13) Discriminant Analysis

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February 2018

Reference

Tables, Graphics, and Figures from

An Introduction to Statistical Learning

James et al. (2017): Chapters: 4.4, 4.6.3, 4.6.4

Bayes Rule

$$P(Y = k|X = x) = \frac{P(Y=k)P(X=x|Y=k)}{P(X=x)}$$

$$= \frac{P(Y=k)P(X=x|Y=k)}{\sum_{l=1}^{K} P(Y=l)P(X=x|Y=l)}$$

$$Posterior = \frac{Prior \times Likelihood}{Marginal}$$

Bayes' Theorem for Classification

$$p_k(X) = Pr(Y = k | X = x)$$

$$= \frac{\pi_k f_k(x)}{\sum_{l=1}^{K} \pi_l f_l(x)}$$

$$f_k(x) = Pr(X = x | Y = k)$$

Linear Discriminant Analysis (LDA)

$$f_k(x) = \frac{1}{\sqrt{2\pi}\sigma_k} exp[-\frac{1}{2\sigma_k^2}(x-\mu_k)^2]$$

$$p_k(x) = \frac{\pi_k \frac{1}{\sqrt{2\pi}\sigma} exp[-\frac{1}{2\sigma^2}(x-\mu_k)^2]}{\sum_{l=1}^{K} \pi_l \frac{1}{\sqrt{2\pi}\sigma} exp[-\frac{1}{2\sigma^2}(x-\mu_l)^2]}$$

$$\delta_k(x) = x\frac{\mu_k}{\sigma^2} - \frac{\mu_k^2}{2\sigma^2} + log(\pi_k)$$

LDA Vector Notation

$$X \sim N(\mu, \Sigma)$$

$$f(x) = \frac{1}{(2\pi)^{\frac{p}{2}} |\Sigma|^{\frac{1}{2}}} exp[-\frac{1}{2}(x-\mu)^T \Sigma^{-1} (x-\mu)]$$

$$\delta_k(x) = x^T \Sigma^{-1} \mu_k - \frac{1}{2} \mu_k^T \Sigma^{-1} \mu_k + log \pi_k$$

K=2, and $\pi_1 = \pi_2$

$$\delta_k(x) = x\frac{\mu_k}{\sigma^2} - \frac{\mu_k^2}{2\sigma^2} + log(\pi_k)$$

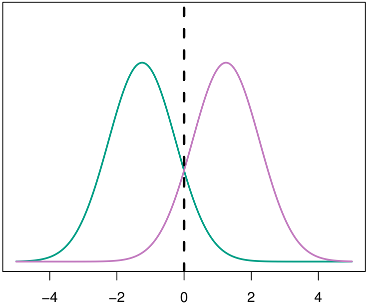
$$\delta_1 = \delta_2$$

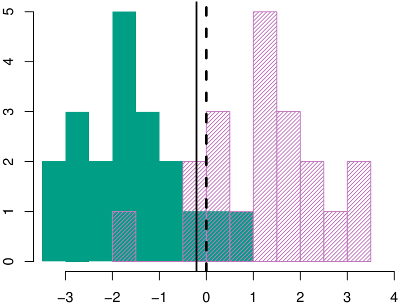
$$2x\mu_1 - \mu_1^2 = 2x\mu_2 - \mu_2^2$$

$$2x(\mu_1 - \mu_2) = \mu_1^2 - \mu_2^2$$

$$x = \frac{\mu_1^2 - \mu_2^2}{2(\mu_1 - \mu_2)} = \frac{\mu_1 + \mu_2}{2}$$

$\mu_1 = -1.25$, $\mu_2 = 1.25$, $\sigma_1^2 = \sigma_2^2 = 1$, **and** $\pi_1 = \pi_2 = 0.5$





Bayes Classifier: Linear Discriminant Analysis

$$\hat{\delta}_k(x) = x\frac{\hat{\mu}_k}{\hat{\sigma}^2} - \frac{\hat{\mu}_k^2}{2\hat{\sigma}^2} + log(\hat{\pi}_k)$$

$$\hat{\mu}_k = \frac{1}{n_k}\sum_{i:y_i=k} x_i$$

$$\hat{\sigma}^2 = \frac{1}{n-K}\sum_{k=1}^{K}\sum_{i:y_i=k}(x_i - \hat{\mu}_k)^2$$

$$\hat{\pi}_k = \frac{n_k}{n}$$

Default Data

		True default status		
		No	Yes	Total
Predicted default status	No	9,644	252	9,896
	Yes	23	81	104
	Total	9,667	333	10,000

Training Error: $\frac{23+252}{10000} = 2.75\%$

Specificity: $\frac{9644}{9667} \cong 99.8\%$

Sensitivity: $\frac{81}{333} \cong 24.3\%$

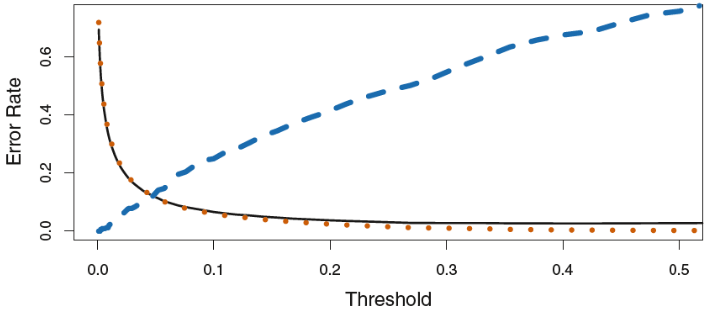
$Pr(default = Yes|X = x) > 0.2$

		True default status		
		No	Yes	Total
Predicted default status	No	9,432	138	9,570
	Yes	235	195	430
	Total	9,667	333	10,000

Training Error: $\frac{235+138}{10000} = 3.73\%$

Sensitivity: $\frac{195}{333} \cong 58.5\%$

Error Rates as Function of Posterior Probability



Black: overall error rate

Blue: fraction of defaulting that are incorrectly classified

Orange: fraction of errors among the non-defaulting

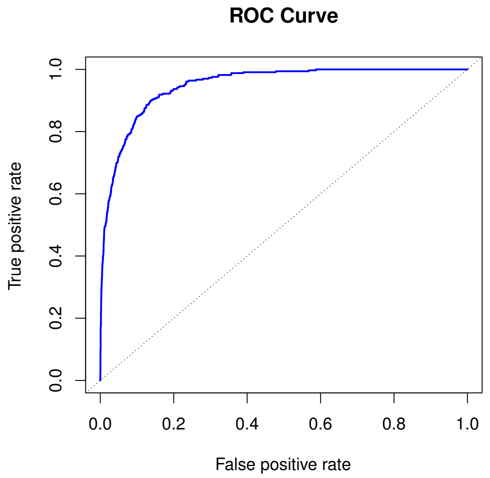
Terminology

		Predicted class		
		− or Null	+ or Non-null	Total
True	− or Null	True Neg. (TN)	False Pos. (FP)	N
class	+ or Non-null	False Neg. (FN)	True Pos. (TP)	P
	Total	N^*	P^*	

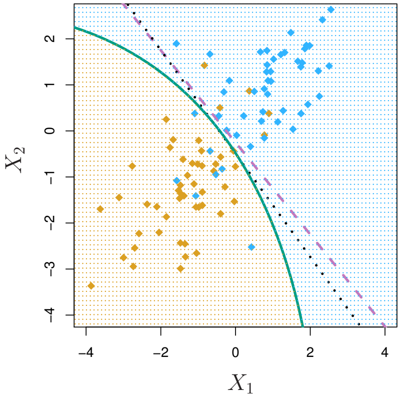
Name	Definition	Synonyms
False Pos. rate	FP/N	Type I error, 1−Specificity
True Pos. rate	TP/P	1−Type II error, power, sensitivity, recall
Pos. Pred. value	TP/P^*	Precision, 1−false discovery proportion
Neg. Pred. value	TN/N^*	

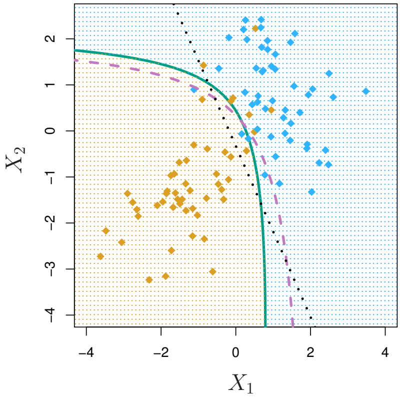
Receiver Operating Characteristics (ROC)

$$AUC = 0.95$$



LDA vs Quadratic Discriminant Analysis (QDA)





Quadratic Discriminant Analysis (QDA)

$$X \sim N(\mu_k, \Sigma_k)$$

$$-\frac{1}{2}(x-\mu_k)^T\Sigma_k^{-1}(x-\mu_k) - \frac{1}{2}log|\Sigma_k| + log\pi_k$$

$$-\frac{1}{2}x^T\Sigma_k^{-1}x + x^T\Sigma_k^{-1}\mu_k - \frac{1}{2}\mu_k^T\Sigma_k^{-1}\mu_k - \frac{1}{2}log|\Sigma_k| + log\pi_k$$

Linear Discriminant Analysis in R

```
library(MASS); library(ISLR); attach(Smarket)
train=(Year<2005)
Smarket.2005=Smarket[!train,]
Direction.2005=Direction[!train]
lda.fit=lda(Direction~Lag1+Lag2,data=Smarket,
subset=train)
```

lda.fit

```
Prior probabilities of groups:
    Down       Up
0.491984 0.508016

Group means:
            Lag1        Lag2
Down  0.04279022  0.03389409
Up   -0.03954635 -0.03132544

Coefficients of linear discriminants:
            LD1
Lag1 -0.6420190
Lag2 -0.5135293
```

lda.pred=predict(lda.fit, Smarket.2005)

lda.class=lda.pred$class

table(lda.class,Direction.2005)

```
           Direction.2005
lda.class  Down  Up
     Down    35  35
     Up      76 106
```

mean(lda.class==Direction.2005)

55.95%

lda.pred=predict(lda.fit, Smarket.2005)

lda.class=lda.pred$class

table(lda.class,Direction.2005)

```
           Direction.2005
lda.class Down  Up
     Down   35  35
     Up     76 106
```

mean(lda.class==Direction.2005)

55.95%

qda.fit=qda(Direction~Lag1+Lag2,data=Smarket, subset=train)

qda.fit

```
Prior probabilities of groups:
    Down       Up
0.491984 0.508016

Group means:
            Lag1        Lag2
Down  0.04279022  0.03389409
Up   -0.03954635 -0.03132544
```

qda.class=predict(qda.fit,Smarket.2005)$class

table(qda.class,Direction.2005)

```
           Direction.2005
qda.class Down  Up
     Down   30  20
     Up     81 121
```

mean(qda.class==Direction.2005)

59.92%