Math Methods in Financial Economics - Spring 2021

Prof. Alex Himonas

Siena Gruler, 2021, Economics and ACMS

Bo Heatherman, 2021, Mathematics and Economics

Vitor Furtado Farias, 2022, Mathematics and Economics

Campbell Goff, 2022, Mathematics and Economics

Gabriel Silva Simoes, 2022, Computer Science

A Theoretical Analysis of Stock Prices and Their Volatility

I. Abstract

Throughout this paper, we analyze the background and foundations for pricing stocks as well as examining the causes and consequences of volatility. Starting at the origins of publicly traded companies, this paper builds from acknowledging the initial purposes of stocks to explaining the construction of modern stock prices seen on the DOW Jones and S&P 500. Examining several different methods of stock pricing, we aim to provide clarity on the rationality of stock prices by comparing the changes in stock price volatility to changes in a company's financial standing.

II. Historical Background

The goal of this section is to provide historical context and background on the origins of stocks, their original purpose and pricing format, as well as changes in the pricing and trading of stocks which affected the purpose and led to distortion of prices. Advancements in technology, mathematics, and public access have made great changes in the trading and investment sectors. Understanding the history behind the market is essential to procuring an answer to the question of if stock prices are still rational in the modern trading system.

Trading in commodities and futures actually existed far before the origination of stocks and publicly traded companies. In the early 1400's, European merchants would buy goods in anticipation of future price increases, foreshadowing the modern day futures trading system. This practice was common in Antwerp, then the center of international trade, which led to the dissemination of trade practices throughout the Old and New Worlds. The Dutch again paved the way in trading innovation when the Dutch East India company became the first publicly traded company in the world (Social Finance). Valued at around $7.9 trillion at its peak (modern dollars), this titan of trade was worth around the same as modern day Germany and Japan combined ($3.4 and $4.8 trillion respectively) (Business Insider). The original idea was to issue stock as a means to raise capital for explorations, ship construction, and other large investments that the company would not have the cash to pay for until after long trade expeditions were completed. In return for buying a share of stock, investors were promised dividends of future profits earned by the Dutch East India Company.

Eventually this practice caught fire as more countries realized that stock issuance was an effective and efficient way to raise capital. For several hundred years, the idea was most applicable to international trade and exploration companies since the time frame between project costs and eventual income could be several years. At the turn of the century in 1800, more and more companies across a variety of industries began to issue stock. Thus the stock market was born, and the finance industry would continue to grow over the next several decades. The Industrial Revolution of the mid to late 1800s only helped fuel this rise (Small Business Chronicle). The idea of a stock market would mark a subtle shift in the purpose and usage of stock. Investors now were not only interested in buying a stock with the purpose of securing the future dividends of a firm, but they were purchasing stock in the hopes for stock market fluctuations that would in turn make their holdings more favorable for sale.

Public perception of the economy, and not necessarily perception of individual firms, has distorted stock prices on many occasions. Economic crashes in the Great Depression in the late 1920's, Black Monday in 1987, and the Great Recession at the start of the early 21st century all saw plummeting stock prices in companies whose change in solvency due to hard times was vastly overexaggerated by the market (Atlanta Journal-Constitution). As the trading of stocks has continued to evolve, new entrants to the market bring new levels of volatility and sometimes less educated strategies on when to buy and sell. Easy access to trading, affordable stock prices, and even glorification of the finance industry in the media (think *Wall Street* and *The Wolf of Wall Street*), have increased the number of passive investors and dangerously confident passive day traders. Most recently, the financial services company Robinhood found itself in hot water due to a stock price manipulation scandal with the companies GameStop (GME) and AMC Theaters (AMC) that was enabled by the Robinhood platform. Many question whether or not the wildly distorted prices caused by the Robinhood craze violate the Efficient Market Hypothesis as well as how easy access to trading could affect and possibly increase the distortion of stock prices in the future.

Expanding upon the Gamestop trading craze of early 2021, it is hard to justify an argument for the rational pricing of Gamestop's stock. As of the end of Q3 2020, Gamestop was continuing on a downward trend from a company perspective. CNN Business reported that "between 400 and 450 stores globally will close this year, which is more than the 320 stores GameStop originally said in March," as the retail gaming giant was taking a hit from the pandemic whilst already struggling (CNN). Much of these struggles pre-pandemic were the result of a lack of innovation in response to the gaming industry's shift away from in person shopping as well as shifting away from disc requiring games. With this slow spiral in mind, how could rational investors purchase Gamestop stock as an expectation of strong future dividends?

Between January 4th and January 25th of 2021, Gamestop saw an over 1800% increase in its stock price as the stock rose from a measly $17.69 to a whopping $325.00 (Yahoo Finance). Much of this increase was due to an organized targeting of the stock by an online community called "Wall Street Bets" found on the social media site Reddit. This group admittedly did not see any improvement in Gamestop's financial situation, and had no reason to expect any future, bottomline saving innovations. They had merely noticed that several large hedge funds had positioned themselves to short Gamestop, and the online community wanted to cause losses to 'elites' by making "an attempt to 'short-squeeze' those hedge funds" (Screen Rant). This gamification of the stock market resulted in widespread uncertainty and incredible volatility in Gamestop's stock value as well as the market on the whole. On January 28, 2021, CNBC reported that the DOW Jones plummeted "on Friday [Jan. 22, 2021], wrapping up a roller-coaster week on Wall Street as heightened speculative trading by retail investors continued to unnerve the market" which saw the DOW drop "620.74 points, or 2%" (CNBC). This incident displays the market's need for rational stock prices as irrational and gamified trading led to upheaval and tremendous losses. Simultaneously, it showcases the imperfections of the stock market and how a relatively small, cartel-like action can transcend the predetermined rules of rational stock prices and trading.

III. The Efficient Market Hypothesis

In order to understand if stock prices are rational, one first must understand how stocks come to be priced. With an ever fluctuating stock market, it is hard to pinpoint exactly what is used to determine an accurate and fair price of a stock. Information about the sales and productivity of a company, information about changes in management or the leadership structure within a company, and even information about the general economy of a country are all important factors in determining stock prices. The common theme around these and other factors is simply information—the information about a company, an industry, a country, etc is often the most important piece of the puzzle in the analysis of stock prices. This idea that information is heavily reflected in the stock market is known as the Efficient Market Hypothesis (EMH) (Ross et al. 408).

The Efficient Market Hypothesis is the understanding that stocks are priced based on available information and these prices are accurate, given said information. With any new information, prices will automatically adjust, therefore there will be no overreaction or delayed reaction to stock prices (Ross et al. 411). The question then is what type of information is being reflected, all information or just public? There exists three forms of market efficiency which describe different levels of available information: weak form efficient, semistrong form efficient, and strong form efficient (Ross et al. 411). The first of these forms indicates "the current price of a stock reflects the stock's own past prices" (Ross et al. 412). However, this does not mean that one is able to determine the future stock price by studying past prices—the method of using lagged stock prices to predict future ones is common but it does not produce good forecasts (Ross et al. 412). Semistrong form efficient markets, the second form of efficiency, means that market prices reflect known public information about a stock or company (Ross et al. 412). The emphasis here is on the public part of information, no insider information about a company is incorporated into stock prices which differentiates semi strong efficiency with strong efficiency. Therefore, a strong efficient market is defined when “all information of every kind is reflected in stock prices" (Ross et al. 411). When considering the rationality of stock prices, the three forms of market efficiency are important to include in the development of one’s argument.

While the Efficient Market Hypothesis is a great place to start in the understanding of stock pricing, we need to also understand some rebuttals to its proposition. One such counterargument of the idea of efficient markets is highly successful investors—think Warren Buffet and George Soros. Under the EMH, it would be practically impossible for them to make millions of dollars from investing because there would be no way for them to know which stocks are undervalued. Despite this fact, they still manage to earn extraordinary returns, so why is this not evidence to suggest that markets are not efficient? Simply put “markets can be efficient even if many market participants are quite irrational” (Malkiel 60). The reason successful investors have high returns is because they take on high risks. For example, throwing a dart randomly at a number of stocks would not produce the same return as the “pros” because the dart would produce a portfolio with average risk and average returns; on the other hand, the professional investors would often be choosing stocks that have higher risks associated with them, and thus higher returns (Ross et al. 410).

Another rejoinder to the Efficient Market Hypothesis is regarding the consistent daily fluctuation of stock prices. How can markets be efficient and stocks be priced correctly if stock prices are ever changing? Varying prices actually further prove the EMH; a predictable or consistent pattern of stock prices would imply in itself a non-efficient market. This would allow investors to exploit the expected prices to earn more return with a lower risk (Malkiel 63). Due to the perpetual stream of information, it corresponds to a perpetually changing stock market (Ross et al. 411). From the definition of the Efficient Market Hypothesis, any of the new data and information that has been discovered would immediately be factored into stock prices thus they would always be at an efficient level.

IV. Various Methods of Pricing Stocks

From a theoretical perspective, stocks are primarily priced by valuation by fundamentals. This involves analyzing a firm's fundamentals (products, operations, financial statements, etc) in order to estimate the future cash flow of the firm which is then discounted back to its present value. This approach is useful in many ways. Several examples of its uses are by active fund managers, firms engaging in Mergers and Acquisitions (M&A), and firms about to do an Initial Public Offering (IPO). Active fund managers want to find (and either buy or sell) stocks which they believe to be mispriced by markets. Firms engaging in M&A want to either find the appropriate price at which to sell themselves or the appropriate price at which to buy another firm. Firms about to do an IPO want to find the appropriate price at which to list themselves. It is of note, though, that it is almost always Investment Banks who, as a service for a fee, value firms and then help then engage in M&A or do an IPO.

There are several difficulties with valuation by fundamentals. It is often hard to estimate future cash flows precisely. It may not be obvious what the right measure of future cash flows is (is it cash returned to the shareholders or cash that is earned by the firm?). It also may be difficult to determine the correct discount rate (which should depend on the risk of the investment and hence the firm). Under the Efficient Market Hypothesis, the market would factor in prices all the necessary information regarding expected future cash flows and expected discount rates. In order to analyze the consistency of the EMH, it is important to look at how such valuations are generally done.

Now, we explain six different methods of valuing a stock given the assumption that future dividends, share prices, share repurchases, and firm cash flows are certain. If they are not, then the right hand side of the formulas in methods I, II, V, and VI would be in expected value. In III and IV, the dividends are assumed to be certain.

4.1. Method I

$$P_0 = \frac{D_1 + P_1}{1 + r_E}$$

Where:

P_0: current stock price

P_1: stock price at the next period

D_1: dividend paid in the next period

r_E: the discount rate appropriate for this equity

Assuming that the stock is purchased at the current time and sold in the next period after receiving a dividend, the future cash flow of the stock is $D_1 + P_1$ in the next period. We assume that these are the only cash flows to the investor. This then needs to be discounted at the appropriate rate which is how we arrive at the formula.

4.2. Method II

$$P_0 = \sum_{k=1}^{\infty} \frac{D_k}{(1 + r_E)^k}$$

Where:

D_k: dividend paid in period k

In this model, instead of assuming that the stock is sold after one period, we assume that the investor holds it forever. We still, however, assume that the only cash flows to the investor are the dividends paid out by the company.

We know $P_1 = \frac{D_2+P_2}{1+r_E}$, so then by substitution $P_0 = \frac{D_1+\frac{D_2+P_2}{1+r_E}}{1+r_E} = \frac{D_1}{1+r_E} + \frac{D_2+P_2}{(1+r_E)^2}$,

but now $P_2 = \frac{D_3+P_3}{1+r_E}$, so we can substitute again to get $P_0 = \frac{D_1}{1+r_E} + \frac{D_2}{(1+r_E)^2} + \frac{D_3+P_3}{(1+r_E)^3}$.

We can now proceed by induction. Suppose $P_0 = \sum_{k=1}^{n-1} \frac{D_k}{(1+r_E)^k} + \frac{D_n+P_n}{(1+r_E)^n}$ for some $n \in N$. We know $P_t = \frac{D_{t+1}+P_{t+1}}{1+r_E}$, so $P_0 = \sum_{k=1}^{n-1} \frac{D_k}{(1+r_E)^k} + \frac{D_n+\frac{D_{n+1}+P_{n+1}}{1+r_E}}{(1+r_E)^n} = \sum_{k=1}^{n} \frac{D_k}{(1+r_E)^k} + \frac{D_{n+1}+P_{n+1}}{1+r_E}$,

so by induction $P_0 = \sum_{k=1}^{n-1} \frac{D_k}{(1+r_E)^k} + \frac{D_n+P_n}{(1+r_E)^n}$, $\forall n \in N$. Now, we can take the limit as n approaches infinity to get

$$P_0 = \lim_{n\to\infty} \left(\sum_{k=1}^{n-1} \frac{D_k}{(1+r_E)^k} + \frac{D_n+P_n}{(1+r_E)^n}\right) = \lim_{n\to\infty} \left(\sum_{k=1}^{n-1} \frac{D_k}{(1+r_E)^k}\right) + \lim_{n\to\infty} \left(\frac{D_n+P_n}{(1+r_E)^n}\right)$$

$= \sum_{k=1}^{\infty} \frac{D_k}{(1+r_E)^k} + 0 = \sum_{k=1}^{\infty} \frac{D_k}{(1+r_E)^k}$, because we assume that both D_n and P_n grow slower than $(1 + r_e)^n$.

4.3. Method III

$$P_0 = \frac{D}{r_E}$$

Where:

D: the constant dividend paid in all periods

If the company announces that the dividend that they pay will be constant in all periods (call this constant D), then we may use this information and plug it into II. From II, we get $P_0 = \sum_{k=1}^{\infty} \frac{D}{(1+r_E)^k} = (\sum_{k=1}^{\infty} D(\frac{1}{1+r_E})^{k-1}) - D$. Now, from the infinite geometric sum formula we get $P_0 = \frac{D}{1-\frac{1}{1+r_E}} - D = \frac{D}{\frac{r_E}{1+r_E}} - D = \frac{D}{r_E}(1 + r_E) - D = \frac{D}{r_E} + D - D = \frac{D}{r_E}$.

4.4. Method IV

$$P_0 = \frac{D_1}{r_E - g}$$

Where:

D_1: the dividend paid in the first period

g: the constant periodly growth rate of the dividend

If the company announces that the dividend that they pay will have constant growth in all periods, then similarly to III we may use this information and plug it into II. Here, we assume the the dividend paid in period t will be a function of t, some specified growth rate g, and the dividend paid in the initial period D_1. Specifically,$D_t = D_1(1 + g)^{t-1}$which we now substitute into II. We must note that we assume $g < r_E$. Now, $P_0 = \sum_{k=1}^{\infty} \frac{D_1(1+g)^{k-1}}{(1+r_E)^k} = \sum_{k=1}^{\infty} \frac{D_1}{1+r_E} (\frac{1+g}{1+r_E})^{k-1}$. Again using the infinite geometric sum formula we get $P_0 = \frac{\frac{D_1}{1+r_E}}{1-\frac{1+g}{1+r_E}} = \frac{\frac{D_1}{1+r_E}}{\frac{r_E-g}{1+r_E}} = \frac{D_1}{r_E-g}$.

4.5. Method V

$$P_0 = \frac{PV(all\ future\ dividends\ and\ share\ repurchases)}{\#\ of\ shares\ outstanding}$$

Where share repurchase is a transaction in which the firm buys back its own shares from the marketplace. This is called the Total Payout Model of stock pricing. Similarly to I-IV above, this model also revolves around discounting the future cash flows of the stock. The difference, though, is that we relax our assumption that the only cash flows to the investor are the dividends paid out by the company. In addition to paying dividends, some companies also return value to their shareholders in the form of share repurchases. A share repurchase is a transaction where the firm buys back its own shares from the marketplace. Assuming that an investor holds a stock forever, receives dividends, and only sells in the case of a share repurchase, the future cash flows of the stock to the investor are those dividends and share repurchases. Thus, the value of the stock is the present value of those dividends and share repurchases. Now, we can take the present value of *all* future dividends and share repurchases and divide that by the total number of shares outstanding to get the value of one share (which gives V).

4.6. Method VI

$$P_0 = \frac{Value_0 - Debt_0 + Cash_0}{\# \ of \ shares \ outstanding}$$

Where:

$Value_0$: Enterprise Value of the firm at time t = 0

$Debt_0$: Debt of the firm at time t = 0

$Cash_0$: Cash held by the firm at time t = 0

Some firms never pay dividends or engage in share repurchases, and this model provides a way to value stocks in the absence of those two things. At a high level, this model essentially involves valuing the total amount of money that *could* be distributed to all shareholders and dividing that by the number of shares outstanding to determine the value of one share. The Equity Value of a firm is the value of the total amount of money that could be distributed to all shareholders. This should make intuitive sense because the value of a firm's equity should (similarly to the above cases) be the value of the cash flows that equity holders will (or could) receive in the future. The Enterprise Value of a firm is the value of the total amount of money that could be distributed to *all claimants* (excluding currently held cash). To get to Equity Value from Enterprise Value we then must subtract the value of the firm's debt because that money is owed to debtors and thus not available to equity holders. We must also add the cash held by the firm because this was not reflected in the Enterprise Value and could be distributed to equity holders. This gives us the above formula. However, when actually applying the formula we need a way to calculate the firm's Enterprise Value. The total amount of money that could be distributed to all claimants (excluding currently held cash) is the total future free cash flows of the firm, so then the Enterprise Value is the present value of those free cash flows which we find by, again, using the correct discount rate. Free cash flow is the cash that the firm generates left over after accounting for cash outflows to support its operations and maintain its capital assets. There are several ways of calculating future free cash flows, but they rely on a fair amount of accounting knowledge, and hence, they are beyond our scope here.

V. Statistical Analysis Over Historical Data

Under the Efficient Market Hypothesis, the current market price accounts for all the available information regarding the future value of dividends and interest rates. We want to consider whether this is actually true. As we alluded to in the previous section, this corresponds to taking the future values in the valuation methods above as random variables and determining the price as the expected value of those formulas. In other words, the Efficient Market Hypothesis states that:

$$P_t := E_t\left[P_t^*\right]$$

Where:

P_t: price of the stock at the end of period t corrected by inflation

P_t^*: the *ex post* rational price, the price if we had knowledge of the actual future

E_t: the expectation conditioned on the data available at time t

We will model the price P_t^* in terms of future dividends, as described in method II from section 4, where we regarded the price of the stock to be the present discounted value of future cash flows, which we assumed to be composed solely of the issued dividends. Thus:

$$P_t^* := \sum_{k=1}^{\infty} \frac{D_{t+k}}{(1 + r_E)^k}$$

Where:

D_t: dividends paid at the end of period t corrected by inflation

r_E: real discount rate

To estimate the real discount rate implicit in the market pricing, we note that taking the expected value of the return from t to $t + 1$:

$$\frac{P_{t+1} - P_t + D_{t+1}}{P_t} = \frac{\sum_{k=1}^{\infty} \frac{D_{t+1+k}}{(1+r_E)^k} - \sum_{k=1}^{\infty} \frac{D_{t+k}}{(1+r_E)^k} + D_{t+1}}{P_t}$$

$$= \frac{\left(\sum_{k=2}^{\infty} \frac{D_{t+k}}{(1+r_E)^{k-1}}\right) - \sum_{k=1}^{\infty} \frac{D_{t+k}}{(1+r_E)^k} + D_{t+1}}{P_t}$$

$$= \frac{\left(1+r_E\right)\left(\sum_{k=2}^{\infty} \frac{D_{t+k}}{(1+r_E)^k}\right) - \sum_{k=1}^{\infty} \frac{D_{t+k}}{(1+r_E)^k} + D_{t+1}}{P_t}$$

$$= \frac{\left(1+r_E\right)\left(\sum_{k=1}^{\infty} \frac{D_{t+k}}{(1+r_E)^k} - \frac{D_{t+1}}{1+r_E}\right) - \sum_{k=1}^{\infty} \frac{D_{t+k}}{(1+r_E)^k} + D_{t+1}}{P_t}$$

$$= \frac{\left(1+r_E\right)\left(\sum_{k=1}^{\infty} \frac{D_{t+k}}{(1+r_E)^k} - \frac{D_{t+1}}{1+r_E}\right) - \sum_{k=1}^{\infty} \frac{D_{t+k}}{(1+r_E)^k} + D_{t+1}}{P_t}$$

$$\left(1 + r_E\right) - 1 + \frac{\left(1+r_E\right)\left(-\frac{D_{t+1}}{1+r_E}\right) + D_{t+1}}{P_t} = r_E$$

As a result, we can simply use the geometric average of returns to estimate the real discount rate. As elaborated in method II of section 4, we may write the formula recursively:

$$P_t^* = \frac{P_{t+1}^* + D_{t+1}}{1 + r_E}$$

Since we do not have data all the way to infinite, we approximate the *ex post* rational price by making assumptions about the last point $P^*_{t_{last}}$ for which we still have market predictions. In other words, we must make an assumption about the future dividends for which we do not have data. Note however, that as we compute recursively prices in the past we successively divide $P^*_{t_{last}}$ by $1 + r_E$, such that its influence in the series P^*_t for becomes less and less significant. As a result, for the purposes of this paper, we will simply assume $P^*_{t_{last}}$ to be equal to the observed market price $P_{t_{last}}$. When comparing P_t and P^*_t far in the past, this assumption should have almost no effect on our results.

Computing the *ex post* rational price series for the S&P 500, we obtain the following plot:

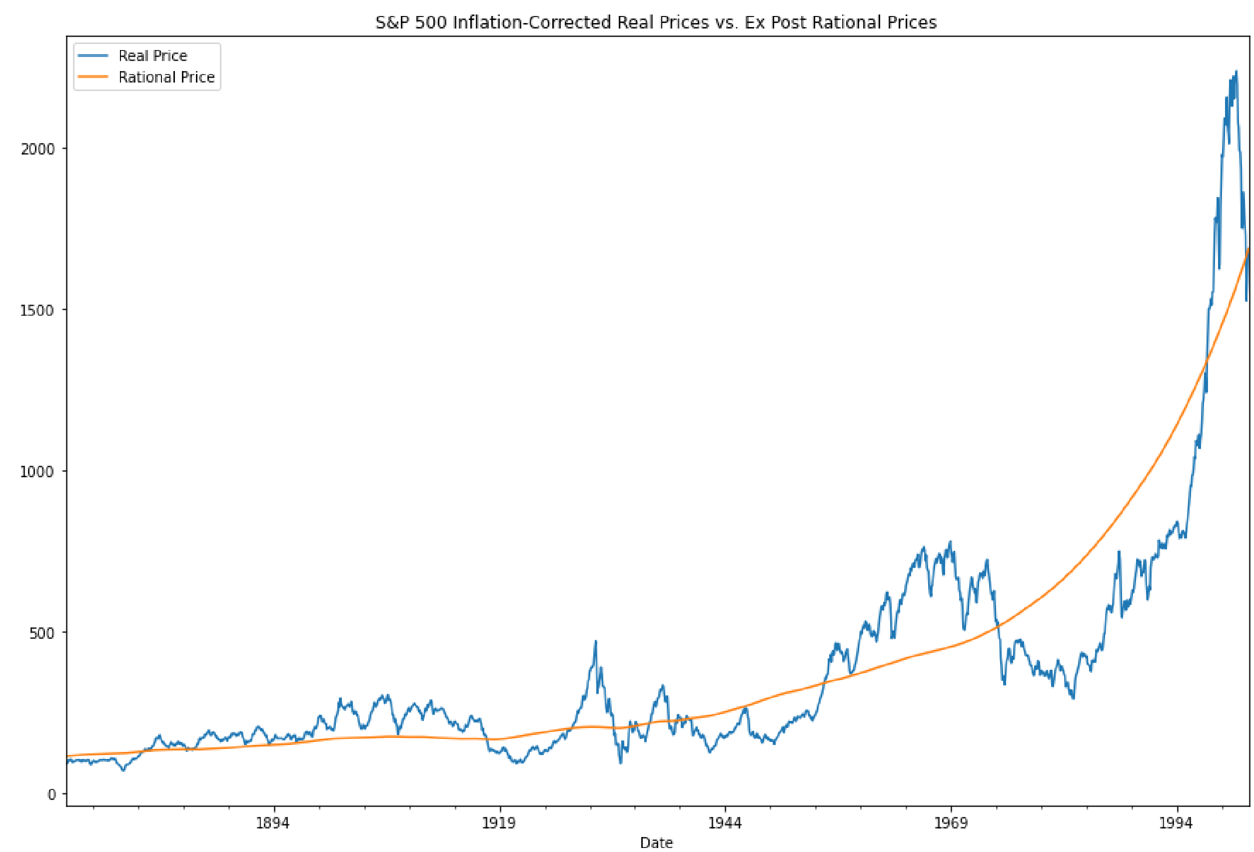

Figure 1

While we can already compare the two series and see that the volatility of the real market prices is much larger than the volatility of *ex post* rational prices, we would like to be able to actually compute the standard deviation of each series with respect to the growth trend. To visualize the detrending method, let us first look at the same graph as above in logarithmic scale:

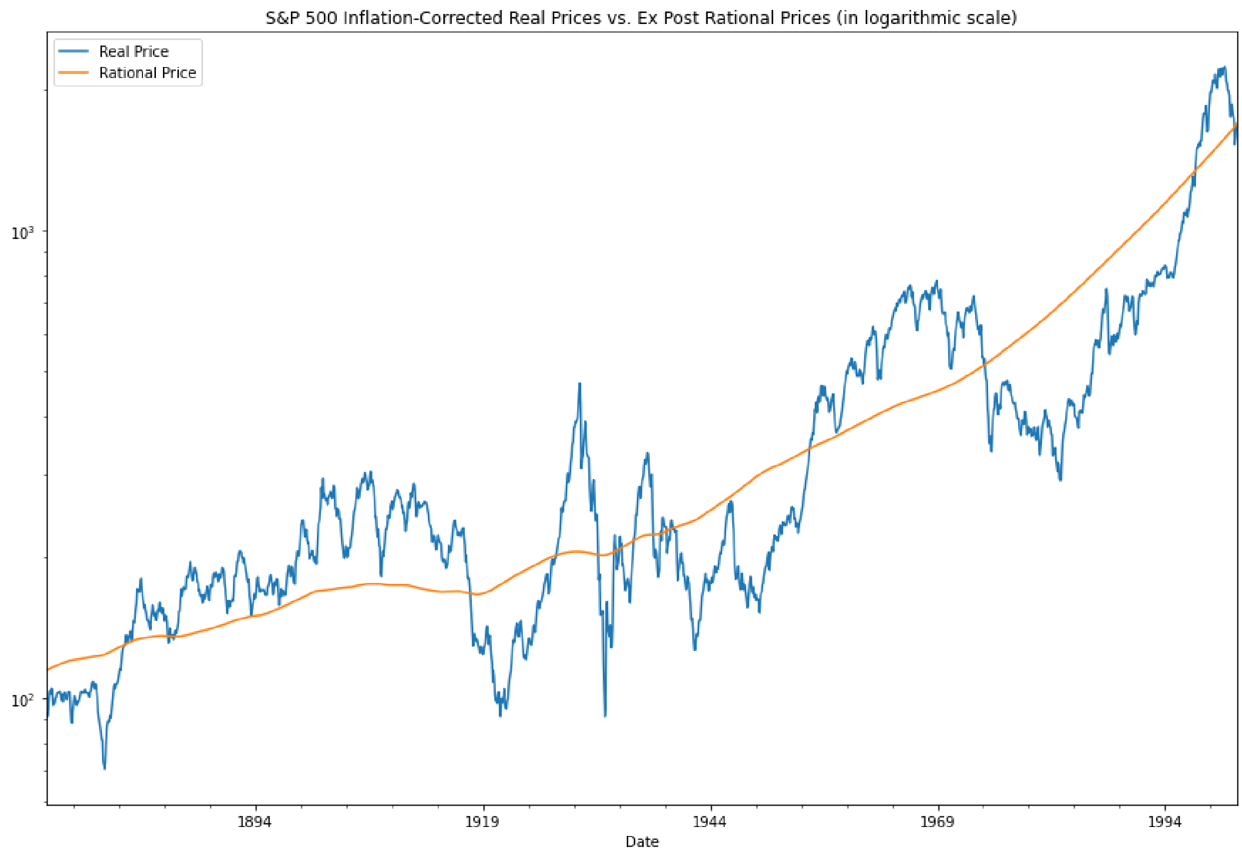

Figure 2

We can see that exponential growth is roughly constant by noticing that the series form a roughly linear shape in logarithmic scale. This prompts a detrending based on an exponential growth factor. Following Shiller, we define the rate of growth λ , a trend factor for price and dividend series. We can use this to normalize stock prices with respect to growth by multiplying the equation for P_t^* by λ^{T-t}, where T is the base year, or the first period in our data series. We call this growth-detrended *ex post* rational price p_t^*, which is written in terms of the growth-normalized dividend d_t as defined below:

$$p_t^* := \lambda^{T-t} P_t^*$$

$$= \sum_{k=1}^{\infty} \lambda^{T-t} \frac{D_{t+k}}{(1+r_E)^k}$$

$$= \sum_{k=1}^{\infty} \frac{\lambda^{T-t}}{\lambda^{T-(t+k)}} \frac{\lambda^{T-(t+k)} D_{t+k}}{(1+r_E)^k}$$

$$= \sum_{k=1}^{\infty} \frac{\lambda^{T-t}}{\lambda^{T-(t+k)}} \frac{d_{t+k}}{(1+r_E)^k} \qquad \text{(we define } d_t := \lambda^{T-t} D_t\text{)}$$

$$= \sum_{k=1}^{\infty} \lambda^k \frac{d_{t+k}}{(1+r_E)^k}$$

$$= \sum_{k=1}^{\infty} \frac{\lambda^k}{(1+r_E)^k} d_{t+k}$$

We define the growth-normalized discount rate to be $\bar{r}$:

$$\bar{r} := \frac{\lambda}{1+r_E} - 1$$

Which produces the simplified expression:

$$p_t^{\,*} = \sum_{k=1}^{\infty} \frac{d_{t+k}}{(1+\overline{r})^k}$$

We also define the detrended real price series:

$$p_t := E_t\left[p_t^{\,*}\right]$$

Alternatively, we may write the recursive formula, as done previously:

$$p_t^{\,*} = \frac{p_{t+1}^{\,*} + d_{t+1}}{1+\overline{r}}$$

Then, by fitting the price series to an exponential so as to determine the growth factor λ, and then detrending the data, we obtain the following plot:

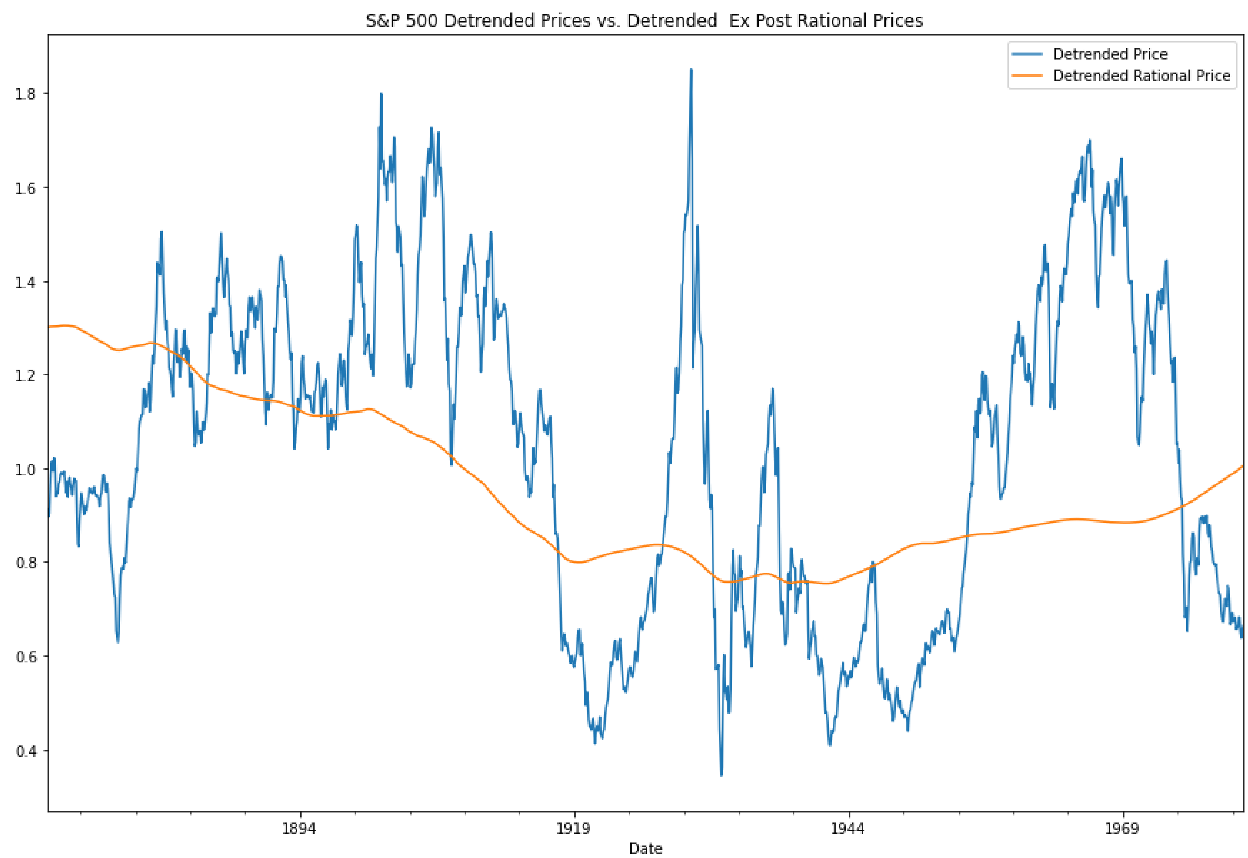

Figure 3

And we also compute the standard deviation for the two series:

Detrended Price: $\sigma(p_t) = 0.426$

Detrended Rational Price: $\sigma(p_t^*) = 0.230$

This allows us to make a much more robust comparison of the volatility of market prices and *ex post* rational prices, as shall follow in the next section. The code for analysing the data can be found in Appendix A.

VI. Volatility Comparison

As explained in the section above, the efficient market model can be described as asserting that:

$$p_t := \mathrm{E}\,[p_t^*]$$

In other words, p_t is the optimal forecasting of p_t^* according to its expected value. To measure how well the p_t "forecast" predicts p_t^*, we can define the forecast error:

$$u_t := p_t^* - p_t$$

One key fact of the efficient markets model is that the forecast error must be uncorrelated to p_t. In other words, the covariance of p_t with u_t must be equal to zero. The reason behind this statements is that we assume that p_t is the best forecast to p_t^*. If the forecast error u_t was consistently correlated with the forecast itself, then the forecast p_t could be improved. Therefore, markets would not be efficient. By contradiction, we determine that covariance(u_t, p_t) = 0.

Rearranging the equation above, we get:

$$p_t^* = u_t + p_t$$

Once the variables in the right hand side are independent, we can use the principle of statistics that the variance of the sum of two uncorrelated variables is the sum of their variances:

$$\mathrm{var}(p_t^*) = \mathrm{var}(u_t + p_t)$$

Since variances are non-negative, we arrive at $\mathrm{var}(p_t) \leq \mathrm{var}(p_t^*)$, which can be restated in terms of standard deviations as:

$$\sigma(p_t) \leq \sigma(p_t^*)$$

This conclusion might be surprising for some readers, as this means that the price of stock p_t should fluctuate less than the *ex post* rational price series p_t^*. Although our day-to-day experience with stock price fluctuations might have led us to believe that p_t would have a higher variance in the efficient market model, it is possible to create a simple model that illustrates why p_t^* must fluctuate more in the model.

Imagine a world in which the price of a stock can be either 1 (state 1) or 2 (state 2) in period t. The probability that state 1 happens is p=0.5, while the probability that state 2 happens is q = (1 - p) = 0.5. Let's simulate the price of the stock from period 1 to 10, and output the result in a graph:

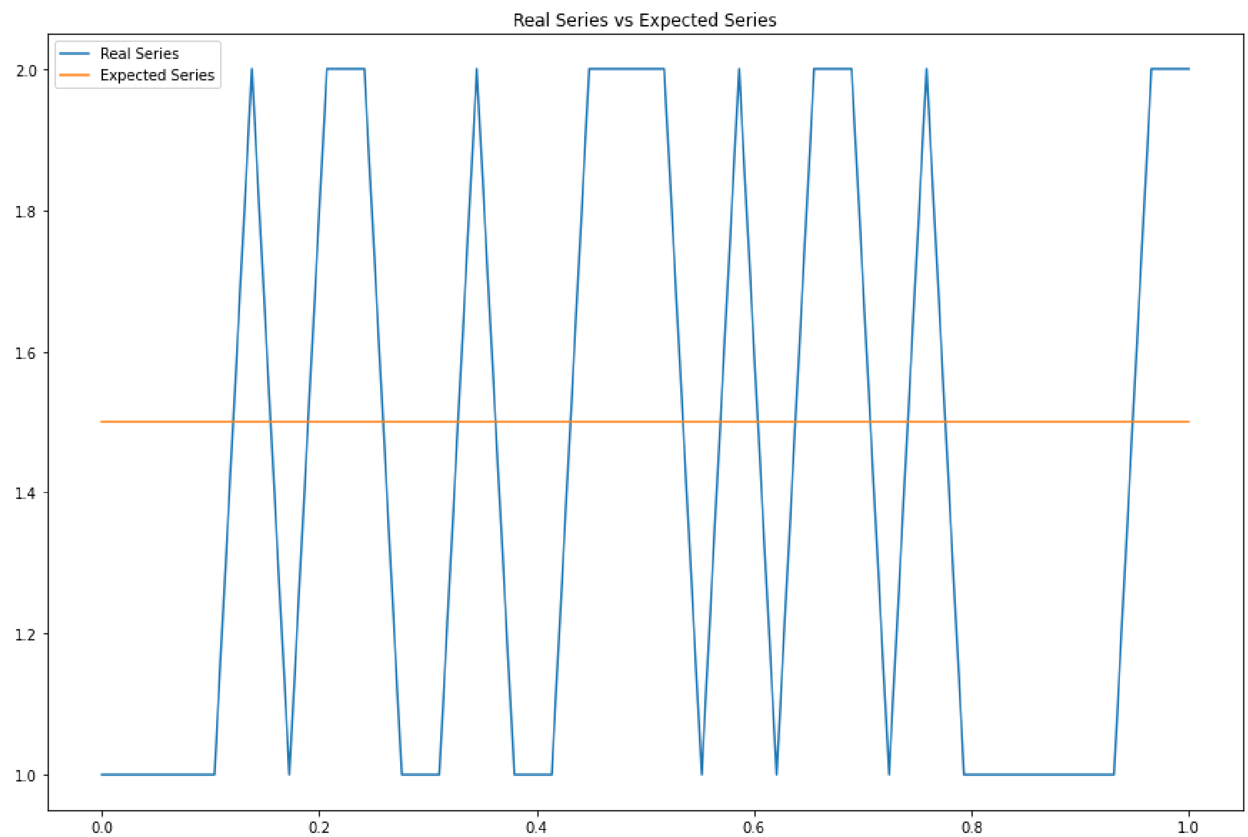

Figure 4

As we can see from the graph above, p_t^* remained constant over the whole period, as:

$$p_t = \mathrm{E}[p_t^*] = 1.5$$

Therefore, if the Efficient Market Hypothesis is correct under our assumptions, we would expect that $\mathrm{var}(p_t) \leq \mathrm{var}(p_t^*)$. However, empirical evidence does not support this conclusion. We calculated that the variance of p_t^* from 1870 to 1980 was 0.230, two times lower than the variance found for p_t, which was 0.426.

VII. Can We Reconcile Volatility With The Efficient Market Hypothesis?

The above result can be found in Shiller's 1981 paper. Unsurprisingly, the Efficient Market Model does not fully explain the data. Therefore, we are forced to either give up the model completely (which is usually not ideal) or we can try to make adjustments on our initial assumptions to account for previously unforeseen scenarios.

One possible explanation for this phenomenon is that the market is constantly (and rightfully) expecting that real interest rates will fluctuate drastically. This explanation is valid because we cannot directly measure the expected real interest rates. Nevertheless, this means that we cannot evaluate this hypothesis statistically. What we can do, however, is to calculate how much the expected real interest rate would have to fluctuate in order to fully explain the discrepancy between p_t^* and p_t. Shiller does precisely that in his paper in section IV. Time-Varying Real Discount Rates. He changed the Efficient Market Model in order to calculate the variance in stock prices given a fluctuation in the expectation of real interest rates $\bar{r}$. Using this model, we can estimate that expected real interest rates from 1960 to 1980 would have to fluctuate between -8.16 to 17.27 percent! This is the lowest possible fluctuation of $\bar{r}$ that can explain the price fluctuations. However, this is much greater than the actual real interest rates fluctuations in the period (Figure 5), so market expectations would have to be far from the actual values.

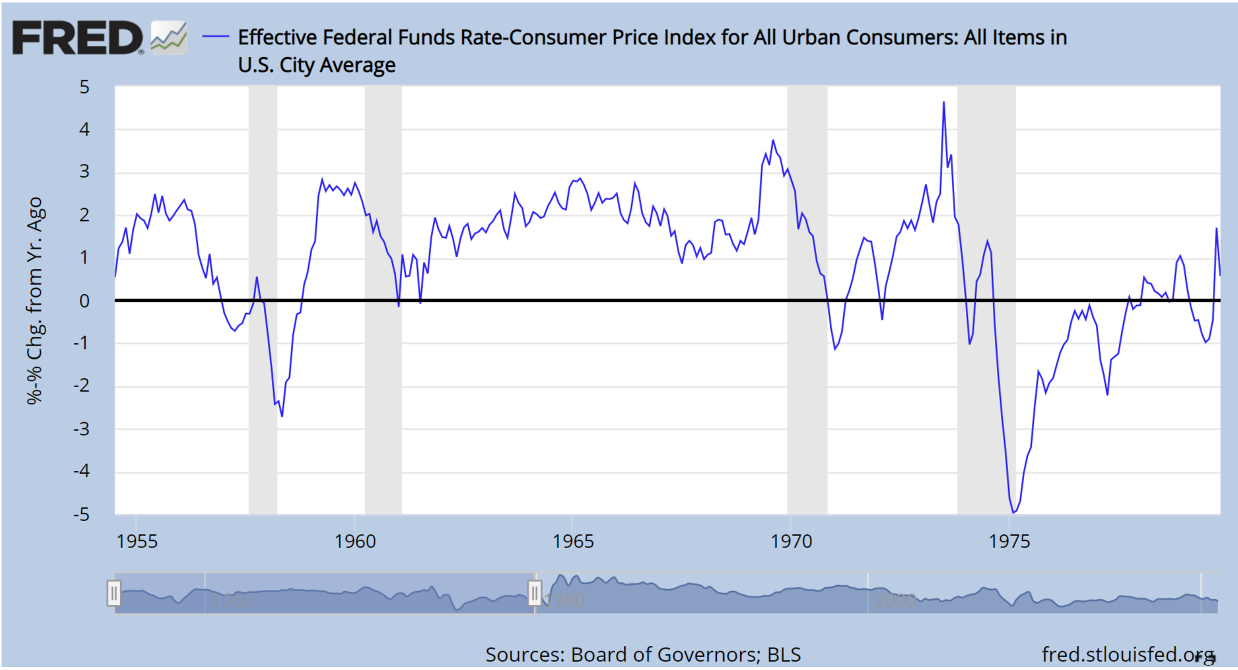
FRED
Effective Federal Funds Rate-Consumer Price Index for All Urban Consumers: All Items in U.S. City Average
%-% Chg. from Yr. Ago
5
4
3
2
1
0
-1
-2
-3
-4
-5
1955
1960
1965
1970
1975
Sources: Board of Governors; BLS
fred.stlouisfed.org

Figure 5

VIII. Conclusion

In this paper, we have looked at the discussion of whether market volatility can be reconciled with the Efficient Market Hypothesis. We have started from a brief overview of the history of stock markets and recent developments that have produced increased volatility in markets such as the Gamestop trading craze and moved into examining the claims of the Efficient Market Hypothesis. We have considered multiple valuation techniques used by financial institutions in determining the expected price of a stock based on the fundamental data available about a company. Using such methods and data-analysis code written in Python, we have reproduced results from Shiller that show that market volatility is in fact much higher than what rationally expected by future changes in fundamentals. This would seem to indicate a serious failure of the Efficient Market Hypothesis, but we do also consider different arguments for why such a discrepancy would not contradict the EMH.

Bibliography

Contributor, Chron. “How the Stock Market Was Started & by Whom.” *Small Business - Chron.com*, Chron.com, 24 Nov. 2020, smallbusiness.chron.com/stock-market-started-whom-14745.html.

Contributor, Social Finance. “A Brief History of the Stock Market.” *SoFi*, SoFi, 20 Jan. 2021, www.sofi.com/learn/content/history-of-the-stock-market/.

Desjardins, Jeff. “How Today's Tech Giants Compare to the Massive Companies of Empires Past.” *Business Insider*, Business Insider, 12 Dec. 2017, www.businessinsider.com/how-todays-tech-giants-compare-to-massive-companies-of-empires-past-2017-12.

Malkiel, Burton G. "The efficient market hypothesis and its critics." *Journal of economic perspectives* 17.1 (2003): 59-82.

Pirani, Fiza. “These Are the 5 Biggest Stock Market Crashes in US History.” *Ajc*, The Atlanta Journal-Constitution, 6 Feb. 2018, www.ajc.com/news/national/these-are-the-biggest-stock-market-crashes-history/XAS0qPBaFbTY25OGZV4jXK/.

Ross, Stephen A., Randolph Westerfield, and Bradford D. Jordan. "Some Lessons from Capital Market History." *Fundamentals of Corporate Finance*. 12th ed. New York: McGraw-Hill Education, 2019. 382-412. Print.

Shiller, Robert J. “Do Stock Prices Move Too Much to Be Justified by Subsequent Changes in Dividends?” *The American Economic Review*, vol. 71, no. 3, 1981, pp. 421–436. *JSTOR*, www.jstor.org/stable/1802789. Accessed 13 Apr. 2021.

“GameStop Corporation (GME) Stock Price, News, Quote & History.” *Yahoo! Finance*, Yahoo!, 3 May 2021, finance.yahoo.com/quote/GME/.

Li, Yun. “Dow Drops More than 600 Points Friday, Suffers Worst Week since October amid GameStop Trading Frenzy.” *CNBC*, CNBC, 1 Feb. 2021, www.cnbc.com/2021/01/28/stock-futures-decline-as-volatile-wall-street-week-continues.html.

Tyler, Joshua. “GameStop Stock Fluctuations Could Save Company From Bankruptcy, Says Analyst.” *ScreenRant*, 28 Jan. 2021, screenrant.com/gamestop-stock-changes-save-bankruptcy/.

Valinsky, Jordan. “GameStop Is Closing Hundreds More Stores.” *CNN*, Cable News Network, 10 Sept. 2020, www.cnn.com/2020/09/10/investing/gamestop-store-closures/index.html.

Appendix A: Historical Data Analysis and Plot Generation Code

The following Python 3 code was written from scratch. It reproduces data analysis by Shiller and includes data analysis using our own derivations. The following packages must be installed before running:

```
pip install pandas, numpy, matplotlib, scipy
```

```
#!/usr/bin/env python3

import pandas as pd
import numpy as np
from matplotlib import pyplot as plt
from scipy.stats.mstats import gmean
from scipy.signal import detrend

df = pd.read_csv('historical-data.csv')

# parse date from 2019.01, 2019.02, ... 2019.1, 2019.11, 2019.12 format
def read_date(x):
    x = str(x)
    if len(x) == 6:
        x += "0"
    return pd.to_datetime(x, format="%Y.%m")

df['Date'] = pd.to_datetime(df['Date'].apply(read_date), errors='coerce')
df.set_index("Date")

# interpolate interest rates (some values are missing for the first few
years)
df['Long Interest Rate GS10'] = df['Long Interest Rate GS10'].interpolate()

# interpolate monthly dividends based on quarterly dividends
df['Dividends'] = df['Dividends'].interpolate()

# interpolate monthly earnings based on quarterly earnings
df['Earnings'] = df['Earnings'].interpolate()

# restrict to up to previous year, 2021 doesn't have all the necessary data
```

```
df = df.loc[df["Date"] < "2021-01-01"]

# use Consumer Price Index to compute how many dollars at the time would
correspond to today's dollars due to inflation
# most_recent_cpi = 264.09
most_recent_cpi = df.iloc[-1]["Consumer Price Index"]
df["Inflation Adjustment"] = most_recent_cpi/df["Consumer Price Index"]

# compute prices adjusted by inflation
df["Real Price"] = df["S&P Composite Price"] * df["Inflation Adjustment"]

# compute dividends adjusted by inflation
df["Real Dividends"] = df["Dividends"] * df["Inflation Adjustment"]

# compute earnings adjusted by inflation
df["Real Earnings"] = df["Earnings"] * df["Inflation Adjustment"]

# compute real total return price
df.loc[0, "Real Total Return Price"] = df.loc[0, "Real Price"]
for i in range(1, len(df)):
    df.loc[i, "Real Total Return Price"] = df.loc[i - 1, "Real Total Return
Price"] * (df.loc[i, "Real Price"] + df.loc[i, "Real Dividends"]/12) /
df.loc[i - 1, "Real Price"]

# compute 10-year annualized stock real return
for i in range(0, len(df) - 12):
    df.loc[i, "Real Return"] = (df.loc[i + 12, "Real Total Return
Price"]/df.loc[i, "Real Total Return Price"]) - 1

# restrict to data up until 2002:
df = df.loc[df["Date"] < "2002-01-01"]

# determine the implicit discount rate r_e
r_e = gmean(1 + df["Real Return"]) - 1

print(f"r_e = {r_e}")

# determine the ex post rational price series
df.loc[len(df) - 1, "Rational Price"] = df.loc[len(df) - 1, "Real Price"]
for i in range(len(df) - 2, -1, -1):
    df.loc[i, "Rational Price"] = (df.loc[i+1, "Rational Price"] +
df.loc[i+1, "Real Dividends"]/12) * (1 + r_e)**(-1/12)
```

```
desired_dates = (df["Date"] >= "1870-01-01") & (df["Date"] <= "1980-01-01")
df.plot(x="Date", y=["Real Price", "Rational Price"], figsize=(15, 10),
title="S&P 500 Inflation-Corrected Real Prices vs. Ex Post Rational Prices")

df.plot(x="Date", y=["Real Price", "Rational Price"], figsize=(15, 10),
title="S&P 500 Inflation-Corrected Real Prices vs. Ex Post Rational Prices
(in logarithmic scale)", logy=True)

# detrend price series
df["Detrended Price"] = np.exp(detrend(np.log(df["Real Price"])))
df["Detrended Rational Price"] = np.exp(detrend(np.log(df["Rational
Price"])))

df.loc[desired_dates].plot(x="Date", y=["Detrended Price", "Detrended
Rational Price"], figsize=(15, 10), title="S&P 500 Detrended Prices vs.
Detrended  Ex Post Rational Prices")

# compute standard deviation of the two series
stddev_p = np.std(df["Detrended Price"])
stddev_p_star = np.std(df["Detrended Rational Price"])

print(f"sigma(p) = {stddev_p}, sigma(p*) = {stddev_p_star}")

# generate graph for standard deviation comparison explanation
n = 30
x = np.linspace(0, 1, n)
y_real = np.random.choice([1, 2], n)
y_pred = np.full(n, 1.5)

plt.subplots(figsize=(15, 10))
plt.title("Real Series vs Expected Series")
plt.plot(x, y_real, label="Real Series")
plt.plot(x, y_pred, label="Expected Series")
plt.legend()
plt.show()
```