

Chapter 3

The Input: Waves

Learning Objectives:

1. For Regular Waves:

- (a) Identify wave length, wave frequency, and wave celerity given the water elevation measurements over time.
- (b) Explain the distinction between wave celerity and group velocity.
- (c) Calculate the theoretical wave celerity and group speed given wave frequency and water depth.
- (d) Write the equation for regular sinusoidal water elevation as a function of time given the wave frequency (ω) or wave length (λ) and the wave height or amplitude (H_0 or ζ_0).
- (e) Calculate the average energy per unit area of the wave surface.
- (f) Calculate the pressure at a point beneath the water surface, accounting for wave motion.
- (g) Describe the particle motion below a water wave.

2. For Irregular Waves:

- (a) Find the mean water elevation, wave height, zero-crossing period, and peak period for a given irregular wave time history.
- (b) Find the variance of a water elevation time history.
- (c) Explain what the significant wave height is and how to find it from the irregular *wave time history*.
- (d) Explain the concept of linear superposition and how it applies to modeling irregular waves.
- (e) Describe how the Fourier Transform can be used to find the frequency content of an irregular wave time history.
- (f) Describe how to use a Fourier Transform in finding the wave energy spectrum of an irregular wave time history.
- (g) State what information is shown in a wave energy spectrum.

- (h) State the equation for the spectral ordinate of the wave energy spectrum and how the spectral ordinate relates to the wave amplitude.
- (i) Explain how to find a spectral moment and what the zeroth, second, and fourth spectral moments correspond to with respect to the water surface.
- (j) Find the mean water elevation, wave height, zero-crossing period, and peak period for a given irregular wave energy spectrum.
- (k) Explain what the significant wave height is and how to find it from the irregular *wave energy spectrum*.
- (l) State the three idealized wave spectra used in this course, what parameters each depends on, and their equations.
- (m) Identify what ocean conditions each idealized wave spectrum is most suited to model.
- (n) Explain what a “probability of exceedance” is and be able to calculate it for a given sea condition.
- (o) State which probability distribution can be used to describe probabilities related to **water elevation** and **wave heights**.

3. Laboratory Objectives:

- (a) Calculate the wave length and frequency of a regular wave using wave elevation data (using both digital signal processing techniques and equations)
- (b) Calculate the wave celerity and group velocity for regular waves using wave elevation time history at two different, known locations.
- (c) Identify the effect of water depth on wave celerity and group velocity.
- (d) Describe the purpose of the discrete Fourier Transform (DFT)for analyzing wave measurements.
- (e) Explain the connection between an irregular wave time history and the related wave energy spectrum.
- (f) Create, measure, and analyze complex waves using a FFT.

What are water waves? How do they move? How do we describe them? To tackle this topic, I am going to start with the simple (yet unrealistic) concept of “regular” waves. Once we have an understanding of how these hypothetical waves (that can be accurately produced in the laboratory) work, we can move to discussing how to deal with more realistic wave conditions.

3.1 Regular Waves

Regular waves are shaped like a sine wave moving along the surface of the water. We are going to start this discussion with long-crested, deep water waves. This type of wave is *periodic*, meaning it has a consistent frequency or period of occurrence. Table 3.1 gives the

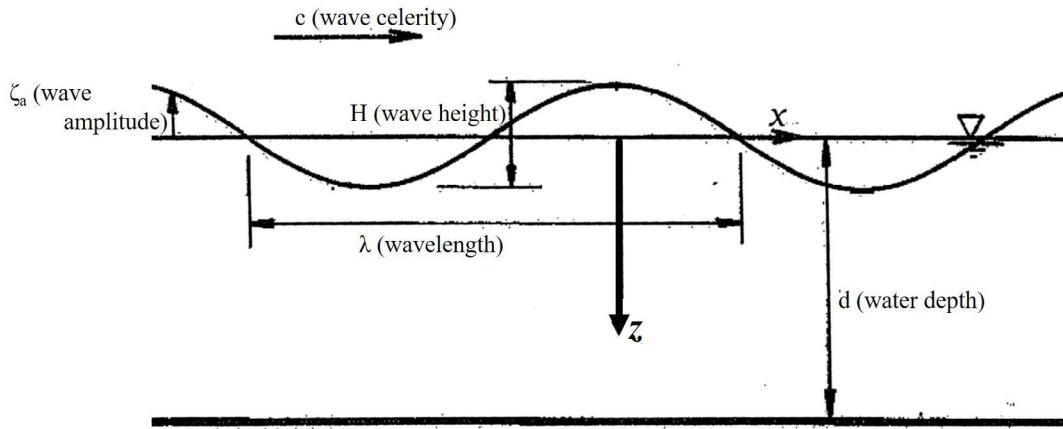


Figure 3.1: Include an image that shows a regular wave with labels

notation we will use to describe the characteristics of our regular wave and Figure 3.1 shows a wave with the components labeled. Our wave is a progressive wave, meaning it moves horizontally over the water surface (a wave that merely oscillates up and down is considered a standing wave). The shape of each wave passing by looks the same and the whole wave train can be viewed as an advancing rigid corrugated sheet.

The following notation (following reference 2) is used for describing water waves:

ζ	the instantaneous depression of the water surface below the mean level ($y = 0$)
ζ_0	the wave amplitude from the mean level ($y = 0$) to a crest or trough
H	wave height (always twice the wave amplitude)
λ	wave length (distance from one crest - or trough - to the next)
c	wave celerity
T	wave period (time interval between successive crests - or troughs - passing a fixed point)
α	the instantaneous wave slope (gradient of the surface profile)
α_0	maximum wave slope or wave slope amplitude
H/λ	wave steepness

Table 3.1: Nomenclature for describing waves

If we were to consider only a single point in space and describe the water surface elevation at that point as the wave moves past, we have the following mathematical expression:

$$\zeta(t) = \zeta_0 \sin(\omega t - \epsilon)$$

In this expression, t is the variable for time, ζ_0 is the wave amplitude, ϵ is the phase angle (the degrees the shape is different from a perfect sine wave) and ω is the wave frequency (in radians/second) - i.e. a measure of the oscillations that pass this point in one second. If we instead consider the entire wave train in space, but only for a single moment in *time*, the mathematical expression is:

$$\zeta(x) = \zeta_0 \sin(kx)$$

In this expression x is the variable for position and k is the **wave number** (in m^{-1} or ft^{-1}). The wave number represents the frequency as a function of wavelength - the number of cycles that occur over a unit of length. The expression for wave number is

$$k = \frac{2\pi}{\lambda}.$$

But water waves exist and change in both *time* and *space* - you can stay with a wave and move through space in time **or** you can stay at one location and see the wave move past in time. So the equation for the water elevation must account for the point at which we are measuring (where are you standing?) and the time the measurement is made (what time are you looking?).

$$\zeta(x, t) = \zeta_0 \sin(kx - \omega t - \epsilon)$$

For regular waves in deep water, there is a fixed relationship between the frequency of the wave, the length of the wave, and the speed that the wave travels. For a high frequency wave, there is only a short time between peaks and, therefore, the wave length is very short. For a low frequency wave, there is a long time between peaks and the wave length is long. *For deep water*, these relationships are as follows:

$$T = \frac{2\pi}{\omega}$$

$$\lambda = \frac{2\pi g}{\omega^2} = \frac{gT^2}{2\pi}$$

As the water depth becomes shallower, the relationship between wave length and wave frequency changes. In shallow water the wave length depends only on the water depth. So, *for shallow water*:

$$\lambda = 2\pi d$$

where d is the water depth.

The general relationship between wave frequency, wave length, and water depth is given by the “**Dispersion Equation**”, which includes a hyperbolic tanh:

$$\omega^2 = gk \tanh kd.$$

Hyperbolic Functions

Hyperbolic functions are combinations of exponentials:

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

For water waves, we are using the kd term (or $2\pi d/\lambda$) in the hyperbolic function. When the water depth is very large relative to the wave length, kd is large. When depth is small relative to the wave length, kd becomes very small. We can simplify the hyperbolic expression in these situations:

Function	Large kd	Small kd
$\sinh kd$	$e^{kd}/2$	1
$\cosh kd$	$e^{kd}/2$	kd
$\tanh kd$	1	kd

Using these simplifications of the hyperbolic functions, we can show that in deep water the wave frequency depends only on the wave length,

$$\omega = \sqrt{gk}$$

while in shallow water the wave frequency also depends on the water depth,

$$\omega = \sqrt{gk^2d}.$$

Wave Celerity

The wave celerity, that is the speed of the wave traveling over the water surface, is given by:

$$c = \sqrt{\frac{g}{k} \tanh kd}.$$

As with the wave length, the equation for the velocity of the wave in deep and shallow water can be found by simplifying this equation. In deep water the wave celerity is:

$$c = \sqrt{\frac{g}{k}}$$

or

$$c = \frac{g}{\omega}.$$

In shallow water the wave celerity is:

$$c = \sqrt{gd}.$$

This relationship helps explain why tsunamis are difficult to observe out in the open ocean yet develop into towering waves as they approach the shore. When in deep water the tsunami wave has a very long wave length and is traveling extremely fast. However, as the wave approaches the shore the wave speed (and length) becomes determined by the water depth and the wave has to slow down. Thus, the energy in the wave that was stored in the speed (kinetic energy) is transformed into energy stored in the wave amplitude (potential energy).

Energy Transmission and Group Velocity

While celerity can give you the velocity of the crest of a wave moving over the water surface, the group velocity gives the velocity of the energy associated with the wave. This is a strange concept, but it can be demonstrated in a wave tank or tow tank. When the wave maker sends the first wave of a regular wave train down the tank, you can witness the height of the wave decreasing as it travels and eventually that first wave disappears. This is because the energy in the wave (which is seen in the wave height) is traveling half as fast as the wave crest (for waves in deep water). How does this work? Consider one wave in the middle of a train of waves, all nominally the same height and wavelength. This one wave moves one wavelength forward in one wave period, but it only takes 1/2 of the energy with it. However, the wave directly in front also took only 1/2 of its energy, so our wave gets the left over energy (returning to full energy level) and maintains its current wave amplitude. Now, what happens to the wave at the front of the wave train? There is no wave in front of it to leave any energy. So, each wavelength this wave moves forward decreases its energy level by 1/2, which results in a reduced wave amplitude! And what happens to the wave at the back of the wave train? When it travels forward it leaves 1/2 of its energy behind, but there is no wave behind to make use of that energy. So, the 1/2 energy of the last wave that is left behind creates a new wave at a lower amplitude! An observer thus sees waves disappear from the front of a wave train and appear at the back. Be sure to observe this phenomenon when we have the regular waves lab!

For any depth water, the mathematical expression for group velocity is:

$$u_G = \frac{c}{2} \left(1 + \frac{2kd}{\sinh 2kd} \right).$$

Just as for the wave celerity, we can simplify this expression for considering just deep or shallow water. For deep water the group velocity is equal to half the wave celerity:

$$u_G = \frac{c}{2} = \frac{g}{2\omega}.$$

In shallow water the group velocity is equal to the wave celerity (the wave crest and the energy of the wave travel at the same speed):

$$u_G = c = \frac{g}{\omega}.$$

The energy associated with a train of regular waves includes contributions from both potential and kinetic energy. Consider a vertical chunk of wave that has a height of ζ . The center of gravity of this chunk is located in the middle $\zeta/2$, and a mass of $\rho g \delta x$. The potential energy of this chunk (mgh) is thus,

$$(\rho g \zeta \delta x) \frac{\zeta}{2} = \frac{\rho g \zeta^2 \delta x}{2}.$$

If we integrate this energy over the entire wavelength, we get the potential energy of the wave (per unit width),

$$E_{PE} = \frac{\rho g \lambda \zeta_0^2}{4}.$$

The wave also has energy stored as kinetic energy ($\frac{1}{2}mv^2$). If the total velocity of a segment is q , the kinetic energy of a segment is

$$\frac{\rho q^2 \delta x \delta z}{2}.$$

Integrating this over the full wavelength gives the kinetic energy of the wave (per unit width) and using the relationship between wave speed and wavelength,

$$E_{KE} = \frac{\rho g \lambda \zeta_0^2}{4}.$$

So, adding the potential and kinetic energy contributions together provides the total energy for a wave per unit area of sea surface,

$$\bar{E} = \frac{\rho g \zeta_0^2}{2}$$

Deep Water

How do we determine if water depth is “deep” or not? Water is deep when the water particles involved in the wave motion do not detect the bottom. For deep water waves, the water particles move in a *circular* motion. This means that the particles are NOT traveling with the wave, but the wave passes along while the particles stay in pretty much the same spot. You have experienced this if you have ever bounced up and down in the ocean as waves pass you by. Although the waves move you up and down, there is very little sideways motion. The particles near the surface of the water make large circular motions, but as you go deeper in the water the particle motion decreases in amplitude (see Figure 3.2). The motion of water particles at a depth of z is given by,

$$\zeta(x, z, \omega, t) = e^{-kz} \zeta_0 \sin(kx - \omega t - \epsilon)$$

Eventually the circular motion becomes so small that the water particles don’t move as the wave passes by. You have experienced something like this if you have every tried to avoid a wave by diving deep as it travels past. If you had stayed on the surface you would have been moved all over by the breaking wave, but by diving deep you feel only a slight push or pull as the wave travels past. The water depth is considered deep if the particles near the bottom don’t react to the wave moving past. If the water particles near the bottom move back and forth as the wave passes by, the water depth cannot be considered deep. Typically, we can assume water depth is deep if the water depth is greater than half the wave length:

$$d > \lambda/2 \text{ for deep water.}$$

For comparison, we consider truly shallow water to be when the water depth is 1/20th of the wavelength.

Table 3.2 shows the information given in reference 2 relating the different wave characteristics to each other for deep water waves. The water is considered deep when it is greater than half the wave length.

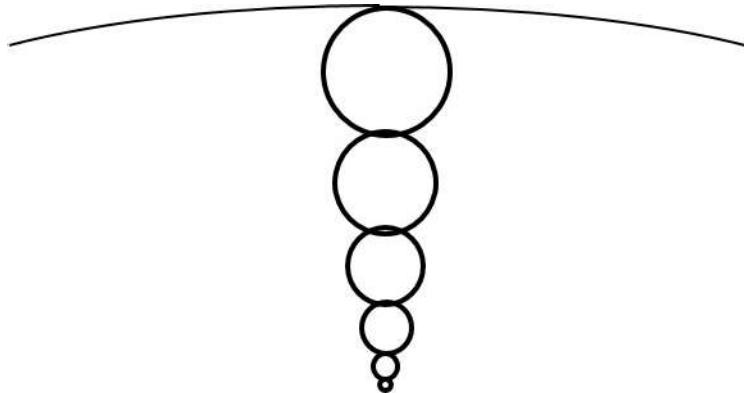


Figure 3.2: Decreasing Particle Motion as a Function of Depth

-	ω	T	k	λ	c	u_G
ω	-	$T = \frac{2\pi}{\omega}$	$k = \frac{\omega^2}{g}$	$\lambda = \frac{2\pi g}{\omega^2}$	$c = \frac{g}{\omega}$	$u_G = \frac{g}{2\omega}$
T	$\omega = \frac{2\pi}{T}$	-	$k = \frac{4\pi^2}{gT^2}$	$\lambda = \frac{gT^2}{2\pi}$	$c = \frac{gT}{2\pi}$	$u_G = \frac{gT}{4\pi}$
k	$\omega = \sqrt{gk}$	$T = \frac{2\pi}{\sqrt{gk}}$	-	$\lambda = \frac{2\pi}{k}$	$c = \sqrt{\frac{g}{k}}$	$u_G = \sqrt{\frac{g}{4k}}$
λ	$\omega = \sqrt{\frac{2\pi g}{\lambda}}$	$T = \sqrt{\frac{2\pi\lambda}{g}}$	$k = \frac{2\pi}{\lambda}$	-	$c = \sqrt{\frac{g\lambda}{2\pi}}$	$u_G = \sqrt{\frac{g\lambda}{8\pi}}$
c	$\omega = \frac{g}{c}$	$T = \frac{2\pi c}{g}$	$k = \frac{g}{c^2}$	$\lambda = \frac{2\pi c^2}{g}$	-	$u_G = \frac{c}{2}$
u_G	$\omega = \frac{g}{2u_G}$	$T = \frac{4\pi u_G}{g}$	$k = \frac{g}{4u_G^2}$	$\lambda = \frac{8\pi u_G^2}{g}$	$c = 2u_G$	-

Table 3.2: Linear Wave Relationships

Wave Slope

The *slope* of a wave surface, α , is the angle the water surface makes with the calm water surface. The slope changes as you move along the wave reaching a maximum as you travel from the trough to the crest and at a minimum when measured at the crest or trough. The maximum wave slope is,

$$\alpha_0 = \frac{2\pi}{\lambda} \zeta_0.$$

Pressure under a Wave

What is the pressure at a point under the wave? If the water surface was level (calm water), the pressure at a point beneath that calm surface would depend on the depth, z_B ,

$$P_B = \rho g z_B.$$

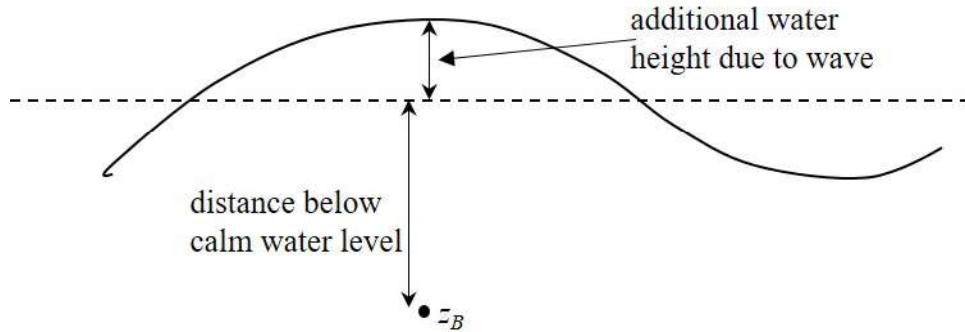


Figure 3.3: Figure showing a point z_B beneath a wave

The wave height would appear to add an additional hydrostatic component based on the water surface elevation above (or below) the calm water level (see Figure 3.3). However, due to the dynamic nature of the wave and the decreasing effect of the particle motion, the pressure fluctuation due to the wave is **less** than the additional hydrostatic contribution! The hydrostatic contribution due to the wave is actually

$$\tilde{P} = -(\rho g \zeta_0) e^{-kz_B} \sin(kx - \omega t - \epsilon).$$

where the e^{-kz_B} term accounts for the decreasing effect of the wave with depth. Adding this to the calm surface hydrostatic pressure gives the pressure at a point under a wave as

$$P = \rho g z_B - (\rho g \zeta_0) e^{-kz_B} \sin(kx - \omega t - \epsilon).$$

3.2 Irregular Waves

We are now going to move on to waves that have characteristics more similar to naturally occurring “real” waves. Waves are formed from wind blowing over the surface of the water. This energy transfer continues after the waves have formed. The energy absorption of the waves is countered by wave breaking and viscosity. Consider the time-history of water elevation as measured by a buoy in the ocean shown in Figure 3.4. A histogram can be used to evaluate the range of water elevation variation. To create such a histogram, water surface measurements at a particular location are made at regular intervals (say, every 1 minute). The measurements are then grouped into elevation ranges. For example, the number of measurements between 0 ft (the calm water surface) and 0.25 ft are recorded, then the number of measurements between 0.25 ft and 0.5 ft are recorded, and so on for all measurements. If we divide the number of measurements in each group by the total number of measurements, it gives a percentage of measurements (or occurrences) within each elevation range. Figure 3.5 shows a typical histogram for water elevation measurements made in a seaway. As is generally the case, the histogram of water elevation has the shape of a Gaussian (or normal) curve. If, however, instead of plotting all the water elevation measurements, only the *wave heights* are

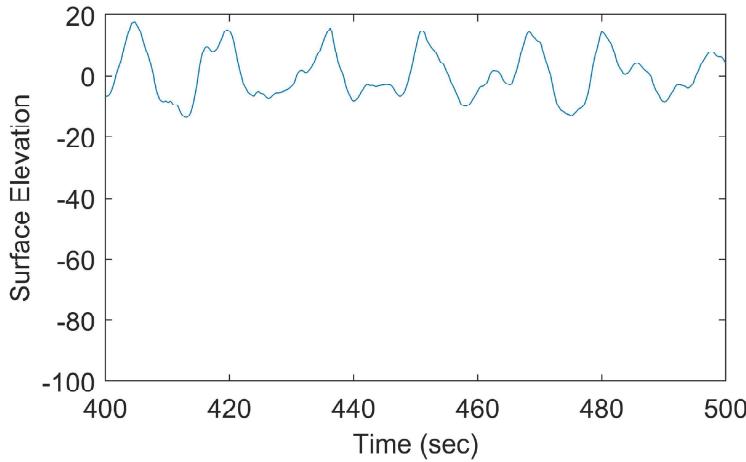


Figure 3.4: Water Surface Elevation from Hurricane Andrew (1992)

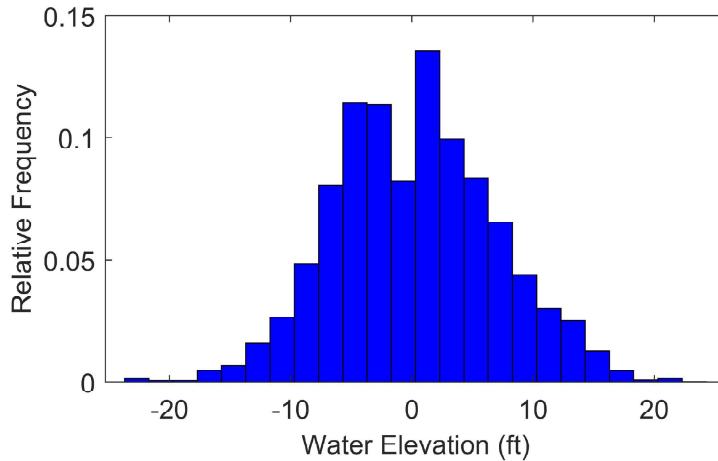


Figure 3.5: Hurricane Andrew Water Elevation Histogram

made into a histogram, the shape of the histogram is of a Rayleigh curve. Figure 3.6 shows the wave height histogram for the same data set as in Figure 3.5.

Unlike with regular waves, there are no global parameters that make describing the shape of an irregular sea straight-forward. The shape is generally sinusoidal, but each oscillation has a different amplitude and different period. Some of the oscillations don't even go below the calm water surface, but ride on other waves! To be able to summarize such a system, we need to use the tools provided by *statistics*.

Useful Statistical Measures

The easiest statistical measures to work with are means or averages. There are different characteristics of the wave signal that we can average. For example, we can average all the measurements of the water elevation, ζ . This will give us the mean water level. For notation, averages are written with a bar, so the mean water level is written as $\bar{\zeta}$. The mean water

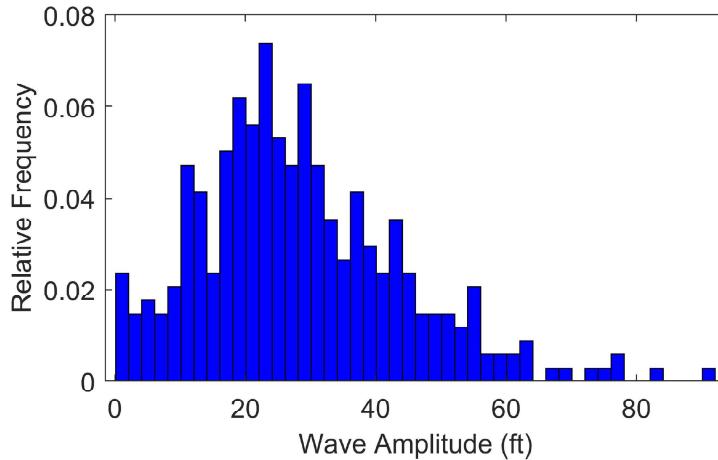


Figure 3.6: Hurricane Andrew Wave Height Histogram

level for the data shown in Figure 3.4 is 0.25 ft. We can also measure the peak amplitude for every wave in the signal, ζ_a . The mean of all the peaks would be the mean wave amplitude, $\bar{\zeta}_a$. The mean wave height would be twice the mean wave amplitude,

$$\bar{H}_a = 2\bar{\zeta}_a.$$

The time between each peak can also be measured, T_p and averaged, giving the mean period of the peaks, \bar{T}_p . The time between zero-crossings (the time between the water surface passing up through the nominal zero water level) can be measured for all waves, T_z and averaged, \bar{T}_z . The mean of any N set of numbers (x) is given by

$$\bar{x} = \sum_{n=1}^N \frac{x_n}{N}. \quad (3.1)$$

The average value is also referred to as the “expected” value. So, the mean water elevation is

$$\bar{\zeta} = \sum_{n=1}^N \frac{\zeta_n}{N}$$

where N is the number of measurements and ζ_n is each measurement of the water surface.

The variance of a set of numbers is a measure of how spread out the data is (i.e. how far the numbers lie from the mean). The variance of the water elevation is given by

$$m_0 = \sum_{n=1}^N \frac{(\zeta_n - \bar{\zeta})^2}{N}. \quad (3.2)$$

The standard deviation is another measure of the dispersion of the data. If the standard deviation is small, the data points are close to the mean. If the standard deviation is large,

the data points are spread out over a large range of values. The standard deviation is equal to the square root of the variance,

$$\sigma_0 = \sqrt{m_0}.$$

For meaningful results, the data must include **AT LEAST** 100 pairs of peaks and troughs.

Another useful concept when explaining waves is the value of the *significant wave height*. This value is the average of the highest 1/3rd of the wave heights in the measurements. To find the significant wave height, you measure all of the wave amplitudes and order them by magnitude. For example, from the data shown in Figure 3.4 the largest 10 wave amplitudes are 31.4, 31.8, 33.6, 34.8, 36.4, 37.6, 38.6, 38.6, 41.6, and 45.2. Next, identify the set of amplitudes that consist of the top third and take the average. For the data set shown in Figure 3.4, the average of the top 1/3rd of the wave amplitudes is 23.2 ft. To find the significant wave height, we double this number,

$$\bar{H}_{1/3} = 2\bar{\zeta}_{1/3},$$

so for our data set the significant wave height is 46.4 ft.

3.2.1 Superposition and Fourier Analysis

Another way to describe the waves shown in Figure 3.4 is to look at the frequency information in the signal. The frequency components of the wave exist because of the principle of superposition.

Principle of Superposition

Consider two waves traveling past the same point ($x = 0$) that have the same amplitude, ζ_0 , but different frequencies, ω_1 and ω_2 :

$$\begin{aligned}\zeta_1(t) &= \zeta_0 \sin(\omega_1 t) \\ \zeta_2(t) &= \zeta_0 \sin(\omega_2 t).\end{aligned}$$

The water elevation at this point will be the sum of the two waves traveling past:

$$\begin{aligned}\zeta(t) &= \zeta_1(t) + \zeta_2(t) \\ \zeta(t) &= \zeta_0 \sin(\omega_1 t) + \zeta_0 \sin(\omega_2 t).\end{aligned}$$

The top image in Figure 3.7 shows how two waves both with a wave amplitude of 1 ft but different wave frequencies of 2π rad/sec and 3π rad/sec, respectively, combine to form a new wave. If we also change the wave amplitude (the second wave now has an amplitude of 0.5 ft) and add a phase angle (the second wave now has a phase angle of $\epsilon = \pi/2$), the combined wave changes. This is shown in the bottom image of Figure 3.7. The more waves of different frequencies, amplitudes, and phase angles are used, the more complex the resulting water elevation becomes. Figure 3.8 shows a wave consisting of 50 component frequencies.

Just as we can create an irregular wave train by combining many components of different frequencies, we can identify the different frequency components in a given wave train using

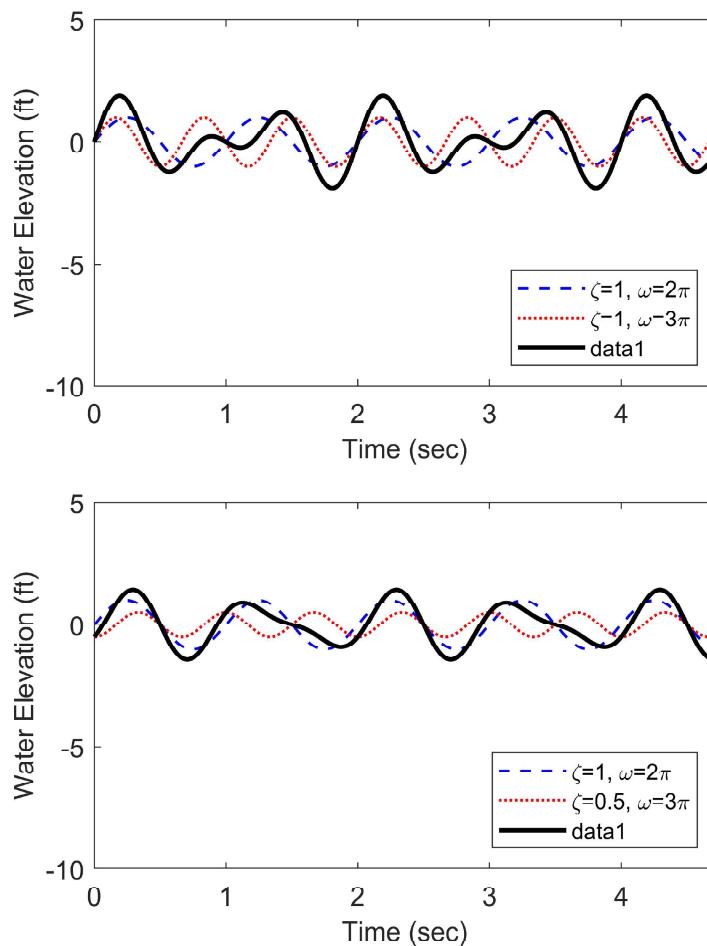


Figure 3.7: Superposition of two waves

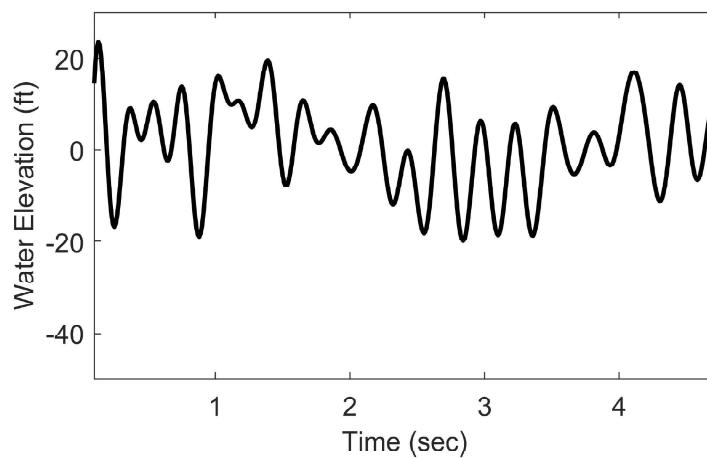


Figure 3.8: Superposition of 50 components

a Fourier Transform. The Fourier Transform identifies the amplitude and phase angle associated with each frequency component in the wave. We can consider the wave time history to be represented by the sum of all these components added to the mean water elevation,

$$\zeta(t) = \bar{\zeta} + \sum_{n=1}^N \zeta_{n0} \sin(\omega_n t + \epsilon_n)$$

where ω_n represents each wave frequency in the signal and ζ_{n0} and ϵ_n are the associated wave amplitude and phase angle for each frequency.

Discrete Fourier Transform

If a signal consists of a single sinusoid, finding the frequency content of that signal will result in a single number. We typically represent the frequency content as a plot of amplitude versus frequency, so for this case there would be a single spike with a height equal to the wave amplitude and the spike occurring at the frequency of the sinusoid. However, most wave signals do not look like perfect sinusoids and measured systems of all kinds typically are contaminated by some amount of noise.

Our data always consists of a sequence of measurements, X_1, X_2, X_3, \dots . The *discrete fourier transform* (DFT) converts a sequence of values corresponding to certain times into a sequence of values corresponding to specific frequencies. Computers are very good at this and the most commonly used algorithm for computing a DFT quickly is called the Fast Fourier Transform (FFT). The FFT is so common that the terms DFT and FFT are often used interchangeably.

The **time domain** consists of a set of numbers (x_0, x_1, x_2, \dots) each measured at a particular time (t_0, t_1, t_2, \dots). The DFT provides the **frequency domain** information in the form of numbers (X_0, X_1, X_2, \dots) where each X_k represents a portion of the signal that occurs at f_k . By convention, n refers to the time domain data point and k refers to the frequency domain data point. The frequency values X_k are typically complex numbers (expressed as real and imaginary components).

If we have N data points in time (x_n), we will only have $N/2$ points in the frequency domain. This is because for each frequency we have two pieces of information - the magnitude and phase - while at each time we only have one piece of information: magnitude. Since we can't create new information out of nothing, with N points in the time domain we can only have enough information for $N/2$ points in the frequency domain.

In the time domain, the data has an associated sampling frequency (samples/second). This is related to the time interval between data points (Δt),

$$\text{sample frequency} = \frac{1}{\Delta t}.$$

For example, if we take 100 regularly spaced measurements in 0.5 sec we have a sample of 100 data points ($N = 100$). The time between samples is found by dividing the total time by the total number of samples. We are taking a sample every 0.005 seconds (0.5 seconds divided by 100 samples):

$$\Delta t = \frac{0.5 \text{ sec}}{100 \text{ samples}} = 0.005 \text{ seconds/sample.}$$

Our sampling frequency is, therefore,

$$f_s = \frac{1}{0.005} = 200 \text{ samples/sec or Hz.}$$

The resolution with which we can get frequency information from the DFT depends on the sampling frequency **AND** the total number of points

$$\Delta f = \frac{f_s}{N}$$

where Δf is the frequency resolution. So, in this example we have a frequency resolution of 2 Hz (=200 Hz/100 samples).

The raw form of the frequency information that the FFT delivers is not in a physically meaningful form (for example, the values are just complex numbers). We need to scale the results to find the amplitudes of the peaks (power spectrum) and the phase information. To find the amplitude of response at a particular frequency, we need to take the absolute magnitude of the complex number, multiply it by 2, and divide by the total number of points.

$$\text{Magnitude} = 2 \frac{|X_k|}{N}$$

where X_k is the complex number at frequency k . The phase angle information is determined from taking the tangent of the real and complex parts of the FFT output. The process for the FFT is mathematically complicated, but algorithms exist using software like MATLAB that we can use to get X_k . The scaling to find the magnitude (and also the phase information) must be done manually.

The fastest oscillation (i.e. the highest frequency) that can be observed depends on how rapidly the data is sampled. The time resolution dictates the highest observable frequency. Remember, N samples in the time domain only gives frequency information up to $N/2$. Therefore, the highest frequency that can be measured is

$$f_{\max} = \frac{N}{2} \cdot \Delta f$$

which is known as the Nyquist frequency. To summarize, the highest frequency you can observe depends on the total number of points. So, if you want to observe 100 Hz responses and only are currently able to observe 50 Hz responses you need to collect more points. You can do this by collecting data for a longer total time or for the same amount of time, but at a higher sampling frequency. If you want to be able to resolve your frequency more (for example, you only have a resolution of 2 Hz but want to be able to distinguish results at a resolution of 0.5 Hz), you must increase the total number of time domain data points by increasing the total time you take measurements. If I currently have 100 samples taken at a 200 Hz sampling frequency, my frequency resolution is 2 Hz and my Nyquist frequency is 100 Hz. If I want to increase my Nyquist frequency to 200 Hz, I could keep my total time the same and double my sampling frequency (to 400 Hz). My frequency resolution would then be $400 \text{ Hz}/200 \text{ samples} = 2 \text{ Hz}$ (same as before), but my Nyquist frequency is now $(200 \text{ samples}/2) \cdot (2 \text{ Hz}) = 200 \text{ Hz}$. If I want to have a finer frequency resolution, I

need to take more data points at the same sampling frequency. If I continue at my 400 Hz sampling frequency for a full second (as opposed to the previous 0.5 second), I will get $N = 400$ samples. Now my frequency resolution is $400 \text{ Hz}/400 \text{ samples} = 1 \text{ Hz}$ and my Nyquist frequency is $(400 \text{ samples}/2) \cdot (1 \text{ Hz}) = 200 \text{ Hz}$ (same as before).

3.2.2 Wave Energy Spectrum

Using the principle of superposition, an irregular wave pattern can be created as a large number of sinusoidal waves of different frequencies and heights that are superimposed on each other. One way to summarize a set of waves is to determine the total energy. This is found by adding together the energies of all of the waves that produced the irregular wave pattern. The roughness of the seaway is then decided by the total energy content of all the waves present. As shown above when considering regular waves, the energy content for a sinusoidal wave is $\frac{1}{2}\rho g\zeta_a^2$ per square area of sea surface. To account for all the sinusoidal waves superimposed, every wave amplitude must be accounted for, so

$$E_T = \frac{1}{2}\rho g(\zeta_{a1}^2 + \zeta_{a2}^2 + \dots + \zeta_{an}^2).$$

In essence, the total energy is made up of the energy contribution of each wave component. Each wave component (the wave amplitude for a particular wave frequency) has an energy component. The frequency distribution of this energy is called the *energy spectrum* of the seaway.

Wave Energy Spectrum Example ¹

Consider an irregular seaway composed of four wave components:

Wave component	1	2	3	4
Length (ft)	1265	562	316	202
Freq (rad/sec)	0.4	0.6	0.8	1.0
Height (ft)	3	5	4	2
Amplitude (ft)	1.5	2.5	2	1

The total energy per square foot of the wave surface is

$$\begin{aligned} E_T &= \frac{\rho g}{2}(\zeta_{a1}^2 + \zeta_{a2}^2 + \zeta_{a3}^2 + \zeta_{a4}^2) \\ &= \frac{64}{2}(1.5^2 + 2.5^2 + 2^2 + 1^2) \\ &= 432 \text{ lb/ft} \end{aligned}$$

The individual energy components can be graphed by taking the energy contribution for each component and dividing by the frequency spacing. For example, the first wave component has an energy of 72 lb/ft. The ordinate on the wave energy spectrum plot (see Figure 3.9) is $(72 \text{ lb/ft})/(0.2 \text{ rad/sec}) = 360 \text{ lb-sec/ft}$. Figure 3.9 shows the energy component contributions. The area under this ‘curve’ is the total energy of the irregular seaway, E_T . The Wave energy spectrum curve, Figure 3.10, shows the Wave Energy Spectrum curve with the spectral ordinate (equation 3.4) on the vertical axis. The area under this curve is proportional to the total energy: total energy = (water specific gravity) · (area under curve).

¹taken from reference 3

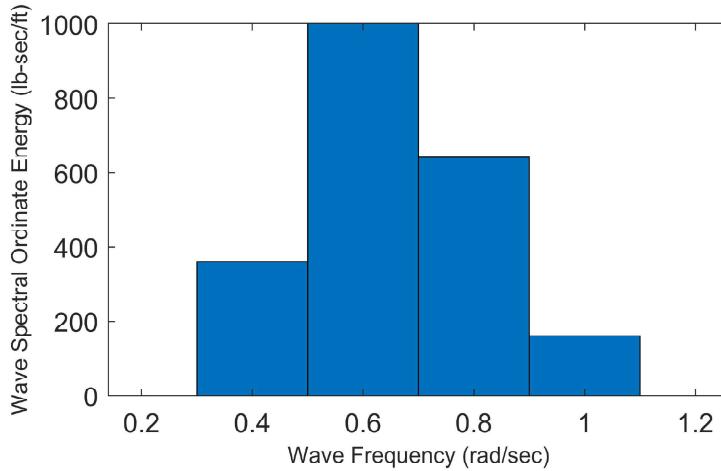


Figure 3.9: Wave Energy Ordinates Spectrum example

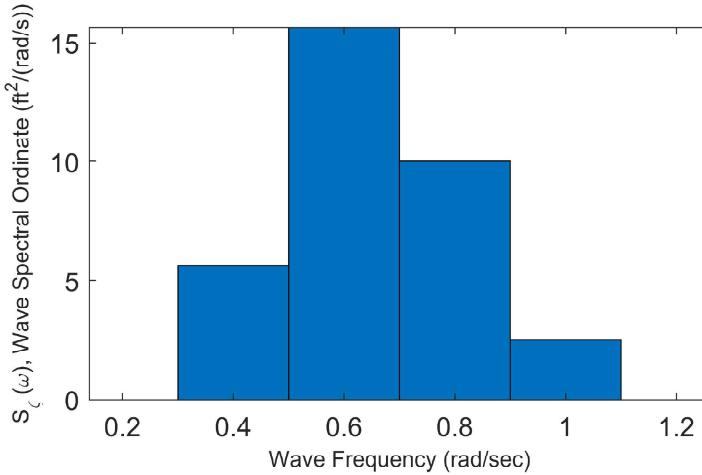


Figure 3.10: Wave Spectral Ordinates Spectrum example

The relative importance of the wave components (each a sinusoidal wave) making up an irregular wave pattern (usually recorded as a time history) may be quantified in terms of a wave amplitude energy density spectrum, also called a **wave energy spectrum**. The time history of the water elevation can be expressed as

$$\zeta(t) = \bar{\zeta} + \sum_{n=1}^{\infty} \zeta_{i0} \sin(\omega_i t + \epsilon_i) \quad (3.3)$$

where the magnitude of ζ_{i0} represents the significance of each frequency component amplitude. Figure 3.11 shows a wave energy curve when the number of components is high and the frequency spacing approaches zero. The spectral ordinate ($S_\zeta(\omega_n)$) is the value on the vertical axis. The area under a segment of the curve equals the energy of that frequency

component wave, so the spectral ordinate for each frequency is

$$\begin{aligned}\rho g S_\zeta(\omega_i) \delta\omega &= \frac{\rho g \zeta_{i0}^2}{2} \\ S_\zeta(\omega_i) &= \frac{\zeta_{i0}^2}{2\delta\omega}.\end{aligned}\quad (3.4)$$

If the energy spectrum is known, it is possible to reverse the spectral analysis process and generate a corresponding time history by adding a large number of component sine waves (see equation 3.3). In principle, an infinite number of sine wave components is required. However, a limited number (≈ 50) will work since we can generally ignore components that have small spectral ordinate values. Manipulating the spectral ordinate equation (3.4) gives the wave amplitude for each frequency component:

$$\zeta_{i0} = \sqrt{2S_\zeta(\omega_i)\delta\omega}. \quad (3.5)$$

To recreate an irregular wave energy time history we also need the phase angle for each component (ϵ_i). If the phase angles for each component are chosen at random, the recreated time history will not match the original (from which the wave energy spectrum was generated), but the seaway will have the same energy as the original waves.

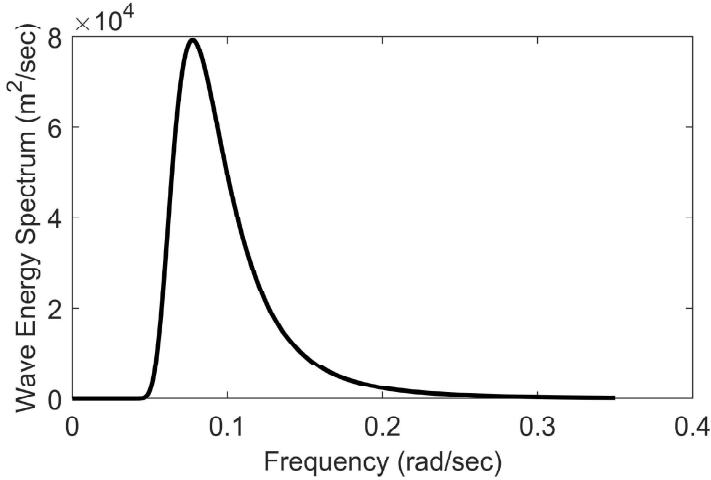


Figure 3.11: Wave Energy Spectrum as number of components approaches ∞

Relationship between Water Elevation Time History and the Wave Energy Spectrum

It is important to understand that the measured time history of the water surface elevation and the wave energy spectrum are both representations of the same information - the seaway. We can get the same information from either source, although the calculation method is different. Consider the **variance** of the water surface time history. The variance is a measure of the degree of “spread” in the wave surface and the wave amplitudes are a measure of the wave energy. For our water elevation, the larger the waves, the larger the variance and the

higher the energy in the seaway. The equation for the variance (represented as m_0) is given in equation 3.2 ($m_0 = \sum_{i=1}^N \frac{(\zeta_i - \bar{\zeta})^2}{N}$). It turns out that the variance of the time history is also equal to the area under the wave energy spectrum:

$$m_0 = \int_0^\infty S_\zeta(\omega) d\omega.$$

To restate, **the variance of the irregular wave time history is equal to the area under the wave energy spectrum.**

The **wave energy spectrum**, $S_\zeta(\omega)$, is determined from the wave amplitudes and frequencies from a DFT of the wave time history. Therefore, the **wave velocity spectrum**, $S_{\dot{\zeta}}(\omega)$, and the **wave acceleration spectrum**, $S_{\ddot{\zeta}}(\omega)$, can be found using the velocity and acceleration amplitudes and frequencies.

As shown before, the water surface elevation (position) is given by equation 3.3. The velocity ($\dot{\zeta}$) and acceleration ($\ddot{\zeta}$) of the water surface can be found from the derivatives of the water surface elevation. So,

$$\begin{aligned} |\dot{\zeta}(t)| &= \sum_{i=1}^{\infty} \zeta_{i0} \omega_i \cos(\omega_i t + \epsilon_i) \\ |\ddot{\zeta}(t)| &= \sum_{i=1}^{\infty} \zeta_{i0} \omega_i^2 \cos(\omega_i t + \epsilon_i). \end{aligned}$$

Similarly, the velocity and acceleration spectra ordinates can be related to the wave energy ordinates:

$$\begin{aligned} S_{\dot{\zeta}}(\omega_i) &= \frac{\omega_i^2 \zeta_{i0}}{2\delta\omega} = \omega_i^2 S_\zeta(\omega_i) \\ S_{\ddot{\zeta}}(\omega_i) &= \frac{\omega_i^4 \zeta_{i0}}{2\delta\omega} = \omega_i^4 S_\zeta(\omega_i). \end{aligned}$$

Notice the patterns with respect to the frequency (ω_i) terms. The velocity and acceleration spectral ordinates can be obtained by multiplying the position spectral ordinates by the 2nd and 4th powers of the frequency. As for the wave energy spectrum, the area under the velocity and acceleration spectra is equivalent to the variances of these time histories, respectively,

$$\begin{aligned} m_2 &= \int_0^\infty S_{\dot{\zeta}}(\omega) d\omega = \int_0^\infty \omega^2 S_\zeta(\omega) d\omega \\ m_4 &= \int_0^\infty S_{\ddot{\zeta}}(\omega) d\omega = \int_0^\infty \omega^4 S_\zeta(\omega) d\omega. \end{aligned}$$

These areas (m_0 , m_2 , and m_4) are called **spectral moments**. In general, the relationship between a spectral moment and the wave energy spectrum curve is

$$m_n = \int_0^\infty \omega^n S_\zeta(\omega) d\omega. \quad (3.6)$$

These spectral moments can be used to link the spectra to statistical characteristics of the time history such as the mean wave period (\bar{T}) and frequency ($\bar{\omega}$), zero-crossing mean period (\bar{T}_z), and peak mean period (\bar{T}_p). These characteristics can be determined directly from the time history of the water surface elevation or from the wave energy spectrum for that time history using the spectral moments.

$$\begin{aligned}\text{mean frequency:} \quad & \bar{\omega} = \frac{m_1}{m_0} \\ \text{mean period:} \quad & \bar{T} = \frac{2\pi}{\bar{\omega}} = 2\pi \frac{m_0}{m_1} \\ \text{mean peak period:} \quad & \bar{T}_p = 2\pi \sqrt{\frac{m_2}{m_4}} \\ \text{mean zero-crossing period:} \quad & \bar{T}_z = 2\pi \sqrt{\frac{m_0}{m_2}}\end{aligned}$$

Spectrum Bandwidth The spectrum bandwidth describes the relative width of the wave energy spectrum compared to the height. A **Narrow Band** spectrum has a sharp spike that covers only a small range of frequencies, see Figure 3.12. A narrow-banded water surface elevation time history can loosely be described as a sine wave of varying amplitude. The wave energy is concentrated in a narrow band of frequencies. Essentially, for every peak or trough there is a zero-crossing. So, $\bar{T}_p \approx \bar{T}_z$.

A **Wide Band** spectrum has a flatter curve the covers a large range of frequencies, see Figure 3.13. A wide-banded water surface elevation time history has many peaks and troughs not followed immediately by a zero crossing. The wave energy is spread over a wide band of frequencies. Essentially, there are many local peaks and troughs between each zero-crossing. So, $\bar{T}_p \ll \bar{T}_z$.

The ratio between the average period of the peaks and the average zero-crossing period can be regarded as a measure of the “narrow-bandedness.” This is quantified by the **Bandwidth Parameter**.

Bandwidth Parameter

$$\epsilon = \sqrt{1 - \frac{\bar{T}_p^2}{\bar{T}_z^2}} = \sqrt{1 - \frac{m_2^2}{m_0 m_4}}. \quad (3.7)$$

As the bandwidth parameter approaches zero ($\epsilon = 0$), the spectrum is very narrow-banded ($\bar{T}_z = \bar{T}_p$) and the wave surface approaches a regular wave. In general, the spectra associated with waves and ship motions are narrow-banded. As the bandwidth parameter approaches 1 ($\epsilon = 1$), the spectrum is very wide-banded ($\bar{T}_p \approx 0$).

The bandwidth parameter together with the spectral moments allows the significant wave height of the irregular seaway to be calculated. Remember that the *significant wave height* is the average of the highest one-third of all wave heights in the seaway. Using the bandwidth parameter, ϵ , and the spectral moment m_0 , the significant wave height is equal to

$$\bar{H}_{1/3} = 4.00 \sqrt{m_0} \sqrt{1 - \frac{\epsilon^2}{2}}.$$

If the wave spectrum is narrow-banded, such that ϵ approaches 0, the significant wave height can be found from

$$\bar{H}_{1/3} = 4.00 \sqrt{m_0}. \quad (3.8)$$

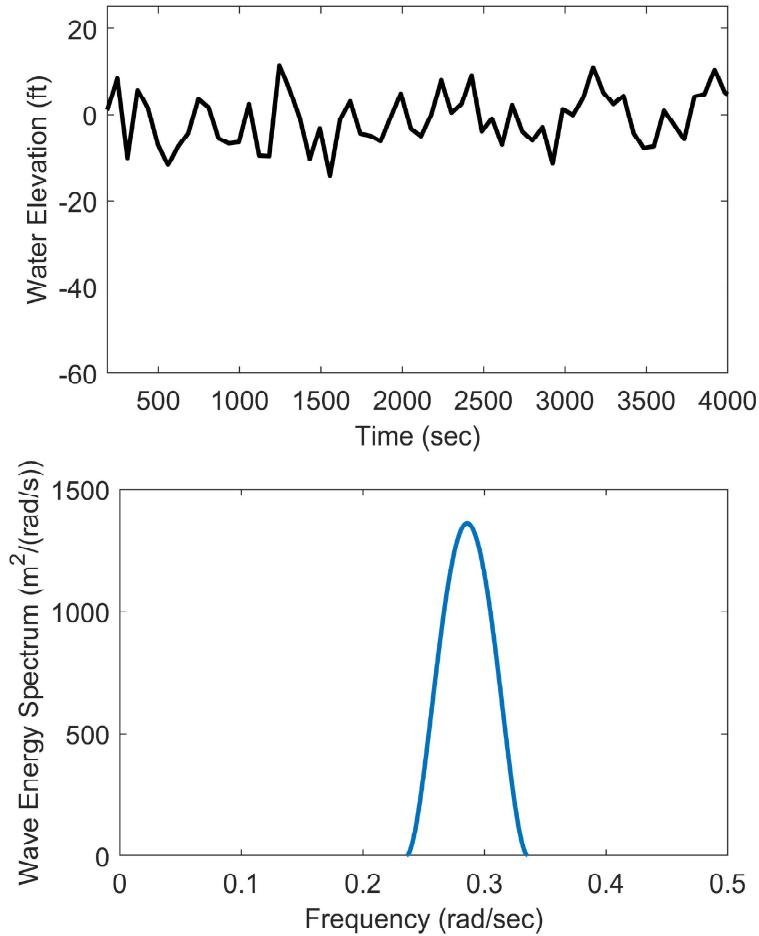


Figure 3.12: Sample Narrow-Banded Wave Energy Spectrum and Water Surface Time History

If the wave spectrum is wide-banded, such that ϵ approaches 1, the significant wave height can be found from

$$\bar{H}_{1/3} = 2.83\sqrt{m_0}.$$

Since m_0 is equal to the area under the wave energy spectrum curve, the significant wave height can be estimated by integrating $S_\zeta(\omega)$.

3.2.3 Idealized Wave Spectra

Sea waves are primarily the result of wind transferring energy to the sea surface. The kinetic energy of the wind (wind speed) creates potential energy of the water (waves). The height and length of the generated waves depends on the wind velocity, the length of time the wind blows over the water surface, and the *fetch*. Fetch is the distance of water over which the wind blows before reaching land. Open ocean has effectively infinite fetch, while bays, lakes, and coastal areas are considered “limited fetch” areas. Fetch-limited areas have waves

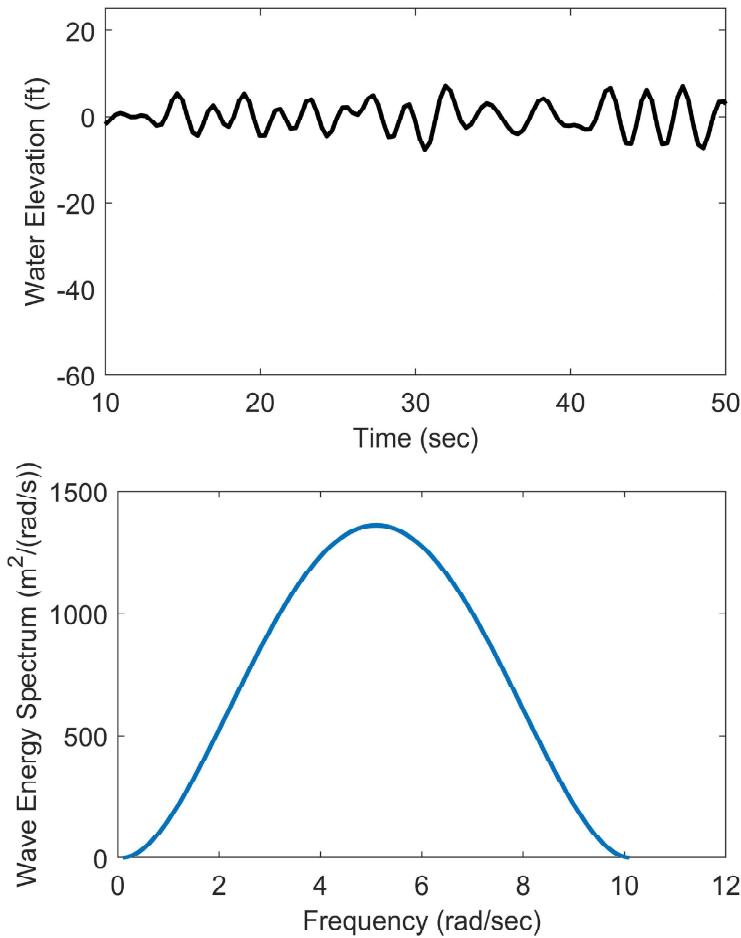


Figure 3.13: Sample Wide-Banded Wave Energy Spectrum and Water Surface Time History

that are shorter (higher frequency) and choppier (steeper). The waves often have white caps and there is not as much of an underlying swell like you see in open ocean waves. If there are no fetch limitations, the waves eventually reach an equilibrium where the amount of energy transferred from the wind maintains the wave heights, but the dissipation of the water (viscous and wave breaking) prevents additional amplitude growth. A sea in this condition is considered *fully developed*.

For ship design purposes, we use different formulae to represent open ocean and coastal (limited fetch) wave conditions. There are two types of open ocean spectra that we will be considering. The first is the Pierson-Moskowitz (1964) one parameter spectrum. This wave field has only one input and that is the wind speed. The spectrum assumes there is plenty of area for the wind/water interface and that the wind has been blowing for long enough that the wave field has reached equilibrium. The Pierson-Moskowitz spectrum was based off of extensive measurements in the North Atlantic Ocean and is intended to represent the spectrum of a fully developed sea.

Pierson-Moskowitz Wave Spectrum - one parameter spectrum, wind speed

$$S_\zeta(\omega) = \frac{0.0081g^2}{\omega^5} e^{-0.74(\frac{g}{W_{19.5}\omega})^4} \quad (3.9)$$

where $W_{19.5}$ is the average wind speed (in m/s) at 19.5 meters above the sea surface. The units for this spectrum are m^2/sec . Since 19.5 meters is not a typical height for wind measurements, this can be related to the more standard 10 meters by $W_{19.5} = 1.026W_{10}$. Figure 3.14 shows several generated Pierson-Moskowitz spectra for different wind speeds.

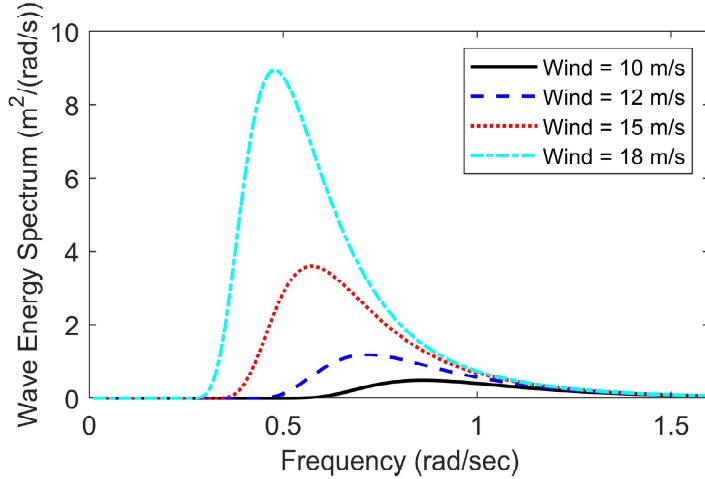


Figure 3.14: Example Pierson Moskowitz spectra for different wind speeds

Another open-ocean wave spectrum is the ITTC (International Towing Tank Conference) 1973 wave spectrum. This is also known as the **Bretschneider Wave Spectrum**. This is a two-parameter spectrum and depends on the given significant wave height and modal period. The **modal period** is the wave period that coincides with the peak of the wave energy spectrum. The ITTC wave spectrum can exactly replicate the Pierson-Moskowitz spectrum if the significant wave height and modal period that results from the wind speed are used.

ITTC 1973 (or Bretschneider) Wave Spectrum - two parameter spectrum, significant wave height and modal wave period

$$S_\zeta(\omega) = \frac{1.25}{4} \left(\frac{\omega_0}{\omega} \right)^4 \frac{\bar{H}_{1/3}^2}{\omega} e^{-1.25(\frac{\omega_0}{\omega})^4} \quad (3.10)$$

where $\bar{H}_{1/3}^2$ is the significant wave height and ω_0 is the modal wave frequency ($\omega_0 = \frac{2\pi}{T_0}$ where T_0 is the modal wave period). Figure 3.15 shows several generated ITTC spectra for different wind speeds.

For limited fetch conditions, we typically use the JONSWAP (Joint North Sea Wave Observation Project) spectrum. This spectrum is intended to represent open wave conditions, but where the fetch is limited (i.e., like the North Sea). The spectrum is more narrow (peakier) than the pure open ocean spectra.

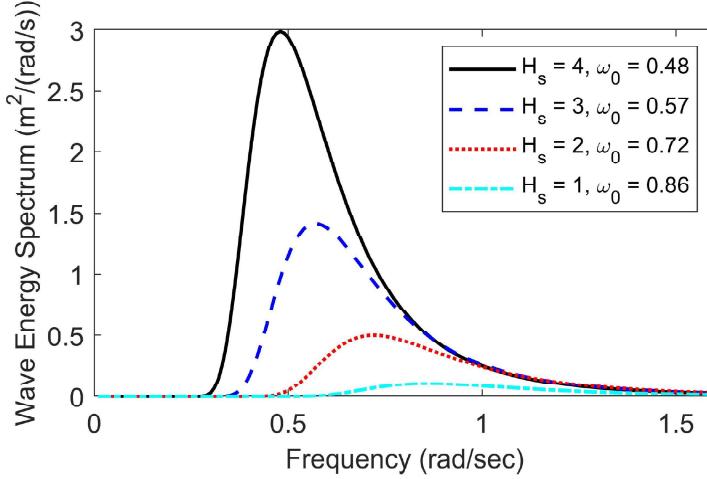


Figure 3.15: Example ITTC (Bretschneider) spectra for different significant wave height/modal period combinations.

JONSWAP Wave Spectrum - three parameter spectrum, wind speed and fetch and peakness factor (γ)

$$S_\zeta(\omega) = \frac{\alpha g^2}{\omega^5} e^{-(\frac{5\omega^4}{4\omega_0^4})} \gamma^r \quad (3.11)$$

where α depends on the wind speed (W_{10}) and fetch (F) and equals $\alpha = 0.076 \left(\frac{W_{10}^2}{Fg} \right)^{0.22}$, the modal wave frequency (ω_0) depends on the wind speed and fetch and equals

$$\omega_0 = 22 \left(\frac{g^2}{W_{10} F} \right)^{1/3}$$

, and r depends on the frequencies and a σ parameter and equals

$$r = e^{-\left[\frac{(\omega - \omega_0)^2}{2\sigma^2 \omega_0^2} \right]}$$

. The σ parameter depends on whether the spectral ordinate being calculated is less than or greater than the wave modal frequency,

$$\sigma = \begin{cases} 0.07 & \text{if } \omega \leq \omega_0 \\ 0.09 & \text{if } \omega > \omega_0 \end{cases}$$

The parameter γ is the “peakness factor” and can be modified to meet the needs of the sea conditions. It ranges from about 1 to about 7, but a typical value for γ is 3.3. The JONSWAP spectrum can also be expressed as a function of significant wave height, modal frequency and the peakness factor. This formulation makes it easier to see how the limited fetch changes the wave energy distribution compared with the fully open ocean ITTC spectrum. This JONSWAP formula is

$$S_\zeta(\omega) = B_J \bar{H}_{1/3}^2 \frac{2\pi}{\omega} \left(\frac{\omega_0}{\omega} \right)^4 e^{-\left[\frac{5}{4} \left(\frac{\omega_0}{\omega} \right)^4 \right]} \gamma^r$$

where

$$B_J = \frac{0.06238}{0.230 + 0.0336\gamma - \frac{0.0185}{1.9+\gamma}} [1.094 - 0.01915 \ln \gamma]$$

and r and γ are the same as defined above. Figure 3.16 shows several generated JONSWAP spectra for the same significant wave heights and modal wave periods as used for the ITTC spectra (Figure 3.15) and with a peakness factor of 2.

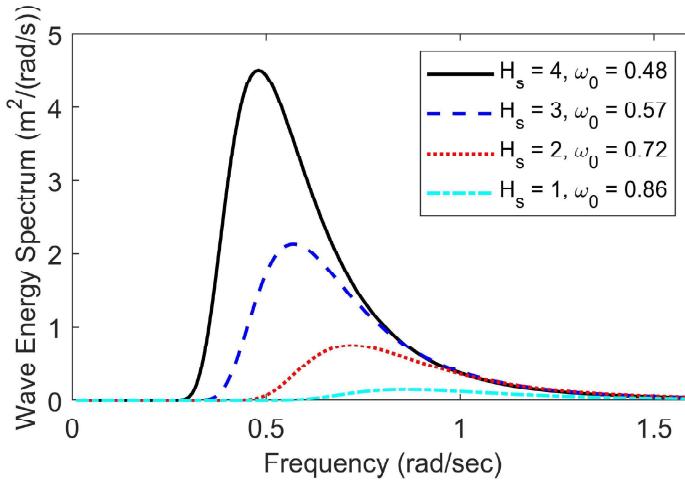


Figure 3.16: Example JONSWAP spectra for different wind speed and fetch combinations.

Recap

Okay, so we have covered a LOT of material so far in this chapter. Before we continue, let's do a short review.

We started with a discussion of **regular waves**. These waves have a single amplitude and period and we can calculate the wave celerity and group velocity as well as the energy of such a wave. We also learned how to calculate the pressure of a point under such a wave.

Next we used the principle of superposition to create a set of (more realistic) **irregular waves**. These waves have varying wave amplitudes and periods and, in a real seaway, do not have any repeating patterns. Although we can't create an infinite discrete irregular wave field, we can create a wave field that approximates the chaotic environment typically found in the ocean. These waves can't be described using a single amplitude and period, so we turn to averages. The important quantities we take averages of include mean wave frequency, the zero-crossing wave period, the peak wave period, the wave amplitude, and the wave height. We also can find **significant wave amplitude** or **significant wave height**, which are the average of the 1/3rd highest wave amplitudes and heights, respectively.

Using a **Fourier Transform**, we can transform the irregular wave time history into a **wave energy spectrum**. This shows the energy distribution of the irregular waves over a range of frequencies. The area under this curve, also known as the **zeroth spectral moment** or m_0 is equal to the variance (σ^2) of the irregular wave time history. Using other spectral moments and a bandwidth parameter we can use the wave energy spectrum to find

the statistical wave properties mentioned above (mean wave frequency, the zero-crossing wave period, the peak wave period, the wave amplitude, the wave height, and the significant wave height).

Okay, so why did we just work our way in a circle (start with finding these properties from the irregular wave time history, transform the time history into a wave energy spectrum, then find these properties from the wave energy spectrum)? Because, as ship designers, we want to create realistic random sea states to design from and to do this we use **idealized wave spectra** such as the Pierson-Moskowitz, ITTC (Bretschneider), or JONSWAP spectra. In the next chapter we are going to learn about how ships respond to regular waves and then put that together with these realistic sea environments to make predictions about how a ship may respond when encountering realistic sea conditions.

Before we move on to discussing ship responses, however, we have one more wave related issue to tackle. All of these energy wave spectra give information about what frequencies contain the main energy of the sea surface. What we don't know is what the specific waves the ship will encounter will be. Since the sea is random, we must approach this from a probabilistic perspective. So, to finish out the chapter on waves, we will now do a review of the relevant Probability and Statistics topics you have, hopefully, seen before!

3.2.4 Review of Probability and Statistics with Marine Applications

We have shown that the irregular time histories of waves can be characterized in terms of energy spectra and various statistical quantities. Seakeeping studies, however, often demand a more intimate knowledge of waves. In particular, we need to be able to answer questions like "What is the likelihood of a particular wave height being exceeded?" We can use wave energy spectra and probability distributions to answer this type of question.

Probability Density Function (PDF)

The probability density function is defined such that the area enclosed by the PDF curve over a bin is equal to the probability of the measurement falling within that bin. So, the probability of the x -axis value falling between a and b is equal to the area under the curve from a to b . Figure 3.17 shows the area from a to b for a normal probability distribution curve. The probability the water elevation falls between these two limits is equal to the shaded area on the plot. The area under the entire probability density function equals one, since there is 100% probability that any measurement falls within the set of collected measurements.

Water elevation typically follows a Gaussian or normal distribution. This is the typical "bell" curve, see Figure 3.18. The empirical rule states that there is about a 68% probability a measurement will fall between $\pm\sigma$ (one standard deviation), there is about a 95% probability a measurement will fall between $\pm 2\sigma$, and a 99% probability any measurement will fall between $\pm 3\sigma$.

While *water elevation* typically follows a Gaussian distribution, *wave heights* (and amplitudes) follow a Rayleigh distribution for narrow-banded wave spectra. The probability for

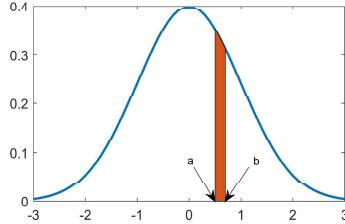


Figure 3.17: The probability of the wave elevation falling between a and b equals the area under the pdf curve between these two values

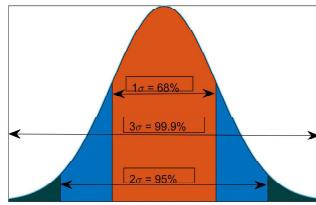


Figure 3.18: Gaussian or Normal Probability Distribution

the wave amplitudes depends on the variance of the water elevation. Figure 3.19 shows a typical Rayleigh distribution. The probability a wave amplitude would fall between two heights is equal to the area under the curve between those two points. The Rayleigh probability distribution equals

$$f = \frac{\zeta_a}{m_0} e^{-\frac{\zeta_a^2}{2m_0}}$$

where ζ_a is the wave amplitude and m_0 is the *variance from the water elevation time history* or area under the wave energy spectrum curve.

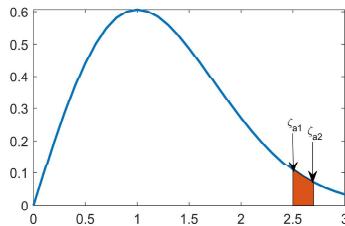


Figure 3.19: Rayleigh Probability Distribution

Considering the wave amplitudes (Rayleigh probability distribution), the probability that an amplitude ζ_a will exceed a specific amplitude, ζ_{A1} is

$$P(\zeta_a > \zeta_{A1}) = e^{-\frac{\zeta_{A1}^2}{2m_0}}. \quad (3.12)$$

The probability that the wave amplitude will fall *between* amplitudes ζ_{A1} and ζ_{A2} is

$$P(\zeta_{A1} < \zeta_a < \zeta_{A2}) = e^{-\frac{\zeta_{A2}^2}{2m_0}} - e^{-\frac{\zeta_{A1}^2}{2m_0}},$$

i.e. the probability of exceeding ζ_{A2} minus the probability of exceeding ζ_{A1} .

Significant Wave Height and Related Statistics

The significant wave height is the mean of the highest 1/3rd of the heights recorded in a wave time history. It closely correlates with the average wave height estimated visually by an experienced observer. It is expected that the experienced sailor's estimates of "average" wave heights might be similar to the significant wave height. The Rayleigh formula for the mean value of the highest 1/ n th of all observations is

$$\zeta_{\frac{1}{n}} = \sqrt{-2m_0 \ln \frac{1}{n}}$$

So, for $n = 1$, the mean of all amplitudes, $\bar{\zeta}_a = 1.25\sigma_0$ where σ_0 is the standard deviation from the water surface elevation ($\sigma_0 = \sqrt{m_0}$). For the significant wave amplitude,

$$\bar{\zeta}_{1/3} = 2.00\sigma_0.$$

Remember from earlier that the significant wave height, $\bar{H}_{1/3}$ equaled $4.00\sqrt{m_0}$. This is the same as saying that the significant wave *height* is equal to twice the significant wave *amplitude*. These results are widely assumed to apply to all wave records. However, this is only **strictly** true if the Rayleigh formula applies. Table 3.3 shows the values that can be multiplied by the water elevation standard deviation (σ_0) to determine the average of the highest 1/ n th amplitudes.

n	$\frac{\zeta_{1/n}}{\sigma_0}$	n	$\frac{\zeta_{1/n}}{\sigma_0}$
1	1.25	10	2.54
2	1.77	100	3.34
3	2.00	1000	3.72

Table 3.3: Mean of the highest 1/ n / amplitudes (Rayleigh formula)

Probability of Exceedance

So, what is the procedure for predicting the probability of the wave amplitude exceeding a particular value (ζ_B) in a specific sea state? First we have to decide which idealized spectra we want to use to create our sea state. Since the NATO sea state conditions provide a mean significant wave height and most probable modal wave period, using the ITTC (Bretschneider) spectrum makes sense.

1. First we build an ITTC wave energy spectrum for our sea state (using the mean significant wave height and most probable modal period). We will need to convert the modal period, T_0 , into modal frequency, $\omega_0 = 2\pi/T_0$.

$$S_\zeta(\omega) = \frac{1.25}{4} \left(\frac{\omega_0}{\omega} \right)^4 \frac{\bar{H}_{1/3}^2}{\omega} e^{-1.25 \left(\frac{\omega_0}{\omega} \right)^4}.$$

2. We can find the variance, m_0 from this wave energy spectrum.

$$m_0 = \int_0^\infty S_\zeta(\omega) d\omega.$$

3. Then, we use the variance and the value of interest to calculate the probability of exceedance.

$$P(\zeta_a > \zeta_B) = e^{-\frac{\zeta_B^2}{2m_0}}.$$

Calculation of Probability of Exceedance Example Consider the ocean spectrum for a Bretschneider sea state 6. For the time history recorded (a total of 23.5 minutes), the variance of the water elevation was 16.81 ft^2 . Find the significant wave height and the probability of a wave height exceeding 25 ft. Find the probability of exceeding the significant wave height.

How did we get m_0 (variance of the water elevation)? It was found either by taking the variance of the time history (as in this problem) or by finding the area under the wave energy spectrum (as explained in the procedure above). Once we have it we can find the significant wave height directly

$$\bar{H}_{1/3} = 4.00\sqrt{m_0} = 4.00\sqrt{16.81} = 16.4 \text{ ft.}$$

To find the probability of exceedance we plug this into the equation 3.12. This equation requires us to use the wave amplitude. Since we want the probability of exceedance for a wave *height* of 25 ft, the corresponding wave amplitude is $25/2 = 12.5 \text{ ft}$.

$$P(\zeta_a > 12.5) = e^{-\frac{12.5^2}{2(16.81)}} = 0.0096 = 0.96\%$$

So, there is a 0.96% probability that we will encounter a wave height over 25 ft. The probability we will exceed the significant wave height of 16.4 ft (amplitude of 8.2 ft) is

$$P(\zeta_a > 8.2) = e^{-\frac{8.2^2}{2(16.81)}} = 0.1353 = 13.53\%.$$

Problems

Problems on Regular Waves

1. What is the equation for the wave celerity based on wave frequency in deep water?
2. What is the equation for the wave celerity in shallow water?
3. What is the equation for the group velocity based on wave frequency in deep water?
4. What is the equation for the group velocity in shallow water?
5. What is the relationship between the wave celerity and the group velocity in deep water?
6. Is the relationship between wave celerity and group velocity the same in deep and shallow water? If not, what is the relationship in shallow water?
7. In a model basin ($g = 9.81 \text{ m/s}^2$, $\rho = 1000 \text{ kg/m}^3$), three pictures are taken with a high-speed camera of a traveling wave. Based on the pictures, graphs of the wave profile are established and provided below. The wave profile from the first picture is labeled $t = 0$, although the motion was initiated earlier, because the timing device was started at this time.
 - (a) From the graph provided, determine the wave celerity and wave length.
 - (b) Using the celerity and wavelength from part (a), calculate the period and frequency (in rad/sec).
 - (c) Give the mathematical expression for the wave (*be sure to include an amplitude, frequency, and wave number in your expression*).
 - (d) Calculate the average energy per square meter of wave surface (in Joules/m²).

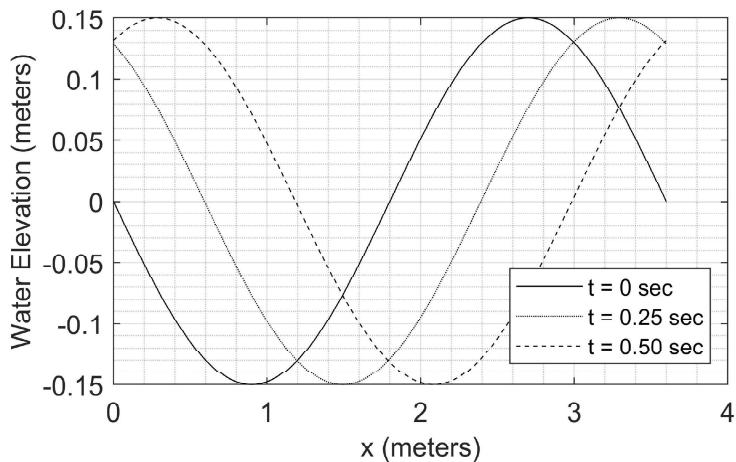


Figure 3.20: Figure from Regular Waves PreLab 2014

8. For a wave traveling in deep water, what happens to the wavelength and the wave celerity as the wave frequency decreases?
9. Consider a wave with a wave period of 5 seconds and a wave amplitude of 2 ft traveling in deep water.
 - (a) What is the wave celerity?
 - (b) What is the group velocity?
10. Consider a wave with a wavelength of 20 ft and a wave amplitude of 1 ft traveling in water of depth 30 ft.
 - (a) What is the wave frequency (in rad/sec)?
 - (b) What is the wave period?
 - (c) What is the wave slope?
 - (d) What is the wave celerity?
 - (e) What is the group velocity?
11. The speed of the energy transmission of the wave is the:
 - (i) wave celerity
 - (ii) group velocity
 - (iii) wave speed
 - (iv) ship speed
12. A wave has a wavelength of 32 ft, an amplitude of 2 ft, and a wave celerity of 12.8 ft/s. The period of the wave is:
 - (i) 0.4 sec
 - (ii) 2.0 sec
 - (iii) 2.5 sec
 - (iv) 6.25 sec
13. The group velocity refers to the energy transmission speed and is
 - (i) faster than the wave celerity
 - (ii) slower than the wave celerity
 - (iii) the same speed as the wave celerity
 - (iv) less than or equal to the wave speed, depending on the water depth
14. Explain the advantage to a submarine for going deep during a large storm when out at sea.

15. Label all 5 elements on the figure (wave celerity, wave amplitude, wave height, water depth, and wavelength).

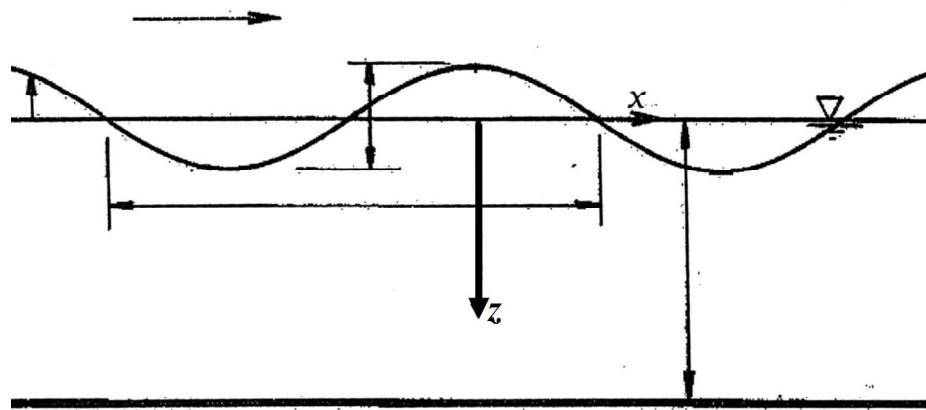


Figure 3.21: Figure of wave with labels missing

16. At sea in deep water, you observe a wave passing through a field of wave buoys. At time $t = 0$ the wave passes buoy #1 with an amplitude of 3.5 ft. You follow the wave crest and note that it passes buoy #2 with the same 3.5 ft amplitude 5.08 seconds later. The buoys are 230 ft apart. Using $\rho = 1.9936 \text{ slugs}/\text{ft}^3$ for the water density and $g = 32.174 \text{ ft}/\text{s}^2$ for gravitational acceleration, determine the wave speed and period. Is this speed the wave celerity or the group speed? What is the wavelength and frequency of the wave?
17. A wave of frequency 1.3 rad/sec is traveling in a water depth of 5 ft. Find the wave celerity and group speed.
18. **True or False:** For a wave traveling in shallow water, the water particles near the surface travel with the wave, while the particles near the sea floor make ellipses.
19. Sketch a wave and label the wave height, wavelength, water depth, and wave slope.
20. What is the speed and period of a deep-sea wave that is 1000 ft long?

Problems on Irregular Waves

21. The wave height characteristics from wave records are shown in the table below. Find the total number of waves measured, the average height, the significant height, the average one-tenth and one-hundredth highest waves.

Wave Height (ft)	# of waves having this height
1	4
2	40
3	31
4	25
5	2

22. The wave spectrum of the sea surface, $S_\zeta(\omega)$, for the South Atlantic Ocean off the coast of South Africa is idealized as shown below. Determine the following for this particular seaway:
- the spectrum bandwidth
 - the significant wave height
 - the mean zero up-crossing
 - the mean wave frequency

Suggestion: use $\Delta\omega$ rad/sec and Simpson's Rule.

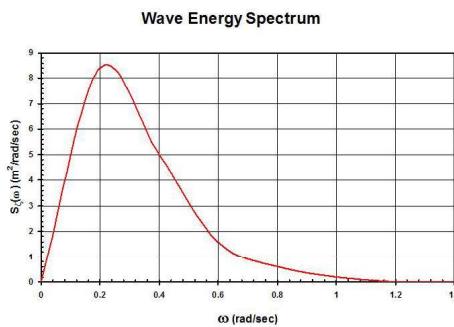
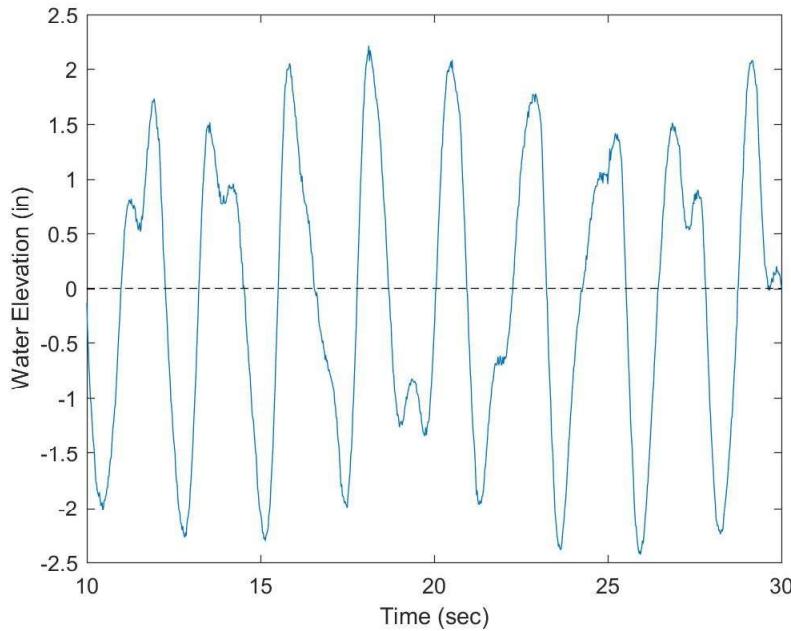


Figure 3.22: Wave Energy Spectrum

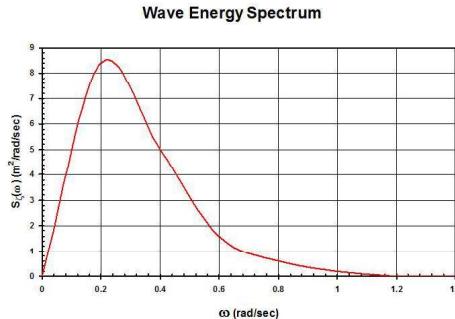
23. Briefly explain what a Wave Energy Spectrum is, where it comes from, and how it can be related to a Rayleigh Probability Distribution.
24. What is the relationship between the variance of the irregular wave signal and the wave energy spectrum curve?
25. The spectral moment m_2 is
- the variance of the velocity of the irregular wave signal
 - the variance of the acceleration of the irregular wave signal
 - the variance of the motion of the irregular wave signal
 - the variance of the mean of the motion time history of the irregular wave signal
26. A narrow spectrum bandwidth for a wave energy spectrum indicates
- the mean period is zero
 - the mean period and the zero-crossing period are the same
 - the mean period and the peak periods are the same
 - the zero-crossing period and the peak periods are the same

27. The wave energy spectrum spectral ordinate is proportional to
- (i) the wave amplitude
 - (ii) the wave frequency
 - (iii) the wave amplitude squared
 - (iv) the wave frequency squared
28. The expected or average value of the wave surface is
- (i) the variance
 - (ii) the mean
 - (iii) the standard deviation
 - (iv) the root mean square
29. The measure of how far a set of numbers is spread out is
- (i) the variance
 - (ii) the mean
 - (iii) the standard deviation
 - (iv) the root mean square
30. The wave energy spectrum is the
- (i) measure of the distribution of energy across possible wave amplitudes
 - (ii) measure of the distribution of energy across the sea surface
 - (iii) measure of the distribution of energy across the frequency spectrum
 - (iv) measure of the distribution of energy across the time history
31. The energy spectrum spectral ordinate refers to
- (i) the energy per unit length of sea surface
 - (ii) the total energy per square unit area of sea surface at a wave frequency component
 - (iii) the total energy per square unit area of sea surface
 - (iv) the energy per unit length of sea surface at a wave amplitude
32. For an irregular sea, the mean period of the peaks is equal to $2\pi\sqrt{\frac{m_2}{m_4}}$. m_2 and m_4 are the variances of the
- (i) irregular wave time history and velocity, respectively
 - (ii) irregular wave time history and acceleration, respectively
 - (iii) irregular wave time velocity and acceleration, respectively
33. **True or False:** The period between peaks is about equal to the period between zero-crossings for a wide-banded time history.



34. Which idealized wave energy spectrum would be best to use when creating an open ocean sea state with a given significant wave height and modal period?
35. On the irregular wave time history shown, label an example of ζ_a , T_z , T_p , and H_a .
36. Sketch a sample plot of wave height (ζ_a) versus wave frequency (ω).
37. State the relationship between the spectral ordinate, $S_\zeta(\omega_i)$, representing the wave energy, and the wave amplitude, ζ_a .
38. How do you find the variance of the time history of the water elevation?
39. How does the variance relate to the $S_\zeta(\omega)$ curve?
40. What is the equation for the spectral moments? What is the zeroth spectral moment (m_0) equal to?
41. How do you find $\bar{\omega}$, \bar{T}_z , \bar{T}_p , and $\bar{H}_{1/3}$ if
 - (a) you are given the time history of the water elevation?
 - (b) you are given the wave energy spectrum?
42. List the three models for ocean wave spectra and identify what parameters they use and what ocean situations they are the most suited to model.

43. Consider the spectrum below. The area under the curve is 3.47 m^2 . What is the significant wave height?



Problems on Waves and Probability

44. Using the ITTC spectrum, determine the probability a wave of 50 ft will be exceeded in a sea state 8 (NATO guidelines: mean wave height = 37.7 ft, most probable modal wave period = 16.4 sec)
45. The ten year extreme wave height (H) at a site of interest for offshore drilling and extraction can be assumed to follow a Rayleigh distribution with a mean wave height of 16 ft (mean wave height = 2.5σ). Note: remember the relationship between wave height and wave amplitude! Determine the following:
- Probability of a wave height greater than 24 feet
 - Probability of a wave height less than or equal to 18 feet
 - Probability of seeing a wave of less than or equal to 24 feet, but greater than 18 feet
 - What is the one in 50,000 wave height (i.e. the wave height that has a probability of 0.002% of being exceeded)?
46. For a wave height histogram that follows the ideal Rayleigh distribution, for a value of $m_0 = 80.70 \text{ ft}^2$, what is the probability of exceedance for a wave with a wave amplitude of 24 ft.

Problems on Computer Analysis

47. What is the purpose of the *Fast Fourier Transform* (FFT)? What are some of the limitations and restrictions you should be aware of when using the FFT? How can you control the size of the frequency interval the procedure will use?

48. Consider the water surface elevations measured in the Gulf of Mexico during Hurricane Andrew in August 1992 (total number of measurements $N = 6498$). The data are in units of feet and were sampled at 4 Hz. Consider the range of frequencies that can be resolved using a Fourier Transform:
- (a) What is the highest frequency wave that can be resolved using this data?
 - (b) What is the lowest frequency wave that can be resolved using this data?