos:

$$c_3 = \frac{i\hbar\partial_t c_1 - e_1 c_1}{\gamma e^{i\omega t}}$$

$$c_1 = \frac{i\hbar\partial_t c_3 - e_3 c_3}{\gamma e^{-i\omega t}}$$

e substituindo nas equações anteriores:

$$\begin{cases} i\hbar\partial_{tt}c_1 = e_1\partial_t c_1 + \left(\frac{i\hbar\partial_t c_1 - e_1 c_1}{\gamma e^{i\omega t}}\right)\gamma e^{i\omega t}\left(i\omega - \frac{ie_3}{\hbar}\right) - \frac{i\gamma^2}{\hbar}c_1 \\ i\hbar\partial_{tt}c_3 = e_3\partial_t c_3 + \left(\frac{i\hbar\partial_t c_3 - e_3 c_3}{\gamma e^{-i\omega t}}\right)\gamma e^{-i\omega t}\left(-i\omega - \frac{ie_1}{\hbar}\right) - \frac{i\gamma^2}{\hbar}c_3 \end{cases}$$

$$\Rightarrow \begin{cases} i\hbar\partial_{tt}c_1 = e_1\partial_t c_1 + (i\hbar\partial_t c_1 - e_1 c_1)\left(i\omega - \frac{ie_3}{\hbar}\right) - \frac{i\gamma^2}{\hbar}c_1 \\ i\hbar\partial_{tt}c_3 = e_3\partial_t c_3 + (i\hbar\partial_t c_3 - e_3 c_3)\left(-i\omega - \frac{ie_1}{\hbar}\right) - \frac{i\gamma^2}{\hbar}c_3 \end{cases}$$

$$\Rightarrow \begin{cases} i\hbar\partial_{tt}c_1 = e_1\partial_t c_1 - \omega\hbar\partial_t c_1 + e_3\partial_t c_1 - i\omega e_1 c_1 + \frac{ie_1e_3c_1}{\hbar} - \frac{i\gamma^2}{\hbar}c_1 \\ i\hbar\partial_{tt}c_3 = e_3\partial_t c_3 + \omega\hbar\partial_t c_3 + e_1\partial_t c_3 + i\omega e_3 c_3 + \frac{ie_1e_3c_3}{\hbar} - \frac{i\gamma^2}{\hbar}c_3 \end{cases}$$

$$\Rightarrow \begin{cases} i\hbar\partial_{tt}c_1 = \partial_t c_1(e_1 - \omega\hbar + e_3) + c_1\left(-i\omega e_1 + \frac{ie_1e_3 - i\gamma^2}{\hbar}\right) \\ i\hbar\partial_{tt}c_3 = \partial_t c_3(e_3 + \omega\hbar + e_1) + c_3\left(i\omega e_3 + \frac{ie_1e_3 - i\gamma^2}{\hbar}\right) \end{cases}$$

$$\Rightarrow \begin{cases} \partial_{tt}c_1 + i\left(\frac{e_1 + e_3}{\hbar} - \omega\right)\partial_t c_1 + c_1\left(\frac{\omega e_1}{\hbar} + \frac{\gamma^2 - e_1e_3}{\hbar^2}\right) = 0\ (VI) \\ \partial_{tt}c_3 + i\left(\frac{e_1 + e_3}{\hbar} + \omega\right)\partial_t c_3 + c_3\left(-\frac{\omega e_3}{\hbar} + \frac{\gamma^2 - e_1e_3}{\hbar^2}\right) = 0\ (VII) \end{cases}$$

Para resolver as equações diferenciais acima analiticamente, será tomado o caso particular $e_1 = e_3 = 0$, de modo que:

$$\Rightarrow \begin{cases} \partial_{tt}c_1 - i\omega\partial_t c_1 + c_1\left(\frac{\gamma^2}{\hbar^2}\right) = 0\ (VI) \\ \partial_{tt}c_3 + i\omega\partial_t c_3 + c_3\left(\frac{\gamma^2}{\hbar^2}\right) = 0\ (VII) \end{cases}$$

Podemos resolver (VI) como segue:

$$\lambda^2 + \frac{i}{\hbar}(e_1 - \omega\hbar + e_3) + \left(\frac{\omega e_1}{\hbar} + \frac{\gamma^2 - e_1e_3}{\hbar^2}\right) = 0$$

$$\Rightarrow \lambda_\pm = \frac{i}{2\hbar}(\omega\hbar - e_1 - e_3) \pm \frac{1}{2}\sqrt{\left(\frac{2\gamma}{\hbar}\right)^2 - \left(\frac{e_1 + e_3 - \omega\hbar}{\hbar}\right)^2 - \frac{4e_1}{\hbar}\left(\omega - \frac{e_3}{\hbar}\right)}$$

e:

$$c_1(t) = \alpha_1 e^{\lambda_+ t} + \alpha_2 e^{\lambda_- t}$$

Nesse caso, vamos tomar o caso particular $e_1 = e_3 = 0$, de modo que:

$$\lambda_\pm = \frac{i\omega}{2} \pm \frac{1}{2}\sqrt{\left(\frac{2\gamma}{\hbar}\right)^2 - \omega^2}$$

e:

$$\widetilde{c_1}(t) =$$

3. No caso geral de um Hamiltoniano dependente do tempo com $[H_t, H_{t'}] \neq 0$, o operador de evolução temporal é dado por uma série de Dyson

$$U_t = \mathbb{I} + \sum_{j=1}^{\infty}\left(\frac{-i}{\hbar}\right)^j \int_0^t dt_1 H_{t_1} \int_0^{t_1} dt_2 H_{t_2} \cdots \int_0^{t_{j-1}} dt_j H_{t_j} = \mathbb{I} + \sum_{j=1}^{\infty} U_t^{(j)}$$

O estado evoluído correspondente pode ser escrito como

$$|\psi_t\rangle = U_t|\psi_0\rangle = \left(\mathbb{I} + \sum_{j=1}^{\infty} U_t^{(j)}\right)|\psi_0\rangle = |\psi_0\rangle + \sum_{j=1}^{\infty} U_t^{(j)}|\psi_0\rangle = |\psi_0\rangle + \sum_{j=1}^{\infty}\left|\psi_t^{(j)}\right\rangle$$

com $\left|\psi_t^{(j)}\right\rangle \coloneqq U_t^{(j)}|\psi_0\rangle$. Considere o Hamiltoniano que usamos para um sistema de dois estados, $H_t = \begin{bmatrix} e_1 & \gamma e^{i\omega t} \\ \gamma e^{-i\omega t} & e_2 \end{bmatrix}$, com o sistema preparado inicialmente no estado de mais baixa energia: $|\psi_0\rangle = |1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Obtenha $U_t^{(1)}$ e $\left|\psi_t^{(1)}\right\rangle$. Normalize o estado $|\psi_t\rangle \approx |\psi_0\rangle + \left|\psi_t^{(1)}\right\rangle$ e calcule a probabilidade de encontrarmos o sistema no estado $|2\rangle$ no instante de tempo t. Faça um gráfico comparativo dessa probabilidade com o resultado exato obtido usando a representação de interação.

R: Operador evolução temporal:

$$U_t^{(1)} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \mathbb{I} - \frac{i}{\hbar}\int_0^t H_{t_1} dt_1 = \mathbb{I} - \frac{i}{\hbar}\int_0^t \begin{bmatrix} e_1 & \gamma e^{i\omega t_1} \\ \gamma e^{-i\omega t_1} & e_2 \end{bmatrix} dt_1$$

$$\Rightarrow \begin{cases} u_{11} = 1 + \frac{i}{\hbar} e_1 t \\ u_{12} = \frac{\gamma}{\hbar\omega}\left(1 - e^{i\omega t}\right) \\ u_{21} = -\frac{\gamma}{\hbar\omega}\left(1 - e^{-i\omega t}\right) \\ u_{22} = 1 + \frac{i}{\hbar} e_2 t \end{cases}$$

$$\therefore \boxed{U_t^{(1)} = \begin{bmatrix} 1 + e_1 t & \frac{\gamma}{\hbar\omega}\left(1 - e^{i\omega t}\right) \\ -\frac{\gamma}{\hbar\omega}\left(1 - e^{-i\omega t}\right) & 1 + e_2 t \end{bmatrix}}$$

Estado evoluído:

$$\left|\psi_t^{(1)}\right\rangle = U_t^{(1)}|\psi_0\rangle = \begin{bmatrix} 1 + e_1 t & \frac{\gamma}{\hbar\omega}\left(1 - e^{i\omega t}\right) \\ -\frac{\gamma}{\hbar\omega}\left(1 - e^{-i\omega t}\right) & 1 + e_2 t \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\therefore \boxed{\left|\psi_t^{(1)}\right\rangle = \begin{bmatrix} 1 + e_1 t \\ -\frac{\gamma}{\hbar\omega}\left(1 - e^{-i\omega t}\right) \end{bmatrix} = (1 + e_1 t)|1\rangle - \frac{\gamma}{\hbar\omega}\left(1 - e^{-i\omega t}\right)|2\rangle}$$

Normalização do estado:

$$|\psi_t\rangle \approx \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 + e_1 t \\ -\frac{\gamma}{\hbar\omega}\left(1 - e^{-i\omega t}\right) \end{bmatrix} = \begin{bmatrix} 2 + e_1 t \\ -\frac{\gamma}{\hbar\omega}\left(1 - e^{-i\omega t}\right) \end{bmatrix} = (2 + e_1 t)|1\rangle - \frac{\gamma}{\hbar\omega}\left(1 - e^{-i\omega t}\right)|2\rangle$$

Calculemos:

$$\|\psi_t\| = \sqrt{|2 + e_1 t|^2 + \left|-\frac{\gamma}{\hbar\omega}(1 - e^{-i\omega t})\right|^2} = \sqrt{(2 + e_1 t)^2 + \left(\frac{\gamma}{\hbar\omega}\right)^2 |1 - e^{-i\omega t}|^2} =$$

$$= \sqrt{(2 + e_1 t)^2 + \left(\frac{\gamma}{\hbar\omega}\right)^2 (2 - 2\cos(\omega t))}$$

Assim, a normalização fica:

$$|\psi_t^{\text{norm}}\rangle = \frac{1}{\sqrt{(2 + e_1 t)^2 + \left(\frac{\gamma}{\hbar\omega}\right)^2 (2 - 2\cos(\omega t))}}\left((2 + e_1 t)|1\rangle - \frac{\gamma}{\hbar\omega}\left(1 - e^{-i\omega t}\right)|2\rangle\right)$$

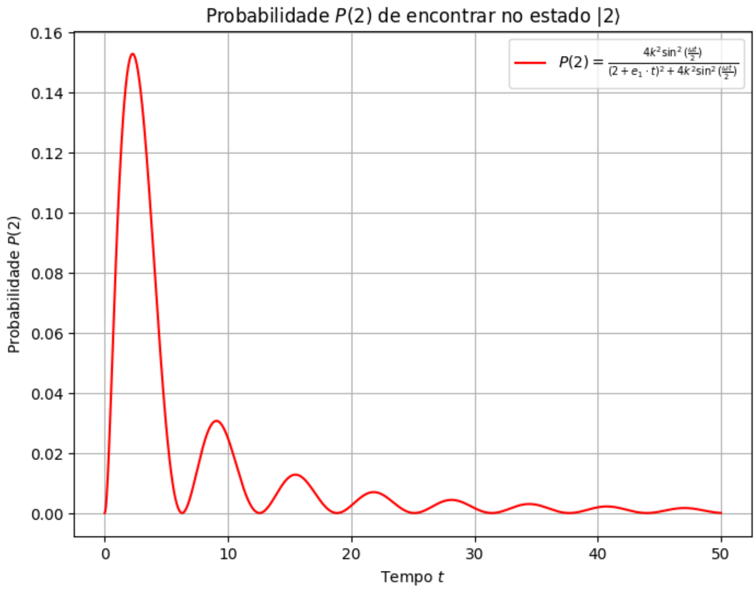
Probabilidade de encontrar no estado $|2\rangle$:

$$\Pr(2) = |\langle 2|\psi_t^{\text{norm}}\rangle|^2 = |c_2|^2 = \frac{\left|-\frac{\gamma}{\hbar\omega}\left(1 - e^{-i\omega t}\right)\right|^2}{\|\psi_t\|^2} = \frac{\left(\frac{\gamma}{\hbar\omega}\right)^2 (2 - 2\cos(\omega t))}{(2 + e_1 t)^2 + \left(\frac{\gamma}{\hbar\omega}\right)^2 (2 - 2\cos(\omega t))}$$

Para simplificar, tomemos $k = \gamma/\hbar\omega$ e substituímos $2 - 2\cos(\omega t) = 4\sin^2\left(\frac{\omega t}{2}\right)$:

$$\boxed{\Pr(2) = \frac{4k^2 \sin^2\left(\frac{\omega t}{2}\right)}{(2 + e_1 t)^2 + 4k^2 \sin^2\left(\frac{\omega t}{2}\right)}}$$

Gráfico:



4. Considere a Mecânica Quântica de sistemas compostos.

a) Para dois vetores quaisquer $|\psi\rangle$ e $|\phi\rangle$ de um espaço de Hilbert discreto de dimensão finita, verifique que $|\psi\rangle\langle\phi| = |\psi\rangle \otimes \langle\phi|$.

R: Tomemos $|a\rangle, |b\rangle \in \mathbb{C}^n$, tal que:

$$|a\rangle = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

$$|b\rangle = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Assim:

$$|a\rangle\langle b| = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \begin{bmatrix} b_1^* & b_2^* & \cdots & b_n^* \end{bmatrix} = \begin{bmatrix} a_1 b_1^* & a_1 b_2^* & \cdots & a_1 b_n^* \\ a_2 b_1^* & a_2 b_2^* & \cdots & a_2 b_n^* \\ \vdots & \vdots & \ddots & \vdots \\ a_n b_1^* & a_n b_2^* & \cdots & a_n b_n^* \end{bmatrix}$$

e

$$|a\rangle \otimes \langle b| = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \otimes \begin{bmatrix} b_1^* & b_2^* & \cdots & b_n^* \end{bmatrix} = \begin{bmatrix} a_1 [b_1^* & b_2^* & \cdots & b_n^*] \\ a_2 [b_1^* & b_2^* & \cdots & b_n^*] \\ & \vdots & & \\ a_n [b_1^* & b_2^* & \cdots & b_n^*] \end{bmatrix} = \begin{bmatrix} a_1 b_1^* & a_1 b_2^* & \cdots & a_1 b_n^* \\ a_2 b_1^* & a_2 b_2^* & \cdots & a_2 b_n^* \\ \vdots & \vdots & \ddots & \vdots \\ a_n b_1^* & a_n b_2^* & \cdots & a_n b_n^* \end{bmatrix}$$

c.q.d.

Considere dois neutrões, que tem spin $1/2$, preparados no estado

$$|\Psi\rangle_{ab} = \frac{1}{\sqrt{3}}|z_+\rangle_a \otimes |z_+\rangle_b + \frac{1}{\sqrt{3}}|z_+\rangle_a \otimes |z_-\rangle_b + \frac{1}{\sqrt{3}}|z_-\rangle_a \otimes |z_+\rangle_b$$

b) Quais são os possíveis resultados para medidas da componente de spin na direção z aplicadas no spin da esquerda (σ_z^a)?

R: Nesse caso, podemos obter medidas de $+1$, para o estado $|z_+\rangle$, e de -1 para o estado $|z_-\rangle$, dados pelos autovalores do operador σ_z. O operador de Pauli

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

é diagonal, donde sai diretamente seus autovalores.

c) Quais são as probabilidades correspondentes?

R: Para o primeiro autovalor, temos:

$$\Pr(z_+, \Psi_{ab}) = \Pr(z_+, z_+, \Psi_{ab}) + \Pr(z_+, z_-, \Psi_{ab}) = |(\langle z_+|_a \otimes \langle z_+|_b)|\Psi_{ab}\rangle|^2 + |(\langle z_+|_a \otimes \langle z_-|_b)|\Psi_{ab}\rangle|^2$$

Calculando cada projeção separadamente:

$$(\langle z_+|_a \otimes \langle z_+|_b)\left(\frac{1}{\sqrt{3}}|z_+\rangle_a \otimes |z_+\rangle_b + \frac{1}{\sqrt{3}}|z_+\rangle_a \otimes |z_-\rangle_b + \frac{1}{\sqrt{3}}|z_-\rangle_a \otimes |z_+\rangle_b\right)$$

$$\Rightarrow \frac{1}{\sqrt{3}}\left(\underbrace{\langle z_+|_a|z_+\rangle_a}_{=1} \cdot \underbrace{\langle z_+|_b|z_+\rangle_b}_{=1}\right) + \frac{1}{\sqrt{3}}\left(\underbrace{\langle z_+|_a|z_+\rangle_a}_{=1} \cdot \underbrace{\langle z_+|_b|z_-\rangle_b}_{=0}\right) + \frac{1}{\sqrt{3}}\left(\underbrace{\langle z_+|_a|z_-\rangle_a}_{=0} \cdot \underbrace{\langle z_+|_b|z_+\rangle_b}_{=1}\right) = \frac{1}{\sqrt{3}}$$

e temos:

$$\Pr(z_+, z_+, \Psi_{ab}) = \left|\frac{1}{\sqrt{3}}\right|^2 = \frac{1}{3}$$

Seguindo os cálculos:

$$\Pr(z_+, z_-, \Psi_{ab}) = (\langle z_+|_a \cdot \langle z_-|_b)\left(\frac{1}{\sqrt{3}}|z_+\rangle_a \otimes |z_+\rangle_b + \frac{1}{\sqrt{3}}|z_+\rangle_a \otimes |z_-\rangle_b + \frac{1}{\sqrt{3}}|z_-\rangle_a \otimes |z_+\rangle_b\right) =$$

$$= \frac{1}{\sqrt{3}}\left(\underbrace{\langle z_+|_a|z_+\rangle_a}_{=1} \cdot \underbrace{\langle z_-|_b|z_+\rangle_b}_{=0}\right) + \frac{1}{\sqrt{3}}\left(\underbrace{\langle z_+|_a|z_+\rangle_a}_{=1} \cdot \underbrace{\langle z_-|_b|z_-\rangle_b}_{=1}\right) + \frac{1}{\sqrt{3}}\left(\underbrace{\langle z_+|_a|z_-\rangle_a}_{=0} \cdot \underbrace{\langle z_-|_b|z_+\rangle_b}_{=0}\right) = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \Pr(z_+, z_-, \Psi_{ab}) = \frac{1}{3}$$

$$\therefore \boxed{\Pr(z_+, \Psi_{ab}) = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}}$$

Para o segundo autovalor:

$$\Pr(z_-, \Psi_{ab}) = \Pr(z_-, z_+, \Psi_{ab}) =$$

$$= \left|(\langle z_-|_a \otimes \langle z_+|_b)\left(\frac{1}{\sqrt{3}}|z_+\rangle_a \otimes |z_+\rangle_b + \frac{1}{\sqrt{3}}|z_+\rangle_a \otimes |z_-\rangle_b + \frac{1}{\sqrt{3}}|z_-\rangle_a \otimes |z_+\rangle_b\right)\right|^2 =$$

$$= \left|\frac{1}{\sqrt{3}}\left(\underbrace{\langle z_-|_a|z_+\rangle_a}_{=0} \cdot \underbrace{\langle z_+|_b|z_+\rangle_b}_{=1}\right) + \frac{1}{\sqrt{3}}\left(\underbrace{\langle z_-|_a|z_+\rangle_a}_{=0} \cdot \underbrace{\langle z_+|_b|z_-\rangle_b}_{=0}\right) + \frac{1}{\sqrt{3}}\left(\underbrace{\langle z_-|_a|z_-\rangle_a}_{=1} \cdot \underbrace{\langle z_+|_b|z_+\rangle_b}_{=1}\right)\right|^2 = \left|\frac{1}{\sqrt{3}}\right|^2 = \frac{1}{3}$$

$$\therefore \boxed{\Pr(z_-, \Psi_{ab}) = \frac{1}{3}}$$

d) Quais são os estados pós-medida correspondentes dos dois spins?

R: os estados pós-medida normalizados são dados por:

$$\left|\Psi_j^{\text{norm}}\right\rangle = \frac{\prod_{z_j}|\Psi_{ab}\rangle}{\left\|\prod_{z_j}|\Psi_{ab}\rangle\right\|} = \frac{\prod_{z_j}|\Psi_{ab}\rangle}{\sqrt{\langle\Psi_{ab}|\prod_{z_j}|\Psi_{ab}\rangle}}$$

Temos:

$$\prod_{z_+^a}|\Psi_{ab}\rangle = (|z_+\rangle_a\langle z_+|_a \otimes \mathbb{I}_b)\left(\frac{1}{\sqrt{3}}|z_+\rangle_a \otimes |z_+\rangle_b + \frac{1}{\sqrt{3}}|z_+\rangle_a \otimes |z_-\rangle_b + \frac{1}{\sqrt{3}}|z_-\rangle_a \otimes |z_+\rangle_b\right) =$$

$$= \frac{1}{\sqrt{3}}(|z_+\rangle_a\langle z_+|_a|z_+\rangle_a) \otimes |z_+\rangle_b + \frac{1}{\sqrt{3}}(|z_+\rangle_a\langle z_+|_a|z_+\rangle_a) \otimes |z_-\rangle_b + \frac{1}{\sqrt{3}}(|z_+\rangle_a\langle z_+|_a|z_-\rangle_a) \otimes |z_+\rangle_b$$

$$= \frac{1}{\sqrt{3}}|z_+\rangle_a \otimes |z_+\rangle_b + \frac{1}{\sqrt{3}}|z_+\rangle_a \otimes |z_-\rangle_b + 0$$

Norma:

$$\left\|\prod_{z_+^a}|\Psi_{ab}\rangle\right\| = \sqrt{\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}}} = \sqrt{\frac{2}{3}}$$

Assim:

$$|\Psi_+^{\text{norm}}\rangle = \sqrt{\frac{3}{2}}\left(\frac{1}{\sqrt{3}}|z_+\rangle_a \otimes |z_+\rangle_b + \frac{1}{\sqrt{3}}|z_+\rangle_a \otimes |z_-\rangle_b\right)$$

$$\therefore \boxed{|\Psi_+^{\text{norm}}\rangle = \frac{1}{\sqrt{2}}(|z_+\rangle_a \otimes |z_+\rangle_b + |z_+\rangle_a \otimes |z_-\rangle_b)}$$

Continuando:

$$\prod_{z_-^a}|\Psi_{ab}\rangle = (|z_-\rangle_a\langle z_-|_a \otimes \mathbb{I}_b)\left(\frac{1}{\sqrt{3}}|z_+\rangle_a \otimes |z_+\rangle_b + \frac{1}{\sqrt{3}}|z_+\rangle_a \otimes |z_-\rangle_b + \frac{1}{\sqrt{3}}|z_-\rangle_a \otimes |z_+\rangle_b\right)$$

$$= \frac{1}{\sqrt{3}}(|z_-\rangle_a\langle z_-|_a|z_+\rangle_a) \otimes |z_+\rangle_b + \frac{1}{\sqrt{3}}(|z_-\rangle_a\langle z_-|_a|z_+\rangle_a) \otimes |z_-\rangle_b + \frac{1}{\sqrt{3}}(|z_-\rangle_a\langle z_-|_a|z_-\rangle_a) \otimes |z_+\rangle_b$$

$$= \frac{1}{\sqrt{3}}|z_-\rangle_a \otimes |z_+\rangle_b$$

Norma:

$$\left\|\prod_{z_-^a}|\Psi_{ab}\rangle\right\| = \sqrt{\frac{1}{3}}$$

E:

$$|\Psi_-^{\text{norm}}\rangle = \sqrt{3} \cdot \frac{1}{\sqrt{3}}|z_-\rangle_a \otimes |z_+\rangle_b$$

$$\therefore \boxed{|\Psi_-^{\text{norm}}\rangle = |z_-\rangle_a \otimes |z_+\rangle_b}$$

e) Para os estados pós-medida, quais são as probabilidades para medidas de σ_x^b?

R: Considerando a matriz de Pauli

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

e seus autovetores normalizados:

$$|x_+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$|x_-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Podemos reescrevê-los como combinação dos estados $|z_+\rangle$ e $|z_-\rangle$:

$$|x_+\rangle_b = \frac{1}{\sqrt{2}}(|z_+\rangle_b + |z_-\rangle_b)$$

$$|x_-\rangle_b = \frac{1}{\sqrt{2}}(|z_+\rangle_b - |z_-\rangle_b)$$

Podemos, a partir disso, obter as probabilidades para cada estado, $|\Psi_+^{\mathrm{norm}}\rangle$ e $|\Psi_-^{\mathrm{norm}}\rangle$:

$$\Pr(x_+, \Psi_+^{\mathrm{norm}}) = \sum_{j\in\{z_+,z_-\}} |(\langle x_+|_b \otimes \langle j|_a)|\Psi_+^{\mathrm{norm}}\rangle|^2$$

$$\Pr(x_+, \Psi_-^{\mathrm{norm}}) = \sum_{j\in\{z_+,z_-\}} |(\langle x_+|_b \otimes \langle j|_a)|\Psi_-^{\mathrm{norm}}\rangle|^2$$

$$\Pr(x_-, \Psi_+^{\mathrm{norm}}) = \sum_{j\in\{z_+,z_-\}} |(\langle x_-|_b \otimes \langle j|_a)|\Psi_+^{\mathrm{norm}}\rangle|^2$$

$$\Pr(x_-, \Psi_-^{\mathrm{norm}}) = \sum_{j\in\{z_+,z_-\}} |(\langle x_-|_b \otimes \langle j|_a)|\Psi_-^{\mathrm{norm}}\rangle|^2$$

i) $\Pr(x_+, \Psi_+^{\mathrm{norm}})$

$$\left|\frac{1}{\sqrt{2}}[(\langle x_+|_b \otimes \langle z_+|_a)(|z_+\rangle_a \otimes |z_+\rangle_b) + (\langle x_+|_b \otimes \langle z_+|_a)(|z_+\rangle_a \otimes |z_-\rangle_b)]\right|^2 + \left|\frac{1}{\sqrt{2}}[(\langle x_+|_b \otimes \langle z_-|_a)(|z_+\rangle_a \otimes |z_+\rangle_b) + (\langle x_+|_b \otimes \langle z_-|_a)(|z_+\rangle_a \otimes |z_-\rangle_b)]\right|^2$$

$$\Rightarrow \left|\frac{1}{\sqrt{2}}(\langle z_+|_a|z_+\rangle_a \cdot \langle x_+|_b|z_+\rangle_b + \langle z_+|_a|z_+\rangle_a \cdot \langle x_+|_b|z_-\rangle_b)\right|^2 + \left|\frac{1}{\sqrt{2}}(\langle z_-|_a|z_+\rangle_a \cdot \langle x_+|_b|z_+\rangle_b + \langle z_-|_a|z_+\rangle_a \cdot \langle x_+|_b|z_-\rangle_b)\right|^2$$

$$\Rightarrow \left|\frac{1}{\sqrt{2}}\langle x_+|_b|z_+\rangle_b + \frac{1}{\sqrt{2}}\langle x_+|_b|z_-\rangle_b\right|^2 + 0 = \left|\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}(\langle z_+|_b + \langle z_-|_b)|z_+\rangle_b + \frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}(\langle z_+|_b + \langle z_-|_b)|z_-\rangle_b\right|^2 = \left|\frac{1}{2}+\frac{1}{2}\right|^2 = 1$$

ii) $\Pr(x_-, \Psi_+^{\mathrm{norm}})$

Repetindo os cálculos acima para o caso de $|x_-\rangle$, chegamos em:

$$\left|\frac{1}{\sqrt{2}}\langle x_-|_b|z_+\rangle_b + \frac{1}{\sqrt{2}}\langle x_-|_b|z_-\rangle_b\right|^2 = \left|\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}(\langle z_+|_b - \langle z_-|_b)|z_+\rangle_b + \frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}(\langle z_+|_b - \langle z_-|_b)|z_-\rangle_b\right|^2 = \left|\frac{1}{2}-\frac{1}{2}\right|^2 = 0$$

iii) $\Pr(x_+, \Psi_-^{\mathrm{norm}})$

$$|(\langle x_+|_b \otimes \langle z_+|_a)(|z_-\rangle_a \otimes |z_+\rangle_b)|^2 + |(\langle x_+|_b \otimes \langle z_-|_a)(|z_-\rangle_a \otimes |z_+\rangle_b)|^2 = |\langle z_+|_a|z_-\rangle_a \cdot \langle x_+|_b|z_+\rangle_b|^2 + |\langle z_-|_a|z_-\rangle_a \cdot \langle x_+|_b|z_+\rangle_b|^2 = 0 + \left|\frac{1}{\sqrt{2}}(\langle z_+|_b + \langle z_-|_b)|z_+\rangle_b\right|^2 = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$$

iv) $\Pr(x_-, \Psi_-^{\mathrm{norm}})$

$$|(\langle x_-|_b \otimes \langle z_+|_a)(|z_-\rangle_a \otimes |z_+\rangle_b)|^2 + |(\langle x_-|_b \otimes \langle z_-|_a)(|z_-\rangle_a \otimes |z_+\rangle_b)|^2$$
$$= |\langle z_+|_a|z_-\rangle_a \cdot \langle x_-|_b|z_+\rangle_b|^2 + |\langle z_-|_a|z_-\rangle_a \cdot \langle x_-|_b|z_+\rangle_b|^2 = 0 + \left|\frac{1}{\sqrt{2}}(\langle z_+|_b - \langle z_-|_b)|z_+\rangle_b\right|^2$$
$$= \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$$

Resumindo:

$$\begin{cases} \Pr(x_+, \Psi_+^{\text{norm}}) = 1 \\ \Pr(x_-, \Psi_+^{\text{norm}}) = 0 \\ \Pr(x_+, \Psi_-^{\text{norm}}) = \dfrac{1}{2} \\ \Pr(x_-, \Psi_-^{\text{norm}}) = \dfrac{1}{2} \end{cases}$$

f) Verifique que $[\sigma_z^a, \sigma_z^b] = 0$.

R: Considerando que as matrizes de spin atuam em espaços de Hilbert diferentes, sabemos diretamente que elas comutam.

Formalmente, podemos calcular, para o sistema composto $\mathcal{H}_a \otimes \mathcal{H}_b$:

$$[\sigma_z^a, \sigma_z^b] = \sigma_z^a\sigma_z^b - \sigma_z^b\sigma_z^a = (\sigma_z^a \otimes \mathbb{I}_b)\left(\mathbb{I}_a \otimes \sigma_z^b\right) - \left(\mathbb{I}_a \otimes \sigma_z^b\right)(\sigma_z^a \otimes \mathbb{I}_b)$$
$$= (\sigma_z^a \mathbb{I}_a) \otimes \left(\mathbb{I}_b \sigma_z^b\right) - (\mathbb{I}_a \sigma_z^a) \otimes \left(\sigma_z^b \mathbb{I}_b\right) = \sigma_z^a \otimes \sigma_z^b - \sigma_z^a \otimes \sigma_z^b = 0$$

onde $\mathbb{I}_a$ e $\mathbb{I}_b$ são os operadores identidade de seus respectivos espaços de Hilbert.

□