8.1) Maximum Likelihood Estimator (MLE)

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Reference

Tables, Graphics, and Figures from:

Hansen (2018). **Econometrics.** Ch 5.

Normal Regression Model

$$y = x'\beta + \epsilon$$

$$\epsilon \sim N(0, \sigma^2)$$

$$f(y|x) = \frac{1}{\sqrt{2\pi\sigma^2}} exp[-\frac{1}{2\sigma^2}(y - x'\beta)^2]$$

$$f(y_1, ..., y_n|x_1, ..., x_n) = \prod_{i=1}^{n} f(y_i|x_i)$$

Maximum Likelihood Estimator (MLE)

$$\mathcal{L}(\beta, \sigma^2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} exp[-\frac{1}{2\sigma^2}(y_i - x_i'\beta)^2]$$

$$ln\mathcal{L} = -\frac{n}{2} ln(2\pi\sigma^2) - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (y_i - x_i'\beta)^2$$

$$\max_{\beta,\sigma^2} ln[\mathcal{L}(\beta, \sigma^2)]$$

$$\frac{\partial ln\mathcal{L}}{\partial\beta} = \frac{1}{\hat{\sigma}^2} \sum_{i=1}^{n} x_i(y_i - x_i'\hat{\beta}) = 0$$

$$\hat{\beta}_{mle} = (\sum_{i=1}^{n} x_i x_i')^{-1}(\sum_{i=1}^{n} x_i y_i) = \hat{\beta}_{ols}$$

Log-Likelihood

$$ln\mathcal{L} = -\frac{n}{2}ln(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i - x_i'\beta)^2$$

$$\frac{\partial ln\mathcal{L}}{\partial\sigma^2} = -\frac{n}{2\hat{\sigma}^2} + \frac{1}{2\hat{\sigma}^4}\sum_{i=1}^{n}(y_i - x_i'\hat{\beta})^2 = 0$$

$$\hat{\sigma}^2_{mle} = \frac{1}{n}\sum_{i=1}^{n}(y_i - x_i'\hat{\beta})^2 = \frac{1}{n}\sum_{i=1}^{n}\hat{e}_i^2 = \hat{\sigma}^2_{ols}$$

$$ln\mathcal{L}(\hat{\beta}_{mle}, \hat{\sigma}^2_{mle}) = -\frac{n}{2}ln(2\pi\hat{\sigma}_{mle}) - \frac{n}{2}$$

Likelihood Ratio Test

$$y_i = x'_{1i}\beta_1 + x'_{2i}\beta_2 + \epsilon$$

$$H_0 : \beta_2 = 0$$

$$ln\mathcal{L}(\hat{\beta}, \hat{\sigma}^2) = -\frac{n}{2}ln(2\pi\hat{\sigma}^2) - \frac{n}{2}$$

$$ln\mathcal{L}(\tilde{\beta}_1, \tilde{\sigma}^2) = -\frac{n}{2}ln(2\pi\tilde{\sigma}^2) - \frac{n}{2}$$

$$LR = n \times ln(\frac{\tilde{\sigma}^2}{\hat{\sigma}^2})$$

Information Bound for Normal Regression

$$\frac{\partial ln\mathcal{L}}{\partial \beta} = \frac{1}{\hat{\sigma}^2}\sum_{i=1}^{n} x_i(y_i - x_i'\hat{\beta}) = 0$$

$$\frac{\partial^2 ln\mathcal{L}}{\partial \beta \partial \beta'} = \frac{1}{\hat{\sigma}^2} X'X$$

$$\frac{\partial ln\mathcal{L}}{\partial \sigma^2} = -\frac{n}{2\hat{\sigma}^2} + \frac{1}{2\hat{\sigma}^4}\sum_{i=1}^{n} (y_i - x_i'\hat{\beta})^2 = 0$$

$$\frac{\partial^2 ln\mathcal{L}}{\partial \sigma^2 \partial \sigma^{2\prime}} = \frac{n}{2\hat{\sigma}^4}$$

The Cramér-Rao Lower Bound

$$I = \begin{pmatrix} \frac{1}{\hat{\sigma}^2}X'X & 0 \\ 0 & \frac{n}{2\hat{\sigma}^4} \end{pmatrix}$$

$$I^{-1} = \begin{pmatrix} \hat{\sigma}^2(X'X)^{-1} & 0 \\ 0 & \frac{2\hat{\sigma}^4}{n} \end{pmatrix}$$