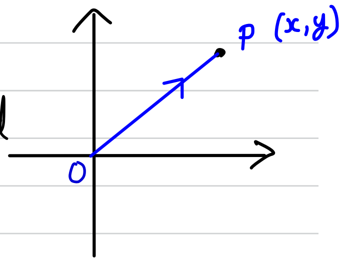
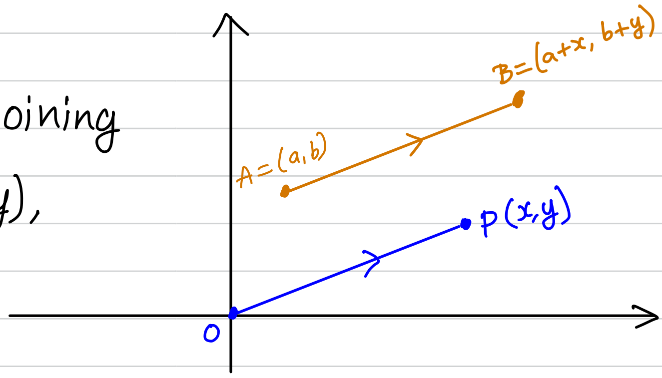
Definition [Vectors]

- Vector is a quantity which has both magnitude and direction.
- Let P be a point in 2-space as shown. The position vector of a point $P(x,y)$ is $\overrightarrow{OP}$ and is referred as $\begin{bmatrix} x \\ y \end{bmatrix}$.



- Any vector in 2-space can be represented by a position vector of a point in the space.

  For instance, if $\overrightarrow{AB}$ is a vector joining the points (a,b) and $(a+x, b+y)$, then $\overrightarrow{AB} = \overrightarrow{OP}$ as shown.



- An ordered pair (x, y) can be interpreted as a point, in which case x and y are the coordinates, or it can be interpreted as a vector, in which case x and y are the components. We prefer to refer vector as $\begin{bmatrix} x \\ y \end{bmatrix}$. The distinction is mathematically unimportant.
- Set of all vectors in 2-space is denoted by $\mathbb{R}^2$

  i.e $$\mathbb{R}^2 = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \,\middle|\, x, y \in \mathbb{R} \right\}$$

- In general, the ordered n-tuple $(x_1, x_2, \cdots, x_n)$ can be viewed as a "generalized point" or a "generalized vector" in n-space.

- The set of a vectors in n-space is denoted by $\mathbb{R}^n$

i.e $\mathbb{R}^n = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \middle| \; x_i \in \mathbb{R} \right\}$

- Let $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and $Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$ be any vectors in $\mathbb{R}^n$

  - The addition $X+Y$ is defined as

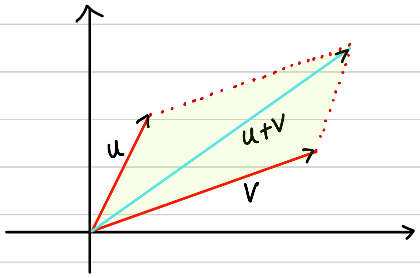
$$X + Y = \begin{bmatrix} x_1+y_1 \\ x_2+y_2 \\ \vdots \\ x_n+y_n \end{bmatrix}$$

  - Let α be a scalar (a real no.). Then the scalar multiple $\alpha \cdot X$ is defined by

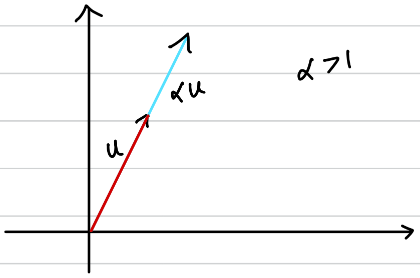
$$\alpha \cdot X = \begin{bmatrix} \alpha x_1 \\ \alpha x_2 \\ \vdots \\ \alpha x_n \end{bmatrix}$$

- Geometrically, suppose $u = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $v = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ are vectors in $\mathbb{R}^2$.

Addition



Scalar multiplication



- Scalar multiplication of a vector is a vector in the same direction or a vector in the opposite direction.

- Addition of two vectors is a vector that is in the same plane as the original two vectors.

- These two operations of addition and scalar multiplication are called the standard operations (or usual operations) on $\mathbb{R}^n$.

The set $\mathbb{R}^n$ under the binary operations $+$ and $\cdot$ as defined above is called a vector space. It is usually referred as $\mathbb{R}^n$ - vectorspace.

Arithmetic properties of addition and scalar multiplication of vectors in $\mathbb{R}^n$ are listed as follows:

If $u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$, $v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ and $w = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}$ are vectors in $\mathbb{R}^n$

and α and β are scalars, then

1) $u+v = v+u$

2) $u+(v+w) = (u+v)+w$

3) $u+0 = 0+u = 0$ (additive identity) ; $0 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

4) $u+(-u) = 0$ (additive inverse)

5) $\alpha\cdot(\beta u) = (\alpha\beta)\cdot u$; $-u = \begin{bmatrix} -u_1 \\ -u_2 \\ \vdots \\ -u_n \end{bmatrix}$

6) $\alpha\cdot(u+v) = \alpha\cdot u + \alpha\cdot v$

7) $(\alpha+\beta)\cdot u = \alpha\cdot u + \beta\cdot u$

8) $1\cdot u = u$, $1 \in \mathbb{R}$

Generalized Vector Space

The time has now come to generalize the concept of a vector.

We can think of a vector space in general, as a collection of objects that behave as vectors do in $\mathbb{R}^n$. The objects of such a set are called vectors.

Defn [Field]: A set $\mathbb{F}$ with operations $+$ and $\times$ is called a Field if

for any $\alpha, \beta, \gamma \in \mathbb{F}$

i) $\alpha+\beta \in \mathbb{F}$ (closed under addition)

ii) $\alpha+\beta = \beta+\alpha$

iii) $(\alpha+\beta)+\gamma = \alpha+(\beta+\gamma)$

iv) $\alpha+0 = 0+\alpha = \alpha$ (additive identity exist)

v) $\alpha+(-\alpha) = (-\alpha)+\alpha = 0$ (additive inverse)

vi) $\alpha \times \beta \in \mathbb{F}$

vii) $\alpha \times \beta = \beta \times \alpha$

viii) $(\alpha \times \beta) \times \gamma = \alpha \times (\beta \times \gamma)$

ix) $\alpha \times 1 = 1 \times \alpha = \alpha$ (Multiplicative identity)

x) $\alpha \times \alpha^{-1} = \alpha^{-1} \times \alpha = 1$ (Multiplicative inverse)

Ex 1: Under addition $+$ and multiplication $\times$,

i) $\mathbb{R}$ set of all real numbers is a Field

ii) $\mathbb{C}$ set of all complex no.s is a Field

iii) $\mathbb{Q}$ set of rational no.s is a Field

Ex 2: Set of integers $\mathbb{Z}$ is not a field.

Since every no. doesnot have multiplicative inverse.

Ex 3: Set of irrational no.s $\mathbb{Q}^c$ is not a field.

Since it is not closed under multiplication.

Axiomatic definition of a Vector Space

Any collection of mathematical objects W with binary operations $+$ on W and $\cdot$ on scalars (Fields, $\mathbb{F}$) with W such that

i) $w_1 + w_2 \in W$ for any $w_1, w_2 \in W$

ii) $\alpha \cdot w_1 \in W$ for any $w_1 \in W$ and $\alpha \in \mathbb{F}$

is a vector space over $\mathbb{F}$ if it satisfies the below eight axioms: For any $w_1, w_2, w_3 \in W$ and $\alpha, \beta \in \mathbb{F}$

VS1) $w_1 + w_2 = w_2 + w_1$ (Commutative)

VS2) $w_1 + (w_2 + w_3) = (w_1 + w_2) + w_3$ (Associative)

VS3) There exist an element $0 \in W$ such that

$$w_1 + 0 = 0 + w_1 = w_1$$ (Existence of additive identity)

VS4) For every $w_1 \in W$ there exist $-w_1 \in W$ such that

$$w_1 + (-w_1) = (-w_1) + w_1 = 0$$ (Existence of additive inverse)

VS5) $\alpha \cdot (\beta \cdot w_1) = (\alpha\beta) \cdot w_1$

VS6) $\alpha \cdot (w_1 + w_2) = \alpha \cdot w_1 + \alpha \cdot w_2$ (distributive law 1)

VS7) $(\alpha + \beta) \cdot w_1 = \alpha \cdot w_1 + \alpha \cdot w_2$ (distributive law 2)

VS8) $1 \cdot \alpha = \alpha$ (Multiplicative identity)

Notation:

A vector space W over $\mathbb{F}$ under $+$ and $\cdot$ is denoted by $(W(\mathbb{F}), +, \cdot)$

Ex 0: $(\mathbb{R}, +, \cdot)$ is a vector space with usual addition and scalar multiplication.

Ex 1: $(\mathbb{R}^n, +, \cdot)$ is a vector space under standard addition and scalar multiplication.

Ex 2: Let $\mathbb{C} = \{ a+ib \mid a, b \in \mathbb{R},\ i = \sqrt{-1} \}$ with binary composition $+$ and $\cdot$ defined by

i) $(a+ib) + (c+id) = (a+c) + i(b+d)$

ii) $\alpha \cdot (a+ib) = \alpha a + i \alpha b$

$(\mathbb{C}, +, \cdot)$ is a vector space. (Verify)

Ex 3: Let $M_{2\times2}(\mathbb{R})$ be set of all matrices with real enteries

i.e $M_{2\times2}(\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \;\middle|\; a, b, c, d \in \mathbb{R} \right\}$ is a vector space with usual matrix addition and scalar multiplication defined by (it is usually denoted by $\mathbb{R}^{2\times2}$)

i) $\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} = \begin{bmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & d_1+d_2 \end{bmatrix}$

ii) $\alpha \cdot \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} = \begin{bmatrix} \alpha a_1 & \alpha b_1 \\ \alpha c_1 & \alpha d_1 \end{bmatrix}$

for every $\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}, \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \in M_2$ and $\alpha \in \mathbb{R}$.

Ex 4: The set of all $m\times n$ matrices with real enteries denoted by $M_{m\times n}(\mathbb{R})$ (or $\mathbb{R}^{m\times n}$) is also a vector space under usual addition and scalar multiplication.

Ex5: The set of all polynomials of degree $\leq n$ with coefficients from the field $\mathbb{R}$, denoted by $P_n(\mathbb{R})$,

i.e $P_n(\mathbb{R}) = \{ a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \mid a_i \in \mathbb{R} \}$

is a vector space under addition and scalar multiplication defined as below:

For any $A = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$, $B = b_0 + b_1 x + \cdots + b_r x^r$ $(r \leq n)$

in $P_n(\mathbb{R})$ and $\alpha \in \mathbb{R}$

i) $A + B = (a_0 + b_0) + (a_1 + b_1) x + (a_2 + b_2) + \cdots + (a_r + b_r) x^r + \cdots + a_n x^n$

ii) $\alpha \cdot A = \alpha a_0 + \alpha a_1 x + \alpha a_2 x^2 + \cdots + \alpha a_n x^n$

Here zero vector is $0 + 0x + \cdots + 0 x^n = 0$

Ex6: The set of all polynomials with real co-efficients denoted by $P(\mathbb{R})$

i.e $P(\mathbb{R}^n) = \{ a_0 + a_1 x + a_2 x^2 + \cdots + a_r x^r \mid a_i \in \mathbb{R} \}$

is also a vector space under the operations defined as in Ex5.

Ex7: The set of all real valued functions defined on $[a, b]$

i.e $\mathcal{F} = \{ f \mid f : [a, b] \to \mathbb{R} \}$

is a vector space over $\mathbb{R}$ under pointwise addition and scalar multiplication defined as

for any $f_1, f_2 \in \mathcal{F}$ and $\alpha \in \mathbb{R}$

i) $(f_1 + f_2)(x) = f_1(x) + f_2(x)$

ii) $(\alpha f_1)(x) = \alpha f_1(x)$

Clearly it is closed under addition and scalar multiplication

Since for $f_1, f_2 \in \mathcal{F}$

$(f_1+f_2)(x) = f_1(x) + f_2(x) \in \mathbb{R}$; $(\alpha f_1)(x) = \alpha f_1(x) \in \mathbb{R}$

$\Rightarrow f_1+f_2 \in \mathcal{F}$ $\Rightarrow \alpha f_1 \in \mathcal{F}$

VS1) $(f_1+f_2)(x) = f_1(x) + f_2(x) = f_2(x) + f_1(x) = (f_2+f_1)(x)$

$\Rightarrow f_1+f_2 = f_2+f_1$

VS2) $\left((f_1+f_2)+f_3\right)(x) = (f_1+f_2)(x) + f_3(x)$

$= \left(f_1(x) + f_2(x)\right) + f_3(x)$

$= f_1(x) + \left(f_2(x) + f_3(x)\right)$

$= \left(f_1 + (f_2+f_3)\right)(x)$

$\Rightarrow (f_1+f_2)+f_3 = f_1 + (f_2+f_3)$

VS3) We define $\mathbf{0} \in \mathcal{F}$ as a zero function that sends each element in $[a,b]$ to 0 in $\mathbb{R}$

$(f+\mathbf{0})(x) = f(x) + \mathbf{0}(x) = f(x) + 0 = f(x)$

$\Rightarrow f+\mathbf{0} = f$

Additive identity exist

VS4) $(f_1 + (-1)f_1)(x) = f_1(x) + (-1) f_1(x) = f_1(x) - f_1(x) = 0 = \mathbf{0}(x)$

$\Rightarrow f_1 + (-1) f_1 = \mathbf{0}$

$\therefore (-1)f_1 = -f_1$ is additive inverse of f_1

VS5) $\left(\alpha\cdot(\beta\cdot f_1)\right)(x) = \alpha\left(\beta f_1(x)\right) = (\alpha\beta) f_1(x) = \left((\alpha\beta) f_1\right)(x)$

$\Rightarrow \alpha(\beta f_1) = (\alpha\beta) f_1$

VS6) $\left((\alpha+\beta)\cdot f_1\right)(x) = (\alpha+\beta) f_1(x) = \alpha f_1(x) + \beta f_1(x) = (\alpha f_1 + \beta f_1)(x)$

$\Rightarrow (\alpha+\beta)\cdot f_1 = \alpha f_1 + \beta f_1$

VS7) $(\alpha\cdot(f_1+f_2))(x) = \alpha(f_1+f_2)(x) = \alpha(f_1(x)+f_2(x))$

$= (\alpha f_1(x) + \alpha f_2(x)) = (\alpha f_1 + \alpha f_2)(x)$

$\Rightarrow \alpha\cdot(f_1+f_2) = \alpha f_1 + \alpha f_2$

VS8) $(1\cdot f_1)x = f_1(x) \Rightarrow 1\cdot f_1 = f_1$

Thus $\mathcal{F}$ is a vector space under pointwise addition and scalar multiplication.

Ex 8: [Counter example]

Let $\mathbb{R}^2 = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| \; x, y \in \mathbb{R}^2 \right\}$. Define binary operations as below:

For any $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ in $\mathbb{R}^n$ and $\alpha \in \mathbb{R}$

i) $u+v = \begin{bmatrix} u_1+v_1 \\ u_2+v_2 \end{bmatrix}$; ii) $\alpha\cdot u = \begin{bmatrix} \alpha u_1 \\ 0 \end{bmatrix}$

Clearly $\mathbb{R}^2$ under above mentioned binary operations is not a vector space. Since $1\cdot u = \begin{bmatrix} u_1 \\ 0 \end{bmatrix} \neq u$, doesnot satisfy VS8.

Ex 9: Let $S = \{(a_1,a_2) \mid a_1, a_2 \in \mathbb{R}\}$. For any $(a_1,a_2), (b_1,b_2) \in S$ and $c \in \mathbb{R}$, define $+$ and $\cdot$ as

i) $(a_1,a_2) + (b_1,b_2) = (a_1+b_1, a_2-b_2)$

ii) $c\cdot(a_1,a_2) = (ca_1, ca_2)$. Is $(S,+,\cdot)$ a vector space?

Soln: No.

VS1) Let $(a_1,a_2), (b_1,b_2), (c_1,c_2) \in S$.

Consider $(a_1,a_2) + \underbrace{((b_1,b_2) + (c_1,c_2))}$

$= (a_1,a_2) + (b_1+c_1, b_2-c_2)$

$= (a_1+b_1+c_1,\ a_2-(b_2-c_2))$

Now, $((a_1,a_2)+(b_1,b_2))+(c_1,c_2)$

$= (a_1+b_1,\ a_2-b_2)+(c_1,c_2)$

$= (a_1+b_1,\ a_2-b_2-c_2)$

Thus, it doesnot satify associative law.

Even VS4 fails to hold.

Proposition: Let V be a vector space over $\mathbb{R}$, $u \in V$ and $\alpha \in \mathbb{R}$. Then

a) $0u = \mathbf{0}$ — 0 is additive identity in $\mathbb{R}$ and $\mathbf{0}$ is additive identity in V

b) $\alpha \cdot \mathbf{0} = \mathbf{0}$

c) $(-\alpha)u = -(\alpha u) = \alpha(-u)$

d) If $\alpha u = 0 \Rightarrow \alpha = 0$ or $u = \mathbf{0}$

True or False

a) Every vector space contains a zero vector

Ans : True

b) Every vector space contain atleast two vectors.

Ans: False.

c) The rational numbers $\mathbb{Q}$ form a vector space over $\mathbb{Q}$.

Ans: True

d) Set of all functions $\mathcal{F} = \{ f \mid f : \mathbb{Z} \to \mathbb{Z} \}$ form a vector space over $\mathbb{R}$ under pointwise addition and scalar multi.

Ans: False. Since not closed under scalar multiplication.

i.e for any $f \in F$ and $\alpha \in \mathbb{R}$, $(\alpha f)(x) = \alpha(f(x)) \notin \mathbb{Z}$

e) In any vector space $V(\mathbb{R})$, $\alpha \cdot u = \beta \cdot u \Rightarrow \alpha = \beta$ for any $u \in V$ $(u \neq 0)$ and $\alpha, \beta \in \mathbb{R}$

Ans: True (Verify)

f) A vector space may have more than one zero vector.

Ans: No. Zero vector is unique. (prove it)

g) For any v in a vector space V has unique additive inverse.

Ans: Yes. (prove it)

Exercise:

1) Determine if the set $\mathbb{T} := \mathbb{R} \cup \{\infty\}$ with addition and scalar multiplication defined for all $v, w \in \mathbb{T}$ and $\alpha \in \mathbb{R}$ by

i) $v + w = \min(v, w)$

ii) $\alpha \cdot v = \alpha + v$

is a vector space over $\mathbb{R}$. If it is not, then list all of the defining axioms that fail to hold.

Subspace (a vector space inside a vector space)

Definition [Subspace]

Let $V(\mathbb{R})$ be a vector space over $\mathbb{R}$, let W be a non empty subset of V. A subset W is called a subspace of V if W is a vector space over $\mathbb{R}$ with the operations of addition and scalar multiplication defined on V.

It is usually denoted by $W \leq V$.

Ex 1: $P_n(\mathbb{R})$ is a subspace of $P(\mathbb{R})$.

Ex 2: Let $C[a,b]$ be set of all real valued continuous fns.

i.e $C[a,b] = \{ f \mid f : [a,b] \to \mathbb{R} \text{ and } f \text{ is continous} \}$

It is a vector space under pointwise addition and scalar multiplication and it is a subset of set of all real valued fn. $\mathcal{F}$ on $[a,b]$

$\therefore \; C[a,b] \leq \mathcal{F}$

Ex 3: Let $C^{(n)}[a,b]$ be set of all real valued functions on $[a,b]$ such that $f', f'', f''', \cdots f^{(n-1)}, f^{(n)}$ exist and are continuous. (Here n is any non negative integer).

i.e, $C^{(n)}[a,b] = \{ f : [a,b] \to \mathbb{R} \mid f', f'', f''', \cdots, f^{(n)} \text{ exist and continuous} \}$

It is a vector space over $\mathbb{R}$ under pointwise addition and scalar multiplication and $C^{(n)}[a,b] \subseteq C[a,b] \subseteq \mathcal{F}$

$\therefore \; C^{(n)}[a,b] \leq C[a,b] \leq \mathcal{F}$, where $\mathcal{F}$ is set of all real valued functions on $[a,b]$.

Below theorem can be used to verify which subset inherit the structure of a vector space.

non empty

Thm 1: Let $(V, +, \cdot)$ be a vector space. Let $W \subseteq V$. Then $(W, +, \cdot)$ is said to be a subspace of $(V, +, \cdot)$ iff

$$\left.\begin{array}{lll} \text{i)} & w_1 + w_2 \in W & \text{for all } w_1, w_2 \in W \\ \text{ii)} & \alpha . w_1 \in W & \text{for all } \alpha \in \mathbb{R} \text{ and } w_1 \in W. \end{array}\right\} \text{Closure conditions}$$

OR

linear combination of w_1 and w_2

$\alpha w_1 + \beta w_2 \in W$ for all $\alpha, \beta \in \mathbb{R}$ and $w_1, w_2 \in W$

pf: Let $(W, +, \cdot)$ be a subspace of $(V, +, \cdot)$.

Since W is a subspace, from defn W is by itself a vector space.

Thus i) and ii) hold.

Conversely, let $W \subseteq V$, and i) and ii) hold, we have to show that W is a subspace of V.

That is, we need to show that W by itself a vector space with same binary operations.

For all $w \in W$, $\alpha w \in W$ (from ii))

a) Let $\alpha = -1$, implies $-w \in W$ (Existence of inverse)

b) Let $\alpha = 0$, implies $0 . w = 0 \in W$ (Existence of additive identity)

All other 6 axioms are true for any element in V, thus it is true for elements in W as well.

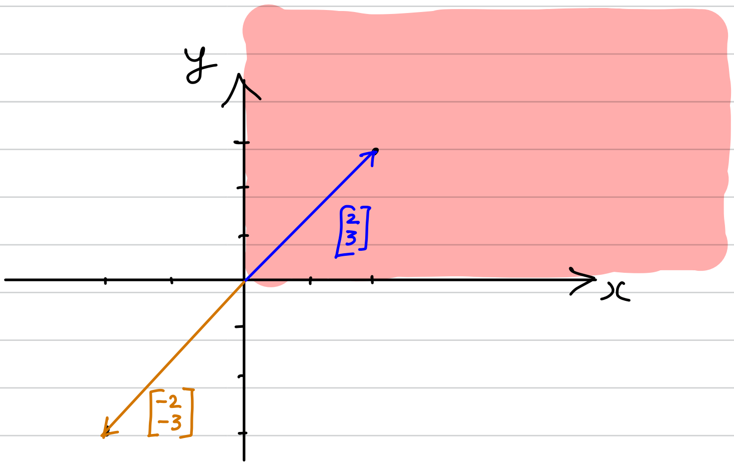
Q: Is $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| \; x \geq 0 \text{ and } y \geq 0 \right\}$ a subspace of $\mathbb{R}^2$ under standard addition and scalar multiplicatn.

Ans: No

$\because$ if we consider $\alpha = -1$,

then $\alpha . \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \end{bmatrix} \notin V$.

Not closed under scalar multiplication.



Q: Is $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| \; x \geq 0 \text{ and } y \geq 0 \text{ or } x \leq 0 \text{ and } y \leq 0 \right\}$

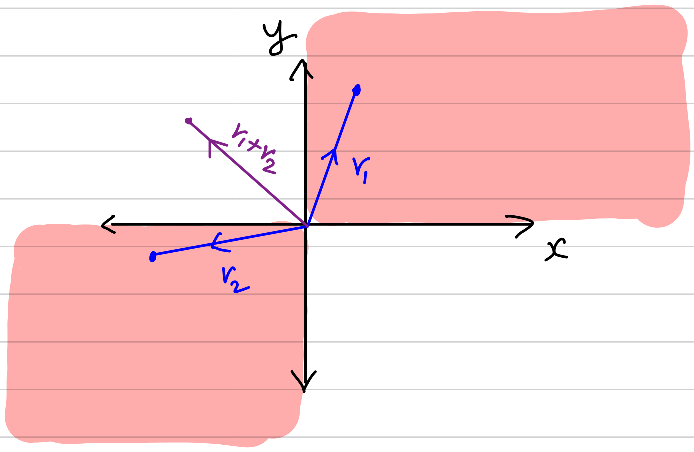
a subspace of $\mathbb{R}^2$?

Ans: No

See fig v_1 and $v_2 \in V$, but not $v_1 + v_2$.

Consider $v_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, $v_2 = \begin{bmatrix} -3 \\ -2 \end{bmatrix} \in V$.

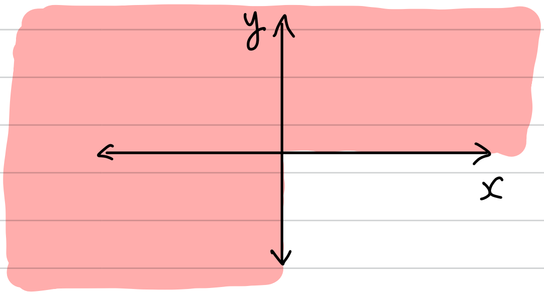
$v_1 + v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \notin V$



Q: Is $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| \; x \geq 0 \text{ and } y \geq 0 \text{ or } x \leq 0 \text{ and } y \leq 0 \text{ or } x \leq 0 \text{ and } y \geq 0 \right\}$

a subspace of $\mathbb{R}^2$?

Ans: No (Why?)



Ex4: Obtain all possible subspaces of the vector space $\mathbb{R}^2$ under standard $+$ and $\cdot$

Soln: i) $\mathbb{R}^2$ itself (we call improper subspace)

ii) The singleton set, $W = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$ (It is called a trivial subspace)

iii) All lines through $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Since linear combination of any two vectors on a line through $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is a vector on the same line.

Ex5: Obtain all possible subspaces of $\mathbb{R}^3$ under standard addition and scalar multi.

Soln: i) $\mathbb{R}^3$ itself, improper subspace

ii) $W = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$, trivial subspace

iii) All lines through $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

iv) All planes through $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Remarks:

i) Every subspace of a vector space is a vector space in its own right

ii) Any vector space V automatically contains two subspaces

a) The set $\{0\}$, set consisting of only zero vector, is called trivial subspace.

b) V itself is called improper subspace.

Ex6 : Check whether the following are subspace of $M_{2\times 2}(\mathbb{R})$ under usual matrix addition and scalar multiplication.

i) $M_1 = \left\{ \begin{bmatrix} a & b \\ c & 1 \end{bmatrix} \;\middle|\; a, b, c \in \mathbb{R} \right\}$

ii) $M_1 = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \;\middle|\; \det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = 0 \right\}$

Ans i) : No. Since $\begin{bmatrix} a_1 & b_1 \\ c_1 & 1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & 1 \end{bmatrix} = \begin{bmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & 2 \end{bmatrix} \notin M.$

Ans ii) : No.

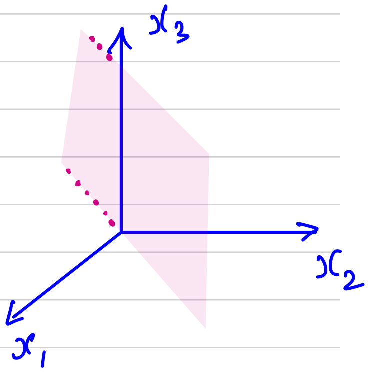
Because, $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in M$, and $A + B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \notin M$

$(\det(A+B) \neq 0)$

Ex7 : Let $S = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \;\middle|\; \begin{matrix} x_1 = x_2 \\ \forall\, x_i \in \mathbb{R} \end{matrix} \right\}$. S.T S is a subspace of $\mathbb{R}^3$.

Soln : We need to verify that the two closure properties hold :

Let $\begin{bmatrix} a \\ a \\ b \end{bmatrix}$ and $\begin{bmatrix} c \\ c \\ d \end{bmatrix}$ be any vectors in S.

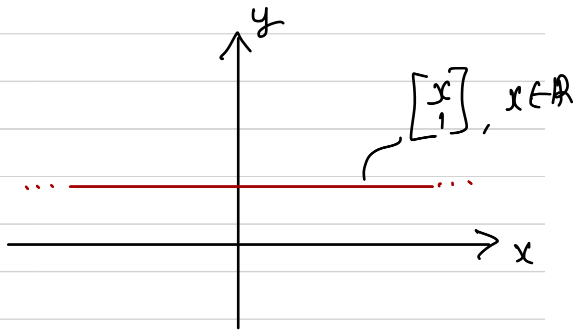


i) $\begin{bmatrix} a \\ a \\ b \end{bmatrix} + \begin{bmatrix} c \\ c \\ d \end{bmatrix} = \begin{bmatrix} a+c \\ a+c \\ b+d \end{bmatrix} \in S.$

ii) $\alpha . \begin{bmatrix} a \\ a \\ b \end{bmatrix} = \begin{bmatrix} \alpha a \\ \alpha a \\ \alpha b \end{bmatrix} \in S,$ for any $\alpha \in \mathbb{R}$.

$\because$ S is non-empty and satisfies two closure conditions, it follows that S is a subspace of $\mathbb{R}^3$.

Ex 8: Let $S = \left\{ \begin{bmatrix} x \\ 1 \end{bmatrix} \middle| \, x \in \mathbb{R} \right\}$. Check whether S is a subspace of $\mathbb{R}^2$ or not.



Soln: For any $\begin{bmatrix} x_1 \\ 1 \end{bmatrix}, \begin{bmatrix} x_2 \\ 1 \end{bmatrix}$ in S,

$$\begin{bmatrix} x_1 \\ 1 \end{bmatrix} + \begin{bmatrix} x_2 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ 2 \end{bmatrix} \notin S$$

Since it fails to satisfy closure conditions, S is not a subspace of $\mathbb{R}^2$.

or

Since $\begin{bmatrix} 0 \\ 0 \end{bmatrix} \notin S$, S is not a subspace of $\mathbb{R}^2$.

Ex 9: Let $S = \left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \middle| \begin{array}{l} a_{12} = -a_{21} \\ a_{ij} \in \mathbb{R} \end{array} \right\}$.

Is this a subspace of $\mathbb{R}^{2\times 2}$. ($\mathbb{R}^{2\times 2}$ is set of matrices of order 2)

Soln: S is non-empty because $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in S$.

We verify two closure conditions:

Let $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and $\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ be any two elements in S.

i) $$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{pmatrix} \in S$$

$$\left(\begin{array}{l} \because \; a_{21} = -a_{12} \text{ and } b_{21} = -b_{12} \\ \Rightarrow \; a_{21} + b_{21} = -(a_{12} + b_{12}) \end{array} \right)$$

ii) $\alpha \cdot \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} \alpha a_{11} & \alpha a_{12} \\ \alpha a_{21} & \alpha a_{22} \end{pmatrix} \in S$ (because $\alpha a_{21} = -(\alpha a_{12})$)

$\therefore$ S is a subspace of $\mathbb{R}^{2\times 2}$.

The Null Space of a Matrix

Defn: Let A be $m\times n$ matrix. The null space of A, denoted by $N(A)$ is the set of all solutions of the homogenous system $Ax = 0$

i.e $N(A) = \{ x \in \mathbb{R}^n \mid Ax = 0 \}$

Clearly $N(A)$ is a subset of $\mathbb{R}^n$.

Ex: If A is $m\times n$ matrix, then $N(A)$ is a subspace of $\mathbb{R}^n$.

Soln: Clearly, $0 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in N(A)$, so $N(A)$ is nonempty.

Let x_1 and x_2 be any vectors in $N(A)$.

i) $A(x_1 + x_2) = Ax_1 + Ax_2 = 0 + 0 = 0$.

Thus $x_1 + x_2 \in N(A)$.

ii) For any scalar α,

$A(\alpha x_1) = \alpha A(x_1) = \alpha . 0 = 0$.

implies that $\alpha x_1 \in N(A)$

It follows that $N(A)$ is a subspace of $\mathbb{R}^n$.

Ex: Determine $N(A)$ if

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix}$$

Soln: Consider Homogenous System

$$Ax = 0 \,, \quad \text{where} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Consider, Augmented matrix

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 \end{array}\right]$$

$R_2 \to R_2 - 2R_1$

$$\sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -2 & 1 & 0 \end{array}\right]$$

$R_1 \to R_1 + R_2$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & -1 & 1 & 0 \\ 0 & -1 & -2 & 1 & 0 \end{array}\right]$$

$R_2 \to -R_2$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 \end{array}\right] \quad \text{is in rref.}$$

Equivalent system

$$x_1 - x_3 + x_4 = 0$$

$$x_2 + 2x_3 + x_4 = 0$$

No. of free variable $= n - r$ (r is rank of A)

$= 4 - 2 = 2$

free variables are x_3 and x_4. Let $x_3 = k_1$ and $x_4 = k_2$

$\Rightarrow x_1 = k_1 - k_2$ and $x_2 = -2k_1 - k_2$

Thus, general soln

$$X = \begin{bmatrix} k_1 - k_2 \\ -2k_1 - k_2 \\ k_1 \\ k_2 \end{bmatrix} = k_1 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \quad (k_1, k_2 \text{ are scalars})$$

Null space of A,

$$N(A) = \left\{ k_1 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \;\middle|\; \text{for all } k_1, k_2 \in \mathbb{R} \right\}$$

Here $N(A)$ is a set of all linear combinations of $\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$

Thm 2: The intersection of two subspaces of a vector space V, is again a subspace.

or

If W_1 and W_2 are subspaces of a vector space V, then $W_1 \cap W_2$ is also a subspace V.

pf: Let V_1 and V_2 be any vectors in $W_1 \cap W_2$

Since $V_1, V_2 \in W_1 \cap W_2$

$\Rightarrow \quad V_1, V_2 \in W_1$ and $V_1, V_2 \in W_2$

$\Rightarrow \quad V_1 + V_2 \in W_1$ and $V_1 + V_2 \in W_2$ ($\because$ W_1 and W_2 are subspaces)

Also,

$\alpha V_1 \in W_1$ and $\alpha V_1 \in W_2$ (")

(for any scalar α)

$\Rightarrow \quad V_1 + V_2 \in W_1 \cap W_2$ and $\alpha V_1 \in W_1 \cap W_2$

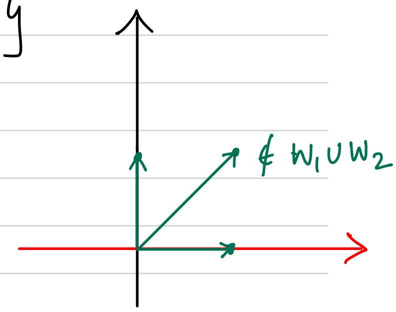
Thus, $W_1 \cap W_2$ is a subspace of V.

Note: Union of two subspaces of a vector space V need not be a subspace of V.

Counter example:

Consider $W_1 = \left\{ \begin{bmatrix} x \\ 0 \end{bmatrix} : x \in \mathbb{R} \right\}$, $W_2 = \left\{ \begin{bmatrix} 0 \\ y \end{bmatrix} : y \in \mathbb{R} \right\}$

are subspaces of $\mathbb{R}^2$



But $W_1 \cup W_2$ is not a subspace of $\mathbb{R}^2$

$\left(\because \begin{bmatrix} x \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \notin W_1 \cup W_2 \right)$

Thm 3: Let W_1 and W_2 be subspaces of a vector space V. Then $W_1 \cup W_2$ is a subspace of V iff $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

Exercise:

True or false?

(a) If V is a vector space and W is a subset of V that is also a vector space, then W is a subspace of V. T

(b) The empty set is a subspace of every vector space. F

(c) If V is a vector space other than the zero vector space, then V contains a subspace W such that $W \neq V$. T

(d) The intersection of any two subsets of V is a subspace of V. F

(e) Any union of subspaces of a vector space V is a subspace of V. F

2) Which of the following subset are subspace of $\mathbb{R}^3$.

i) $S_1 = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \middle| \; x_1 + x_2 - x_3 = 0 \right\}$ Subspace

ii) $S_2 = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \;\middle|\; x_1 x_2 - x_3 = 0 \right\}$ Not a Subspace

iii) $S_3 = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \;\middle|\; \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ Subspace

iv) $S_4 = \left\{ \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \;\middle|\; \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

where c_1 and c_2 are scalars. Not a Subspace.

Linear combination

Let V be a vector space, let $V_1, V_2, V_3, \dots, V_n \in V$.

Then sum of the form

$$\alpha_1 V_1 + \alpha_2 V_2 + \dots + \alpha_n V_n, \quad \text{where } \alpha_i \text{ are scalars}$$

$$i \in \{1, 2, \dots, n\}$$

is called a linear combination of $V_1, V_2, \dots V_n$.

Ex: Let $(1, 2, 1)$, $(5, -3, -9)$, and $(-22, 21, 48)$ be three vectors in $\mathbb{R}^3$. Here, $(-22, 21, 46)$ is linear combination of $(1, -2, 1)$ and $(5, -3, -9)$.

Because $(-22, 21, 48) = 3(1, 2, 1) + (-5)(5, -3, -9)$.

Span (linear span)

Let V be a vector space and $v_1, v_2, v_3, \dots, v_n$ be vectors in V.

The set of all linear combinations of $v_1, v_2, v_3, \dots, v_n$ is called a linear span of $v_1, v_2, v_3, \dots, v_n$

It is denoted by

$$\text{Span}(v_1, v_2, v_3, \dots, v_n) \text{ or } L(v_1, v_2, v_3, \dots, v_n)$$

In other words,

$$\text{Span}(v_1, v_2, \dots, v_n) = \{\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n \mid \alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{R}\}$$

Ex: In $\mathbb{R}^3$,

$$(-22, 21, 46) \in \text{Span}((1, -2, 1), (5, -3, -9))$$

Notation: In $\mathbb{R}^n$,

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{n\times 1}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{n\times 1}, \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{n\times 1}, \ \dots \ e_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}_{n\times 1}$$

Ex 1: Find linear span of e_1 and e_2 in $\mathbb{R}^3$.

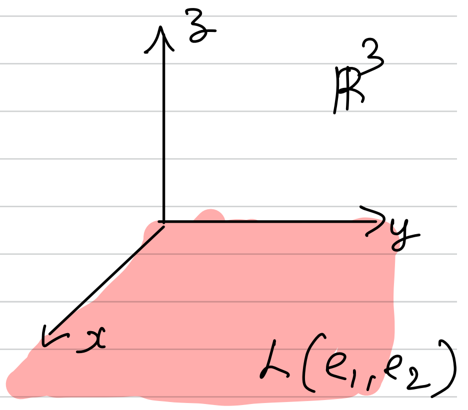
Soln: $\text{Span}(e_1, e_2) = \{\alpha_1 e_1 + \alpha_2 e_2 \mid \forall\, \alpha_1, \alpha_2 \in \mathbb{R}\}$

$$= \left\{ \alpha_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \,\middle|\, \forall\, \alpha_1, \alpha_2 \in \mathbb{R} \right\}$$

$$= \left\{ \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ 0 \end{bmatrix} \,\middle|\, \forall\, \alpha_1, \alpha_2 \in \mathbb{R} \right\}$$

is set of all points in the xy-plane.

It is a subspace of $\mathbb{R}^3$.



Theorem 4: If $v_1, v_2, \dots, v_n$ are elements of a vector space V, then $\text{Span}(v_1, v_2, \dots, v_n)$ is a subspace of V.

Pf: WKT

$$\text{span}(v_1, v_2, \dots, v_n) = \{\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n \mid \alpha_i \in \mathbb{R}\}$$

Let w_1 and w_2 be any two vectors in $\text{span}(v_1, v_2, \dots v_n)$,

let $w_1 = \beta_1 v_1 + \beta_2 v_2 + \dots + \beta_n v_n$ and $w_2 = \gamma_1 v_1 + \gamma_2 v_2 + \dots + \gamma_n v_n$

Consider $w_1 + w_2 = (\beta_1+\gamma_1)v_1 + (\beta_2+\gamma_2)v_2 + \cdots + (\beta_n+\gamma_n)v_n$
$\in \text{Span}(v_1, v_2, \dots v_n)$

Also
$\alpha \cdot w_1 = \alpha\beta_1 v_1 + \alpha\beta_2 v_2 + \cdots + \alpha\beta_n v_n \in \text{Space}(v_1, v_2, \dots v_n)$

Thus, $\text{Span}(v_1, v_2, \dots, v_n)$ is a subspace of V.

Ex 2: Is $(1,-3,4)^T$ a linear combination of $(0,-1,5)^T$ and $(-2,3,4)^T$

Soln: Consider

$$\begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix} = \alpha \begin{bmatrix} 0 \\ -1 \\ 5 \end{bmatrix} + \beta \begin{bmatrix} -2 \\ 3 \\ 4 \end{bmatrix}$$

$$\Rightarrow \quad \begin{aligned} 1 &= \alpha \cdot 0 + (-2)\beta \\ -3 &= \alpha \cdot (-1) + 3\beta \\ 4 &= \alpha \cdot 5 + 4\beta \end{aligned}$$

$$\Rightarrow \quad \left.\begin{aligned} -2\beta &= 1 \\ -\alpha + 3\beta &= -3 \\ 5\alpha + 4\beta &= 4 \end{aligned}\right\} \quad (*)$$

$$\Rightarrow \underbrace{\begin{bmatrix} 0 & -2 \\ -1 & 3 \\ 5 & 4 \end{bmatrix}}_{A} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}}_{b}$$

Consider, Augmented matrix

$[A, b]$

$$= \left[\begin{array}{cc|c} 0 & -2 & 1 \\ -1 & 3 & -3 \\ 5 & 4 & 4 \end{array}\right]$$

$R_1 \leftrightarrow R_3$

$$\sim \left[\begin{array}{cc|c} 5 & 4 & 4 \\ -1 & 3 & -3 \\ 0 & -2 & 1 \end{array}\right]$$

$R_2 \to 5R_2 + R_1$

$$\sim \left[\begin{array}{cc|c} 5 & 4 & 4 \\ 0 & 19 & -11 \\ 0 & -2 & 1 \end{array}\right]$$

$R_3 \to 19R_3 + 2R_2$

$$\sim \left[\begin{array}{cc|c} 5 & 4 & 4 \\ 0 & 19 & -11 \\ 0 & 0 & -3 \end{array}\right]$$

This is in row echelon form.

rank $(A) = 2$, rank $([A:b]) = 3$

rank $(A) \neq$ rank$([A:b])$

$\therefore$ System of eqns $(*)$ is inconsistent.

Thus $\begin{bmatrix}1\\-3\\4\end{bmatrix}$ is not linear combination of $\begin{bmatrix}0\\-1\\5\end{bmatrix}$ and $\begin{bmatrix}-2\\3\\4\end{bmatrix}$.

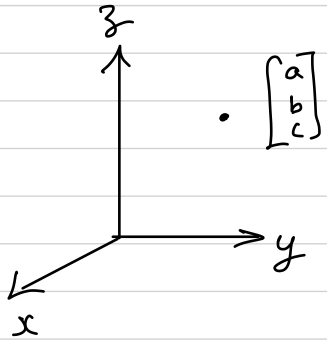
Defn: The set $S = \{v_1, v_2, \ldots, v_n\}$ is called a spanning set for a vector space V iff Span $(S) = V$

In otherwords,

If every vector in V can be written as linear Combination of $S = \{v_1, v_2, \ldots, v_n\}$, then S is called a spanning set of V.

Ex 3: Which of the foll. are spanning sets of $\mathbb{R}^3$?

a) $S = \{e_1, e_2, e_3, (1,2,3)^T\}$



Let $\begin{bmatrix}a\\b\\c\end{bmatrix}$ be an arbitrary vector in $\mathbb{R}^3$.

If $\begin{bmatrix}a\\b\\c\end{bmatrix}$ can be written as linear combination of the vectors in the set S, then set S is called spanning set of $\mathbb{R}^3$.

Consider $$\begin{bmatrix}a\\b\\c\end{bmatrix} = \alpha_1\begin{bmatrix}1\\0\\0\end{bmatrix} + \alpha_2\begin{bmatrix}0\\1\\0\end{bmatrix} + \alpha_3\begin{bmatrix}0\\0\\1\end{bmatrix} + \alpha_4\begin{bmatrix}1\\2\\3\end{bmatrix}$$

$\Rightarrow \quad \alpha_1 = a, \quad \alpha_2 = b, \quad \alpha_3 = c, \quad \alpha_4 = 0$

Thus S is spanning set of $\mathbb{R}^3$.

b) $S = \{ (1,1,1)^T, \ (1,1,0)^T, \ (1,0,0)^T \}$

Consider

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

This leads to the system of eqns

$$\begin{aligned} \alpha_1 + \alpha_2 + \alpha_3 &= a \\ \alpha_1 + \alpha_2 + 0 &= b \\ \alpha_1 + 0 + 0 &= c \end{aligned} \quad \uparrow$$

$\alpha_1 = c$

$\alpha_2 = b - c$

$\alpha_3 = a - b - c$,

Thus $$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = c \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + (b-c) \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + (a-b-c) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

So the three vectors span $\mathbb{R}^3$.

c) $S = \{ (1,0,1)^T, \ (0,1,0)^T \}$

Consider $$\alpha_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_1 \end{bmatrix}$$

Thus, S cannot span $\mathbb{R}^3$. (For instance,

$$\begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix} \neq \alpha_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

for any α_1 and α_2)

Defn: [Linear Independence, LI] The vectors $v_1, v_2, v_3, \dots, v_n$ in a vector space V are said to be linearly independent if no vector v_i, $i \in \{1, 2, \dots, n\}$ can be written as a linearly combination of others.

In other words,

The vectors $v_1, v_2, \dots, v_n$ are said to be linearly independent if

$$\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n = 0$$

implies that $\alpha_1 = \alpha_2 = \dots = \alpha_n = 0$.

Defn: [Linear dependence, LD] The vectors $v_1, v_2, v_3, \dots, v_n$ in a vector space V are said to be linealy dependent if one of the vectors $v_1, v_2, \dots, v_n$ can be written as a linear combination of the others.

In other words,

If $v_1, v_2, \dots, v_n$ are linearly dependent, then

$$\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n = 0$$

has non-trivial choices for scalars $\alpha_1, \alpha_2, \dots \alpha_n$.

Ex1: Which of the foll. collection of vectors are linearly independent in $\mathbb{R}^3$.

a) $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

Consider $\alpha_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 0$

$$\Rightarrow \quad \begin{aligned} \alpha_1 + \alpha_2 + \alpha_3 &= 0 \\ \alpha_1 + \alpha_2 &= 0 \\ \alpha_1 &= 0 \end{aligned} \quad \uparrow$$

$\alpha_1 = 0$, $\quad \alpha_1 = -\alpha_2 \Rightarrow \alpha_2 = 0$ and $\alpha_3 = 0$

The only soln of this system is $\alpha_1 = \alpha_2 = \alpha_3 = 0$.
Thus, the vectors are linearly independent.

(b) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix} \right\}$

Consider
$$\alpha_1 \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} + \alpha_2 \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + \alpha_3 \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix} = 0$$

$$\Rightarrow \quad \begin{aligned} \alpha_1 + 2\alpha_2 + 4\alpha_3 &= 0 \\ 2\alpha_1 + \alpha_2 - \alpha_3 &= 0 \\ 3\alpha_1 + 3\alpha_2 + \alpha_3 &= 0 \end{aligned}$$

$$\Rightarrow \underbrace{\begin{bmatrix} 1 & 2 & 4 \\ 2 & 1 & -1 \\ 4 & 3 & 1 \end{bmatrix}}_{A} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}_{b}$$

Consider

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 1 & -1 \\ 4 & 3 & 1 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1, \quad R_3 \to R_3 - 4R_1$

$$\sim \begin{bmatrix} 1 & 2 & 4 \\ 0 & -3 & -9 \\ 0 & -5 & -15 \end{bmatrix}$$

$R_2 \to -\frac{1}{3} R_2, \qquad R_3 \to -\frac{1}{5} R_3$

$$\sim \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 1 & 3 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$ is in row echelon form

$\therefore$ rank$(A) = 2 < 3$ (no. of unknowns)

$\therefore$ the system has infinite (non trivial) soln.

Thus, given vectors are linearly dependent.

Remark:

Let $v_1, v_2, \dots v_n$ be vectors in $\mathbb{R}^m$ and let $A = [v_1 \; v_2 \; v_n]$ be $m \times n$ matrix. Then $v_1, v_2, \dots, v_n$ are linearly independent if rank$(A) = n$ and linearly dependent if rank$(A) < n$. (n being no. of vectors)

Ex 2: Check whether the vectors

$$\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ 3 \\ 11 \\ 9 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

are linearly independent or linearly dependent.

Soln: Consider,

$$A = \begin{bmatrix} -1 & 2 & 1 & -5 & 1 \\ 0 & 4 & 0 & 3 & 0 \\ 1 & 5 & 0 & 11 & 0 \\ 0 & 1 & 1 & 9 & 0 \end{bmatrix}_{4 \times 5}$$

$\because$ rank$(A) \leq 4 < 5$ (no. of vectors)

$\therefore$ Given set of vectors are linearly dependent.

Ex 3: Determine whether the following vectors are linearly indep. in $\mathbb{R}^3$.

$$\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$$

Soln: Consider

$$A = \begin{bmatrix} 2 & 3 & 2 \\ 1 & 2 & 2 \\ -2 & -2 & 0 \end{bmatrix},$$ Obtain row echelon form of A.

$R_1 \leftrightarrow R_2$

$$\sim \begin{bmatrix} 1 & 2 & 2 \\ 2 & 3 & 2 \\ -2 & -2 & 0 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 + 2R_1$

$$\sim \begin{bmatrix} 1 & 2 & 2 \\ 0 & -1 & -2 \\ 0 & 2 & 4 \end{bmatrix}$$

$R_3 \to R_3 + 2R_2$

$$\sim \begin{bmatrix} 1 & 2 & 2 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

rank$(A) = 2 <$ no. of vectors

$\therefore$ Vectors are L.D.

Ex4: Check whether the following vectors are LI or LD in $\mathbb{R}^4$.

$$\begin{bmatrix} -1 \\ 3 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 4 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 3 \\ 0 \end{bmatrix}$$

Soln: Given vectors are L.D.

Note: Set of vectors with zero vector is always L D. Since zero vector can be written as linear combination of others.

Ex5: Determine whether the following vectors are LI in $\mathbb{R}^{2\times 2}$.

$$\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}, \begin{bmatrix}2 & 3\\ 0 & 2\end{bmatrix}$$

Soln: Consider $c_1\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + c_2\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix} + c_3\begin{bmatrix}2 & 3\\ 0 & 2\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$

$$\Rightarrow \begin{bmatrix}c_1 & 0\\ 0 & c_1\end{bmatrix} + \begin{bmatrix}0 & c_2\\ 0 & 0\end{bmatrix} + \begin{bmatrix}2c_3 & 3c_3\\ 0 & 2c_3\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

$\Rightarrow \begin{aligned} c_1 + 2c_3 &= 0 \\ c_2 + 3c_3 &= 0 \end{aligned}$ (comparing each entries)

two eqns and 3 unknowns, the system has infinite solns.

Thus, given vectors are L.D.

Ex6: Let x_1, x_2 and x_3 be linearly independent vectors in $\mathbb{R}^n$ and let

$$y_1 = x_1 + x_2, \quad y_2 = x_2 + x_3, \quad y_3 = x_3 + x_1$$

Are y_1, y_2 and y_3 L I? Prove your answers.

Soln: Given x_1, x_2 and x_3 are L.I.

Thus, $c_1 x_1 + c_2 x_2 + c_3 x_3 = 0$

$\Rightarrow c_1 = c_2 = c_3 = 0.$

Now, consider

$$d_1 y_1 + d_2 y_2 + d_3 y_3 = 0$$

$$\Rightarrow d_1(x_1 + x_2) + d_2(x_2 + x_3) + d_3(x_3 + x_1) = 0$$

$$\Rightarrow (d_1 + d_3)x_1 + (d_1 + d_2)x_2 + (d_2 + d_3)x_3 = 0$$

$$\Rightarrow \quad d_1 + d_3 = 0 \quad -(1) \qquad (\text{Since } x_1, x_2, x_3 \text{ are LI})$$
$$d_1 + d_2 = 0 \quad -(2)$$
$$d_2 + d_3 = 0 \quad -(3)$$

(2) − (3)

$$\Rightarrow \quad d_1 - d_3 = 0 \quad -(4)$$

(4) + (1)

$$\Rightarrow \quad d_1 = 0$$

thus $d_3 = 0$ and $d_2 = 0$.

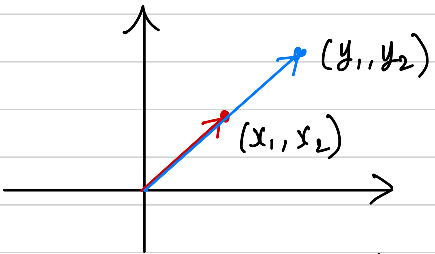
Therefore, y_1, y_2 and y_3 are LI.

Geometric interpretation

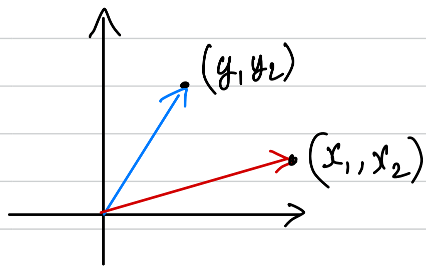
Let $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ be vectors in $\mathbb{R}^2$.

X and Y are dependent if $(0,0)$, (x_1, x_2) and (y_1, y_2) are collinear. (i.e X and Y lie on the same line which passes through origin)

Otherwise, linearly independent



X and Y linearly dependent

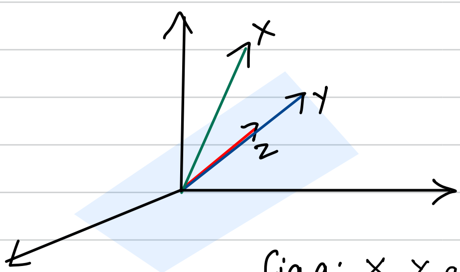


X and Y linearly independent

Let
$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$ be vectors in $\mathbb{R}^3$.

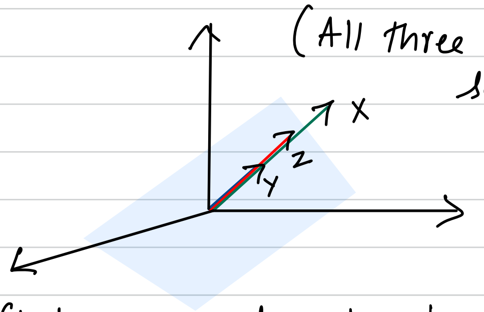
X and Y are LD if $(0,0,0)$, (x_1, x_2, x_3) and (y_1, y_2, y_3) are collinear.

Let $Z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$



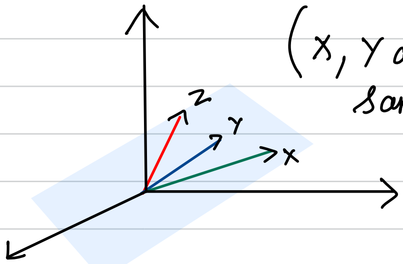
(Y and Z lie on same line)

fig a: X, Y and Z linearly dependent



(All three vectors lie on same line)

fig b: X, Y and Z linearly dependent



(X, Y and Z lie on same plane)

fig c: X, Y and Z linearly dependent

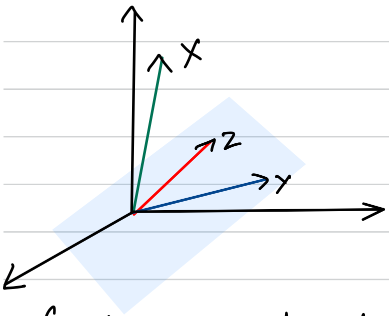


fig d: X, Y and Z linearly independent

In fig a, vector Z is scalar multiple of the vector Y.

In fig b, All three vectors are scalar multiples of each other.

In fig c, One vector is linear combination of the others.

That is, X, Y, and Z are said to be linearly dependent if

i) any two or all vectors lie on same line and/or

ii) vectors X, Y, and Z are coplanar (lie on same plane)

X, Y, and Z are said to be linearly independent if

i) no two vectors lie on a same line and

ii) vectors X, Y, and Z are not coplanar.

Ex 7: Check whether the vectors

$$p_1(x) = x^2 - 2x + 3, \quad p_2(x) = 2x^2 + x + 8, \quad p_3(x) = x^2 + 8x + 7$$

are linearly independent or dependent.

Soln: Consider

$$c_1 p_1(x) + c_2 p_2(x) + c_3 p_3(x) = 0x^2 + 0x + 0 \quad \text{(zero polynomial)}$$

$$\Rightarrow c_1 (x^2 - 2x + 3) + c_2 (2x^2 + x + 8) + c_3 (x^2 + 8x + 7) = 0x^2 + 0x + 0$$

$$\Rightarrow (c_1+2c_2+c_3)x^2 + (-2c_1+c_2+8c_3)x + (3c_1+8c_2+7c_3) = 0$$

Equating coefficients, we obtain

$$c_1+2c_2+c_3=0$$
$$-2c_1+c_2+8c_3=0$$
$$3c_1+8c_2+7c_3=0$$

or

$$\begin{bmatrix} 1 & 2 & 1 \\ -2 & 1 & 8 \\ 3 & 8 & 7 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Consider $\begin{vmatrix} 1 & 2 & 1 \\ -2 & 1 & 8 \\ 3 & 8 & 7 \end{vmatrix} = 1(7-64) - 2(-14-24) + (-16-3) = 0$

$\therefore$ Coefficient matrix for the system is singular, system has has non trivial solns. $\therefore$ $P_1(x)$, $P_2(x)$ and $P_3(x)$ are linearly dependent.

Linear independence and dependence of vectors in $C^{(n-1)}[a,b]$.

Let $C^{(n-1)}[a,b]$ be a vector space and let $f_1, f_2, f_3, \ldots, f_n$ be elements of $C^{(n-1)}[a,b]$.

If $f_1, f_2, f_3, \ldots, f_n$ are linearly dependent, then $\exists$ scalars $c_1, c_2, \ldots, c_n$ not all zero, such that

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0 \quad \forall x \in [a,b].$$

Taking the derivatives wrt x on both sides of the above eqn,

$$c_1 f_1'(x) + c_2 f_2'(x) + \cdots + c_n f_n'(x) = 0$$
$$c_1 f_1''(x) + c_2 f_2''(x) + \cdots + c_n f_n''(x) = 0$$
$$\vdots$$
$$c_1 f_1^{(n-1)}(x) + c_2 f_2^{(n-1)}(x) + \cdots + c_n f_n^{(n-1)}(x) = 0$$

$$\Rightarrow \begin{bmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & & & \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \text{———} \quad (*)$$

For each fixed x in $[a,b]$, the matrix equation will have the same nontrivial soln. $\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$.

Thus, if $f_1, f_2, \cdots, f_n$ are linearly dependent in $C^{(n-1)}[a,b]$, then the coefficient matrix in $(*)$ is singular. (That is determinant is zero)

or

If coefficient matrix in $(*)$ is nonsingular for some $x_0 \in [a,b]$, then $f_1, f_2, \cdots, f_n$ are linearly independent.

Defn: Let $f_1, f_2, \cdots, f_n$ be fns in $C^{(n-1)}[a,b]$, define the function $W[f_1, f_2, \cdots, f_n]$ on $[a,b]$ by

$$W[f_1, f_2, \ldots, f_n] = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & & & \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{vmatrix}$$

the function $W[f_1, f_2, \cdots, f_n]$ is called the Wronskian of $f_1, f_2, \cdots, f_n$

Theorem 5: Let $f_1, f_2, \cdots, f_n$ be elements in $C^{(n-1)}[a,b]$. If $\exists\ x_0 \in [a,b]$ such that $W[f_1, f_2, \cdots, f_n] \neq 0$, then $f_1, f_2, \cdots, f_n$ are linearly independent.

Pf: From previous discussion.

Converse of the above theorem is not true.

Ex: Consider the functions x^2 and $x|x|$ in $C^1[-1,1]$.

$$W(x^2, x|x|) = \begin{vmatrix} x^2 & x|x| \\ 2x & 2|x| \end{vmatrix} \equiv 0$$

$$x|x| = \begin{cases} x^2, & x \geq 0 \\ -x^2, & x < 0 \end{cases}$$

$$\frac{d(x|x|)}{dx} = \begin{cases} 2x & x \geq 0 \\ -2x & x < 0 \end{cases} = 2|x|$$

But x^2 and $x|x|$ are linearly independent.

Consider

$$c_1 x^2 + c_2 x|x| = 0 \quad \forall x \in [-1,1]$$

In particular, for $x=1$ and $x=-1$, we have

$$c_1 + c_2 = 0$$
$$c_1 - c_2 = 0$$

$\Rightarrow c_1 = c_2 = 0$. Thus x^2 and $x|x|$ are L I.

This Ex. shows that the converse of the above theorem is not true.

Ex: Show that e^x and e^{-x} are LI in $C(-\infty, \infty)$.

Soln: Consider

$$W(e^x, e^{-x}) = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -2 \neq 0$$

$\therefore$ e^x and e^{-x} are L.I

Ex: Show that the vectors $1, x, x^2$ and x^3 are LI in $C(-\infty,\infty)$

Soln:

$$W = (1, x, x^2, x^3) = \begin{vmatrix} 1 & x & x^2 & x^3 \\ 0 & 1 & 2x & 3x^2 \\ 0 & 0 & 2 & 6x \\ 0 & 0 & 0 & 6 \end{vmatrix}$$

$$= 6 \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix} = 6(2) = 12 \neq 0$$

$\therefore$ Vectors are L.I

Basis and Dimension

Defn: The vectors $v_1, v_2, \ldots, v_n$ form a basis for a vector space V iff

i) $v_1, v_2, \ldots, v_n$ are linearly independent

ii) $\text{Span}(v_1, v_2, \ldots, v_n) = V$

Ex 1: Let $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Then $\{e_1, e_2, e_3\}$ form a basis for $\mathbb{R}^3$.

Moreover $\{e_1, e_2, e_3\}$ are called standard basis for $\mathbb{R}^3$.

Ex 2: Vectors $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$ is also basis for $\mathbb{R}^3$.

In fact, there are infinitely many bases for a vector space

Defn: [Dimension]: Dimension of a vector space V is equal to the number of vectors in a basis for V.

Ex: Dimension of $\mathbb{R}^2 = 2$.

Dimension of $\mathbb{R}^3 = 3$.

In general, dimension of $\mathbb{R}^n = n$ $\quad (n \geq 1)$

Note: 1) The subspace $\{0\}$ of V is said to have dimension 0.

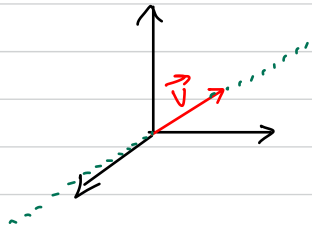
2) V is said to be finite dimensional if there is a finite set of vectors that spans V.

3) If there is no finite set of vectors that spans V, then we say V is infinite dimensional vector space.

For instance, $C^{(n)}[a,b]$ is infinite dimensional vector space.

Ex 3: Let v_1 be a vector in $\mathbb{R}^3$. $\text{Span}(v_1) = \{\alpha v_1 : \alpha \in \mathbb{R}\}$

WKT $\text{span}(v_1)$ is a subspace of $\mathbb{R}^3$.

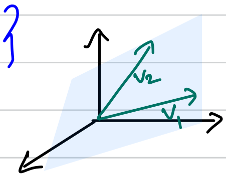


Basis of $\text{Span}(v_1) = \{v_1\}$

dimension of $\text{span}(v_1) = 1$

Ex 4: Let $\{v_1, v_2\}$ be independent vectors in $\mathbb{R}^3$.

$\text{Span}(v_1, v_2) = \{\alpha_1 v_1 + \alpha_2 v_2 : \alpha_1, \alpha_2 \in \mathbb{R}\}$



Basis of $\text{Span}(v_1, v_2) = \{v_1, v_2\}$

dimension is 2.

Note: If W is a subspace of a vector space V, then dimension of $W \le$ dimension of V.

Ex 5: Let P be a vector space of all polynomials. Show that P is infinite dimensional vector space.

Soln: Suppose P is finite dimensional vector space, say of dimension n, then any set of $n+1$ vectors are linearly dependent.

But the set $\{1, x, x^2, \ldots, x^n\}$ ($n+1$ vectors) are linearly independent.

Since $W[1, x, x^2, \ldots, x^n] \neq 0$.

$\therefore$ P cannot be of dimension n. Since n is arbitrary, P must be infinite dimensional.

Note: Same argument shows that $C[a,b]$ is infinite dimensional.

Standard basis

Vectors $\{e_1, e_2\}$ form a basis in $\mathbb{R}^2$; where $e_1 = \begin{bmatrix}1\\0\end{bmatrix}$ and $e_2 = \begin{bmatrix}0\\1\end{bmatrix}$.

Let x be any vector in $\mathbb{R}^2$. Then

$$x = x_1 e_1 + x_2 e_2 .$$

Here x_1 and x_2 are coordinates of x wrt e_1 and e_2.

$\begin{bmatrix}x_1\\x_2\end{bmatrix}$ is called coordinate vector of x wrt $\{e_1, e_2\}$.

Usually any vector in $\mathbb{R}^2$ is represented as a coordinate vector wrt $\{e_1, e_2\}$.

That is why $\{e_1, e_2\}$ is called standard basis of $\mathbb{R}^2$.

In general, standard basis in $\mathbb{R}^n$ is

$$\{e_1, e_2, e_3, \dots, e_n\}.$$

where $e_1 = \begin{bmatrix}1\\0\\0\\\vdots\\0\end{bmatrix}_{n\times 1}$, $e_2 = \begin{bmatrix}0\\1\\0\\\vdots\\0\end{bmatrix}_{n\times 1}$, $\dots$, $e_n = \begin{bmatrix}0\\0\\\vdots\\0\\1\end{bmatrix}_{n\times 1}$

Standard basis of $\mathbb{R}^{2\times 2} = \left\{ \begin{pmatrix}a & b\\c & d\end{pmatrix} \,\middle|\, a, b, c, d \in \mathbb{R} \right\}$

is $\{E_{11}, E_{12}, E_{21}, E_{22}\}$

where $E_{11} = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}$, $E_{12} = \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}$, $E_{21} = \begin{pmatrix}0 & 0\\1 & 0\end{pmatrix}$

$$E_{22} = \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}.$$

Standard basis of $P_2 = \left\{ ax^2 + bx + c \,\middle|\, a, b, c \in \mathbb{R} \right\}$

is $\{1, x, x^2\}$

Coordinate vector

The standard basis for $\mathbb{R}^2$ is $\{e_1, e_2\}$. Any vector X in $\mathbb{R}^2$ can be expressed as linear combination

$$X = x_1 e_1 + x_2 e_2$$

The scalars x_1 and x_2 are coordinates of X wrt $\{e_1, e_2\}$.

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is called coordinate vector of X wrt $\{e_1, e_2\}$ or just coordinate vector of X.

Now, let $\{y, z\}$ be any other basis in $\mathbb{R}^2$. Then vector X can also be represented uniquely as

$$X = \alpha y + \beta z$$

The scalars α and β are called coordinates wrt $\{y, z\}$.

$\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ is called coordinate vector wrt the basis $\{y, z\}$.

Ex: Let $\{y_1, y_2\}$ be the basis in $\mathbb{R}^2$, where $y_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $y_2 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$.

Let $X = \begin{bmatrix} 7 \\ 7 \end{bmatrix}$ be a vector in $\mathbb{R}^2$. Find the coordinate vector of $X = \begin{bmatrix} 7 \\ 7 \end{bmatrix}$ wrt the

i) Standard basis $\{e_1, e_2\}$ ii) the basis $\{y_1, y_2\}$

Soln: i) Coordinate vector of $X = \begin{bmatrix} 7 \\ 7 \end{bmatrix}$ wrt $\{e_1, e_2\}$ is $\begin{bmatrix} 7 \\ 7 \end{bmatrix}$

Since $X = 7e_1 + 7e_2$

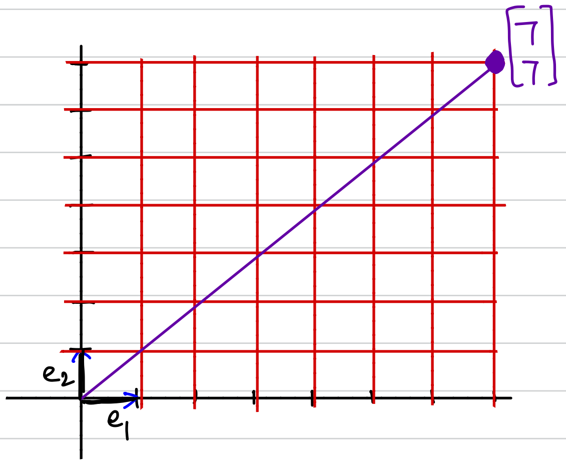
ii) Consider $X = c_1 y_1 + c_2 y_2$

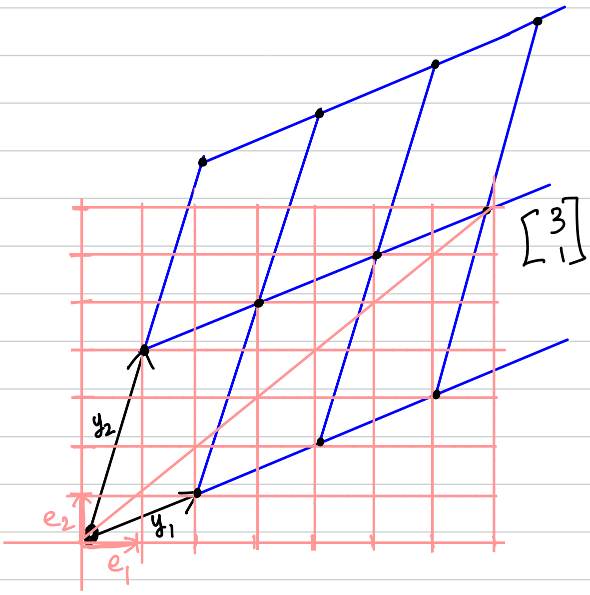
$$\Rightarrow \begin{bmatrix} 7 \\ 7 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 4 \end{bmatrix} \Rightarrow \begin{array}{l} 7 = 2c_1 + c_2 \\ 7 = c_1 + 4c_2 \end{array}$$

$$\Rightarrow c_1 = 3 \text{ and } c_2 = 1.$$

$\therefore$ Coordinate vector of $X = \begin{bmatrix} 7 \\ 7 \end{bmatrix}$ wrt $\{y_1, y_2\}$ is $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

Geometric intepretation





$x = \begin{bmatrix}7\\7\end{bmatrix}$, coordinate vector wrt std basis $\{e_1, e_2\}$.

$\begin{bmatrix}3\\1\end{bmatrix}$ is coordinate vector of x wrt $\{y_1, y_2\}$

Note:

1) For any $a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \in P_n(\mathbb{R})$

The Coordinate vector of $a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ wrt the standard basis $\{1, x, x^2, \ldots, x^n\}$ is $\begin{bmatrix}a_0\\a_1\\a_2\\\vdots\\a_n\end{bmatrix}$

2) For any $\begin{bmatrix}a & b\\c & d\end{bmatrix} \in \mathbb{R}^{2\times 2}$,

The Coordinate vector of $\begin{bmatrix}a & b\\c & d\end{bmatrix}$ wrt the stardard basis $\left\{\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right\}$ is $\begin{bmatrix}a\\b\\c\\d\end{bmatrix}$

$$\because \begin{bmatrix}a & b\\c & d\end{bmatrix} = a\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} + b\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix} + c\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix} + d\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$$

Theorem 6: If $\{v_1, v_2, \dots, v_n\}$ is a spanning set for a vector space V, then any collection of m vectors in V, where $m > n$, is linearly dependent.

Corollary 7: If both $\{v_1, v_2, \dots, v_n\}$ and $\{u_1, u_2, \dots, u_m\}$ are bases for a vector space V, then $n = m$.

Theorem 8: If V is a vector space of dimension $n > 0$, then

i) any set of n linearly independent vectors spans V.

ii) any n vectors that span V are L I.

Theorem 9: If V is a vector space of dimension $n > 0$, then

i) no set of fewer than n vectors can span V.

ii) any subset of fewer than n L I vectors can be extended to form a basis for V.

iii) any spanning set containing more than n vectors can be pared down to form a basis for V.

Ex 1: Let $X_1 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$, $X_2 = \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}$, $X_3 = \begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix}$

a) Show that X_1, X_2 and X_3 are linearly dependent.

b) Show that X_1 and X_2 are linearly independent.

c) What is the dimension of $\text{Span}(X_1, X_2, X_3)$?

d) Give a geometric description of $\text{Span}(X_1, X_2, X_3)$

Soln: a) Conside a matrix

$$A = \begin{bmatrix} 2 & 3 & 2 \\ 1 & -1 & 6 \\ 3 & 4 & 4 \end{bmatrix}$$

We find ref of A.

$R_1 \leftrightarrow R_2$

$$\sim \begin{bmatrix} 1 & -1 & 6 \\ 2 & 3 & 2 \\ 3 & 4 & 4 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$

$$\sim \begin{bmatrix} 1 & -1 & 6 \\ 0 & 5 & -10 \\ 0 & 7 & -14 \end{bmatrix}$$

$R_2 \to \frac{1}{5} R_2$, $R_3 \to \frac{1}{7} R_3$

$$\sim \begin{bmatrix} 1 & -1 & 6 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

$R_3 \to R_3 - R_1$

$$\sim \begin{bmatrix} 1 & -1 & 6 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow$ $\text{rank}(A) = 2 <$ no. of Vectors

$\therefore$ X_1, X_2, X_3 are linearly dependent.

b) Clearly, there is no scalar $\alpha \in \mathbb{R}$ such that $X_2 = \alpha X_1$.

$\therefore$ X_1 and X_2 are LI.

c) From a) and b), we have

Span (x_1, x_2, x_3) = span (x_1, x_2), since $x_3 \in$ Span (x_1, x_2).

Further x_1 and x_2 are LI.

$\therefore$ dimension of Span $(x_1, x_2) = 2$

d) Geometrically span (x_1, x_2) represents a plane through origin on which x_1 and x_2 lie.

Ex2: Find a basis for the subspace S of $\mathbb{R}^4$ consisting of all vectors of the form $(a+b,\ a-b+2c,\ b,\ c)^T$ where a, b, c are real no.s. What is the dimension of S?

Soln: Given

$$S = \left\{ \begin{bmatrix} a+b \\ a-b+2c \\ b \\ c \end{bmatrix} \middle/ \ a, b, c \in \mathbb{R} \right\}$$

Clearly

$$\begin{bmatrix} a+b \\ a-b+2c \\ b \\ c \end{bmatrix} = a \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

and vectors $\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$ are LI (why?)

$\therefore$ These vectors form basis for S.

dimension of S is 3.

Ex 3: The vectors $x_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$, $x_2 = \begin{bmatrix} 2 \\ 5 \\ 4 \end{bmatrix}$, $x_3 = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$, $x_4 = \begin{bmatrix} 2 \\ 7 \\ 4 \end{bmatrix}$ and $x_5 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ span $\mathbb{R}^3$. Pare down the set $\{x_1, x_2, x_3, x_4, x_5\}$ to form basis for $\mathbb{R}^3$.

Soln: Given

$$\text{Span}\{x_1, x_2, x_3, x_4, x_5\} = \mathbb{R}^3$$

Consider the matrix:

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 & 1 \\ 2 & 5 & 3 & 7 & 1 \\ 2 & 4 & 2 & 4 & 0 \end{bmatrix}.$$ Obtain rref of A.

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 2R_1$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 2 & 1 \\ 0 & 1 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix}$$

$R_3 \to \frac{1}{-2} R_3$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 2 & 1 \\ 0 & 1 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_1 \to R_1 - 2R_2$

$$\sim \begin{bmatrix} 1 & 0 & -1 & -4 & 3 \\ 0 & 1 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_1 \to R_1 - 3R_3$, $R_2 \to R_2 + R_3$

$$\sim \begin{array}{c} \begin{bmatrix} 1 & 0 & -1 & -4 & 0 \\ 0 & 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \\ \begin{matrix} u_1 & u_2 & u_3 & u_4 & u_5 \end{matrix} \end{array} = U \text{ (say)}$$

It is in rref.

In the matrix U, the col. vectors u_1, u_2, and u_5 are LI and

$$u_3 = -u_1 + u_2$$

$$u_4 = -4u_1 + 3u_2$$

Same relations hold by col. vectors of A.

$$x_3 = -x_1 + x_2, \quad x_4 = -4x_1 + 3x_2$$

$\therefore$ x_1, x_2 and x_5 are LI

$\Rightarrow$ $\text{Span}\{x_1, x_2, x_3, x_4, x_5\} = \text{Span}\{x_1, x_2, x_5)$

$\therefore$ Basis for $\mathbb{R}^3 = \{x_1, x_2, x_3\}$

Note : column vectors of A and U satisfy same dependency relation.

Ex 4: In $C[-\pi, \pi]$, find the dimension of the subspace spanned by $1, \cos 2x, \cos^2 x$.

Soln: We have $\cos^2 x = \dfrac{\cos 2x + 1}{2}$

$$\Rightarrow \cos^2 x \in \text{span}\{1, \cos 2x\}$$

Further, $1, \cos 2x$ are L.I. Since $W(1, \cos 2x) \neq 0$ for some x.

$\therefore$ $\text{span}\{1, \cos 2x, \cos^2 x\} = \text{span}\{1, \cos 2x)$.

Dimension of $\text{span}\{1, \cos 2x, \cos^2 x\} = 2$

Ex 5: Let S be the subspace of P_3 consisting of all polynomials of the form $ax^2+bx+2a+3b$. Find a basis for S.

Soln: Given $S = \{ax^2+bx+2a+3b \mid a, b \in \mathbb{R}\}$

Clearly, $ax^2+bx+2a+3b = a(x^2+2) + b(x+3)$

$\therefore$ All polynomials are linear combination of x^2+2 and $x+3$.

Also, $W(x^2+2, x+2) = \begin{vmatrix} x^2+2 & x+2 \\ 2x & 1 \end{vmatrix} = x^2+2-2x^2-4x$
$\neq 0 \ \forall x$.

$\therefore$ x^2+2 and $x+2$ are L.I.

Thus $S = \text{span}\{x^2+2, x+2\}$ and basis is $\{x^2+2, x+2\}$.

Ex 6: Find the basis and dimension of the subspace spanned by the subset S of the vector space $\mathbb{R}^{2\times 2}$.

where $S = \left\{ \begin{bmatrix} 1 & -5 \\ -4 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ -1 & 5 \end{bmatrix}, \begin{bmatrix} 2 & -4 \\ -5 & 7 \end{bmatrix}, \begin{bmatrix} 1 & -7 \\ -5 & 1 \end{bmatrix} \right\}$

Soln: Let W be the subspace spanned by S.

i.e $W = \text{Span}(S)$.

Recall that std basis of $\mathbb{R}^{2\times 2}$ is

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

For any $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{R}^{2\times 2}$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$\therefore$ coordinate vector of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ wrt std basis is $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$.

Coordinate vectors of the given vectors are

$$\begin{bmatrix} 1 \\ -5 \\ -4 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \\ -5 \\ 7 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ -7 \\ -5 \\ 1 \end{bmatrix}$$

Now, consider a matrix A such that the row vectors of A are coordinate vectors of the given matrices.

i.e $A = \begin{bmatrix} 1 & -5 & -4 & 2 \\ 1 & 1 & -1 & 5 \\ 2 & -4 & -5 & 7 \\ 1 & -7 & -5 & 1 \end{bmatrix}$. Obtain ref of A

$R_2 \leftarrow R_2 - R_1$, $R_3 \leftarrow R_3 - 2R_1$, $R_4 \leftarrow R_4 - R_1$

$$\sim \begin{bmatrix} 1 & -5 & -4 & 2 \\ 0 & 6 & 3 & 3 \\ 0 & 6 & 3 & 3 \\ 0 & -2 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -5 & -4 & 2 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ It is ref.

Coordinate vectors of basis of W is $\left\{ \begin{bmatrix} 1 \\ -5 \\ -4 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \\ 1 \end{bmatrix} \right\}$

$\therefore$ Basis of the Subspace W is $\left\{ \begin{bmatrix} 1 & -5 \\ -4 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \right\}$

and dim of the subspace $W = 2$

Ex 7: Let $A = \left\{ \begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \right\}$ be a linearly independent subset of $\mathbb{R}^3$. Extend this to a basis of $\mathbb{R}^3$.

Soln: We know that the $\dim(\mathbb{R}^3) = 3$.

Std basis is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

we verify whether $\begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ are LI if not

we consider $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ in place of $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and verify.

Consider $A = \begin{bmatrix} 1 & 2 & 1 \\ -2 & 3 & 0 \\ 5 & 1 & 0 \end{bmatrix}$, find rank of A

$R_2 \leftarrow R_2 + 2R_1$, $R_3 \leftarrow R_3 - 5R_1$

$$\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 7 & 2 \\ 0 & 11 & 5 \end{bmatrix}$$

$R_3 \to 7R_3 - 11R_2$

$$\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 7 & 2 \\ 0 & 0 & 13 \end{bmatrix} \Rightarrow \text{rank}(A) = 3.$$

$\therefore$ $\begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ are LI and form basis.

Row Space and Column Space

Defn: Let A be $m \times n$ matrix:

i) Row space of A is a subspace of $\mathbb{R}^n$ spanned by the row vectors of A. Denoted by $R(A)$.

ii) Column space of A is a subspace of $\mathbb{R}^m$ spanned by the column vectors of A. Denoted by $C(A)$.

Note: $R(A) = C(A^T)$

For instance:

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$ Then

$$R(A) = \text{span}\left\{ \begin{bmatrix} a_{11} \\ a_{12} \\ a_{13} \\ a_{14} \end{bmatrix}, \begin{bmatrix} a_{21} \\ a_{22} \\ a_{23} \\ a_{24} \end{bmatrix}, \begin{bmatrix} a_{31} \\ a_{32} \\ a_{33} \\ a_{34} \end{bmatrix} \right\},$$

$$C(A) = \text{span}\left\{ \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix}, \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix}, \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix}, \begin{bmatrix} a_{14} \\ a_{24} \\ a_{34} \end{bmatrix} \right\}.$$

Four Fundamental subspaces:

Four subspaces of a $m \times n$ matrix A are

i) Row space, $R(A)$. (Subspace of $\mathbb{R}^n$)

ii) Null space, $N(A)$ (Subspace of $\mathbb{R}^n$)

iii) Column space, $C(A)$ an (Subspace of $\mathbb{R}^m$)

iv) Left Null space, $N(A^T)$ (Subspace of $\mathbb{R}^m$)

Recall that, $N(A) = \{ x \in \mathbb{R}^n \mid Ax = 0 \}$

Null space of A^T is set of all solns of $A^T x = 0 \rightarrow$ col. vector

$$\Rightarrow (A^T x)^T = 0 \rightarrow \text{row vector}$$

$$\Rightarrow x^T A = 0$$

x is pre multiplied to A, $\therefore$ it is called left null space.

i.e Null space of A^T = Left null space of A.

Method to find basis and dimension for the row space and the column space of a Matrix.

Let A be any $m \times n$ matrix.

Firstly, obtain rref of A. Say rref$(A) = U$.

- The non-zero row vectors of U form basis for the $R(A)$.
- The rank of A = dimension of $R(A)$

(Since the span of row vectors of U is same as the span of row vectors of A. Further non-zero row vectors of U are LI).

- Determine the columns of U that correspond to the leading 1's. The same columns of A are LI and form a basis for the $C(A)$
- Dimension of the $C(A)$ = Dimension of the $R(A)$

  = The rank of A.

Note: $C(A)$ need not be equal to $C(U)$. But dependency relation of Col. vectors of A is same as the Col. vectors of U.

Method to find basis and dimension for the $N(A)$

Let A be $m \times n$ matrix and U be rref (A).

- Linear system $AX=0$ is equivalent to $UX=0$
- If rank $(A) = r$, then the no. of free variables in $UX=0$ is $n-r$.
- Dimension of $N(A) = n-r$.
- A basis is obtained accordingly by back substitution.

The dimension of $N(A)$ is called the nullity of A.

Rank - Nullity theorem

If A is an $m \times n$ matrix, then the rank of A plus the nullity of A equals n.

Method to find basis and dimension for the $N(A^T)$

Let A be an $m \times n$ matrix.

Consider the matrix $[A, I_m]$, obtained by the concatenation of A and identity matrix I_m.

- Obtain ref of $[A, I_m]$.
- Say $\text{ref}([A, I_m]) = [\underbrace{U}_{\text{ref}(A)}, P]$

  Then $PA = U$

- Suppose rank of $A = r$. Then by the rank-nullity Thm

  Nullity of $A^T = m - r$

  $= \#$ of zero rows in U.

- From $PA = U$, we can easily find X^T such that $X^T A = 0$, which gives basis of $N(A^T)$.

Ex 1: Let
$$A = \begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 4 & -3 & 0 \\ 1 & 2 & 1 & 5 \end{bmatrix}$$

i) Find a basis for the row space of A and a basis for $N(A)$.
ii) Verify the rank - nullity thm.
iii) Find a basis and the dimension of Col. space of A.

Soln: Find reduced row echelon form of A,

$$\text{rref}(A) = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{3\times 4} \text{ (say } U) \qquad \text{(verify)}$$

i) Basis of row space $= \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix} \right\}$

Rank of A = dimension of row space of $A = 2$

To find Null space

Consider the linear system of eqns $AX = 0$ is equivalent to $UX = 0$, it follows

$$1x_1 + 2x_2 + 0x_3 + 3x_4 = 0$$
$$1x_3 + 2x_4 = 0$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

No. of free variables $= n - r$
$= 4 - 2 = 2$

Let x_2 and x_4 be free variables, and

$x_2 = k_1$, $x_4 = k_2$

$\therefore$ $x_3 = -2x_4$ $= -2k_2$ | $x_1 = -2x_2 - 3x_4$ $= -2k_1 - 3k_2$

$$\therefore \text{ Null space } N(A) = \begin{bmatrix} -2k_1 - 3k_2 \\ k_1 \\ -2k_2 \\ k_2 \end{bmatrix}, \quad k_1, k_2 \in \mathbb{R}$$

$$= k_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} -3 \\ 0 \\ -2 \\ 1 \end{bmatrix}, \quad k_1, k_2 \in \mathbb{R}$$

$$\text{Basis of Null space} = \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$$

$$\text{Nullity} = 2$$

ii) Clearly, $\text{Rank}(A) + \text{Nullity}(A) = 2+2 = 4 = \dim(\mathbb{R}^4)$

This verifies rank-nullity thm.

iii) To find basis and dimension of $C(A)$.

From above

$$A = \begin{array}{c} \begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 4 & -3 & 0 \\ 1 & 2 & 1 & 5 \end{bmatrix} \\ \begin{matrix} x_1 & x_2 & x_3 & x_4 \end{matrix} \end{array} \quad \text{and} \quad U = \begin{array}{c} \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{matrix} u_1 & u_2 & u_3 & u_4 \end{matrix} \end{array}$$

WKT $C(A) \neq C(U)$, but the col. vectors of U and col. vectors of A have same dependency relation.

In U, u_1 and u_3 are LI (cols of U with leading 1's)

$u_2 = 2u_1$ and $u_4 = 3u_1 + 2u_3$

Thus, in A, x_1 and x_3 are LI (corresponding cols in A)

Also, $x_2 = 2x_1$ and $x_4 = 3x_1 + 2x_3$

Thus, Basis of $C(A) = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix} \right\}$

$$\text{Dim}(C(A)) = \text{rank}(A) = 2$$

Ex2: Find a basis and the dimension of all four fundamental subspace of the matrix

$$A = \begin{bmatrix} 1 & -2 & 1 & 1 & 2 \\ -1 & 3 & 0 & 2 & -2 \\ 0 & 1 & 1 & 3 & 4 \\ 1 & 2 & 5 & 13 & 5 \end{bmatrix}$$

Soln: Consider the matrix $[A_{4\times5}, I_4]$.

$$[A, I_4] = \left[\begin{array}{ccccc|cccc} 1 & -2 & 1 & 1 & 2 & 1 & 0 & 0 & 0 \\ -1 & 3 & 0 & 2 & -2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 3 & 4 & 0 & 0 & 1 & 0 \\ 1 & 2 & 5 & 13 & 5 & 0 & 0 & 0 & 1 \end{array}\right].$$ Obtain ref.

$R_2 \leftarrow R_2 + R_1$, $R_4 \leftarrow R_4 - R_1$

$$\sim \left[\begin{array}{ccccc|cccc} 1 & -2 & 1 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 3 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 3 & 4 & 0 & 0 & 1 & 0 \\ 0 & 4 & 4 & 12 & 3 & -1 & 0 & 0 & 1 \end{array}\right]$$

$R_3 \leftarrow R_3 - R_2, \quad R_4 \leftarrow R_4 - 4R_2$

$$\sim \left[\begin{array}{ccccc|cccc} 1 & -2 & 1 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 3 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 3 & -5 & -4 & 0 & 1 \end{array}\right]$$

$R_1 \leftarrow R_1 + 2R_2, \quad R_3 \leftarrow R_3/4, \quad R_4 \leftarrow R_4/3$

$$\sim \left[\begin{array}{ccccc|cccc} 1 & 0 & 3 & 7 & 2 & 3 & 2 & 0 & 0 \\ 0 & 1 & 1 & 3 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1/4 & -1/4 & 1/4 & 0 \\ 0 & 0 & 0 & 0 & 1 & -5/3 & -4/3 & 0 & 1/3 \end{array}\right]$$

$R_4 \leftarrow R_4 - R_3$

$$\sim \left[\begin{array}{c|c} \underbrace{\begin{matrix} 1 & 0 & 3 & 7 & 2 \\ 0 & 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{matrix}}_{ref(A)} & \underbrace{\begin{matrix} 3 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -1/4 & -1/4 & 1/4 & 0 \\ -17/12 & -13/12 & -1/4 & 1/3 \end{matrix}}_{P\ (say)} \end{array}\right]$$

$$\Rightarrow \quad ref(A) = \begin{bmatrix} 1 & 0 & 3 & 7 & 2 \\ 0 & 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (\text{say } U)$$

Let
$$P = \begin{bmatrix} 3 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -1/4 & -1/4 & 1/4 & 0 \\ -17/12 & -13/12 & -1/4 & 1/3 \end{bmatrix}$$

Basis for $R(A) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \\ 7 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

Basis for $C(A) = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 4 \\ 5 \end{bmatrix} \right\}$

Cols. of A corresponding to the cols with leading 1's in U.

Dimension of R(A) = Dimension of C(A) = rank(A) = 3

To find N(A), consider $AX = 0$

$$\Rightarrow UX = 0$$

$$\begin{bmatrix} 1 & 0 & 3 & 7 & 2 \\ 0 & 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \uparrow$$

# of free variables $= 5 - 3 = 2$

Let x_3 and x_4 be free variable, let $x_3 = k_1$, $x_4 = k_2$

From backward substitution:

$x_5 = 0$

$x_2 = -x_3 - 3x_4 = -k_1 - 3k_2$

$x_1 = -3x_3 - 7x_4 - 2x_5$

$\quad = -3k_1 - 7k_2$

$$\therefore N(A) = \begin{bmatrix} -3k_1 - 7k_2 \\ -k_1 - 3k_2 \\ k_1 \\ k_2 \\ 0 \end{bmatrix}; \quad k_1, k_2 \in \mathbb{R}$$

$$= k_1 \begin{bmatrix} -3 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} -7 \\ -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}; \quad k_1, k_2 \in \mathbb{R}$$

Basis for $N(A) = \left\{ \begin{bmatrix} -3 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ 3 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

Nullity of $A = 2$

To find basis and dimension of $N(A^T)$

Consider, $PA = U$

$$\Rightarrow \begin{bmatrix} 3 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -1/4 & -1/4 & 1/4 & 0 \\ -17/12 & -13/12 & -1/4 & 1/3 \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 & 1 & 2 \\ -1 & 3 & 0 & 2 & -2 \\ 0 & 1 & 1 & 3 & 4 \\ 1 & 2 & 5 & 13 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 3 & 7 & 2 \\ 0 & 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -17/12 & -13/12 & -1/4 & 1/3 \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 & 1 & 2 \\ -1 & 3 & 0 & 2 & -2 \\ 0 & 1 & 1 & 3 & 4 \\ 1 & 2 & 5 & 13 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Identify row vectors y in P such that $y^T A = 0$. It is basis of $N(A^T)$

It is easy to see that

Basis of $N(A^T) = \left\{ \begin{bmatrix} -17/12 \\ -13/12 \\ -1/4 \\ 1/3 \end{bmatrix} \right\}$

Nullity of $A^T = 1$

Observe: In the above ex:

$R(A),\ N(A) \leq \mathbb{R}^5$

$C(A),\ N(A^T) \leq \mathbb{R}^4$

Further, $\text{Dim}(R(A)) + \text{Dim}(N(A)) = 3+2 = 5 = \text{Dim}(\mathbb{R}^5)$

$\text{Dim}(C(A)) + \text{Dim}(N(A^T)) = 3+1 = 4 = \text{Dim}(\mathbb{R}^4)$

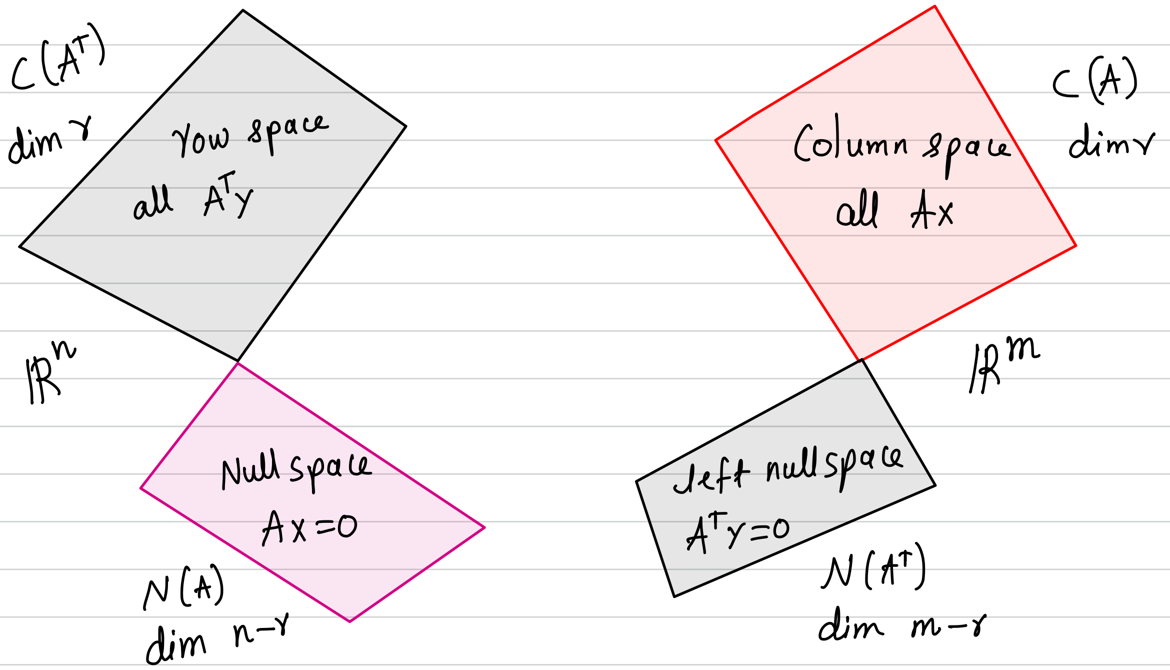
i.e we verified rank-nullity thm.

Note: 1) We can see that every vector $v \in R(A)$ is perpendicular to every vector $w \in N(A)$. $\therefore$ we say $R(A)$ and $N(A)$ are orthogonal subspaces.

2) Further, if A is an $m \times n$ matrix, then $R(A) \oplus N(A) = \mathbb{R}^n$. (Direct Sum).

(i.e for every $x \in \mathbb{R}^n$ there exist $v \in R(A)$ and $w \in N(A)$ such that $x = v + w$, and $R(A) \cap N(A) = \{0\}$)

|||ly $C(A) \oplus N(A^T) = \mathbb{R}^m$.



Ex3: Let $A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}$, $u = \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix}$, $v = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}$

i) Determine if u is in $N(A)$

ii) Could u be in $C(A)$?

iii) Determine if v is in $C(A)$

iv) Could v be in $N(A)$?

Soln: i) Consider

$$Au = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 3 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore u \notin N(A)$

ii) No, since $C(A) \leq \mathbb{R}^3$, but $u \in \mathbb{R}^4$

iii) v is in $C(A)$ if

$Ax = v$ is consistent. It is same as say v is LC of col. vectors of A.

Consider $[A, v] = \left[\begin{array}{cccc|c} 2 & 4 & -2 & 1 & 3 \\ -2 & -5 & 7 & 3 & -1 \\ 3 & 7 & -8 & 6 & 3 \end{array}\right]$

$$\sim \left[\begin{array}{cccc|c} 2 & 4 & -2 & 1 & 3 \\ 0 & -1 & 5 & 4 & 2 \\ 0 & 0 & 0 & 17 & 0 \end{array}\right]$$

$\Rightarrow \text{rank}([A, v]) = \text{rank}(A)$

$\therefore Ax = v$ is consistent. Thus $v \in C(A)$.

iv) No. $v \in \mathbb{R}^3$ and $N(A) \leq \mathbb{R}^4$.

Ex 4: Let A be a 4×5 matrix and let U be the rref of A.

If v_1, v_2, v_3, v_4, v_5 are col. vectors of A,

where $v_1 = \begin{bmatrix} 2 \\ 1 \\ -3 \\ -2 \end{bmatrix}$ and $v_2 = \begin{bmatrix} -1 \\ 2 \\ 3 \\ 1 \end{bmatrix}$

$$U = \begin{bmatrix} 1 & 0 & 2 & 0 & -1 \\ 0 & 1 & 3 & 0 & -2 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

a) find a basis for $N(A)$

b) given that x_0 is the form of $Ax=b$, where

$$b = \begin{bmatrix} 0 \\ 5 \\ 3 \\ 4 \end{bmatrix}, \quad x_0 = \begin{bmatrix} 3 \\ 2 \\ 0 \\ 2 \\ 0 \end{bmatrix}$$

i) find all solns to the system

ii) determine remaining col. vectors of A.

Soln: a) $AX = 0 \implies UX = 0$

$$\implies \begin{bmatrix} 1 & 0 & 2 & 0 & -1 \\ 0 & 1 & 3 & 0 & -2 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \uparrow$$

# of free variables $= 5 - 3 = 2$

Let $x_3 = k_1$, $x_5 = k_2$ be free variables

By backward sub.

$x_4 = -5x_5 = -5k_2$

$x_2 = -3x_3 + 2x_5 = -3k_1 + 2k_2$

$x_1 = -2x_3 + x_5 = -2k_1 + k_2$

$$\therefore N(A) = \left\{ k_1 \begin{bmatrix} -2 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} 1 \\ 2 \\ 0 \\ -5 \\ 1 \end{bmatrix} \;\middle|\; k_1, k_2 \in \mathbb{R} \right\}$$

b) i) Given $Ax_0 = b$

The soln set of A consists of all vectors of the form $x = x_0 + z$, where $z \in N(A)$.

$$\therefore \text{Solns } x = \left\{ \begin{bmatrix} 3 \\ 2 \\ 0 \\ 2 \\ 0 \end{bmatrix} + k_1 \begin{bmatrix} -2 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} 1 \\ 2 \\ 0 \\ -5 \\ 1 \end{bmatrix} \;\middle|\; k_1, k_2 \in \mathbb{R} \right\}$$

ii) $Ax_0 = b \Rightarrow A \begin{bmatrix} 3 \\ 2 \\ 0 \\ 2 \\ 0 \end{bmatrix} = b$

$\Rightarrow 3v_1 + 2v_2 + 2v_4 = b$

($\because v_1, v_2, v_3, v_4, v_5$ are col. vectors of A)

$\Rightarrow 2v_4 = -3v_1 - 2v_2 + b$

$$\Rightarrow 2v_4 = -3\begin{bmatrix} 2 \\ 1 \\ -3 \\ -2 \end{bmatrix} - 2\begin{bmatrix} -1 \\ 2 \\ 3 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 5 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} -4 \\ -2 \\ 6 \\ 8 \end{bmatrix}$$

$$\Rightarrow v_4 = \begin{bmatrix} -2 \\ -1 \\ 3 \\ 4 \end{bmatrix}$$

To determine remaining cols of A, we note that the Col. vectors of A and v satisfy same dependency relations.

From v, $u_3 = 2u_1 + 3u_2$ & $u_5 = -u_1 - 2u_2 + 5u_4$

$$\therefore V_3 = 2V_1 + 3V_2 = \begin{bmatrix} 1 \\ 5 \\ 3 \\ -1 \end{bmatrix}$$

$$V_5 = -V_1 - 2V_2 + 5V_4 = \begin{bmatrix} -10 \\ -10 \\ 12 \\ 20 \end{bmatrix}$$

Remark: Let A be an $m \times n$ matrix:

i) A linear system $Ax = b$ is consistent iff $b \in C(A)$.

ii) A system $Ax = b$ has at most one soln for every $b \in \mathbb{R}^m$ iff Col. vectors of A are LI.

iii) A system $Ax = b$ is Consistent for every $b \in \mathbb{R}^m$ iff $C(A) = \mathbb{R}^m$.

Linear transformation: (LT)

Definition:

A mapping T from a vector space V into a vector space W, denoted by $T: V \to W$, is said to be a linear transformation if

i) $T(v_1 + v_2) = T(v_1) + T(v_2)$, for every $v_1, v_2 \in V$

ii) $T(\alpha v) = \alpha T(v)$, for every $\alpha \in \mathbb{R}$ and $v \in V$

or equivalently

$$T(\alpha v_1 + \beta v_2) = \alpha T(v_1) + \beta T(v_2), \text{ for every } \alpha, \beta \in \mathbb{R} \text{ and } v_1, v_2 \in V.$$

In general,

$$T(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n) = \alpha_1 T(v_1) + \alpha_2 T(v_2) + \cdots + \alpha_n T(v_n)$$

for any $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}$ and $v_1, v_2, \ldots, v_n \in V$

Notation:

A linear transformation from a vector space V into itself, $T: V \to V$, will be referred as a linear operator.

From the definition it is clear that a LT between two vector subspaces preserves the operations of vector addition and scalar multiplication. This ensure that the structure of the vector space is maintained during the transformation.

Ex: Let V be a vector space, then the identity operator I is defined by

$$I(v) = v \quad \text{for all } v \in V.$$

Clearly, I is a LT:

$$I(\alpha v_1 + \beta v_2) = \alpha v_1 + \beta v_2 = \alpha I(v_1) + \beta I(v_2)$$

Ex: Let $T : C[a,b] \to \mathbb{R}$ be a mapping defined by

$$T(f) = \int_a^b f(x)\, dx$$

Then T is a LT.

Soln: For any $f, g \in C[a,b]$ and $\alpha, \beta \in \mathbb{R}$

$$T(\alpha f + \beta g) = \int_a^b (\alpha f(x) + \beta g(x))\, dx$$

$$= \alpha \int_a^b f(x)\, dx + \beta \int_a^b g(x)\, dx = \alpha T(f) + \beta T(g)$$

$\therefore$ T is a LT.

Ex: Let D be a map from $C'[a,b]$ into $C[a,b]$ defined by

$$D(f) = f' \quad (\text{the derivative of } f).$$

Then D is a LT.

Soln: For any $f, g \in C'[a,b]$ and $\alpha, \beta \in \mathbb{R}$

$$D(\alpha f + \beta g) = (\alpha f + \beta g)' = \alpha f' + \beta g'$$

$$= \alpha D(f) + \beta D(g)$$

$\therefore$ D is a LT.

Linear Operators on $\mathbb{R}^2$

Ex 1 : Let T be the operator defined by

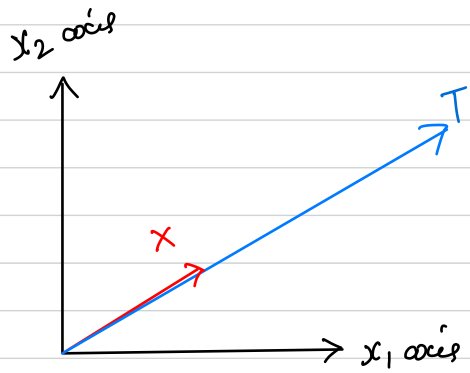
$$T(x) = 3x, \qquad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

for each $x \in \mathbb{R}^2$.

It is a linear operator because

i) $T(x+y) = 3(x+y) = 3x + 3y$
$= T(x) + T(y)$

ii) $T(\alpha x) = 3(\alpha x) = (3\alpha) x$
$= (\alpha 3) x$
$= \alpha (3x)$
$= \alpha T(x).$



In general, the linear operator $T(x) = \alpha x$ can be thought of as a stretching or shrinking by a factor of α.

Ex 2: Consider the mapping L defined by

$$T(x) = x_1 e_1$$

for each $x \in \mathbb{R}^2$. Thus, if $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, then
$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$.

It is a Linear operator because for all $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \in \mathbb{R}^2$ and for all scalars α and β

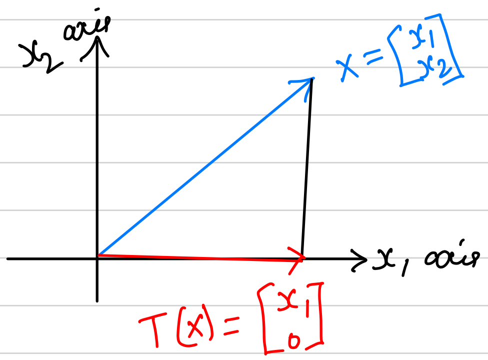
$$T(\alpha X + \beta Y) = T\left(\begin{bmatrix} \alpha x_1 + \beta y_1 \\ \alpha x_2 + \beta y_2 \end{bmatrix}\right) = \begin{bmatrix} \alpha x_1 + \beta y_1 \\ 0 \end{bmatrix}$$

$$= (\alpha x_1 + \beta y_1) e_1$$

$$= \alpha (x_1 e_1) + \beta (y_1 e_1)$$

$$= \alpha T(X) + \beta T(Y)$$

We can think $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$ as a projection onto the x_1-axis



We can think $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = x_1 e_1$ as a projection onto the x_1-axis.

Ex 3: Let T be the operator defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix}$$

S.T T is linear.

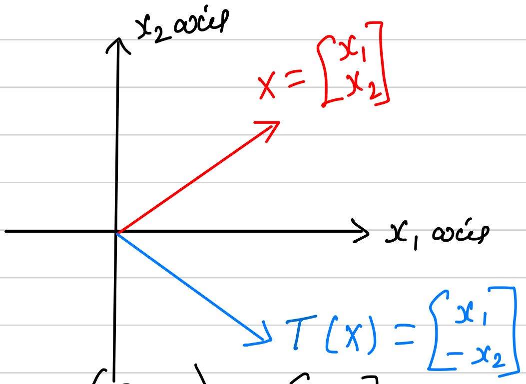
It is a Linear operator in $\mathbb{R}^2$. Since

$$T\left(\alpha \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \beta \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} \alpha x_1 + \beta y_1 \\ \alpha x_2 + \beta y_2 \end{bmatrix}\right) = \begin{bmatrix} \alpha x_1 + \beta y_1 \\ -(\alpha x_2 + \beta y_2) \end{bmatrix}$$

$$= \alpha \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix} + \beta \begin{bmatrix} y_1 \\ -y_2 \end{bmatrix}$$

$$= \alpha \, T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) + \beta \, T\left(\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right)$$

for all $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \in \mathbb{R}^2; \ \alpha, \beta \in \mathbb{R}$



$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix}$, reflects vectors about x_1-axis.

Ex 4: Let operator T be defined by

$$T(X) = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}; \quad X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

S.T it is linear.

Soln: Consider

$$T(\alpha X + \beta Y) = T\left(\alpha \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \beta \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right)$$

$$= T\left\{\begin{bmatrix} \alpha x_1 + \beta y_1 \\ \alpha x_2 + \beta y_2 \end{bmatrix}\right\}$$

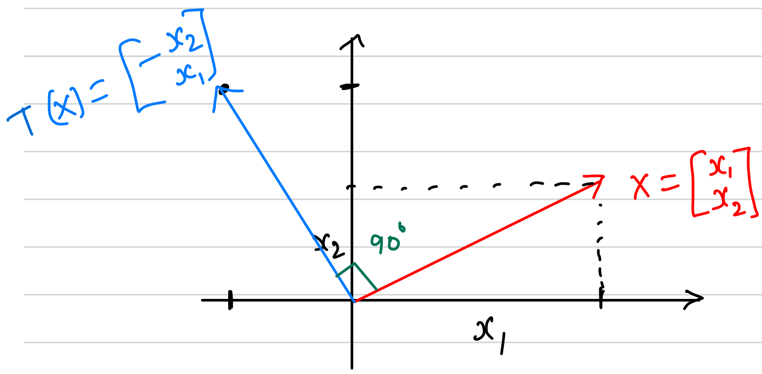
$$= \begin{bmatrix} -(\alpha x_2 + \beta y_2) \\ \alpha x_1 + \beta y_1 \end{bmatrix}$$

$$= \alpha \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix} + \beta \begin{bmatrix} -y_2 \\ y_1 \end{bmatrix}$$

$$= \alpha \, T(X) + \beta \, T(Y)$$

for any $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ in $\mathbb{R}^2$ and scalars α and β.

$\therefore$ T is linear.



The operator T has the effect of rotating each vector in $\mathbb{R}^2$ by 90° in the counterclockwise direction.

Ex5: Show that the translation $T(X) = X + C$, $(C \neq 0)$ is not a Linear transformation in $\mathbb{R}^n$.

Soln: For the zero vector $0 \in \mathbb{R}^n$

$$T(0) \neq 0.$$

$\therefore$ it is not a LT.

Ex6: Consider the mapping $T: \mathbb{R}^2 \to \mathbb{R}$ defined by

$$M\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = (x_1^2 + x_2^2)^{1/2}.$$

Is this a L.T?

Soln: No

Since $M\left(\alpha\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = M\left(\begin{bmatrix} \alpha x_1 \\ \alpha x_2 \end{bmatrix}\right) = (\alpha^2 x_1^2 + \alpha^2 x_2^2)^{1/2} = |\alpha| \, M\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)$

it follow that $M\left(\alpha\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) \neq \alpha M\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)$ whenever $\alpha < 0$ and $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Therefore, M is not a Linear transformatn.

Ex 7: Determine whether the foll are L.T from P_1 to P_2.

a) $T(p(x)) = x^2 + p(x)$

Soln: For any $p(x), q(x) \in P_1(\mathbb{R})$ and $\alpha, \beta \in \mathbb{R}$,

$$T(\alpha p(x) + \beta q(x)) = x^2 + \alpha p(x) + \beta q(x)$$
$$\neq \alpha T(p(x)) + \beta T(q(x))$$

or $T(0) = x^2 \neq 0$

$\therefore$ T is not linear

b) $T(p(x)) = p(x) + x\,p(x) + x^2 p'(x)$

Soln: For any $p(x), q(x) \in P_1(\mathbb{R})$ and $\alpha, \beta \in \mathbb{R}$,

$$T(\alpha p(x) + \beta q(x)) = \alpha p(x) + \beta q(x) + x(\alpha p(x) + \beta q(x)) + x^2(\alpha p'(x) + \beta q'(x))$$

$$= \alpha(p(x) + x\,p(x) + x^2 p'(x)) + \beta(q(x) + x\,q(x) + x^2 q'(x))$$

$$= \alpha T(p(x)) + \beta T(q(x)).$$

$\therefore$ T is linear.

Theorem :

Let V and W be vector spaces, if $T: V \to W$ is a LT, then

i) $T(0_V) = 0_W$, where 0_V and 0_W are zero vectors in V and W, respectively

ii) $T(-v) = -T(v)$ for all $v \in V$

pf: i) We have $T(\alpha \cdot v) = \alpha \cdot T(v)$ for all $v \in V$, $\alpha \in \mathbb{R}$

put $\alpha = 0$,

$$T(0 \cdot v) = 0 \cdot T(v)$$

$$\Rightarrow T(0_V) = 0_W$$

ii) Note that

$$0_W = T(0_V) = T(v + (-v)) = T(v) + T(-v)$$

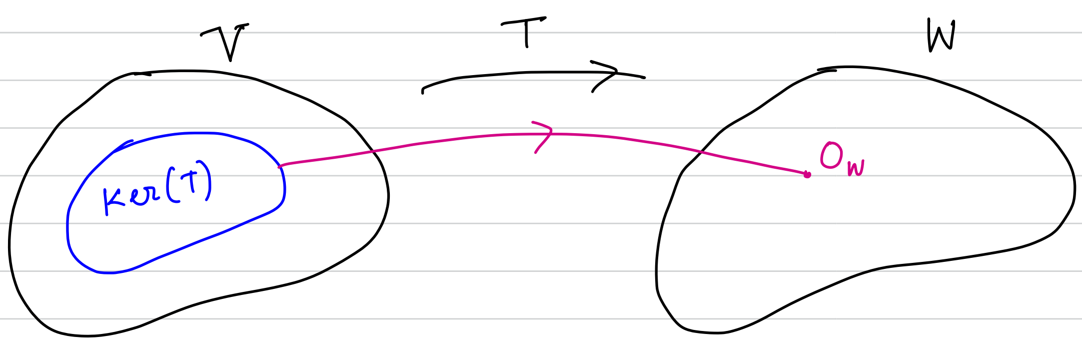
$\Rightarrow$ $T(-v)$ is additive inverse of $T(v)$. That is $T(-v) = -T(v)$.

Image and kernel

Let $T: V \to W$ be a LT

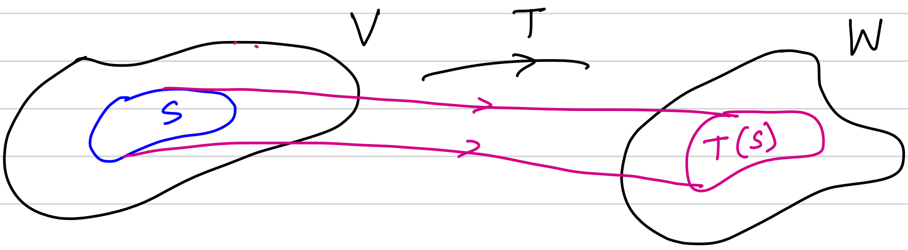
The kernel of T, denoted by $\ker(T)$, is defined as

$$\ker(T) = \{ v \in V \mid T(v) = 0_W, \text{ zero vector in } W \}$$



The image of a subspace S of V under T is

$$T(S) = \{ w \in W \mid w = T(v) \text{ for some } v \in S \}$$



Thm: If $T: V \to W$ is a L.T and S is a subspace of V, then

a) $\ker(T)$ is a subspace of V

b) $T(S)$ is a subspace of W.

pf: a) $\ker(T)$ is a non-empty set because $0_V \in \ker(T)$
$(\because T(0_V) = 0_W)$

Let $v_1, v_2 \in \ker(T)$. Then

i) $T(v_1+v_2) = T(v_1) + T(v_2) = 0_W + 0_W = 0_W$

$\Rightarrow v_1 + v_2 \in \ker(T)$

ii) $T(\alpha v_1) = \alpha T(v_1) = \alpha \cdot 0_W = 0_W$.

$\Rightarrow \alpha v_1 \in \ker(T)$

Thus, $\ker(T)$ is a subspace of V.

pf b) $T(S)$ is nonempty, since $0_W = T(0_V) \in T(S)$.

If $w_1, w_2 \in T(S)$ then there exist $v_1, v_2 \in S$ such that $T(v_1) = w_1$ and $T(v_2) = w_2$. Thus

i) $w_1 + w_2 = T(v_1) + T(v_2) = T(v_1+v_2) \in T(S)$

ii) $\alpha w_1 = \alpha T(v_1) = T(\alpha v_1) \in T(S)$

Hence $T(S)$ is a subspace of W.

Ex 8: Let $D: P_3 \to P_3$ be the L.T defined by

polys of deg ≤ 3

$D(p(x)) = p'(x)$ for every $p(x) \in P_3$

Find kernel and image of D.

Soln: WKT Kernel of D is set of all polynomials of degree 0. Thus $\ker(D) = P_0$.

Derivative of any polynomial $p(x)$ is a polynomial with one degree less than $p(x)$.

$\therefore$ Image of $P_3 = P_2$

Ex 9: Find Image and kernel of the LT

$T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$.

Soln: Image of $\mathbb{R}^2 = \left\{ \begin{bmatrix} x \\ 0 \end{bmatrix} \middle| x \in \mathbb{R} \right\}$

$= \text{span}(e_1)$

Kernel of $\mathbb{R}^2 = \left\{ \begin{bmatrix} 0 \\ y \end{bmatrix} \middle| y \in \mathbb{R} \right\}$

$= \text{span}(e_2)$

Linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$.

Ex10: Determine whether the following are LT from $\mathbb{R}^3$ into $\mathbb{R}^2$.

i) $$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 1+x_1 \\ x_2 \end{bmatrix}$$

Soln: For any $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ in $\mathbb{R}^3$ and $\alpha, \beta \in \mathbb{R}$

$$T\left(\alpha \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \beta \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}\right) = T\left(\begin{bmatrix} \alpha x_1 + \beta y_1 \\ \alpha x_2 + \beta y_2 \\ \alpha x_3 + \beta y_3 \end{bmatrix}\right) = \begin{bmatrix} 1 + \alpha x_1 + \beta y_1 \\ \alpha x_2 + \beta y_2 \end{bmatrix}$$

$$\neq \alpha\, T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) + \beta\, T\left(\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}\right)$$

or

$$T\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\therefore$ T is not linear.

Ex 11: The mapping T from $\mathbb{R}^2$ to $\mathbb{R}^3$ is defined by

$$T(X) = \begin{bmatrix} x_2 \\ x_1 \\ x_1 + x_2 \end{bmatrix}, \quad \text{where } X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

S.T T is linear.

Soln: Consider

$$T(\alpha X + \beta Y) = T\left(\begin{bmatrix} \alpha x_1 + \beta y_1 \\ \alpha x_2 + \beta y_2 \end{bmatrix}\right) = \begin{bmatrix} \alpha x_2 + \beta y_2 \\ \alpha x_1 + \beta y_1 \\ \alpha x_1 + \beta y_1 + \alpha x_2 + \beta y_2 \end{bmatrix}$$

$$= \alpha \begin{bmatrix} x_2 \\ x_1 \\ x_1+x_2 \end{bmatrix} + \beta \begin{bmatrix} y_2 \\ y_1 \\ y_1+y_2 \end{bmatrix}$$

$$= \alpha T(X) + T(Y)$$

for any $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \in \mathbb{R}^2$ and for any $\alpha, \beta \in \mathbb{R}$.

$\therefore$ T is linear.

Note: If we define a matrix A by

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix},$$

then for any $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$AX = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1 \\ x_1+x_2 \end{bmatrix} = T(X)$$

In general, if A is any $m \times n$ matrix, we can defined a linear transformation T_A from $\mathbb{R}^n$ to $\mathbb{R}^m$ by

$$T_A(X) = AX, \quad \text{for each } X \in \mathbb{R}^n$$

This transformation T_A is linear, since

$$\begin{aligned} T_A(\alpha X + \beta Y) &= A(\alpha X + \beta Y) \\ &= \alpha AX + \beta AY \\ &= \alpha T_A(X) + \beta T_A Y. \end{aligned}$$

Also, for any linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$, we can defined an $m \times n$ matrix.

Construction of a matrix representing a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$.

Let T be a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$, there is a matrix of order $m \times n$ such that

$$T(x) = Ax$$

where,
$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m \times n}$$

for any $x \in \mathbb{R}^n$. And column vectors of A are

$$\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} = T(e_1), \quad \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} = T(e_2), \quad \ldots, \quad \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = T(e_n)$$

pf : If $\quad x = x_1 e_1 + x_2 e_2 + x_3 e_3 + \cdots + x_n e_n$

is an arbitrary vector in $\mathbb{R}^n$,

$$T(x) = x_1 T(e_1) + x_2 T(e_2) + \cdots + x_n T(e_n)$$
$$= \begin{bmatrix} T(e_1) & T(e_2) & \cdots & T(e_n) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

if
$$T(e_i) = \begin{bmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{mi} \end{bmatrix},$$

then $T(x) = Ax$.

Ex 12 : For the LT $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 3x+y \\ 2x+2y \end{bmatrix},$$

find a matrix A such that $T(x) = Ax$ for every $x \in \mathbb{R}^2$.

Soln : Given $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 3x+y \\ 2x+2y \end{bmatrix}$

Std basis of $\mathbb{R}^2$ is $\{e_1, e_2\}$ and

$$T(e_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

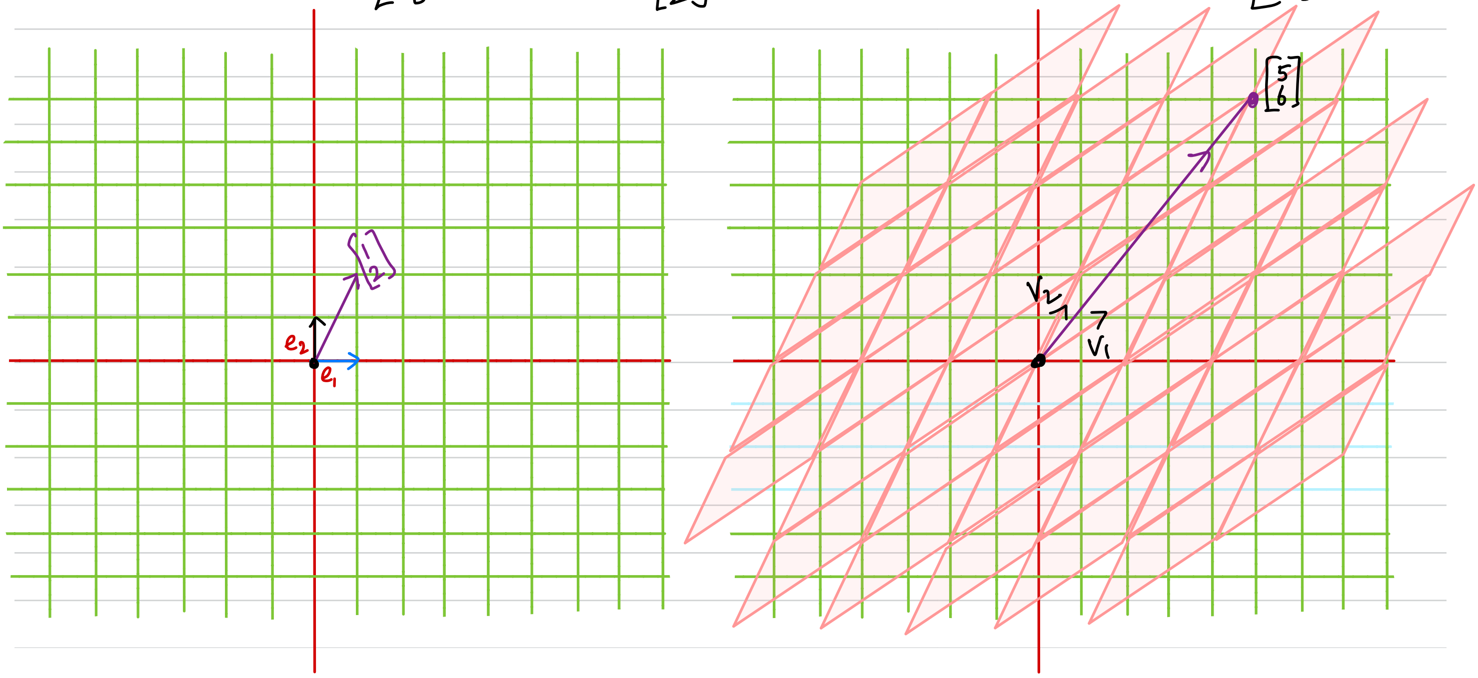
$$T(e_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$\therefore$ Required matrix $A = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}$

From the below picture, it is easy to say that matrix can be interpreted as transformation of space.

Consider the matrix $A = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}$

it maps $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to $v_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ to $v_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.



Let $x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ be a vector in $\mathbb{R}^2$. Then

$$Ax = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1 \begin{bmatrix} 3 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

EX 13: Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be a mapping defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 - x_2 - x_3 \\ 2x_2 - x_1 - x_3 \\ 2x_3 - x_1 - x_2 \end{bmatrix}$$

S.T T is linear and find its matrix representation.

Soln: For any $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \in \mathbb{R}^3$ and $\alpha, \beta \in \mathbb{R}$

$$T\left(\alpha \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \beta \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}\right) = T\left(\begin{bmatrix} \alpha x_1 + \beta y_1 \\ \alpha x_2 + \beta y_2 \\ \alpha x_3 + \beta y_3 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 2(\alpha x_1 + \beta y_1) - (\alpha x_2 + \beta y_2) - (\alpha x_3 + \beta y_3) \\ 2(\alpha x_2 + \beta y_2) - (\alpha x_1 + \beta y_1) - (\alpha x_3 + \beta y_3) \\ 2(\alpha x_3 + \beta y_3) - (\alpha x_1 + \beta y_1) - (\alpha x_2 + \beta y_2) \end{bmatrix}$$

$$= \alpha \begin{bmatrix} 2x_1 - x_2 - x_3 \\ 2x_2 - x_1 - x_3 \\ 2x_3 - x_1 - x_2 \end{bmatrix} + \beta \begin{bmatrix} 2y_1 - y_2 - y_3 \\ 2y_2 - y_1 - y_3 \\ 2y_3 - y_1 - y_2 \end{bmatrix}$$

$$= \alpha\, T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) + \beta\, T\left(\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}\right)$$

$\Rightarrow$ T is linear.

Matrix representation, $A = \begin{bmatrix} T(e_1) & T(e_2) & T(e_3) \end{bmatrix}$

where $T(e_1) = T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}2\\-1\\-1\end{bmatrix}$

$T(e_2) = T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}-1\\2\\-1\end{bmatrix}$

$T(e_3) = T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}-1\\-1\\2\end{bmatrix}$

$$\therefore \quad A = \begin{bmatrix}2 & -1 & -1\\-1 & 2 & -1\\-1 & -1 & 2\end{bmatrix}$$

or

$$T\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right) = \begin{bmatrix}2x_1 - x_2 - x_3\\2x_2 - x_1 - x_3\\2x_3 - x_1 - x_2\end{bmatrix}$$

$$= x_1\begin{bmatrix}2\\-1\\-1\end{bmatrix} + x_2\begin{bmatrix}-1\\2\\-1\end{bmatrix} + x_3\begin{bmatrix}-1\\-1\\2\end{bmatrix}$$

$$\Rightarrow T\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right) = \begin{bmatrix}2 & -1 & -1\\-1 & 2 & -1\\-1 & -1 & 2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$$

$\Rightarrow T(X) = AX$ for every $X \in \mathbb{R}^3$

where $A = \begin{bmatrix}2 & -1 & -1\\-1 & 2 & -1\\-1 & -1 & 2\end{bmatrix}$

Note: Linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ can be entirely determined by how it transforms the basis vectors of $\mathbb{R}^n$. So to find the matrix of LT we see how the basis are transformed.

Ex 14: Find the L.T $T: \mathbb{R}^2 \to \mathbb{R}^3$ such that $T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}0\\1\\2\end{bmatrix}$ $T\left(\begin{bmatrix}-1\\1\end{bmatrix}\right) = \begin{bmatrix}2\\1\\0\end{bmatrix}$.

Soln: $T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = T\left(1.e_1 + 1.e_2\right) = 1.T(e_1) + 1.T(e_2)$

$$\Rightarrow \begin{bmatrix}0\\1\\2\end{bmatrix} = T(e_1) + T(e_2) \quad \text{——— (1)}$$

$T\left(\begin{bmatrix}-1\\1\end{bmatrix}\right) = T\left(-1.e_1 + 1.e_2\right) = -1\cdot T(e_1) + 1\cdot T(e_2)$

$$\Rightarrow \begin{bmatrix}2\\1\\0\end{bmatrix} = -T(e_1) + T(e_2) \quad \text{— (2)}$$

From (1) and (2)

$$2\,T(e_2) = \begin{bmatrix}2\\2\\2\end{bmatrix} \quad \text{or} \quad T(e_2) = \begin{bmatrix}1\\1\\1\end{bmatrix}$$

and $2\,T(e_1) = \begin{bmatrix}-2\\0\\2\end{bmatrix} = \begin{bmatrix}-1\\0\\1\end{bmatrix}$

$\therefore$ Matrix of transformation $A = \begin{bmatrix}-1 & 1\\0 & 1\\1 & 1\end{bmatrix}$

L.T $T\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = A\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}-x+y\\y\\x+y\end{bmatrix}$

Ex 15: Let $T: \mathbb{R}^2 \to \mathbb{R}^3$ be a Linear operator. If $T\left(\begin{bmatrix}1\\2\end{bmatrix}\right) = \begin{bmatrix}2\\3\end{bmatrix}$ and $T\left(\begin{bmatrix}1\\-1\end{bmatrix}\right) = \begin{bmatrix}5\\2\end{bmatrix}$, find the value of $T\left(\begin{bmatrix}7\\5\end{bmatrix}\right)$

Soln: $T\left(\begin{bmatrix}1\\2\end{bmatrix}\right) = T(1\cdot e_1 + 2\cdot e_2) = 1\cdot T(e_1) + 2\cdot T(e_2)$

$\Rightarrow \begin{bmatrix}2\\3\end{bmatrix} = T(e_1) + 2T(e_2)$ —— ①

and $T\left(\begin{bmatrix}1\\-1\end{bmatrix}\right) = T(1\cdot e_1 + (-1)\cdot e_2) = 1\cdot T(e_1) + (-1)\,T(e_2)$

$\Rightarrow \begin{bmatrix}5\\2\end{bmatrix} = T(e_1) - T(e_2)$ —— ②

From ① and ②

$3\,T(e_2) = \begin{bmatrix}-3\\1\end{bmatrix} \Rightarrow T(e_2) = \begin{bmatrix}-1\\1/3\end{bmatrix}$

$3\,T(e_1) = \begin{bmatrix}12\\7\end{bmatrix} \Rightarrow T(e_1) = \begin{bmatrix}4\\7/3\end{bmatrix}$

$\therefore$ Matrix of transformation is $\begin{bmatrix}4 & -1\\7/3 & 1/3\end{bmatrix}$

And $T\left(\begin{bmatrix}7\\5\end{bmatrix}\right) = \begin{bmatrix}4 & -1\\7/3 & 1/3\end{bmatrix}\begin{bmatrix}7\\5\end{bmatrix} = \begin{bmatrix}23\\18\end{bmatrix}$

or

Let $\begin{bmatrix}7\\5\end{bmatrix} = c_1\begin{bmatrix}1\\2\end{bmatrix} + c_2\begin{bmatrix}1\\-1\end{bmatrix}$

$\Rightarrow$ $c_1 + c_2 = 7$

$2c_1 - c_2 = 5$

$\Rightarrow 3c_1 = 12 \Rightarrow c_1 = 4$

$c_2 = 3$

$T\left(\begin{bmatrix}7\\5\end{bmatrix}\right) = c_1\,T\left(\begin{bmatrix}1\\2\end{bmatrix}\right) + c_2\,T\left(\begin{bmatrix}-1\\1\end{bmatrix}\right)$

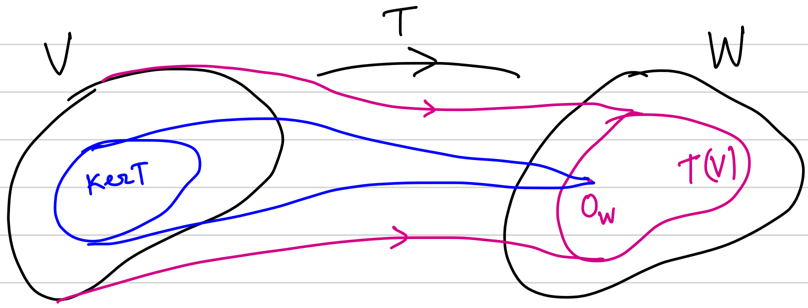
$= 4\begin{bmatrix}2\\3\end{bmatrix} + 3\begin{bmatrix}5\\2\end{bmatrix}$

$= \begin{bmatrix}23\\18\end{bmatrix}$

Rank – Nullity Theorm

Let $T: V \to W$ be a linear transformation and V be a finite dimensional vector space. Then

$$\dim(\underbrace{T(V)}_{\substack{\text{image of } V \\ \text{or Range of } T}}) + \dim(\underbrace{\ker(T)}_{\substack{\text{kernel} \\ \text{of } T}}) = \dim(\underbrace{V}_{\text{domain of } T})$$



Remark: Let $T: V \to W$ be LT and let A be its matrix representation i.e $T(v) = Av$.

Then Range(T) = Column space of A

$\ker(T)$ = Null space of A

$\dim(V)$ = No. of columns in A

$\therefore$ $\dim(\text{Range}(T)) = \text{rank}(A)$

$\dim(\ker(T)) = \text{Nullity}(A)$

Ex 16: Let $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+y \\ x-y \\ 2x+z \end{bmatrix}$ be a L.T. Find Range of T, and Kernel of T, hence verify rank–Nullity thm.

Soln: Clearly $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = x\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + y\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + z\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 2 & 0 & 1 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Find col. space of A and Null space of A.

Matrix representation of some typical Linear transformation in $\mathbb{R}^2$

Scaling transformation: (dilation)

It is a type of transformation that stretches or shrinks objects/vectors in a vector space.

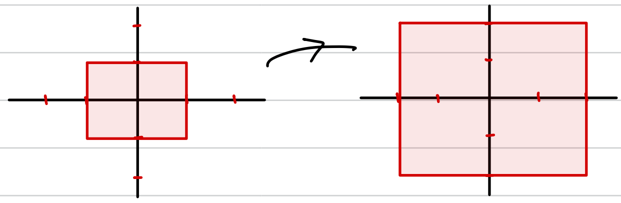
If α is a scaling factor, then

$$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \longrightarrow \begin{bmatrix} \alpha \\ 0 \end{bmatrix} \quad \text{and} \quad e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 \\ \alpha \end{bmatrix}$$

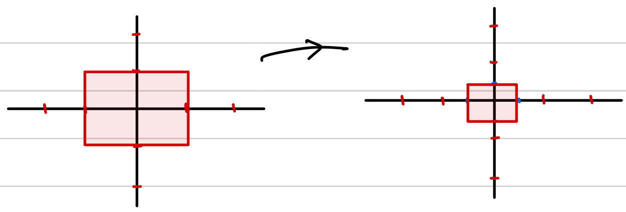
$\therefore$ The matrix $A = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix}$

Ex:

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$



$$A = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$$



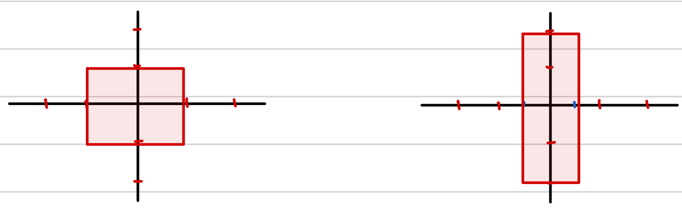
In general, the transformation

$$A = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} \quad \text{i.e} \quad e_1 \to \begin{bmatrix} \alpha \\ 0 \end{bmatrix}, \quad e_2 \to \begin{bmatrix} 0 \\ \beta \end{bmatrix}$$

allows non-uniform scalings in the e_1- and e_2-direction.

Ex:

$$A = \begin{bmatrix} 1/2 & 0 \\ 0 & 2 \end{bmatrix}$$



Reflection

It is a transformation that flips vectors across a specific line, producing a mirror image of a original vector.

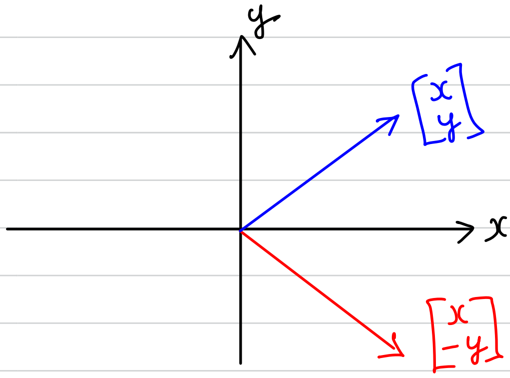
Reflection about the e_1-axis or about the line $y=0$

$$e_1 = \begin{bmatrix}1\\0\end{bmatrix} \to \begin{bmatrix}1\\0\end{bmatrix} \quad \text{and} \quad e_2 = \begin{bmatrix}0\\1\end{bmatrix} \to \begin{bmatrix}0\\-1\end{bmatrix}$$

$\therefore$ The matrix is $A = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$

If $\begin{bmatrix}x\\y\end{bmatrix}$ is any vector,

it transforms to $\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}x\\-y\end{bmatrix}$.



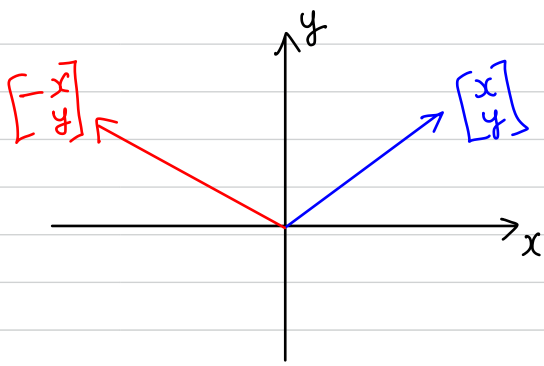
Reflection about e_2-axis or about the line $x=0$.

$$e_1 \to \begin{bmatrix}-1\\0\end{bmatrix}, \quad e_2 \to \begin{bmatrix}0\\1\end{bmatrix}.$$

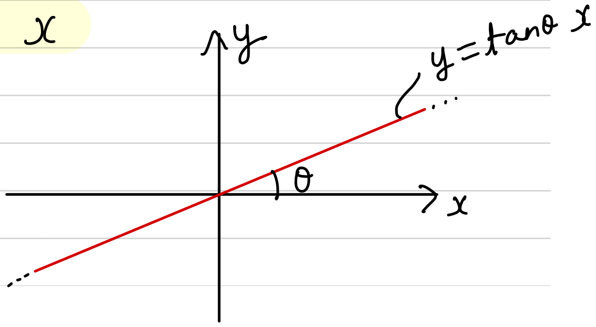
$\therefore$ the matrix $A = \begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}$

Vector $\begin{bmatrix}x\\y\end{bmatrix}$ transforms

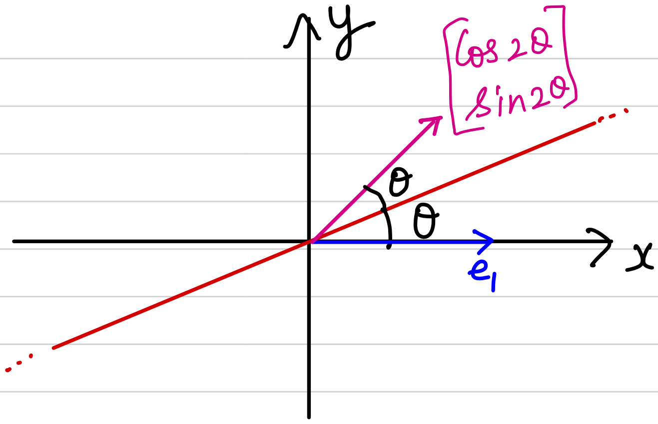
to $\begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}-x\\y\end{bmatrix}$



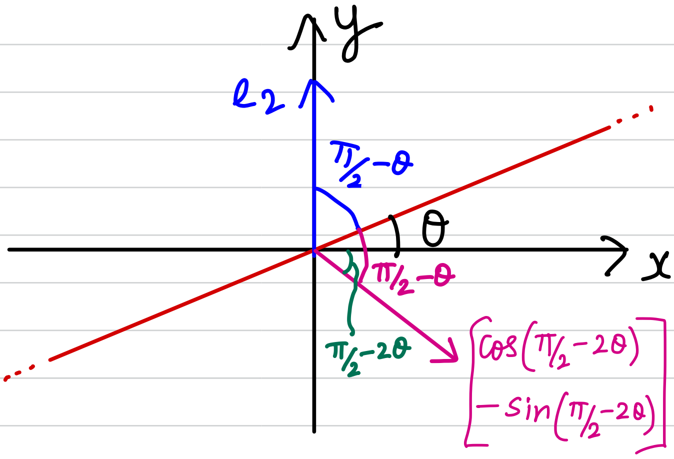
Reflection about the line $y = \tan\theta \; x$



$$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \longrightarrow \begin{bmatrix} \cos 2\theta \\ \sin 2\theta \end{bmatrix}$$



$$e_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \longrightarrow \begin{bmatrix} \sin 2\theta \\ -\cos 2\theta \end{bmatrix}$$



$\therefore$ The matrix $A = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$

In particular, the transformation of reflection about $y = x$ is

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$\because$ $e_1 \to e_2$ and $e_2 \to e_1$

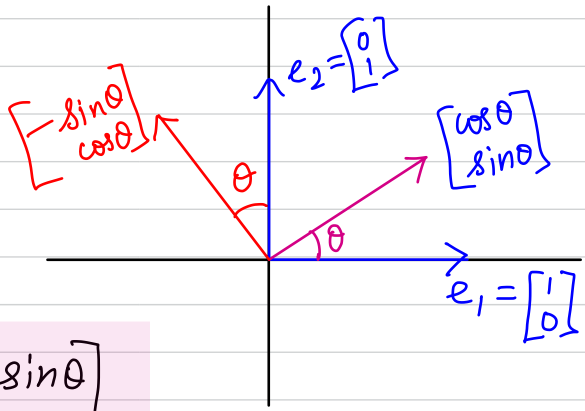
Rotation

It is a transformation that rotates a vector around the origin by a certain angle.

Transformation of rotation by an angle θ in a counterclockwise direction is given by

$$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \longrightarrow \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

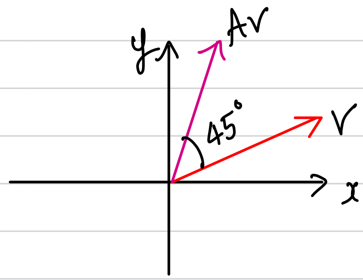
$$e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \longrightarrow \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$



$\therefore$ The matrix $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Ex: Rotation by an angle $45°$ in Counter Clockwise direction:

$$A = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

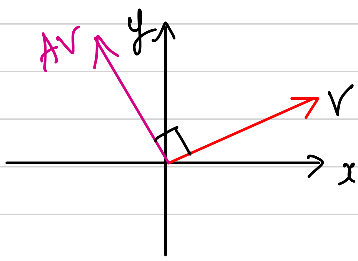


vector $v = \begin{bmatrix} x \\ y \end{bmatrix}$ will be transformed

to $$Av = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} (x-y)/\sqrt{2} \\ (x+y)/\sqrt{2} \end{bmatrix}$$

Ex: Rotation by an angle $90°$ in Counterclockwise direction

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$



vector $v = \begin{bmatrix} x \\ y \end{bmatrix}$ transforms to

$$Av = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}$$

Matrix of rotation by an angle θ in a Clockwise direction is

$$A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

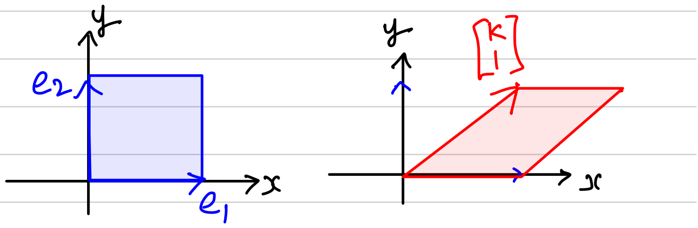
Shear transformation

It is a linear transformation that distorts the shape of an object shifting some of its points along one axis, while keeping other axis fixed. It preserves area.

Shear transformation along x-axis:

$e_1 = \begin{bmatrix}1\\0\end{bmatrix} \longrightarrow e_1$

$e_2 = \begin{bmatrix}0\\1\end{bmatrix} \longrightarrow \begin{bmatrix}k\\1\end{bmatrix}$



$\therefore$ The matrix $A = \begin{bmatrix}1 & k\\0 & 1\end{bmatrix}$

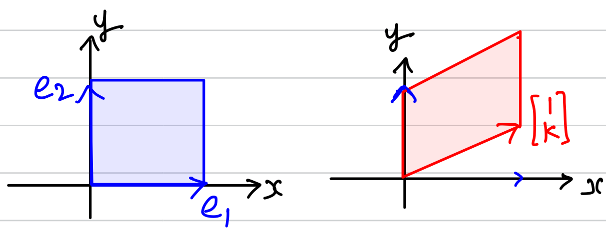
It maps rectangle to parallelogram

Any vector $\begin{bmatrix}x\\y\end{bmatrix}$ transforms to $\begin{bmatrix}1 & k\\0 & 1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}x+yk\\y\end{bmatrix}$

Shear transformation along y-axis

$e_1 \longrightarrow \begin{bmatrix}1\\k\end{bmatrix}$

$e_2 \longrightarrow e_2$



$\therefore$ The matrix $A = \begin{bmatrix}1 & 0\\k & 1\end{bmatrix}$

Any vector $v = \begin{bmatrix}x\\y\end{bmatrix}$ transforms to $Av = \begin{bmatrix}1 & 0\\k & 1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}x\\xk+y\end{bmatrix}$.

Projection

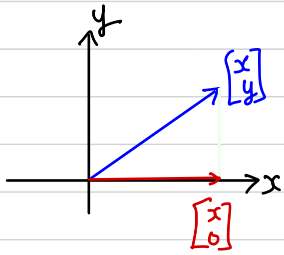
Orthogonal projection matrix onto the e_1-axis. Any vector is flatten onto the e_1-axis.

$e_1 = \begin{bmatrix}1\\0\end{bmatrix} \rightarrow e_1$ and $e_2 = \begin{bmatrix}0\\1\end{bmatrix} \rightarrow \begin{bmatrix}0\\0\end{bmatrix}$

$\therefore$ $\boxed{\text{The matrix } P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}$

if we project $v = \begin{bmatrix} x \\ y \end{bmatrix}$ onto the e_1-axis we get

$$Pv = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}.$$



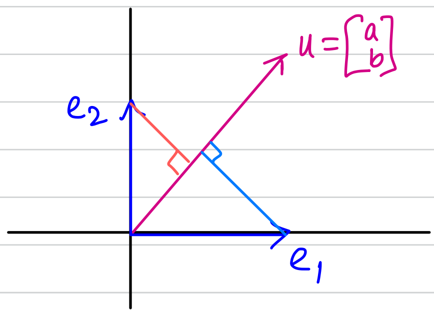
Ill^{ly}, orthogonal projection matrix onto the e_2-axis is

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Orthogonal projection matrix onto any vector $u = \begin{bmatrix} a \\ b \end{bmatrix}$

projection of e_1 onto $u = \dfrac{u^T e_1}{u^T u} u$

$$= \frac{a}{u^T u}\begin{bmatrix} a \\ b \end{bmatrix}$$



projection of e_2 onto $u = \dfrac{u^T e_2}{u^T u} u$

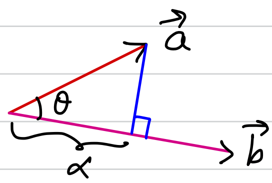
$$= \frac{b}{u^T u}\begin{bmatrix} a \\ b \end{bmatrix}$$

$\therefore$ The matrix $P = \dfrac{1}{u^T u}\begin{bmatrix} a^2 & ab \\ ab & b^2 \end{bmatrix}$

$$= \frac{1}{u^T u}\begin{bmatrix} a \\ b \end{bmatrix}\begin{bmatrix} a & b \end{bmatrix}$$

$$= \frac{u u^T}{u^T u}$$

projection of $\vec{a}$ onto $\vec{b}$, $p = \alpha \hat{b}$



$$\alpha = \cos\theta\, |\vec{a}|$$

$$= \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$$

$$\Rightarrow p = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\,\vec{b}$$

Ex: Determine the matrix that describes a reflection about x-axis, followed by rotation by a angle $\pi/2$ (Counter Clockwise) followed by a dilation of factor 3. Find the image of the point $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$ under this sequence of transformatn.

Soln: Reflection about x-axis

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Rotation by $\pi/2$ in counterclockwise

$$A_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

dilation by the factor 3

$$A_3 = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

Required transformation $\quad A = A_3 A_2 A_1$

$$= \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix}$$

Image of $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$ is $\begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 12 \end{bmatrix}$