4.1) Algebra of Ordinary Least Squares (OLS)

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Reference

Tables, Graphics, and Figures from:

Hansen (2018). **Econometrics.** Ch 3.

Simple Linear Regression

$$\sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} (y_i - \hat{\beta}_0 + \hat{\beta}_1 x_i)^2$$

$$\frac{\partial \sum_{i=1}^{n} e_i^2}{\partial \hat{\beta}_0} = -2 \sum_{i=1}^{n} (y_i - \hat{\beta}_0 + \hat{\beta}_1 x_i) = 0 = \sum_{i=1}^{n} e_i$$

$$\frac{\partial \sum_{i=1}^{n} e_i^2}{\partial \hat{\beta}_1} = -2 \sum_{i=1}^{n} [x_i(y_i - \hat{\beta}_0 + \hat{\beta}_1 x_i)] = 0$$

$$\sum_{i=1}^{n} x_i e_i = 0$$

Matrix Notation

$$\underset{(n\times 1)}{y} = \underset{(n\times K)}{X}\underset{(K\times 1)}{\beta} + \underset{(n\times 1)}{\epsilon}$$

$$\underset{(n\times 1)}{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}, \quad \underset{(n\times K)}{X} = \begin{bmatrix} X_1' \\ \vdots \\ X_n' \end{bmatrix} = \begin{bmatrix} x_{11} & \cdots & x_{1K} \\ \vdots & \cdots & \vdots \\ x_{n1} & \cdots & x_{nK} \end{bmatrix}$$

$$\underset{(K\times 1)}{\beta} = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_K \end{bmatrix}, \quad \underset{(n\times 1)}{\epsilon} = \begin{bmatrix} \epsilon_1 \\ \vdots \\ \epsilon_n \end{bmatrix}$$

Residual Sum of Squares (SSR)

$$SSR(\beta) = \sum_{i=1}^{n} (y_i - x_i'\beta)^2$$

$$(y - X\beta)'(y - X\beta)$$

$$(y' - \beta'X')(y - X\beta)$$

$$y'y - \beta'X'y - y'X\beta + \beta'X'X\beta$$

$$y'y - 2y'X\beta + \beta'X'X\beta$$

First-Order Condition

$$SSR(\beta) = y'y - 2y'X\beta + \beta'X'X\beta$$

$$\frac{\partial SSR(\beta)}{\partial \beta} = X'X\beta - X'y = 0$$

$$\hat{\beta} = (X'X)^{-1}X'y$$

$$X'(y - X\beta) = 0$$

$$X'\epsilon = 0$$

Projection Matrix P

$$\underset{(n \times n)}{P} = X(X'X)^{-1}X'$$

$$P' = P \text{ and } PP = P$$

$$PX = X(X'X)^{-1}X'X = X$$

$$Py = X(X'X)^{-1}X'y = X\hat{\beta} = \hat{y}$$

Orthogonal Projection or Annihilator M

$$\underset{(n\times n)}{M} = I_n - P$$

$$M' = M \text{ and } MM = M$$

$$MX = (I_n - P)X = X - PX = 0$$

$$My = y - Py = y - X\hat{\beta} = e$$

$$e = My = M(X\beta + \epsilon) = M\epsilon$$

Analysis of Variance

$$y = \hat{y} + e = Py + My$$

$$\hat{y}'e = (Py)'(My) = y'PMy = 0$$

$$y'y = \hat{y}'\hat{y} + 2\hat{y}'e + e'e$$

$$\hat{y}'\hat{y} + e'e$$

$$R^2_{uc} = \frac{\hat{y}'\hat{y}}{y'y} = 1 - \frac{e'e}{y'y}$$

$$R^2_c = \frac{\sum_{i=1}^{n}(\hat{y}_i - \bar{y})^2}{\sum_{i=1}^{n}(y_i - \bar{y})} = 1 - \frac{\sum_{i=1}^{n} e_i{}^2}{\sum_{i=1}^{n}(y_i - \bar{y})}$$

Orthogonal Partitioned Regression

$$y = X_1\beta_1 + X_2\beta_2 + \epsilon$$

$$\begin{bmatrix} X_1'X_1 & X_1'X_2 \\ X_2'X_1 & X_2'X_2 \end{bmatrix} \begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{bmatrix} = \begin{bmatrix} X_1'y \\ X_2'y \end{bmatrix}$$

$$\hat{\beta}_1 = (X_1'X_1)^{-1}X_1'y - (X_1'X_1)^{-1}X_1'X_2\hat{\beta}_2$$