# Spot-Vol Model

## 1 Introduction

Change in iv due to the change in forward level, which creates additional sensitivity of option value $V$ to forward $F$, beyond delta, aka $\frac{\partial T}{\partial F}$

$$\left(\frac{\text{skew-delta}}{\text{short-delta}}\right) \Delta_{\text{skew}} = \Delta_{BS} + \text{Vega}_{BS} \times \frac{\partial T}{\partial F}$$

$$\Gamma_{\text{skew}} = \Gamma_{BS} + 2 \times \text{Vanna}_{BS} \times \frac{\partial T}{\partial F} + \text{Volga}_{BS} \times \left(\frac{\partial T}{\partial F}\right)^2$$

Vol curve has the parameterization:

$$\sigma(K) = \sigma_{\text{ref}}(K) + f\left(\frac{z}{K}\right)$$

where $K$ is the strike and $z$ is the normalized strike, $z = \frac{\log(K/F)}{\sigma_0\sqrt{T}}$

## 2 Reference Vol Movement

**(1)** Reference vol moves linearly with spot:

$$\sigma_{\text{ref}}(K) = \sigma_0 + \beta \frac{\partial f}{\partial K}\left(\frac{F}{F_0}\right)$$

where:

$$\beta = \text{Skew-stickiness ratio} \quad (1)$$

$$\frac{\partial f}{\partial K} = \text{ATMF skew} \quad (2)$$

**(2)** Slope of the vol curve is invariant to $F$. Then:

$$\frac{\partial \sigma(K)}{\partial F} = \beta f'(0) \frac{\partial z}{\partial K}\bigg|_{K=F_0} + f(z(F)) \frac{\partial z}{\partial F}$$

$$= (\beta - 1) f'(0) \frac{1}{F_0 \sigma_0 \sqrt{T}}$$

$$\text{When SSR, } \beta = 1, \text{ ATM has } \frac{\partial \sigma(K)}{\partial F} = 0, Up - movecausesvoltogodown$$

but ATM strike moves at the same rate, cancelling each other out.

# 3 Variable Definitions

Given a term $T$ and forward $F = F_T$, let:

$$x = \frac{K}{F}, \quad y = \log x, \quad \text{and} \quad z = \frac{y}{\sigma_0 \sqrt{T}}$$

where:

$$x = \text{moneyness} \tag{3}$$

$$y = \text{log-moneyness} \tag{4}$$

$$z = \text{normalized strike} \tag{5}$$

where $K$ is the strike and $\sigma_0$ is the ATF $(K = F)$ volatility.

# 4 Taylor Series Expansion

Taylor series expansion of the vol curve:

$$\sigma^2(x) = \sigma_0^2 \left(1 + S_1 z + \frac{1}{2} C_1 z^2 + \dots\right)$$

where:

$$S = \text{Skew} \tag{6}$$

$$C = \text{Curvature} \tag{7}$$

Equivalently:

$$\sigma(z) = \sigma_0 \left(1 + S_1 z + \frac{1}{2} C_1 z^2 + \dots\right)$$

where:

$$S_2 = 2S_1 \Rightarrow S_1 = \frac{S_2}{2} \tag{8}$$

$$C_2 = 2C_1 + 2S_1^2 \Rightarrow C_1 = \frac{C_2}{2} - \frac{S_1^2}{4} \tag{9}$$

# 5 Fixed Strike Vol Movement

How does fixed strike vol move?

$$\sigma(K) = \sigma_0 + \frac{S_1}{\sqrt{T}} \ln\left(\frac{K}{F}\right) + \dots$$

Then:

$$\sigma(K/F) = \sigma_0(F) + \frac{S_1(F)}{\sqrt{T}} \ln\left(\frac{K}{F}\right) + \frac{1}{2}\frac{C_1(F)}{\sigma_0\sqrt{T}} \ln^2\left(\frac{K}{F}\right) + \dots$$

# 6 ATF Vol Movement and SSR

ATF vol moves along a slope that is steeper than the actual vol skew slope for the term given. The ratio of these slopes is SSR (denote by $P_\sigma$) and is usually between 1 and 2.

It means given a small forward change $\frac{\delta F}{F}$, ATF vol moves by:

$$\delta\sigma_0 = P_0 \frac{S_1}{\sqrt{T}} \frac{\delta F}{F} \Rightarrow F\delta_F \hat{\sigma}_0(F) = p_0 s_1$$

where $\hat{\sigma}_0 = \sigma_0\sqrt{T}$.

Then:

$$\delta_F \sigma(K|F) = \frac{S_1}{\sqrt{T}F}(P_\sigma - 1) + \frac{1}{\sqrt{T}F}\left(P_{S_1} - \frac{C_1}{\sigma_0\sqrt{T}}\right) \ln\left(\frac{K}{F}\right) + O(\text{Poly } \log\frac{K}{F})$$

$$\Rightarrow F\delta_F \hat{\sigma}(z|F) = s_1(p_0 - 1) + (\hat{\sigma}_0 p_{s_1} - c_1)\, z + O(z^2)$$

Regress implied volatility move on forward move to find $p_\sigma$ empirically, or $(p_\sigma - 1)$

# 7 Forward Returns and Maturity

What this means: forward returns $\rightarrow$ proximity

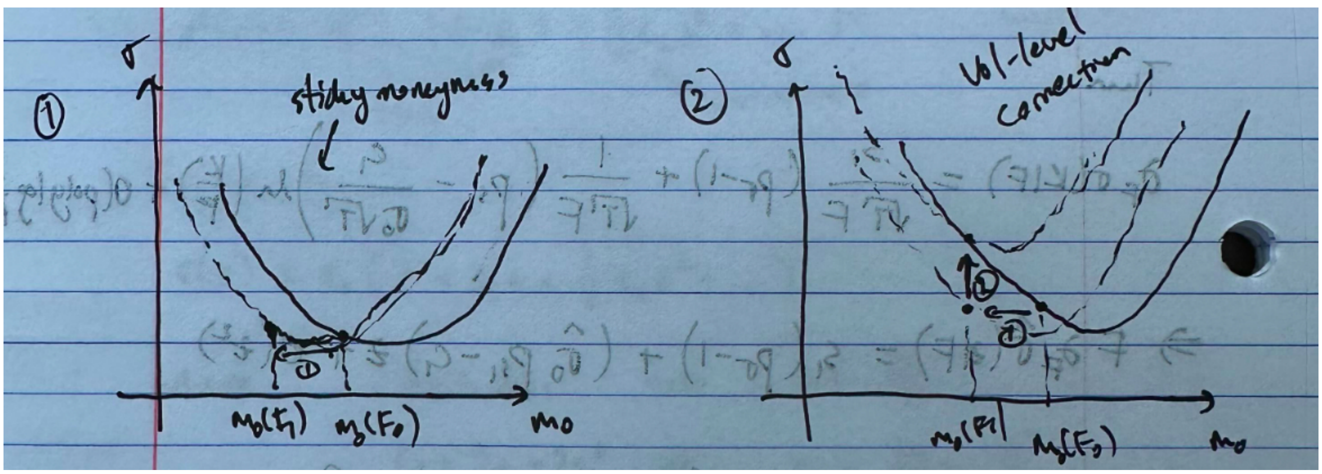
$$d\sigma(r|T) = f_h(r|T) + f_v(r|T)$$

where $f_h(r|T)$ = horizontal update, $f_v(r|T)$ = vertical update

$$f_h = f_{\text{h-stm}}, \quad f_v = f_{\text{h-vc}}$$

where stm = sticky moneyness, vc = vol-level correction

## 7.1 Sticky Moneyness - Vol-Level Correction Diagrams




where $m_i$ = normalized strike at time i, $F_i$ = forward at time i

$$f_v = (p-1)\frac{S_1}{\sqrt{T}}$$

vertical update