1) Law of Large Numbers (LLN), Convergence in Probability and Distribution

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Markov Inequality

Let X be a nonnegative R.V. and $t > 0$.

$$E(X) = \sum_x xf(x) = \sum_{x<t} xf(x) + \sum_{x \geq t} xf(x)$$

$$E(X) \geq \sum_{x \geq t} xf(x) \geq \sum_{x \geq t} tf(x) = tPr(X \geq t)$$

$$-tPr(X \geq t) \geq -E(X)$$

$$\boxed{Pr(X \geq t) \leq \frac{E(X)}{t}}$$

Chebyshev Inequality

Let $Y = [X - E(X)]^2$, then $E(Y) = Var(X)$

$$Pr(X \geq t) \leq \frac{E(X)}{t}$$

$$Pr(|X - E(X)| \geq t)$$

$$= Pr(Y \geq t^2) \leq \frac{Var(X)}{t^2}$$

$$\boxed{Pr(|X - E(X)| \geq t) \leq \frac{Var(X)}{t^2}}$$

Properties of the Sample Mean

$$\bar{X}_n = \frac{1}{n}(X_1 + ... + X_n)$$

$$E(\bar{X}_n) = \frac{1}{n}E(X_1 + ... + X_n) = \frac{1}{n}n\mu = \mu$$

$$Var(\bar{X}_n) = \frac{1}{n^2}Var(\sum_{i=1}^{n} X_i) = \frac{1}{n^2}n\sigma^2 = \frac{\sigma^2}{n}$$

$$Se(\bar{X}_n) = \frac{\sigma}{\sqrt{n}}$$

Weak Law of Large Numbers (WLLN)

$$Pr(|X - E(X)| \geq t) \leq \frac{Var(X)}{t^2}$$

$$Pr(|\bar{X}_n - \mu| < \epsilon) \geq 1 - \frac{\sigma^2}{n\epsilon^2}$$

$$\lim_{n\to\infty} Pr(|\bar{X}_n - \mu| < \epsilon) = 1$$

Convergence in Probability: $\bar{X}_n \xrightarrow{p} \mu$

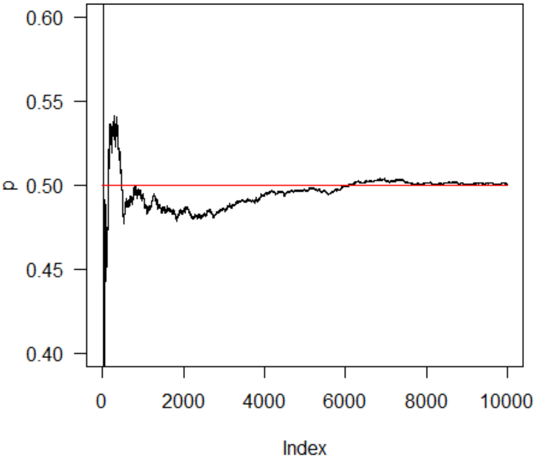
μ is the probability limit (*plim*) of $\bar{X}_n$

Fair Coin Simulation

```
set.seed(7)
n <- 10000
x <- sample(0:1, n, repl=T)
s <- cumsum(x); p <- s/(1:n)
plot(p, ylim=c(.4, .6), type="l")
lines(c(0,n), c(.5,.5), col = "red")
round(cbind(x,s,p), 3)[1:10,]; p[n]
```

LLN: Fair Coin

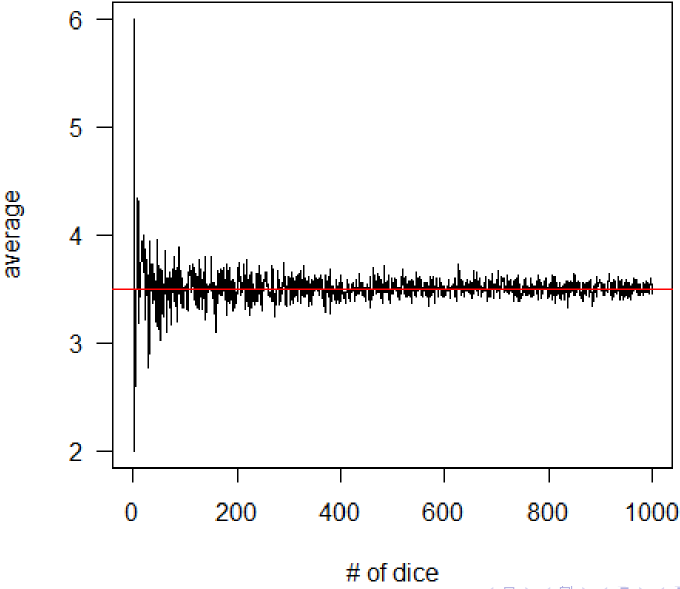
```
        x s     p
 [1,] 1 1 1.000
 [2,] 0 1 0.500
 [3,] 0 1 0.333
 [4,] 0 1 0.250
 [5,] 0 1 0.200
 [6,] 1 2 0.333
 [7,] 0 2 0.286
 [8,] 1 3 0.375
 [9,] 0 3 0.333
[10,] 0 3 0.300
[1] 0.5005
```



Dice Roll Simulation

```
die <- 1:6
roll <- function(n) {
mean(sample(die, size = n, replace = TRUE))
}
plot(sapply(1:1000, roll), type = "l",
xlab = "# of dice", ylab = "average")
abline(h = 3.5, col = "red")
```

Expected Value of Rolling Dice



Convergence in Distribution

A sequence of R.Vs. $\{b_n\}$ converge in distribution to a R.V. b if

$$lim_{n\to\infty} F_n = F$$

at every continuity point of F, where F_n is the distribution of b_n, F is the distribution of b.

$$b_n \xrightarrow{d} b$$

F is the limit distribution of $\{b_n\}$

Continuous Mapping and/or Slutsky's Theorem

$$b_n \xrightarrow{p} b \Rightarrow b_n \xrightarrow{d} b$$

$$b_n \xrightarrow{p} b \Rightarrow g(b_n) \xrightarrow{p} g(b)$$

$$a_n \xrightarrow{d} a \Rightarrow g(a_n) \xrightarrow{d} g(a)$$

$$1)\ a_n + b_n \xrightarrow{d} a + b$$

$$2)\ a_n b_n \xrightarrow{d} ab$$

$$3)\ \frac{a_n}{b_n} \xrightarrow{d} \frac{a}{b} \text{ if } b \neq 0$$