4.2) OLS Gauss-Markov Theorem

Vitor Kamada

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Reference

Tables, Graphics, and Figures from:

Hansen (2018). **Econometrics.** Ch 4.1 to 4.7.

Random Sampling

$$y = X\beta + \epsilon$$

{ $(y_1, x_1), ..., (y_n, x_n)$ }

Independent and Identically Distributed (iid)

Unbiasedness of OLS

$$\hat{\beta} = (X'X)^{-1}X'y$$

$$\hat{\beta} = (X'X)^{-1}X'(X\beta + \epsilon)$$

$$\hat{\beta} = (X'X)^{-1}X'X\beta + (X'X)^{-1}X'\epsilon$$

$$\hat{\beta} = \beta + (X'X)^{-1}X'\epsilon$$

$$E(\hat{\beta}|X) = \beta + (X'X)^{-1}X'E(\epsilon|X)$$

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Consistency of OLS

$$\hat{\beta} = \beta + (X'X)^{-1}X'\epsilon$$

$$\hat{\beta} = \beta + (N^{-1}X'X)^{-1}N^{-1}X'\epsilon$$

$$plim\hat{eta} = eta + (plimN^{-1}X'X)^{-1}(plimN^{-1}X'\epsilon)$$

$$\hat{eta} = eta$$
 if $plimN^{-1}X'u = 0$



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Variance of Error Term (ϵ)

$$Var(\epsilon|X) = E(\epsilon\epsilon'|X)$$

(i)
$$E(\epsilon_i^2|X) = \sigma_i^2$$

(ii)
$$E(\epsilon_i \epsilon_j | X) = E(\epsilon_i | X) E(\epsilon_j | X) = 0$$

$$Var(\epsilon|X) = \begin{bmatrix} \sigma_1^2 & 0 & \cdots & 0 \\ 0 & \sigma_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_n^2 \end{bmatrix}$$

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Variance of Least Squares Estimator

$$\hat{\beta} - \beta = (X'X)^{-1}X'\epsilon$$

$$Var(\hat{\beta}|X) = E[(\hat{\beta} - \beta)(\hat{\beta} - \beta)'|X]$$

$$= E[(X'X)^{-1}X'\epsilon\epsilon'X(X'X)^{-1}|X]$$

$$= (X'X)^{-1}X'[Var(\epsilon|X)]X(X'X)^{-1}$$
If $Var(\epsilon|X) = I_0\sigma^2$, then:

If
$$Var(\epsilon|X) = I_n\sigma^2$$
, then:

$$Var(\hat{\beta}|X) = \sigma^2(X'X)^{-1}X'X(X'X)^{-1}$$
$$= \sigma^2(X'X)^{-1}$$

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Gauss-Markov Theorem

$$ilde{eta}=Cy ext{ and } A=(X'X)^{-1}X'$$
 $D=C-A ext{ or } C=D+A$
 $ilde{eta}=(D+A)y$
 $ilde{eta}=D(Xeta+\epsilon)+\hat{eta}$
If $ilde{eta}$ and \hat{eta} are unbiased, $DX=0$
 $ilde{eta}-eta=D\epsilon+\hat{eta}-eta$
 $ilde{eta}-eta=(D+A)\epsilon$

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OLS is the Best Linear Unbiased Estimator (BLUE)

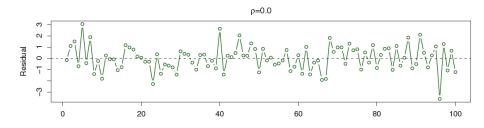
$$Var(\tilde{eta}|X) = (D+A)Var(\epsilon|X)(D'+A')$$
 $\sigma^2(DD'+AD'+DA'+AA')$
 $DA' = DX(X'X)^{-1} = 0$
 $Var(\tilde{eta}|X) = \sigma^2(DD'+(X'X)^{-1})$
 $> Var(eta|X)$

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Uncorrelated Error Terms

$$Cov(\epsilon_t, \epsilon_s | X) = 0$$
, for all $t \neq s$





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Correlated Error Terms

