7) OLS Gauss-Markov Theorem

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Random Sampling

$$y = X\beta + \epsilon$$

$$\{(y_1, x_1), ..., (y_n, x_n)\}$$

Independent and Identically Distributed (iid)

Unbiasedness of OLS

$$\hat{\beta} = (X'X)^{-1}X'y$$

$$\hat{\beta} = (X'X)^{-1}X'(X\beta + \epsilon)$$

$$\hat{\beta} = (X'X)^{-1}X'X\beta + (X'X)^{-1}X'\epsilon$$

$$\hat{\beta} = \beta + (X'X)^{-1}X'\epsilon$$

$$E(\hat{\beta}|X) = \beta + (X'X)^{-1}X'E(\epsilon|X)$$

Consistency of OLS

$$\hat{\beta} = \beta + (X'X)^{-1}X'\epsilon$$

$$\hat{\beta} = \beta + (N^{-1}X'X)^{-1}N^{-1}X'\epsilon$$

$$plim\hat{\beta} = \beta + (plimN^{-1}X'X)^{-1}(plimN^{-1}X'\epsilon)$$

$$\hat{\beta} = \beta \textbf{ if } plimN^{-1}X'u = 0$$

Variance of Error Term (ϵ)

$$Var(\epsilon|X) = E(\epsilon\epsilon'|X)$$

(i) $E(\epsilon_i^2|X) = \sigma_i^2$

(ii) $E(\epsilon_i\epsilon_j|X) = E(\epsilon_i|X)E(\epsilon_j|X) = 0$

$$Var(\epsilon|X) = \begin{bmatrix} \sigma_1^2 & 0 & \cdots & 0 \\ 0 & \sigma_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_n^2 \end{bmatrix}$$

Variance of Least Squares Estimator

$$\hat{\beta} - \beta = (X'X)^{-1}X'\epsilon$$

$$Var(\hat{\beta}|X) = E[(\hat{\beta} - \beta)(\hat{\beta} - \beta)'|X]$$
$$= E[(X'X)^{-1}X'\epsilon\epsilon'X(X'X)^{-1}|X]$$
$$= (X'X)^{-1}X'[Var(\epsilon|X)]X(X'X)^{-1}$$

If $Var(\epsilon|X) = I_n\sigma^2$**, then:**

$$Var(\hat{\beta}|X) = \sigma^2(X'X)^{-1}X'X(X'X)^{-1}$$
$$= \sigma^2(X'X)^{-1}$$

Gauss-Markov Theorem

$$\tilde{\beta} = Cy \textbf{ and } A = (X'X)^{-1}X'$$

$$D = C - A \textbf{ or } C = D + A$$

$$\tilde{\beta} = (D + A)y$$

$$\tilde{\beta} = D(X\beta + \epsilon) + \hat{\beta}$$

If $\tilde{\beta}$ **and** $\hat{\beta}$ **are unbiased,** $DX = 0$

$$\tilde{\beta} - \beta = D\epsilon + \hat{\beta} - \beta$$

$$\tilde{\beta} - \beta = (D + A)\epsilon$$

OLS is the Best Linear Unbiased Estimator (BLUE)

$$Var(\tilde{\beta}|X) = (D + A)Var(\epsilon|X)(D' + A')$$

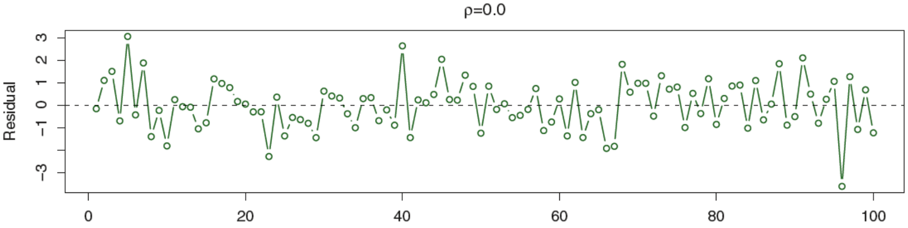
$$\sigma^2(DD' + AD' + DA' + AA')$$

$$DA' = DX(X'X)^{-1} = 0$$

$$Var(\tilde{\beta}|X) = \sigma^2(DD' + (X'X)^{-1})$$

$$\geq Var(\beta|X)$$

$$Cov(\epsilon_t, \epsilon_s | X) = 0, \textbf{ for all } t \neq s$$



Correlated Error Terms