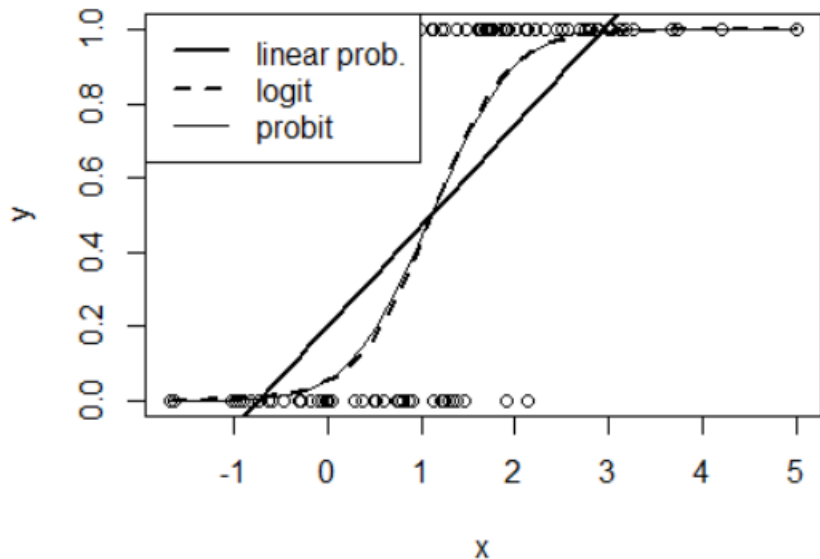# Logit and Probit

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Logit and Probit

Specifying Logit and Probit Models

$$P(y = 1|x) = G(\beta_0 + x\beta)$$

$$0 < G(z) < 1$$

$$G(z) = \frac{e^z}{1+e^z} = \Lambda(z)$$

$$G(z) = \Phi(z) = \int_{-\infty}^{z} \phi(v)dv$$

$$\phi(v) = (2\pi)^{-\frac{1}{2}} exp(-\frac{z^2}{2})$$

Latent Variable Model

$$y^* = \beta_0 + x\beta + e$$

$$y = 1[y^* > 0]$$

$$e \perp x$$

$$e \sim N(0, 1) \text{ or } Logistic(0, 1)$$

$$\therefore 1 - G(-z) = G(z)$$

Maximum Likelihood Estimation (MLE)

$$f(y|x_i; \beta) = [G(x_i\beta)]^y [1 - G(x_i\beta)]^{1-y}$$

$$\ell_i(\beta) = y_i log[G(x_i\beta)] + (1 - y_i) log[1 - G(x_i\beta)]$$

$$\mathcal{L}(\beta) = \sum_{i=1}^{n} \ell_i(\beta)$$

Logistic Regression

$$\ell_i(\beta) = y_i log[G(x_i\beta)] + (1 - y_i)log[1 - G(x_i\beta)]$$

$$\frac{\partial log\mathcal{L}(\beta)}{\partial \beta} = 0$$

$$\sum_{i=1}^{n} \left[\frac{y_i - G(x_i\beta)}{G(x_i\beta)(1 - G(x_i\beta))}g(x_i\beta)\right]x_i = 0$$

$$\sum_{i=1}^{n} \left[y_i - \frac{e^{x_i\beta}}{1 + e^{x_i\beta}}\right]x_i = 0$$

Response Probability for y

$$P(y = 1|x) = P(y^* > 0|x) = P[e > -(\beta_0 + x\beta)|x]$$

$$= 1 - G[-(\beta_0 + x\beta)] = G(\beta_0 + x\beta)$$

$$\frac{\partial p(x)}{\partial x_j} = g(\beta_0 + x\beta)\beta_j$$

$$g(z) = \frac{dG}{dz}(z) > 0$$

$G(\cdot)$ is a strictly increasing cdf

$$\phi(0) = \frac{1}{\sqrt{2\pi}} \approx .4, \qquad \lambda(0) = \frac{exp(0)}{[1+exp(0)]^2} \approx .25$$

Partial Effect of Continuous Variables on the Response Probability

$$\frac{\partial p(x)}{\partial x_j} = g(\beta_0 + x\beta)\beta_j$$

$$\frac{\partial p(x)}{\partial x_j} \Big/ \frac{\partial p(x)}{\partial x_h} = \frac{\beta_j}{\beta_h}$$

Binary: $\frac{\Delta p(x)}{\Delta x_1} = G(\beta_0 + \beta_1 x_1 + \beta_2 x_2 + ... + \beta_k x_k)$

$$-G(\beta_0 + \beta_2 x_2 + ... + \beta_k x_k)$$

Mroz (1987)

library(foreign);library(car); library(lmtest)
mroz<-read.dta("C:/RData/mroz.dta")

Statistic	N	Mean	St. Dev.	Min	Max
inlf	753	0.568	0.496	0	1
kidslt6	753	0.238	0.524	0	3
kidsge6	753	1.353	1.320	0	8
age	753	42.538	8.073	30	60
educ	753	12.287	2.280	5	17
wage	428	4.178	3.310	0.128	25.000
nwifeinc	753	20.129	11.635	−0.029	96.000
lwage	428	1.190	0.723	−2.054	3.219

Linear Probablity Model

```
linprob <- lm(inlf~nwifeinc+educ+exper+I(exper^2)+
age+kidslt6+kidsge6, data=mroz)
```

```
coeftest(linprob,vcov=hccm)
```

```
                Estimate   Std. Error  t value   Pr(>|t|)
(Intercept)   0.58551922   0.15358032   3.8125   0.000149  ***
nwifeinc     -0.00340517   0.00155826  -2.1852   0.029182  *
educ          0.03799530   0.00733982   5.1766  2.909e-07  ***
exper         0.03949239   0.00598359   6.6001  7.800e-11  ***
I(exper^2)   -0.00059631   0.00019895  -2.9973   0.002814  **
age          -0.01609081   0.00241459  -6.6640  5.183e-11  ***
kidslt6      -0.26181047   0.03215160  -8.1430  1.621e-15  ***
kidsge6       0.01301223   0.01366031   0.9526   0.341123
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' '
```

Logit Model

logitres<-glm(inlf~nwifeinc+educ+exper+I(exper^2)
+age+kidslt6+kidsge6, family=binomial(link=logit),data=mroz)

```
              Estimate Std. Error z value Pr(>|z|)
 (Intercept)  0.425452   0.860365   0.495  0.62095
 nwifeinc    -0.021345   0.008421  -2.535  0.01126 *
 educ         0.221170   0.043439   5.091 3.55e-07 ***
 exper        0.205870   0.032057   6.422 1.34e-10 ***
 I(exper^2)  -0.003154   0.001016  -3.104  0.00191 **
 age         -0.088024   0.014573  -6.040 1.54e-09 ***
 kidslt6     -1.443354   0.203583  -7.090 1.34e-12 ***
 kidsge6      0.060112   0.074789   0.804  0.42154
 ---
```

Probit Model

probitres<-glm(inlf~nwifeinc+educ+exper+I(exper^2)+
age+kidslt6+kidsge6, family=binomial(link=probit),data=mroz)

```
             Estimate Std. Error z value Pr(>|z|)
(Intercept)  0.2700736  0.5080782   0.532  0.59503
nwifeinc    -0.0120236  0.0049392  -2.434  0.01492 *
educ         0.1309040  0.0253987   5.154 2.55e-07 ***
exper        0.1233472  0.0187587   6.575 4.85e-11 ***
I(exper^2)  -0.0018871  0.0005999  -3.145  0.00166 **
age         -0.0528524  0.0084624  -6.246 4.22e-10 ***
kidslt6     -0.8683247  0.1183773  -7.335 2.21e-13 ***
kidsge6      0.0360056  0.0440303   0.818  0.41350
```

Pseudo R-squared

$$\text{Pseudo } R^2 = 1 - \frac{\mathcal{L}_{ur}}{\mathcal{L}_0} = 1 - \frac{D_{ur}}{D_0}$$

$\mathcal{L}_0$ the model with only an intercept

$$|\mathcal{L}_{ur}| \leq |\mathcal{L}_0|$$

$$D = -2\mathcal{L}$$

McFadden's Pseudo R^2

logLik(probitres)

$$-401.3022$$

1 - probitres$deviance/probitres$null.deviance

$$0.22$$

$$LR = 2(\mathcal{L}_{ur} - \mathcal{L}_{r})$$

$$LR \overset{a}{\sim} \chi^2_q$$

q # of exclusion restrictions

lrtest(restr,probitres)

```
Model 1: inlf ~ nwifeinc + educ + exper + I(exper^2) + age + kidslt6 +
    kidsge6
Model 2: inlf ~ 1
  #Df  LogLik Df  Chisq Pr(>Chisq)
1    8 -401.30
2    1 -514.87 -7 227.14  < 2.2e-16 ***
```

restr <- glm(inlf~nwifeinc+educ+ kidslt6+kidsge6,
family=binomial(link=logit),data=mroz)

lrtest(restr,probitres)

```
Model 1: inlf ~ nwifeinc + educ + kidslt6 + kidsge6
Model 2: inlf ~ nwifeinc + educ + exper + I(exper^2) + age + kidslt6 +
    kidsge6
  #Df  LogLik Df  Chisq Pr(>Chisq)
1    5 -464.92
2    8 -401.30  3 127.25  < 2.2e-16 ***
```

Partial Effect at the Average (PEA)

$$\Delta \hat{P}(y = 1 | x) \approx [g(\hat{\beta}_0 + x\hat{\beta})\hat{\beta}_j]\Delta x_j$$

$$g(\hat{\beta}_0 + \bar{x}\hat{\beta}) = g(\hat{\beta}_0 + \hat{\beta}_1\bar{x}_1 + \hat{\beta}_2\bar{x}_2 + ... + \hat{\beta}_k\bar{x}_k)$$

Partial Effect of x_j for the "average" person in the sample.

Marginal Effect at the Mean (MEM)

library(mfx)

logitmfx(inlf~nwifeinc+educ+exper+I(exper^2)
+age+kidslt6+kidsge6, data=mroz, atmean=TRUE)

```
Marginal Effects:
               dF/dx      Std. Err.         z      P>|z|
nwifeinc    -0.00519005  0.00204820  -2.5340   0.011278  *
educ         0.05377731  0.01056074   5.0922  3.539e-07  ***
exper        0.05005693  0.00782462   6.3974  1.581e-10  ***
I(exper^2)  -0.00076692  0.00024768  -3.0965   0.001959  **
age         -0.02140302  0.00353973  -6.0465  1.480e-09  ***
kidslt6     -0.35094982  0.04963897  -7.0700  1.549e-12  ***
kidsge6      0.01461621  0.01818832   0.8036   0.421625
---
```

Average Partial Effect (APE) or Average Marginal Effect (AME)

$$n^{-1} \sum_{i=1}^{n} [g(\hat{\beta}_0 + x_i\hat{\beta})\hat{\beta}_j]$$

AME is the average of the nonlinear function rather than the nonlinear function of the average

$$g[E(x\beta)] \neq E[g(x\beta)]$$

Average Marginal Effect (AME)

logitmfx(inlf~nwifeinc+educ+exper+I(exper^2)+age
+kidslt6+kidsge6, data=mroz, atmean=FALSE)

```
Marginal Effects:
                  dF/dx     Std. Err.          z      P>|z|
nwifeinc    -0.00381181   0.00153898    -2.4769   0.013255  *
educ         0.03949652   0.00846811     4.6641  3.099e-06  ***
exper        0.03676411   0.00655577     5.6079  2.048e-08  ***
I(exper^2)  -0.00056326   0.00018795    -2.9968   0.002728  **
age         -0.01571936   0.00293269    -5.3600  8.320e-08  ***
kidslt6     -0.25775366   0.04263493    -6.0456  1.489e-09  ***
kidsge6      0.01073482   0.01339130     0.8016   0.422769
---
```

xpred <- list(nwifeinc=c(100,0),educ=c(5,17),exper=
c(0,30), age=c(20,52),kidslt6=c(2,0),kidsge6=c(0,0))

predict(linprob, xpred,type = "response")

1	2
-0.41	1.04

predict(logitres, xpred,type = "response")

1	2
0.005	0.95

predict(probitres,xpred,type = "response")

1	2
0.001	0.96