12) Fixed Effects (FE) and Random Effects (RE)

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Fixed Effects (Time-Demeaned)

$$y_{it} = x_{it}\beta + c_i + u_{it} \quad \textbf{(1)}$$

$$\bar{y}_i = \bar{x}_i\beta + c_i + \bar{u}_i \quad \textbf{(2)}$$

$$y_{it} - \bar{y}_i = (x_{it} - \bar{x}_i)\beta + u_{it} - \bar{u}_i$$

$$\ddot{y}_{it} = \ddot{x}_{it}\beta + \ddot{u}_{it}$$

Fixed Effects (Estimation)

$$\ddot{y}_{it} = \ddot{x}_{it}\beta + \ddot{u}_{it}$$

FE.1: $E(u_{it}|x_i, c_i) = 0, \quad t = 1, 2, ..., T$

FE.2: $rank[\sum_{t=1}^{T} E(\ddot{x}_i' \ddot{x}_i)] = K$

$$\hat{\beta}_{FE} = \left(\sum_{i=1}^{N} \sum_{t=1}^{T} \ddot{x}_{it}' \ddot{x}_{it}\right)^{-1} \left(\sum_{i=1}^{N} \sum_{t=1}^{T} \ddot{x}_{it}' \ddot{y}_{it}\right)$$

FE.3: $E(u_i u_i' | x_i, c_i) = \sigma_u^2 I_T$

$$Avar(\hat{\beta}_{FE}) = \hat{\sigma}_u^2 (\sum_{i=1}^{N} \sum_{t=1}^{T} \ddot{x}_{it}' \ddot{x}_{it})^{-1}$$

$$Avar(\hat{\beta}_{FE}) = (\ddot{X}'\ddot{X})^{-1} [\sum_{i=1}^{N} \ddot{X}_i' \hat{\ddot{u}}_i \hat{\ddot{u}}_i' \ddot{X}_i] (\ddot{X}'\ddot{X})^{-1}$$

$$y_{it} \text{ on } d1_i, d2_i, \ldots dN_i, x_{it},$$

$$t = 1, 2, \ldots, T; \ i = 1, 2, \ldots, N$$

Random Effects (RE.1)

$$y_{it} = x_{it}\beta + c_i + u_{it}$$

a) $E(u_{it}|x_i, c_i) = 0, \ t = 1, \dots T$

b) $E(c_i|x_i) = E(c_i) = 0$, where
$$x_i = (x_{i1}, x_{i2}, \dots x_{it})$$

Random Effects (RE.2)

$$\text{rank } E(X_i'\Omega^{-1}X_i) = K$$

$$\hat{\beta}_{RE} = (\sum_{i=1}^{N} X_i'\hat{\Omega}^{-1}X_i)^{-1}(\sum_{i=1}^{N} X_i'\hat{\Omega}^{-1}y_i)$$

$$v_{it} = c_i + u_{it}$$

$$\Omega = E(v_i v_i') = \begin{pmatrix} \sigma_c^2 + \sigma_u^2 & \sigma_c^2 & \cdots & \sigma_c^2 \\ \sigma_c^2 & \sigma_c^2 + \sigma_u^2 & \cdots & \vdots \\ \vdots & & \ddots & \vdots \\ \sigma_c^2 & & & \sigma_c^2 + \sigma_u^2 \end{pmatrix}$$

Random Effects (RE.3)

$$a)\ E(u_i u_i' | x_i, c_i) = \sigma_u^2 I_T,$$

$$b)\ E(c_i^2 | x_i) = \sigma_c^2$$

$$v_{it} = c_i + u_{it}$$

$$Corr(v_{it}, v_{is}) = \frac{\sigma_c^2}{\sigma_c^2 + \sigma_u^2},\ t \neq s$$

$$\frac{E(v_{it}, v_{is})}{E(v_{it}^2)} = \frac{E[(c_i + u_{it})(c_i + u_{is})]}{E(c_i^2) + 2E(c_i u_{it}) + E(u_{it}^2)}$$

$$y_{it} - \lambda \bar{y}_i = (x_{it} - \lambda \bar{x}_i)\beta + v_{it} - \lambda \bar{v}_i$$

$$\lambda = 1 - \sqrt{\frac{\sigma_u^2}{\sigma_u^2 + T\sigma_c^2}}$$

Within R^2: $\rho^2\{(y_{it} - \bar{y}_i), (x'_{it}\hat{\beta} - \bar{x}'_i\hat{\beta})\}$

Between R^2: $\rho^2(\bar{y}_i, \bar{x}'_i\hat{\beta})$

Overall R^2: $\rho^2(y_{it}, x'_{it}\hat{\beta})$

State Traffic Fatality Data from Christopher Ruhm

$$Deaths_{st} = \beta_0 + \beta_1 BeerTax_{st} + X_{st} + u_{st}$$

use fatality.dta

xtset state year

vfrall: traffic fatality rate (deaths per 10,000)

unrate: unemployment rate

vmiles: average vehicle miles per driver

perinc: real income per capita (logarithm)

jaild: Mandatory jail dummy

Variable	Obs	Mean	Std. Dev.	Min	Max
vfrall	336	2.040444	.5701938	.82121	4.21784
beertax	336	.513256	.4778442	.0433109	2.720764
unrate	336	7.346726	2.533405	2.4	18
vmiles	336	7.890754	1.475659	4.576346	26.14827
jaild	335	.280597	.449963	0	1
perinc	336	13.88018	2.253046	9.513762	22.19345

Average traffic fatality rate is 2 per 10,000

Average real beer tax is $0.50 per case (in 1988 dollars)

OLS, RE and FE

reg vfrall beertax, vce(cluster state)

estimates store OLS

reg vfrall beertax unrate vmiles jaild perinc i.year, vce(cluster state)

estimates store OLS1

xtreg vfrall beertax unrate vmiles jaild perinc i.year, re vce(cluster state)

estimates store RE

xtreg vfrall beertax, fe vce(cluster state)

estimates store FE1

xtreg vfrall beertax unrate vmiles jaild perinc i.year, fe vce(cluster state)

estimates store FE2

reg vfrall beertax unrate vmiles jaild perinc i.state i.year, vce(cluster state)

estimates store Dummies

estimates table OLS OLS1 RE FE1 FE2 Dummies, b(%7.4f) se

Variable	OLS	OLS1	RE	FE1	FE2	Dummies
beertax	0.3646	0.1306	0.0687	-0.6559	-0.5282	-0.5282
	0.1197	0.0999	0.1226	0.2919	0.3234	0.3499
unrate		0.0060	-0.0830		-0.0767	-0.0767
		0.0249	0.0160		0.0130	0.0141
vmiles		0.1290	0.0209		0.0094	0.0094
		0.0827	0.0153		0.0078	0.0084
jaild		0.2507	0.1263		0.0413	0.0413
		0.1388	0.0920		0.1045	0.1131
perinc		-0.1006	-0.0243		0.0699	0.0699
		0.0314	0.0298		0.0381	0.0413

All models have Time Fixed Effects

xtreg vfrall beertax unrate vmiles jaild perinc i.year, re vce(cluster state) theta

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Random-effects GLS regression              Number of obs      =        335
Group variable: state                      Number of groups   =         48

R-sq:                                      Obs per group:
    within  = 0.2668                                      min =          6
    between = 0.0122                                      avg =        7.0
    overall = 0.0291                                      max =          7

                                           Wald chi2(11)      =      79.98
corr(u_i, X)   = 0  (assumed)              Prob > chi2        =     0.0000

───────────────────── theta ─────────────────────
   min      5%      median       95%       max
 0.8216   0.8344    0.8344     0.8344    0.8344

                        (Std. Err. adjusted for 48 clusters in state)
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Random Effects: $\frac{\sigma_c^2}{\sigma_c^2 + \sigma_u^2} = \frac{0.35^2}{0.35^2 + 0.15^2} = 0.83$

vfrall	Coef.	Std. Err.	z	P>|z|	[95% Conf. Interval]	
beertax	.0686756	.1225986	0.56	0.575	-.1716132	.3089644
unrate	-.0830162	.0159739	-5.20	0.000	-.1143244	-.051708
vmiles	.0208952	.015316	1.36	0.172	-.0091236	.050914
jaild	.1263044	.0920177	1.37	0.170	-.054047	.3066557
perinc	-.0242537	.0298142	-0.81	0.416	-.0826885	.0341811
year						
1983	-.0922118	.0322668	-2.86	0.004	-.1554536	-.02897
1984	-.2550368	.0643028	-3.97	0.000	-.381068	-.1290057
1985	-.3037063	.0687668	-4.42	0.000	-.4384868	-.1689258
1986	-.2209293	.0798166	-2.77	0.006	-.3773669	-.0644917
1987	-.2765892	.1027146	-2.69	0.007	-.4779061	-.0752724
1988	-.3272589	.1140888	-2.87	0.004	-.5508689	-.1036488
_cons	2.962533	.4509262	6.57	0.000	2.078734	3.846332
sigma_u	.35710213					
sigma_e	.15863836					
rho	.83517946	(fraction of variance due to u_i)				

xtreg vfrall beertax unrate vmiles jaild perinc i.year, fe vce(cluster state)

```
Fixed-effects (within) regression          Number of obs     =         335
Group variable: state                      Number of groups  =          48

R-sq:                                       Obs per group:
    within  = 0.3560                                    min =           6
    between = 0.2723                                    avg =         7.0
    overall = 0.1849                                    max =           7

                                            F(11,47)          =       10.89
corr(u_i, Xb)  = -0.8000                    Prob > F          =      0.0000

                                    (Std. Err. adjusted for 48 clusters in state)
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Fixed Effects

vfrall	Coef.	Std. Err.	t	P>|t|	[95% Conf. Interval]	
beertax	-.5282011	.3234169	-1.63	0.109	-1.178832	.1224297
unrate	-.0767362	.0130328	-5.89	0.000	-.1029548	-.0505177
vmiles	.0093521	.0078051	1.20	0.237	-.0063497	.0250539
jaild	.041324	.1045413	0.40	0.694	-.1689861	.251634
perinc	.0698717	.0381331	1.83	0.073	-.0068421	.1465855
year						
1983	-.0924104	.0310347	-2.98	0.005	-.1548441	-.0299767
1984	-.2803558	.0503522	-5.57	0.000	-.3816513	-.1790602
1985	-.3607624	.0546772	-6.60	0.000	-.4707587	-.250766
1986	-.3111751	.065962	-4.72	0.000	-.4438735	-.1784767
1987	-.4033538	.0825874	-4.88	0.000	-.5694982	-.2372095
1988	-.4871078	.0944596	-5.16	0.000	-.6771359	-.2970797
_cons	2.098344	.591725	3.55	0.001	.9079471	3.288741
sigma_u	.83158573					
sigma_e	.15863836					
rho	.96488615	(fraction of variance due to u_i)				