11) Maximum Likelihood Estimator (MLE)

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Normal Regression Model

$$y = x'\beta + \epsilon$$
 $\epsilon \sim N(0, \sigma^2)$

$$f(y|x) = \frac{1}{\sqrt{2\pi\sigma^2}} exp[-\frac{1}{2\sigma^2}(y - x'\beta)^2]$$

$$f(y_1,...,y_n|x_1,...,x_n) = \prod_{i=1}^n f(y_i|x_i)$$

Maximum Likelihood Estimator (MLE)

$$\mathcal{L}(\beta, \sigma^{2}) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^{2}}} exp[-\frac{1}{2\sigma^{2}}(y_{i} - x_{i}'\beta)^{2}]$$

$$In\mathcal{L} = -\frac{n}{2}In(2\pi\sigma^{2}) - \frac{1}{2\sigma^{2}} \sum_{i=1}^{n} (y_{i} - x_{i}'\beta)^{2}$$

$$\max_{\beta, \sigma^{2}} In[\mathcal{L}(\beta, \sigma^{2})]$$

$$\frac{\partial In\mathcal{L}}{\partial \beta} = \frac{1}{\hat{\sigma}^{2}} \sum_{i=1}^{n} x_{i}(y_{i} - x_{i}'\hat{\beta}) = 0$$

$$\hat{\beta}_{mle} = (\sum_{i=1}^{n} x_{i}x_{i}')^{-1}(\sum_{i=1}^{n} x_{i}y_{i}) = \hat{\beta}_{ols}$$

Log-Likelihood

$$ln\mathcal{L} = -\frac{n}{2}ln(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i - x_i'\beta)^2$$

$$\frac{\partial \ln \mathcal{L}}{\partial \sigma^2} = -\frac{n}{2\hat{\sigma}^2} + \frac{1}{2\hat{\sigma}^4} \sum_{i=1}^n (y_i - x_i' \hat{\beta})^2 = 0$$

$$\hat{\sigma}_{mle}^2 = \frac{1}{n} \sum_{i=1}^n (y_i - x_i' \hat{\beta})^2 = \frac{1}{n} \sum_{i=1}^n \hat{e}_i^2 = \hat{\sigma}_{ols}^2$$

$$ln\mathcal{L}(\hat{eta}_{mle},\hat{\sigma}_{mle}^2) = -rac{n}{2}ln(2\pi\hat{\sigma}_{mle}) - rac{n}{2}$$

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Likelihood Ratio Test

$$y_i=x_{1i}'eta_1+x_{2i}'eta_2+\epsilon$$
 $H_0:eta_2=0$ $In\mathcal{L}(\hat{eta},\hat{\sigma}^2)=-rac{n}{2}In(2\pi\hat{\sigma}^2)-rac{n}{2}$

$$\mathit{In}\mathcal{L}(ilde{eta}_1, ilde{\sigma}^2) = -rac{n}{2}\mathit{In}(2\pi ilde{\sigma}^2) - rac{n}{2}$$

$$LR = n \times ln(\frac{\tilde{\sigma}^2}{\hat{\sigma}^2})$$



Information Bound for Normal Regression

$$\frac{\partial \ln \mathcal{L}}{\partial \beta} = \frac{1}{\hat{\sigma}^2} \sum_{i=1}^n x_i (y_i - x_i' \hat{\beta}) = 0$$
$$\frac{\partial^2 \ln \mathcal{L}}{\partial \beta \partial \beta'} = \frac{1}{\hat{\sigma}^2} X' X$$

$$\frac{\partial \ln \mathcal{L}}{\partial \sigma^2} = -\frac{n}{2\hat{\sigma}^2} + \frac{1}{2\hat{\sigma}^4} \sum_{i=1}^n (y_i - x_i' \hat{\beta})^2 = 0$$

$$\frac{\partial^2 \ln \mathcal{L}}{\partial \sigma^2 \partial \sigma^{2\prime}} = \frac{n}{2\hat{\sigma}^4}$$



The Cramér-Rao Lower Bound

$$I = \begin{pmatrix} \frac{1}{\hat{\sigma}^2} X' X & 0 \\ 0 & \frac{n}{2\hat{\sigma}^4} \end{pmatrix}$$

$$I^{-1} = \begin{pmatrix} \hat{\sigma}^2 (X'X)^{-1} & 0 \\ 0 & \frac{2\hat{\sigma}^4}{n} \end{pmatrix}$$