

11) Maximum Likelihood Estimator (MLE)

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Normal Regression Model

$$y = x'\beta + \epsilon$$

$$\epsilon \sim N(0, \sigma^2)$$

$$f(y|x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{1}{2\sigma^2}(y - x'\beta)^2\right]$$

$$f(y_1, \dots, y_n | x_1, \dots, x_n) = \prod_{i=1}^n f(y_i | x_i)$$

Maximum Likelihood Estimator (MLE)

$$\mathcal{L}(\beta, \sigma^2) = \prod_{i=1}^n \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{1}{2\sigma^2}(y_i - x_i'\beta)^2\right]$$

$$\ln \mathcal{L} = -\frac{n}{2} \ln(2\pi\sigma^2) - \frac{1}{2\sigma^2} \sum_{i=1}^n (y_i - x_i'\beta)^2$$

$$\max_{\beta, \sigma^2} \ln[\mathcal{L}(\beta, \sigma^2)]$$

$$\frac{\partial \ln \mathcal{L}}{\partial \beta} = \frac{1}{\hat{\sigma}^2} \sum_{i=1}^n x_i (y_i - x_i'\hat{\beta}) = 0$$

$$\hat{\beta}_{mle} = \left(\sum_{i=1}^n x_i x_i' \right)^{-1} \left(\sum_{i=1}^n x_i y_i \right) = \hat{\beta}_{ols}$$

Log-Likelihood

$$\ln \mathcal{L} = -\frac{n}{2} \ln(2\pi\sigma^2) - \frac{1}{2\sigma^2} \sum_{i=1}^n (y_i - \mathbf{x}_i' \beta)^2$$

$$\frac{\partial \ln \mathcal{L}}{\partial \sigma^2} = -\frac{n}{2\hat{\sigma}^2} + \frac{1}{2\hat{\sigma}^4} \sum_{i=1}^n (y_i - \mathbf{x}_i' \hat{\beta})^2 = 0$$

$$\hat{\sigma}_{mle}^2 = \frac{1}{n} \sum_{i=1}^n (y_i - \mathbf{x}_i' \hat{\beta})^2 = \frac{1}{n} \sum_{i=1}^n \hat{e}_i^2 = \hat{\sigma}_{ols}^2$$

$$\ln \mathcal{L}(\hat{\beta}_{mle}, \hat{\sigma}_{mle}^2) = -\frac{n}{2} \ln(2\pi \hat{\sigma}_{mle}) - \frac{n}{2}$$

Likelihood Ratio Test

$$y_i = \mathbf{x}'_{1i}\beta_1 + \mathbf{x}'_{2i}\beta_2 + \epsilon$$

$$H_0 : \beta_2 = 0$$

$$\ln \mathcal{L}(\hat{\beta}, \hat{\sigma}^2) = -\frac{n}{2} \ln(2\pi \hat{\sigma}^2) - \frac{n}{2}$$

$$\ln \mathcal{L}(\tilde{\beta}_1, \tilde{\sigma}^2) = -\frac{n}{2} \ln(2\pi \tilde{\sigma}^2) - \frac{n}{2}$$

$$LR = n \times \ln\left(\frac{\tilde{\sigma}^2}{\hat{\sigma}^2}\right)$$

Information Bound for Normal Regression

$$\frac{\partial \ln \mathcal{L}}{\partial \beta} = \frac{1}{\hat{\sigma}^2} \sum_{i=1}^n x_i (y_i - x_i' \hat{\beta}) = 0$$

$$\frac{\partial^2 \ln \mathcal{L}}{\partial \beta \partial \beta'} = \frac{1}{\hat{\sigma}^2} X' X$$

$$\frac{\partial \ln \mathcal{L}}{\partial \sigma^2} = -\frac{n}{2\hat{\sigma}^2} + \frac{1}{2\hat{\sigma}^4} \sum_{i=1}^n (y_i - x_i' \hat{\beta})^2 = 0$$

$$\frac{\partial^2 \ln \mathcal{L}}{\partial \sigma^2 \partial \sigma^{2'}} = \frac{n}{2\hat{\sigma}^4}$$

The Cramér-Rao Lower Bound

$$I = \begin{pmatrix} \frac{1}{\hat{\sigma}^2} X'X & 0 \\ 0 & \frac{n}{2\hat{\sigma}^4} \end{pmatrix}$$

$$I^{-1} = \begin{pmatrix} \hat{\sigma}^2 (X'X)^{-1} & 0 \\ 0 & \frac{2\hat{\sigma}^4}{n} \end{pmatrix}$$