2) Central Limit Theorem (CLT), Stochastic Order of Magnitude, and Delta Method

Vitor Kamada

January 2018

Lindeberg–Levy Central Limit Theorem

By construction $Z_n = \frac{\bar{X}_n - E(\bar{X}_n)}{\sqrt{Var(\bar{X}_n)}}$ has mean 0 and variance 1

Let $\{X_i\}$ be *iid* with $E[X_i] = \mu$ and $Var[X_i] = \sigma^2$, then $Z_n \xrightarrow{d} N(0,1)$

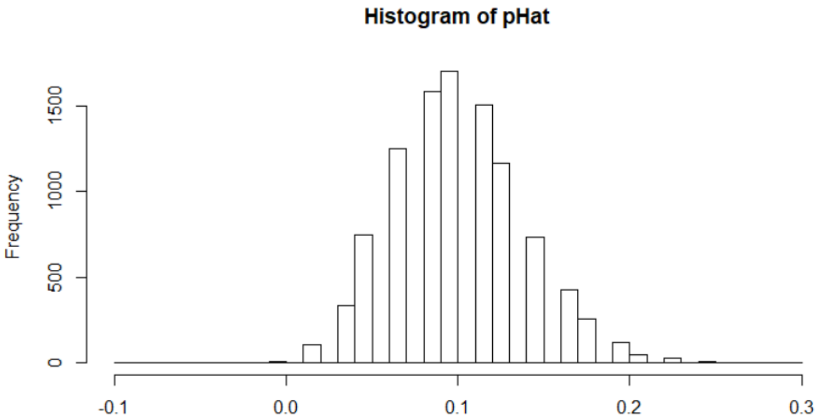
Medical Consultant

Average complication rate for liver donor surgeries in the US is about 10%

A Medical Consultant had only 3 complications in the 62 liver donor surgeries $\hat{p} = \frac{3}{62} \cong 0.048$

```
pHat <- rbinom(10^4, 62, 0.1)/62
sum(pHat<0.048)/10^4
hist(pHat, breaks=seq(-0.1,0.3,0.01))
```

CLT Simulation



$$\hat{p} < 0.048 = 0.0445$$

Univariate Delta Method

$$g(\bar{X}_n) = g(\mu) + g'(\tilde{\mu})(\bar{X}_n - \mu)$$

$$\bar{X}_n < \tilde{\mu} < \mu$$

$$\sqrt{n}[g(\bar{X}_n) - g(\mu)] = \sqrt{n} g'(\tilde{\mu})(\bar{X}_n - \mu)$$

$$\sqrt{n}(\bar{X}_n - \mu) \xrightarrow{d} N(0, \sigma^2)$$

$$\sqrt{n}[g(\bar{X}_n) - g(\mu)] \xrightarrow{d} N(0, \sigma^2[g'(\mu)]^2)$$

Multivariate Delta Method

$$\sqrt{n}(\hat{\beta} - \beta) \xrightarrow{d} N(0, \Sigma)$$

$$\sqrt{n}[h(\hat{\beta}) - h(\beta)]$$

$$\xrightarrow{d}$$

$$N(0, \nabla h(\beta)^T \Sigma \nabla h(\beta))$$

R Simulation: Delta Method

$$X \sim N(5, 3^2)$$

$$F(x) = x^2, \text{ then } F'(x) = 2x$$

$$Var(x^2) \cong 2 * 5 * 9 * 2 * 5 = 900$$

```
set.seed(7)
x=rnorm(1000)*3+5
x2=x^2
var(x2)
sd(x2)
```

Output: 1041.611 and 32.274

Stochastic Order of Magnitude

A sequence of R.Vs. b_n is:

$$o_p(g(n)) \text{ if } plim\frac{b_n}{g(n)} = 0$$

$$O_p(g(n)) \text{ if } 0 < plim\frac{b_n}{g(n)} < \infty$$

$$b_n = o_p(1) \text{ means } b_n \to 0 \text{ as } n \to \infty$$

Continuous Mapping or Slutsky's Theorem

$$o_p(1) + o_p(1) = o_p(1)$$

$$o_p(1) + O_p(1) = O_p(1)$$

$$O_p(1) + O_p(1) = O_p(1)$$

$$o_p(1)o_p(1) = o_p(1)$$

$$o_p(1)O_p(1) = o_p(1)$$

$$O_p(1)O_p(1) = O_p(1)$$

Consistency

$$\text{Unbiasedness: } E(\hat{\beta}) = \beta$$

An estimator $\hat{\beta}$ is consistent for β if

$$plim \hat{\beta} = \beta$$

$$\hat{\beta} = \beta + o_p(1)$$

*Adding $\frac{1}{n}$ to an unbiased and consistent estimator produces a new estimator that is biased but still consistent