## 7) OLS Gauss-Markov Theorem

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## Random Sampling

$$y = X\beta + \epsilon$$
  
{ $(y_1, x_1), ..., (y_n, x_n)$ }

# Independent and Identically Distributed (iid)

#### **Unbiasedness of OLS**

$$\hat{\beta} = (X'X)^{-1}X'y$$

$$\hat{\beta} = (X'X)^{-1}X'(X\beta + \epsilon)$$

$$\hat{\beta} = (X'X)^{-1}X'X\beta + (X'X)^{-1}X'\epsilon$$

$$\hat{\beta} = \beta + (X'X)^{-1}X'\epsilon$$

$$E(\hat{\beta}|X) = \beta + (X'X)^{-1}X'E(\epsilon|X)$$

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## Consistency of OLS

$$\hat{eta} = eta + (X'X)^{-1}X'\epsilon$$
 
$$\hat{eta} = eta + (N^{-1}X'X)^{-1}N^{-1}X'\epsilon$$

$$plim\hat{\beta} = \beta + (plimN^{-1}X'X)^{-1}(plimN^{-1}X'\epsilon)$$

$$\hat{eta} = eta$$
 if  $plimN^{-1}X'u = 0$ 



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## Variance of Error Term $(\epsilon)$

$$Var(\epsilon|X) = E(\epsilon\epsilon'|X)$$

(i) 
$$E(\epsilon_i^2|X) = \sigma_i^2$$

(ii) 
$$E(\epsilon_i \epsilon_j | X) = E(\epsilon_i | X) E(\epsilon_j | X) = 0$$

$$Var(\epsilon|X) = \begin{bmatrix} \sigma_1^2 & 0 & \cdots & 0 \\ 0 & \sigma_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_n^2 \end{bmatrix}$$

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### Variance of Least Squares Estimator

$$\hat{\beta} - \beta = (X'X)^{-1}X'\epsilon$$

$$Var(\hat{\beta}|X) = E[(\hat{\beta} - \beta)(\hat{\beta} - \beta)'|X]$$

$$= E[(X'X)^{-1}X'\epsilon\epsilon'X(X'X)^{-1}|X]$$

$$= (X'X)^{-1}X'[Var(\epsilon|X)]X(X'X)^{-1}$$

If 
$$Var(\epsilon|X) = I_n\sigma^2$$
, then:

$$Var(\hat{\beta}|X) = \sigma^2(X'X)^{-1}X'X(X'X)^{-1}$$
$$= \sigma^2(X'X)^{-1}$$

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#### **Gauss-Markov Theorem**

$$ilde{eta}=Cy ext{ and } A=(X'X)^{-1}X'$$
 $D=C-A ext{ or } C=D+A$ 
 $ilde{eta}=(D+A)y$ 
 $ilde{eta}=D(Xeta+\epsilon)+\hat{eta}$ 
If  $ilde{eta}$  and  $\hat{eta}$  are unbiased,  $DX=0$ 
 $ilde{eta}-eta=D\epsilon+\hat{eta}-eta$ 
 $ilde{eta}-eta=(D+A)\epsilon$ 

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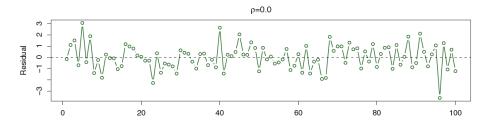
### OLS is the Best Linear Unbiased Estimator (BLUE)

$$Var(\tilde{eta}|X) = (D+A)Var(\epsilon|X)(D'+A')$$
 $\sigma^2(DD'+AD'+DA'+AA')$ 
 $DA' = DX(X'X)^{-1} = 0$ 
 $Var(\tilde{eta}|X) = \sigma^2(DD'+(X'X)^{-1})$ 
 $> Var(eta|X)$ 

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#### **Uncorrelated Error Terms**

$$Cov(\epsilon_t, \epsilon_s | X) = 0$$
, for all  $t \neq s$ 





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#### **Correlated Error Terms**

