9) K-Nearest Neighbors

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Reference

Tables, Graphics, and Figures from

James et al. (2017): Chapters: 2.2.3, 3.5, 4.5, 4.6.5, 4.6.6

Hastie et al. (2017): Chapters: 2.3

Training Error and Test Error

$$Y = f(X) + \epsilon$$

$$\hat{Y} = \hat{f}(X)$$

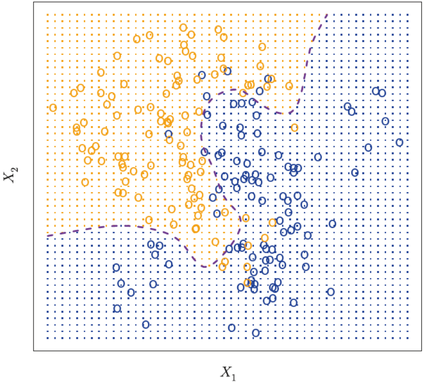
$$\{(x_1, y_1), \ldots, (x_n, y_n)\}$$

$$\frac{1}{n} \sum_{i=1}^{n} I(y_i \neq \hat{y}_i)$$

$$Ave(I(y_0 \neq \hat{y}_0))$$

Bayes Classifier and Error Rate

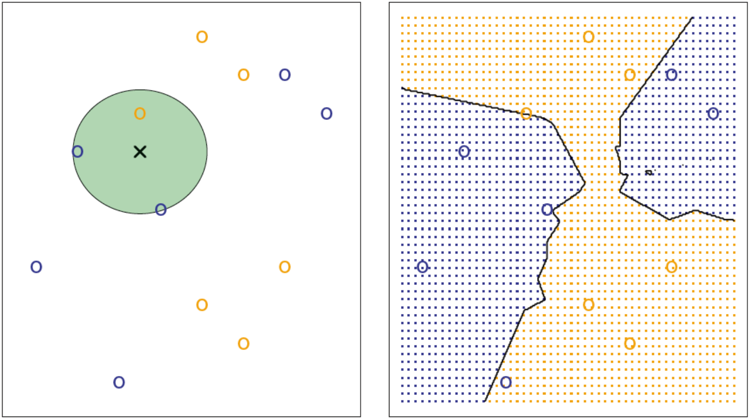
$$Pr(Y = j|X = x_0)$$



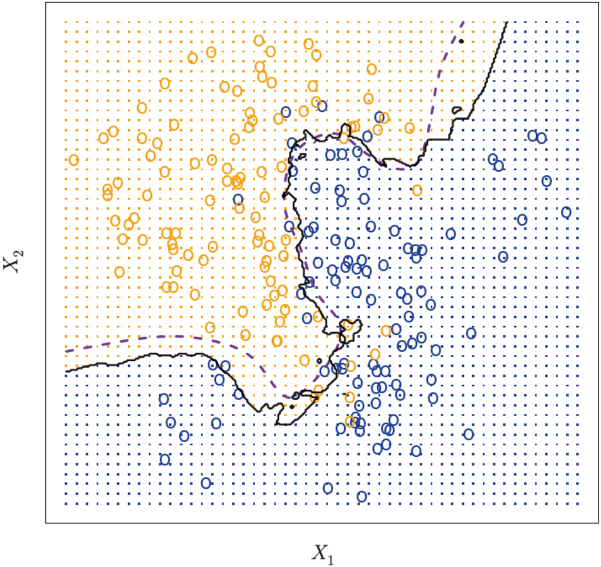
$$1 - E[\max_j Pr(Y = j|X)] = 0.13$$

K-Nearest Neighbors (K=3)

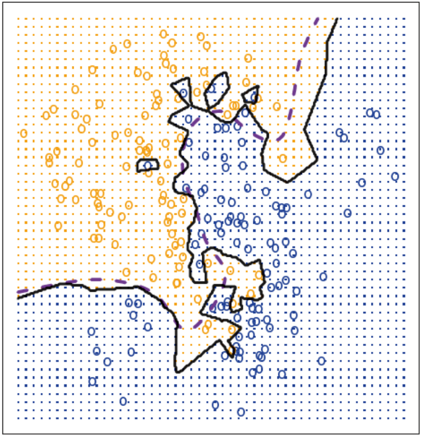
$$Pr(Y = j | X = x_0) = \frac{1}{K} \sum_{i \in N_0} I(y_i = j)$$

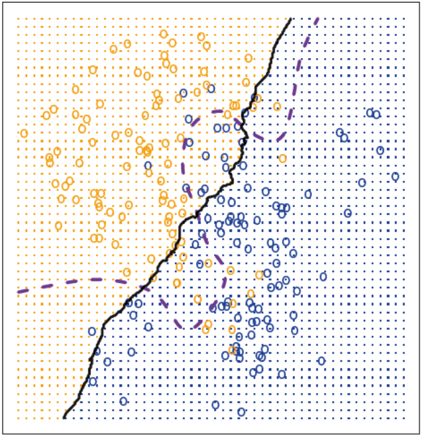


KNN: K=10

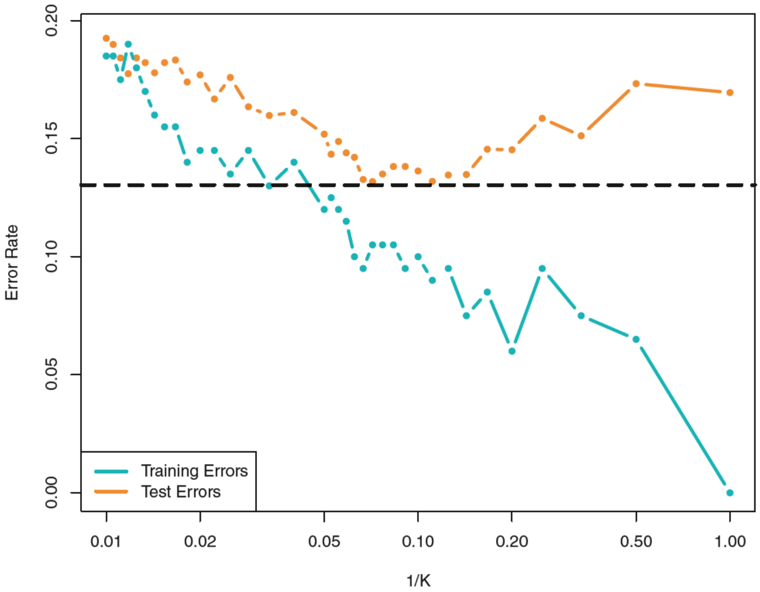


KNN: K=1 and K=100



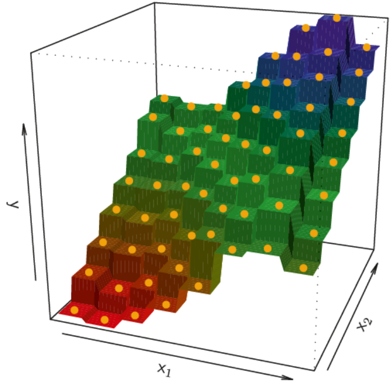


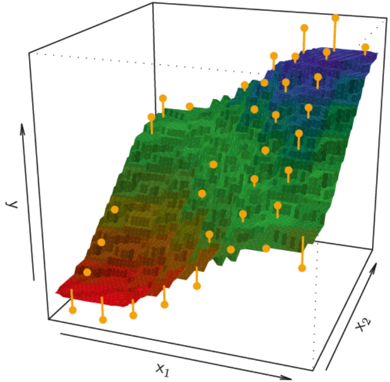
KNN Training and Test Error Rate



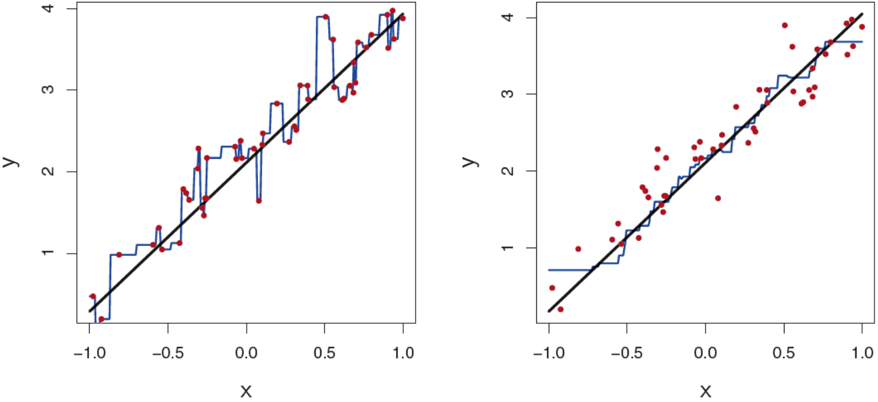
KNN Regression: K=1 and K=9

$$\hat{f}(x_0) = \frac{1}{K} \sum_{x_i \in N_0} y_i$$





One-dimension KNN Regression: K=1 and K=9



Mean Squared Error (MSE)

$$E(Y-\hat{Y})^2 = E[f(X)+\epsilon-\hat{f}(X)]^2$$

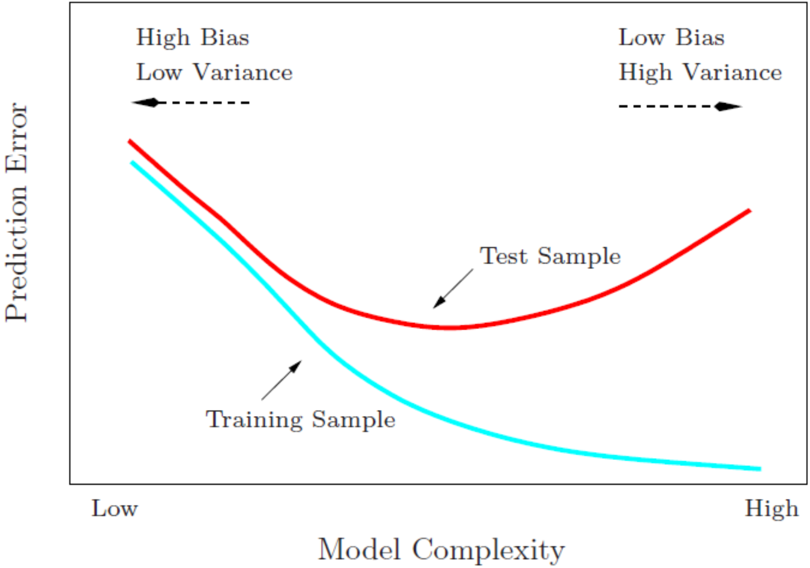
$$= E[f(X)-\hat{f}(X)]^2 + Var(\epsilon)$$

$$MSE(x_0) = E_{\mathcal{T}}[f(x_0)-\hat{y}_0]^2$$

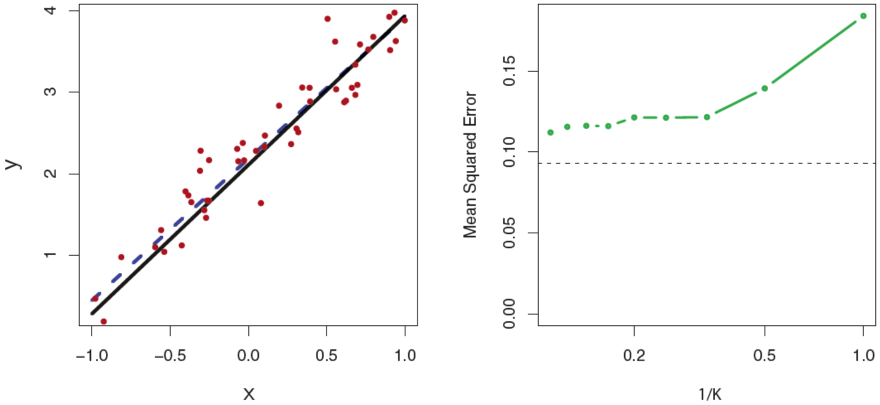
$$E_{\mathcal{T}}[\hat{y}_0 - E_{\mathcal{T}}(\hat{y}_0)]^2 + [E_{\mathcal{T}}(\hat{y}_0) - f(x_0)]^2$$

$$Var_{\mathcal{T}}(\hat{y}_0) + [Bias(\hat{y}_0)]^2$$

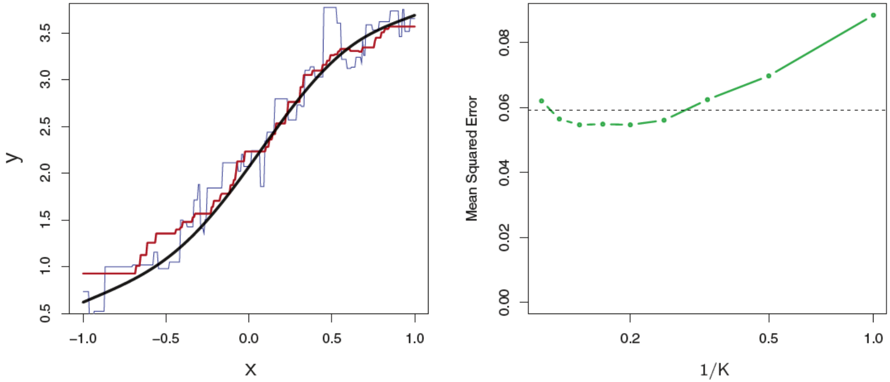
Bias-Variance Tradeoff



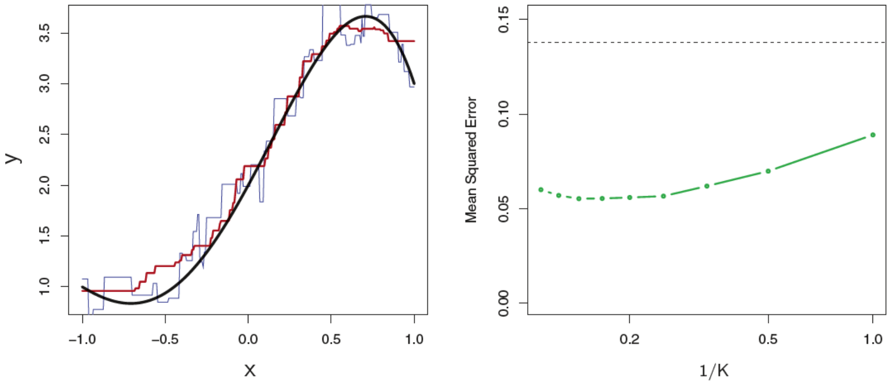
MSE: OLS vs KNN



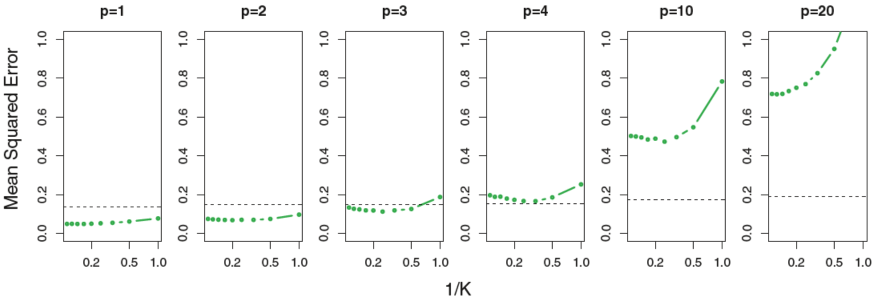
Slightly Non-Linear Relationship



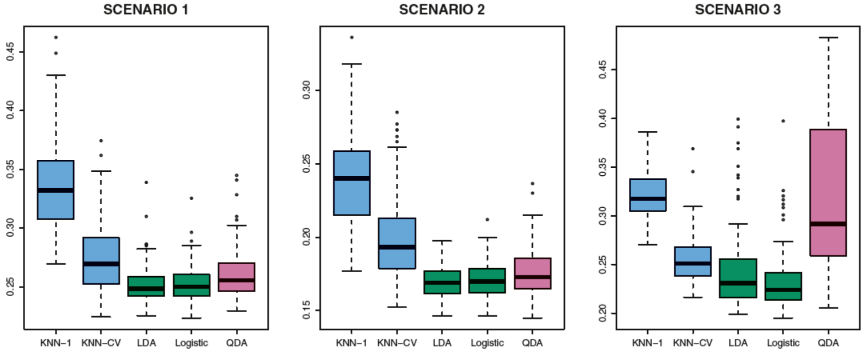
Strongly Non-Linear Relationship



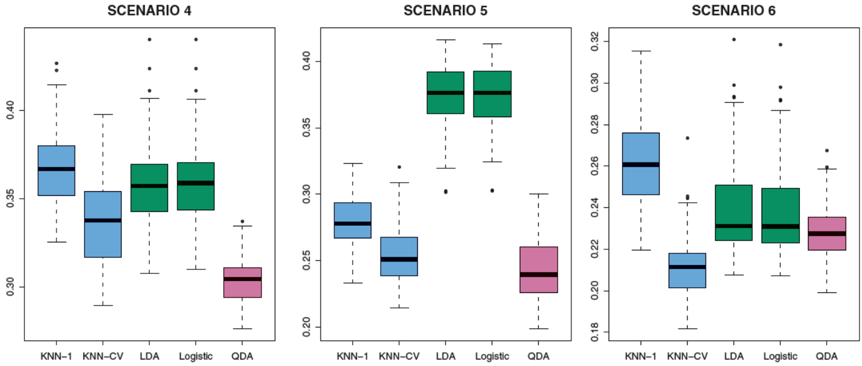
Additional Noise Variables



Test Error Rates: Linear Scenarios



Test Error Rates: Non-Linear Scenarios



library(ISLR); library(class)

dim(Caravan)

5822 86

summary(Purchase)

No	Yes
5474	348

$$\frac{348}{5822} \cong 6\%$$

Standardize the Data

```
standardized.X=scale(Caravan[,-86])
```

var(Caravan[,1])	**165**
var(Caravan[,2])	**0.165**
var(standardized.X[,1])	**1**
var(standardized.X[,2])	**1**

K = 1

```
set.seed(1); test=1:1000
train.X=standardized.X[-test,]
test.X=standardized.X[test,]
train.Y=Purchase[-test]
test.Y=Purchase[test]
knn.pred=knn(train.X,test.X,train.Y,k=1)
mean(test.Y!=knn.pred)          0.118
mean(test.Y!="No")              0.059
```

K = 3 and K=5

knn.pred=knn(train.X,test.X,train.Y,k=3)

table(knn.pred,test.Y)

knn.pred/test.Y	No	Yes
No	920	54
Yes	21	5

$$\frac{5}{26} = 19.2\%$$

knn.pred=knn(train.X,test.X,train.Y,k=5)

table(knn.pred,test.Y)

knn.pred/test.Y	No	Yes
No	930	55
Yes	11	4

$$\frac{4}{15} = 26.7\%$$

Logistic Regression

```
glm.fit=glm(Purchase~.,data=Caravan,family=binomial,
    subset=-test)
glm.probs=predict(glm.fit,Caravan[test,],type="response")
glm.pred=rep("No",1000)
glm.pred[glm.probs>.5]="Yes"
table(glm.pred,test.Y)
```

glm.pred/test.Y	No	Yes
No	934	59
Yes	7	0

Cut-off of 0.25

```
glm.pred=rep("No",1000)
glm.pred[glm.probs>.25]="Yes"
table(glm.pred,test.Y)
```

glm.pred/test.Y	No	Yes
No	919	48
Yes	22	11

$$\frac{11}{33} = 33\%$$