15) Tensor Operations and Stochastic Gradient Descent (SGD)

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Reference

Goodfellow et al. (2016): Ch 2, 4, and 8.

https://www.deeplearningbook.org/

Chollet (2018): Ch 2.

https://www.manning.com/books/deep-learning-with-python

Tensor Rank (R)

Scalar (0R)

$$s$$

Vector (1R)

$$v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_r \end{bmatrix}$$

Matrix (2R)

$$M = \begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1c} \\ m_{12} & m_{22} & \cdots & m_{2c} \\ \vdots & \vdots & \ddots & \vdots \\ m_{r1} & m_{r2} & \cdots & m_{rc} \end{bmatrix}$$

Data Tensors

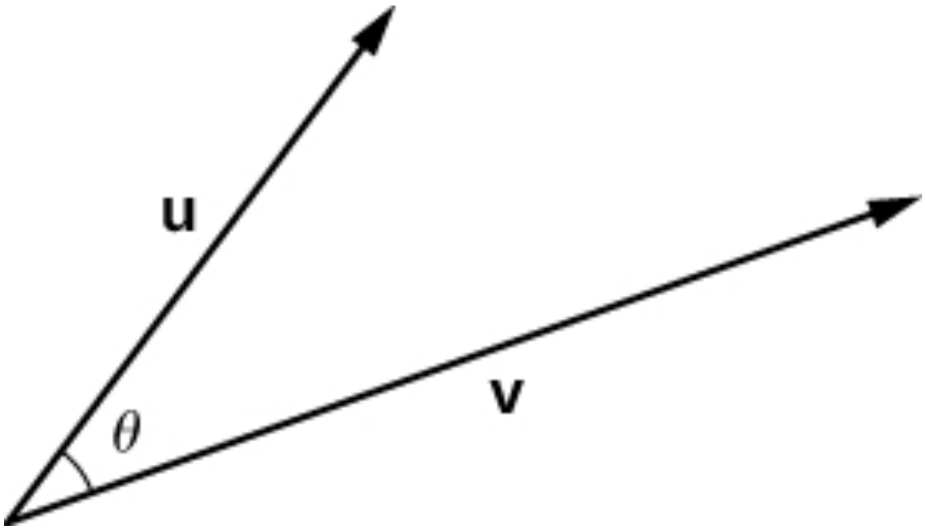
	Cross-Sectional (2R)	**Time Series (3R)**
Sample	100 Firms	100 Firms
Time		10 years
Features	Sales, R&D, Size	Sales, R&D, Size

	Image (4R)	**Video (5R)**
Sample	128	4
Frames		240
Height	256 pixels	256 pixels
Width	256 pixels	256 pixels
Colors	3	3

Dot Product

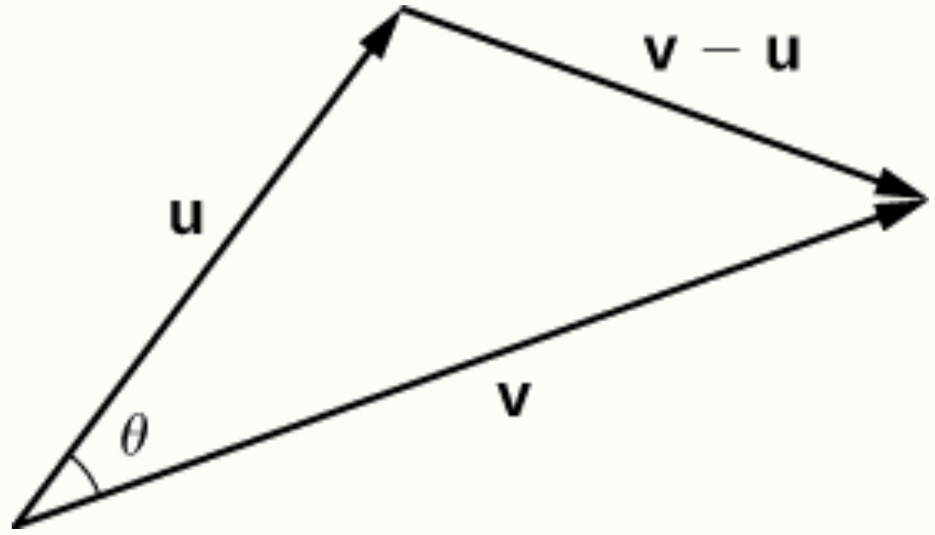
$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2$$

$$\vec{u} \cdot \vec{u} = || \vec{u} ||^2$$



$$\vec{u} \cdot \vec{v} = || \vec{u} || || \vec{v} || cos\theta$$

Law of Cosines

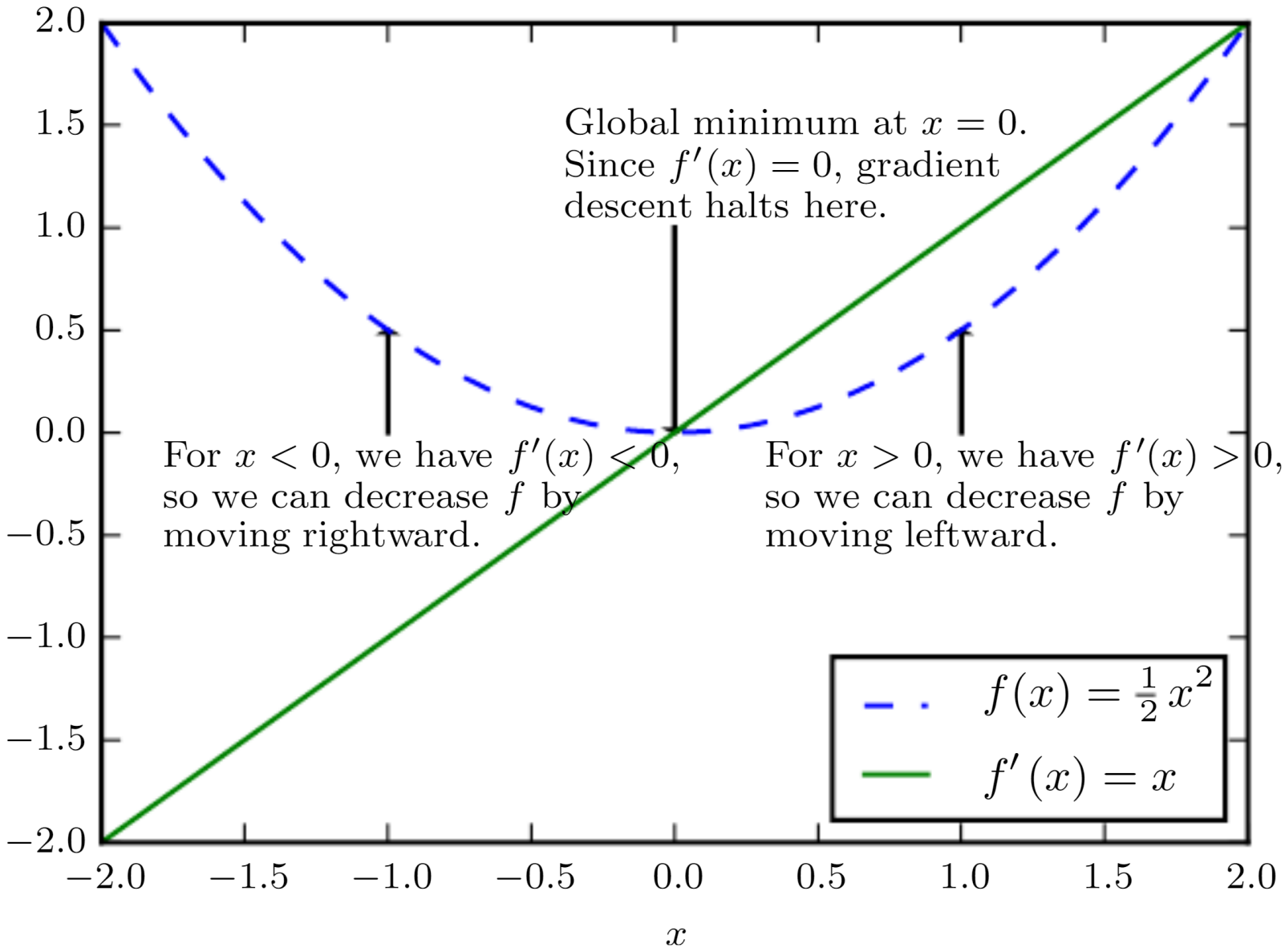


$$||\vec{u} - \vec{v}||^2 = ||\vec{u}||^2 + ||\vec{u}||^2 - 2||\vec{u}||||\vec{v}|| cos\theta$$

$$||\vec{u} - \vec{v}||^2 = ||\vec{u}||^2 + ||\vec{u}||^2 - 2\vec{u} \cdot \vec{v}$$

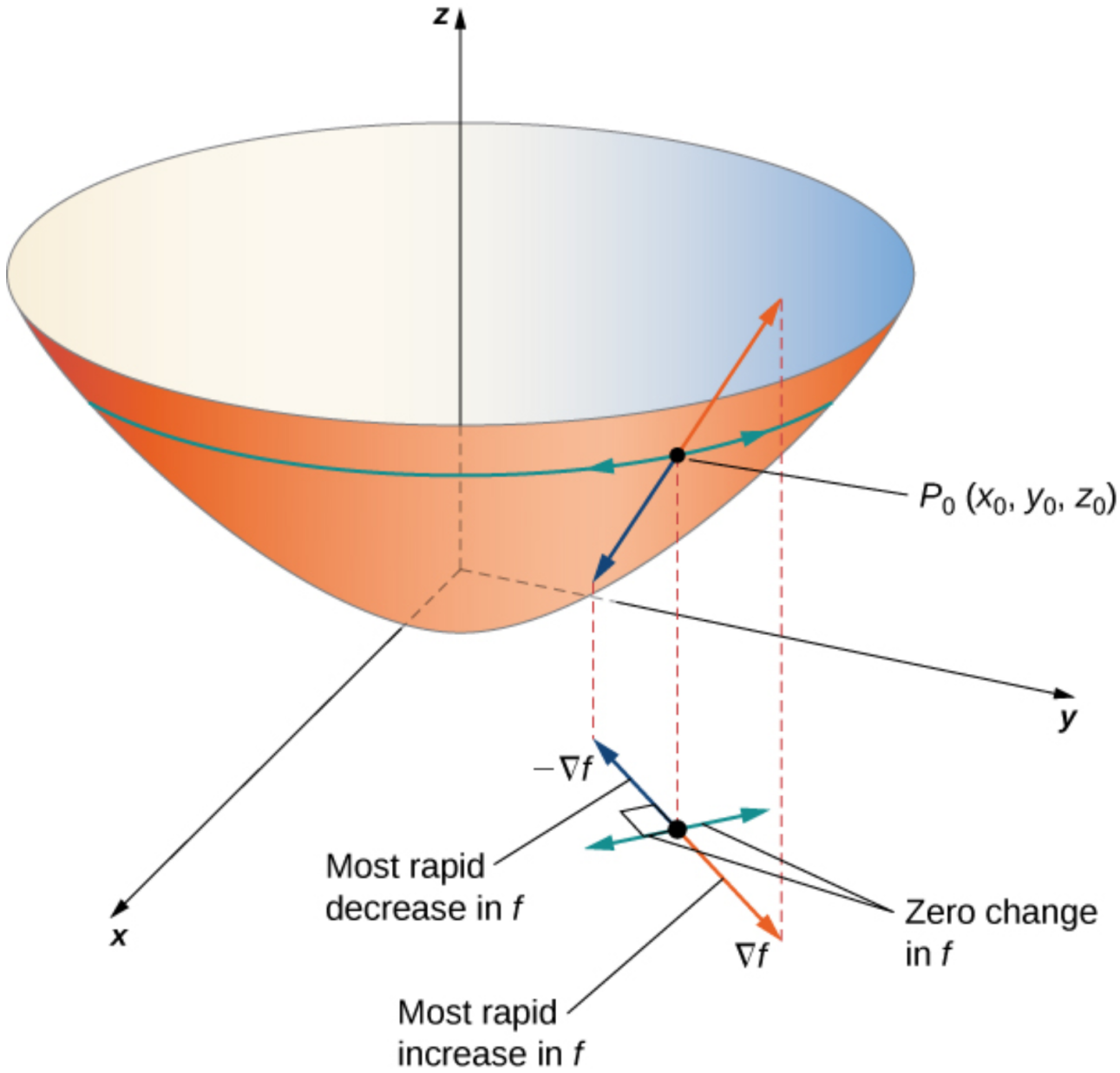
$$\vec{u} \cdot \vec{v} = ||\vec{u}||||\vec{v}|| cos\theta$$

Minimizing $f(x) = \frac{1}{2}x^2$



Goodfellow et al. (2016:81)

Directional Derivative and Gradient



Source: LibreTexts (2019). Calculus - Early Transcendentals (Stewart), Ch 4.1.

Directional Derivatives

$$z = f(x, y)$$

$$\hat{\mathbf{u}} = (cos\theta)\hat{\mathbf{i}} + (sin\theta)\hat{\mathbf{j}}$$

$$||\hat{\mathbf{u}}|| = [cos\theta]^2 + [sin\theta]^2 = 1$$

$$D_{\vec{\boldsymbol{u}}} f(x, y) = \frac{\partial z}{\partial x} cos\theta + \frac{\partial z}{\partial y} sin\theta$$

Gradient

$$\vec{\nabla} f(x,y) = \frac{\partial z}{\partial x}\hat{\mathbf{i}} + \frac{\partial z}{\partial y}\hat{\mathbf{j}}$$

$$D_{\vec{u}} f(x_0,y_0) = \vec{\nabla} f(x_0,y_0) \cdot \hat{\mathbf{u}}$$

$$= \| \vec{\nabla} f(x_0,y_0) \| \| \hat{\mathbf{u}} \| \cos\varphi$$

$$= \| \vec{\nabla} f(x_0,y_0) \| \cos\varphi$$

Gradient Descent

$$D_{\vec{u}} f(x_0, y_0) = \| \vec{\nabla} f(x_0, y_0) \| cos\varphi$$

$$\text{If } \varphi = \pi \text{ then } cos\varphi = -1$$

$$\therefore \vec{\nabla} f(x_0, y_0) \text{ and } \hat{\mathbf{u}} \text{ point in opposite directions}$$

$$\text{Min Value of } D_{\vec{u}} f(x_0, y_0) \text{ is } -\| \vec{\nabla} f(x_0, y_0) \|$$

Deterministic vs SGD

$$J(\theta) = \frac{1}{n} \sum_{i=1}^{n} L(x_i, y_i, \theta)$$

$$\nabla_\theta J(\theta) = \frac{1}{n} \sum_{i=1}^{n} \nabla_\theta L(x_i, y_i, \theta)$$

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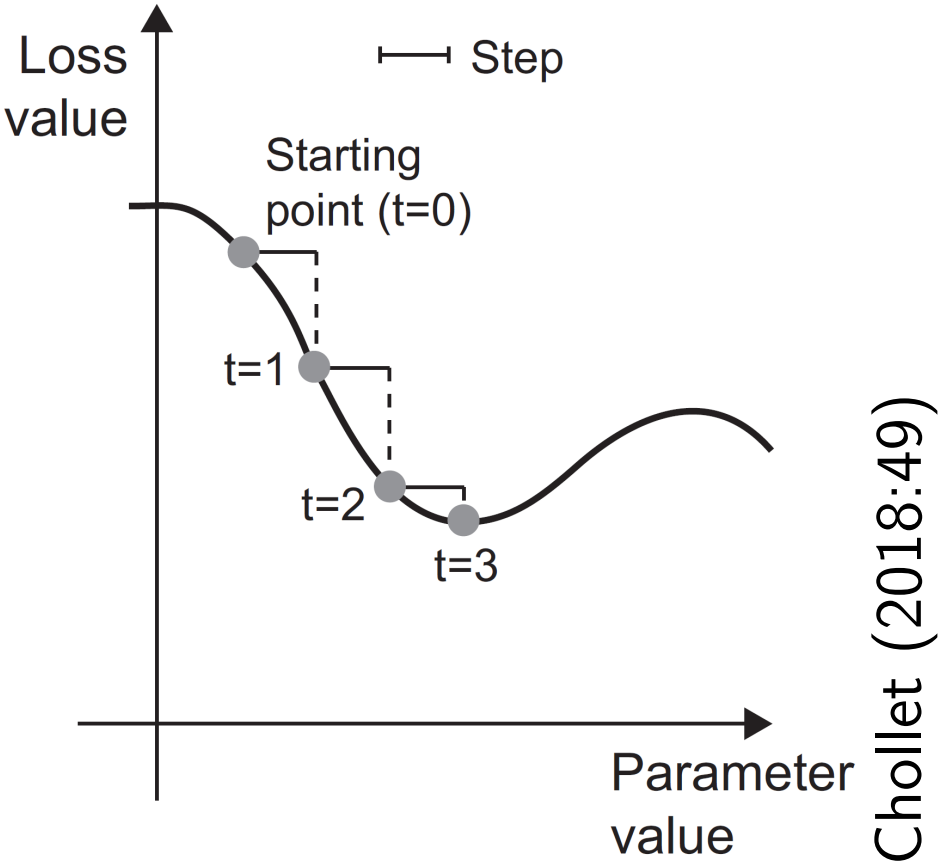
m: minibatch (power of 2)

$$g = \frac{1}{m} \sum_{i=1}^{m} \nabla_\theta L(x_i, y_i, \theta)$$

Epoch: each iteration over all the training data

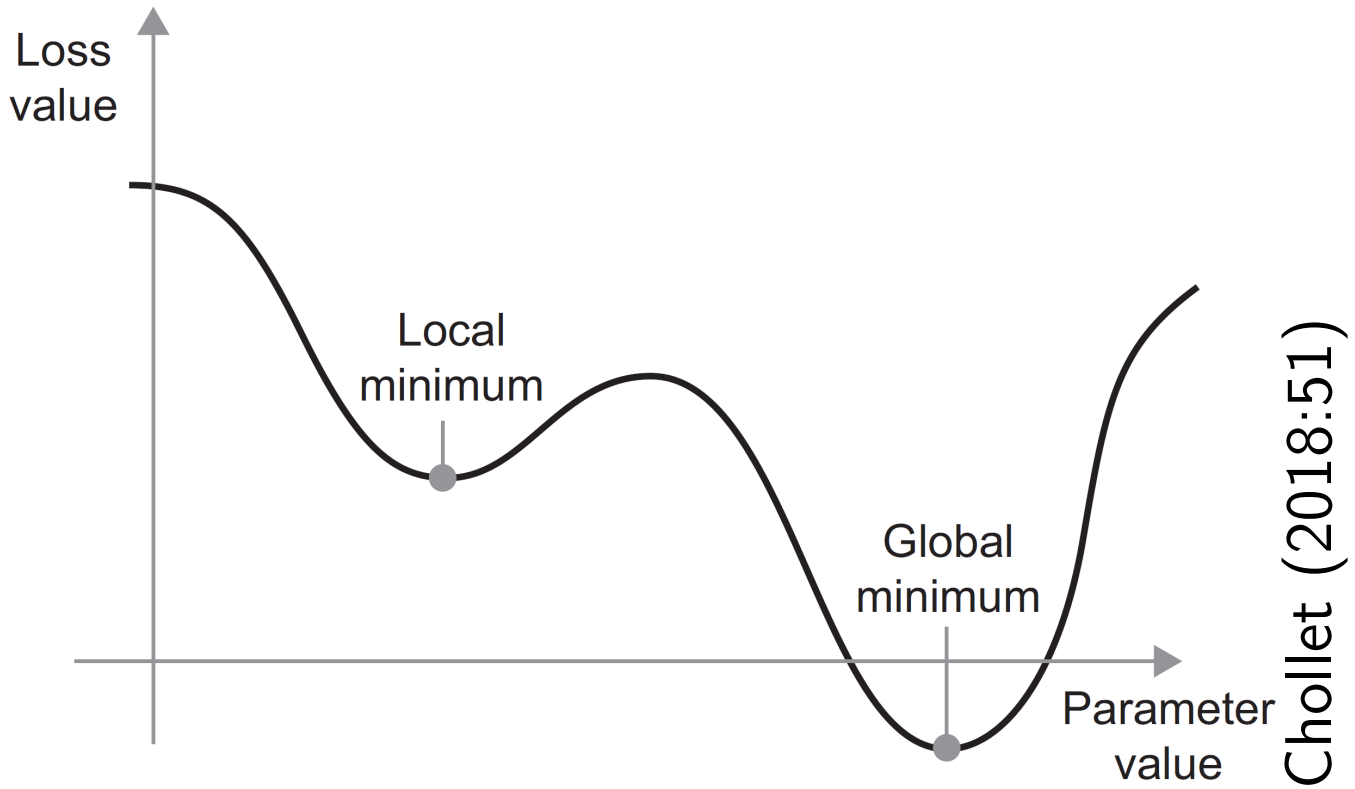
Stochastic Gradient Descent



$$\theta \leftarrow \theta - \epsilon g$$

ϵ : **learning rate or step**

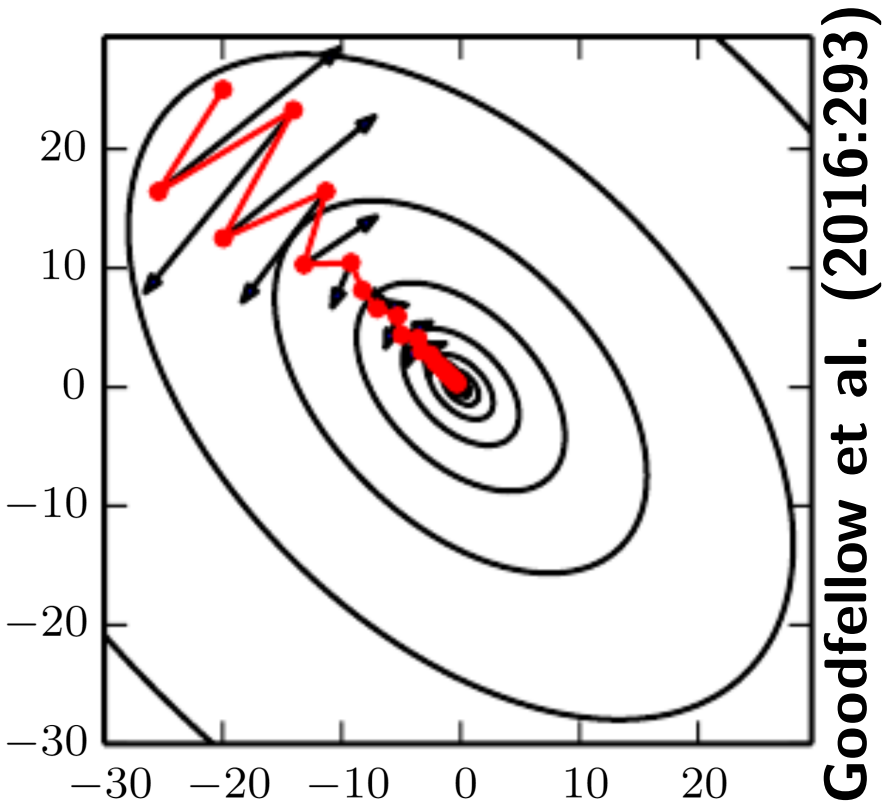
Momentum or Velocity



$$v \leftarrow \alpha v - \epsilon g, \qquad \alpha \in [0, 1)$$

v: Exponentially Decaying Average of the Gradient

Red Path Gradient with Momentum



Black Path Gradient wastes time moving back and forth

Nesterov Momentum

$$g \leftarrow \nabla_\theta \frac{1}{m} \sum_{i=1}^{m} L[(x_i; \theta), y_i]$$

$$g \leftarrow \nabla_\theta \frac{1}{m} \sum_{i=1}^{m} L[(x_i; \theta + \alpha v), y_i]$$

$$v \leftarrow \alpha v - \epsilon g$$

$$\theta \leftarrow \theta + v$$

AdaGrad (Adaptive Gradient)

$$r \leftarrow r + g \odot g$$

$$\Delta\theta \leftarrow -\frac{\epsilon}{\delta + \sqrt{r}} \odot g$$

Adapts learning rates (ϵ), scaling them inversely proportional to past squared values of the gradient (r)

$\delta = 10^{-7}$ for numerical stability

RMSProp

$$r \leftarrow \rho r + (1 - \rho) g \odot g$$

$$\Delta\theta = -\frac{\epsilon}{\sqrt{\delta + r}} \odot g$$

Discard extreme past, using exponentially decaying average (ρ)

$$\theta \leftarrow \theta + \Delta\theta$$

RMSProp + Nesterov Momentum

$$\tilde{\theta} \leftarrow \theta + \alpha \boldsymbol{v}$$

$$g \leftarrow \nabla_{\tilde{\theta}} \frac{1}{m} \sum_{i=1}^{m} L[f(x_i; \tilde{\theta}), y_i]$$

$$r \leftarrow \rho r + (1 - \rho) g \odot g$$

$$\boldsymbol{v} \leftarrow \alpha \boldsymbol{v} - \frac{\epsilon}{\sqrt{r}} \odot g$$

$$\theta \leftarrow \theta + \boldsymbol{v}$$