Unit-2
Linear Algebra-2

Orthogonality

The scalar product in $\mathbb{R}^n$

Let x and y be any vectors in $\mathbb{R}^n$. If

$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and $y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$, then the scalar product of

x and y is

$$x^T y = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n .$$

Ex1 : If

$x = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$ and $y = \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}$ then

$x^T y = 12 - 6 + 2 = 8$

Euclidean length in $\mathbb{R}^n$.

Let $x \in \mathbb{R}^n$ be a vector. If $x = (x_1, x_2, \ldots x_n)^T$, then Euclidean length is

$$\|x\| = (x^T x)^{1/2} = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2}.$$

Defn: Let x and y be any vectors in $\mathbb{R}^n$. Then the distance between x and y is defined to be the number

$$\|x - y\| .$$

Ex2 : If $x = (3,4)^T$ and $y = (-1,7)^T$, then

i) length of x is $\|x\| = \sqrt{3^2 + 4^2} = 5$

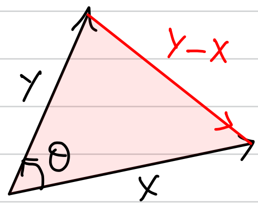
ii) distance between x and y is

$$\|x - y\| = \sqrt{(3-(-1))^2 + (4-7)^2} = 5$$

Theorem: If x and y are two vectors in $\mathbb{R}^2$ (or $\mathbb{R}^3$) and θ is the angle between them, then

$$x^T y = \|x\|\,\|y\| \cos\theta$$

Pf: By the law of cosines, we have



$$\|y-x\|^2 = \|x\|^2 + \|y\|^2 - 2\|x\|\,\|y\|\cos\theta$$

$$\Rightarrow \quad 2\|x\|\|y\|\cos\theta = \frac{1}{2}\left(\|x\|^2 + \|y\|^2 - \|y-x\|^2\right)$$

$$= \frac{1}{2}\left(x^T x + y^T y - (y-x)^T (y-x)\right)$$

$$= \frac{1}{2}\left(\cancel{x^T x} + \cancel{y^T y} - \cancel{y^T y} + y^T x + x^T y - \cancel{x^T x}\right)$$

$$= \frac{1}{2}\left(y^T x + x^T y\right)$$

$$= x^T y \qquad (\because x^T y = y^T x)$$

Note: If x and y are vectors in $\mathbb{R}^n$, then we can specify their directions by forming unit vectors

$$u_1 = \frac{x}{\|x\|}, \quad u_2 = \frac{y}{\|y\|}$$

If θ is the angle b/w x and y, then

$$\cos\theta = \frac{x^T y}{\|x\|\|y\|} = u_1^T u_2$$

Ex 3: Obtain unit vector in the direction of the vector

$$x = \begin{bmatrix} -2 \\ 5 \\ 4 \end{bmatrix}$$

Soln: Unit vector $u = \dfrac{x}{\|x\|}$,

where $\|x\| = \sqrt{4+25+16} = \sqrt{45}$

$$\therefore \quad u = \begin{bmatrix} -2/\sqrt{45} \\ 5/\sqrt{45} \\ 4/\sqrt{45} \end{bmatrix}$$

Ex4: Find the angle between the vectors v and w, where

$$v = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \quad w = \begin{bmatrix} 6 \\ 3 \\ 9 \end{bmatrix}$$

Soln: If θ is angle b/w v and w, then

$$\cos\theta = \frac{v^T w}{\|v\|\|w\|}$$

where $v^T w = 12 + 3 + 27 = 42$

$$\|v\| = \sqrt{4+1+9} = \sqrt{14}$$

$$\|w\| = \sqrt{36+9+81} = \sqrt{126} = 3\sqrt{14}$$

$$\begin{array}{r|l} 2 & 126 \\ 3 & 63 \\ 3 & 21 \\ & 7 \end{array}$$

$$\therefore \quad \cos\theta = \frac{42}{3\sqrt{14}\sqrt{14}} = 1$$

$$\therefore \quad \theta = 0$$

As you can see $w = 3v$, both are in same direction. Thus angle between them is 0.

Corollary: [Cauchy – Schwarz inequality]

If x and y are vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$, then

$$|x^T y| \le \|x\| \|y\|$$

equality holds iff one of the vectors is 0 or one vector is a multiple of the other.

Pf: $\because$ $|\cos\theta| \le 1$, $\|x^T y\| \le \|x\| \|y\|$ (from previous thm)

Definition: The vectors x and y in $\mathbb{R}^2$ (or $\mathbb{R}^3$) are said to be orthogonal if $x^T y = 0$.

Ex 3: Vectors $x = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $y = \begin{bmatrix} -4 \\ 6 \end{bmatrix}$ are orthogonal in $\mathbb{R}^2$.

Since $x^T y = -12 + 12 = 0$

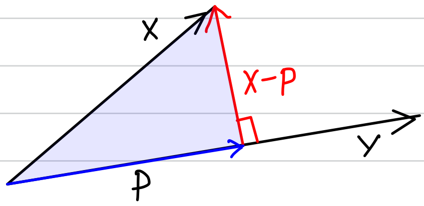
Ex 4: The vector 0 is orthogonal to every vector in $\mathbb{R}^2$.

Scalar and Vector projections

Let x and y be vectors in $\mathbb{R}^2$ (or $\mathbb{R}^3$).

Vector projection of x onto y is

$$p = c\,y, \qquad c \text{ is a scalar.}$$



From fig it is clear that,

y and $x - p$ are orthogonal.

$\therefore \quad y^T(x - p) = 0$

$\Rightarrow y^T(x - cy) = 0$

$\Rightarrow y^T x - c\,y^T y = 0$

or $c = \dfrac{y^T x}{y^T y}$

$\therefore$ Vector projection of x onto y is

$$\boxed{p = y\,\frac{y^T x}{y^T y}}$$

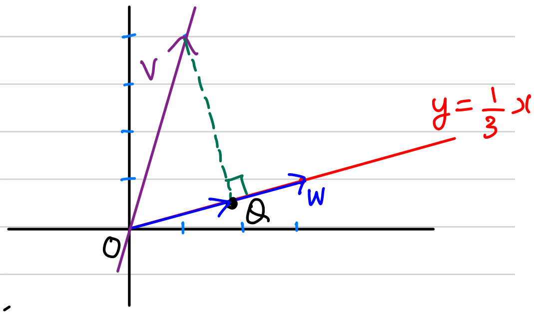
Scalar projection of x onto y is

$$\alpha = c\|y\| = \|y\| \cdot \frac{y^T x}{y^T y} \Rightarrow \boxed{\alpha = \frac{y^T x}{\|y\|}} \quad \left(\because\ y^T y = \|y\|^2\right)$$

Ex5: Determine the point Q on the line $y = \frac{1}{3}x$ that is closest to the point (1, 4).

Soln: The vector $w = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ is a vector in the direction of the line $y = \frac{1}{3}x$.

Let $v = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$



If Q is the desired point, then $\vec{OQ}$ is the vector projection of v onto w.

$$\therefore \quad \vec{OQ} = w \frac{v^T w}{w^T w} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \cdot \left(\frac{3+4}{9+1} \right) = \frac{7}{10} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$\Rightarrow \vec{OQ} = \begin{bmatrix} 0.21 \\ 0.7 \end{bmatrix}$$

Thus, $Q = (0.21, 0.7)$ is the closet point.

Ex6: For each of the following pairs of vectors x and y, Find the vector projection p of x onto y.

a) $x = \begin{bmatrix} 3 \\ 5 \end{bmatrix}, \quad y = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

b) $x = \begin{bmatrix} 2 \\ -5 \\ 4 \end{bmatrix}, \quad y = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

Soln: a) Scalar projection of x onto y is

$$\alpha = \frac{x^T y}{\|y\|} = \frac{3+5}{\sqrt{1+1}} = \frac{8}{\sqrt{2}} = 4\sqrt{2}$$

vector proj. of x onto y is

$$p = y \frac{x^T y}{y^T y} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{8}{2} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

b) Scalar proj. of X onto Y is

$$\alpha = \frac{X^T Y}{\|Y\|} = \frac{2-10-4}{\sqrt{6}} = \frac{-12}{\sqrt{6}} = -2\sqrt{6}$$

Vector proj.

$$p = Y \frac{X^T Y}{Y^T Y} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \cdot \left(\frac{-12}{6}\right) = \begin{bmatrix} -2 \\ -4 \\ 2 \end{bmatrix}$$

Notation: Vector projection of X onto Y is denoted by $Proj_Y(X)$

Inner product space

It is a generalization of scalar products to other vector spaces.

Defn: Let V be a vector space. The inner product on V is a function, $\langle\ \rangle : V \times V \to \mathbb{R}$ satisfying the foll. conditions:

a) $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0$ iff $x = 0$

b) $\langle x, y\rangle = \langle y, x\rangle$ for all $x, y \in V$

c) $\langle \alpha x + \beta y, z\rangle = \alpha\langle x, z\rangle + \beta\langle y, z\rangle$ for all $x, y, z \in V$ and $\alpha, \beta \in \mathbb{R}$.

A vector space with inner product is called an inner product space.

Ex 1: The vector space $\mathbb{R}^n$

i) The standard inner product in $\mathbb{R}^n$ is the scalar product

$$\langle x, y\rangle = x^T y$$

ii) We could also define inner product by

$$\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i w_i ,$$ where vector w is a vector with positive entries, and w_i are referred as weights.

Ex2: The Vector space $\mathbb{R}^{m\times n}$

Let A and B in $\mathbb{R}^{m\times n}$, we define inner product by

$$\langle A, B\rangle = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}\, b_{ij} \qquad \text{——— (1)}$$

Let us verify it.

a) $\langle A, A\rangle = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}^2 \geq 0$ and $\langle A, A\rangle = 0$ iff $A = 0$

b) $\langle A, B\rangle = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}b_{ij} = \sum_{i=1}^{m}\sum_{j=1}^{n} b_{ij}a_{ij} = \langle B, A\rangle.$

c) $\langle \alpha A + \beta B, C\rangle = \sum_{i=1}^{m}\sum_{j=1}^{n} (\alpha a_{ij} + \beta b_{ij}) c_{ij}$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n} \alpha(a_{ij}c_{ij}) + \beta(b_{ij}c_{ij})$$

$$= \alpha\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}c_{ij} + \beta\sum_{i=1}^{m}\sum_{j=1}^{n} b_{ij}c_{ij}$$

$$= \alpha\langle A, C\rangle + \beta\langle B, C\rangle$$

Thus, ① is an inner product.

Ex 3: The Vector Space $C[a,b]$

We define inner product by

$$\boxed{\langle f, g\rangle = \int_a^b f(x)\, g(x)\, dx} \quad\text{——— ②}$$

Let's verify.

a) $\langle f, f\rangle = \int_a^b (f(x))^2\, dx \geq 0$ and $\langle f, f\rangle = 0$ iff $f \equiv 0$.

b) $\langle f, g\rangle = \int_a^b f(x)g(x)dx = \int_a^b g(x) f(x)\, dx$

$$= \langle g, f\rangle$$

c) $\langle \alpha f + \beta g, h\rangle = \int_a^b (\alpha f(x) + \beta g(x))\, h(x)\, dx$

$$= \alpha\int_a^b f(x)h(x)dx + \beta\int_a^b g(x)\, h(x)dx$$

$$= \alpha\langle f, h\rangle + \beta\langle g, h\rangle$$

Hence ② is an inner product.

If $w(x)$ is a positive continuous fn. on $[a,b]$, then

$$\langle f, g\rangle = \int_a^b f(x)\, g(x)\, w(x)\, dx$$

also defines an inner product on $C[a,b]$. The fn. $w(x)$ is called weight fn.

Ex 4: The Vector space P_n.

Let $x_1, x_2, \dots, x_n$ be distinct real numbers. For each pair of polynomials in P_n, we define an inner product by

$$\boxed{\langle p, q\rangle = \sum_{i=1}^{n} p(x_i)\, q(x_i)}$$

a) $\langle p, p\rangle = \sum_{i=1}^{n} (p(x_i))^2 \geq 0$ and $\langle p, p\rangle = 0$ iff

$x_1, x_2, \dots, x_n$ are roots of $p(x) = 0$, but $p(x)$ is a polynomial of degree $< n$, it must be zero polynomial.

b) $\langle p, q\rangle = \sum_{i=1}^{n} p(x_i)\, q(x_i) = \sum_{i=1}^{n} q(x_i)\, p(x_i)$

$$= \langle q, p\rangle$$

c) $\langle \alpha p + \beta q, r\rangle = \sum_{i=1}^{n} (\alpha\, p(x_i) + \beta\, q(x_i))\, r(x_i)$

$$= \alpha \sum_{i=1}^{n} p(x_i)\, r(x_i) + \beta \sum_{i=1}^{n} q(x_i)\, r(x_i)$$

$$= \alpha \langle p, r\rangle + \beta \langle q, r\rangle$$

Basic properties of inner product

1) If v is a vector in an inner product space V, the length or norm of v is given by

$$\|v\| = \sqrt{\langle v, v\rangle}$$

2) Two vectors u and v in an inner product space V is orthogonal if $\langle u, v\rangle = 0$.

3) Unit vector along a vector u is $\dfrac{u}{\|u\|}$.

Theorem [The Pythagorean law]:

If u and v are orthogonal vectors in an inner product space V, then

$$\|u+v\|^2 = \|u\|^2 + \|v\|^2$$

Pf:

$$\begin{aligned}
\|u+v\|^2 &= \langle u+v, u+v\rangle \\
&= \langle u+v, u\rangle + \langle u+v, v\rangle \\
&= \langle u, u\rangle + \langle v, u\rangle + \langle u, v\rangle + \langle v, v\rangle \\
&= \|u\|^2 + 2\langle u, v\rangle + \|v\|^2 \\
&= \|u\|^2 + \|v\|^2 \qquad (\because \langle u, v\rangle = 0)
\end{aligned}$$

Ex 1: Consider the vector space $C[-1,1]$ with inner product defined by

$$\langle f, g\rangle = \int_{-1}^{1} f(x)\, g(x)\, dx.$$

i) Show that vectors 1 and x are orthogonal.

ii) Find length of 1 and length of x.

Soln: i) We show that $\langle 1, x \rangle = 0$.

By defn. $\langle 1, x \rangle = \int_{-1}^{1} x \, dx = \left.\frac{x^2}{2}\right]_{-1}^{1}$

$= \frac{1}{2} - \frac{1}{2} = 0$

$\therefore$ 1 and x are orthogonal.

ii) $\|1\|^2 = \langle 1, 1 \rangle = \int_{-1}^{1} 1 \, dx = x\Big]_{-1}^{1} = 2$

$\therefore \|1\| = \sqrt{2}$

$\|x\|^2 = \langle x, x \rangle = \int_{-1}^{1} x^2 dx = \left.\frac{x^3}{3}\right]_{-1}^{1} = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$

$\therefore \|x\| = \sqrt{\frac{2}{3}}$

Since 1 and x are orthogonal, by pythogorean law

$\|1 + x\|^2 = \|1\|^2 + \|x\|^2$

$= 2 + \frac{2}{3} = \frac{8}{3}$

Ex2: For the vector space $C[-\pi, \pi]$ define inner product by

$\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) g(x) dx$, $\frac{1}{\pi}$ is weight fn.

i) S.T $\cos x$ and $\sin x$ are orthogonal.

ii) Find length of $\cos x$, $\sin x$ and $\cos x + \sin x$.

Soln: i) We S.T $\langle \cos x, \sin x \rangle = 0$.

By defn, $\langle \cos x, \sin x \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos x \sin x \, dx$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sin 2x \, dx$$

$$= 0 \qquad (\because \sin(-2x) = -\sin(2x))$$

ii) $\|\cos x\|^2 = \langle \cos x, \cos x \rangle$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} \cos^2 x \, dx \qquad \left(\cos^2 x = \frac{1+\cos 2x}{2} \right)$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} (1+\cos 2x) \, dx$$

$$= \frac{1}{2\pi} \left[x + \frac{\sin 2x}{2} \right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi} (\pi + \pi) = 1$$

$\therefore \|\cos x\| = 1$

$$\|\sin x\|^2 = \langle \sin x, \sin x \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin^2 x \, dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1-\cos 2x}{2} \, dx$$

$$= \frac{1}{\pi} \left[\frac{x}{2} - \frac{\sin 2x}{4} \right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi} \left[\frac{\pi}{2} + \frac{\pi}{2} \right] = 1$$

$\therefore \|\sin x\| = 1$

By pythagorean law :

$$\|\cos x + \sin x\|^2 = \|\cos x\|^2 + \|\sin x\|^2$$

$$= 1 + 1 = 2$$

$\therefore \|\cos x + \sin x\| = \sqrt{2}.$

Ex3: Let A and B vectors in $\mathbb{R}^{3\times 2}$, where

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 3 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 1 \\ 3 & 0 \\ -3 & 4 \end{bmatrix}.$$

Inner product is define by

$$\langle A, B\rangle = \sum_{i=1}^{3} \sum_{j=1}^{2} a_{ij} b_{ij}$$

i) Find $\langle A, B\rangle$, $\|A\|$ and $\|B\|$

Soln: But defn.

$$\langle A, B\rangle = 1\cdot -1 + 1\cdot 1 + 1\cdot 3 + 2\cdot 0 + 3\cdot -3 + 3\cdot 4$$

$$= -1 + 1 + 3 - 9 + 12$$

$$= 6$$

$$\|A\|^2 = \langle A, A\rangle = 1^2 + 1^2 + 1^2 + 2^2 + 3^2 + 3^2$$

$$= 25$$

$$\therefore \|A\| = 5$$

$$\|B\|^2 = \langle B, B\rangle = 1^2 + 1^2 + 3^2 + 0^2 + (-3)^2 + 4^2$$

$$= 36$$

$$\therefore \|B\| = 6.$$

Note: Norm defined above i.e

$$\|A\| = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}^2\right)^{1/2}$$ is called Frobenius norm

denoted by $\|\cdot\|_F$

Defn: If u and v are vectors in an inner product space V and $v \neq 0$, then

The scalar projection of u onto v is

$$\alpha = \frac{\langle u, v \rangle}{\|v\|} \quad \text{—— (a)}$$

and the vector projection of u onto v is

$$\mathrm{Proj}_v(u) = v \frac{\langle u, v \rangle}{\langle v, v \rangle} \quad \text{—— (b)}$$

From (a) and (b), $\quad \mathrm{Proj}_v(u) = \alpha \dfrac{v}{\|v\|}$

Obervations

If $v \neq 0$ and $p = \mathrm{Proj}_v(u)$, then

1) $u-p$ and p are orthogonal.

2) $u = p$ iff u is a scalar multiple of v.

Pf of 1) : Consider

$$\begin{aligned}
&\langle u-p, p \rangle \\
&= \langle u, p \rangle - \langle p, p \rangle \\
&= \left\langle u, \alpha \frac{v}{\|v\|} \right\rangle - \left\langle \alpha \frac{v}{\|v\|}, \alpha \frac{v}{\|v\|} \right\rangle \\
&= \frac{\alpha}{\|v\|} \langle u, v \rangle - \frac{\alpha^2}{\|v\|^2} \langle v, v \rangle \\
&= \alpha^2 - \alpha^2 \qquad \left(\because \langle v, v \rangle = \|v\|^2 \right) \\
&= 0
\end{aligned}$$

Pf 2) : If $u = \beta v$, then

$$\text{proj}_v(u) = v \frac{\langle \beta v, v \rangle}{\langle v, v \rangle} = \beta v = u$$

Conversely, if $u = \text{proj}_v(u)$, then

$$u = \alpha \frac{v}{\|v\|} = \beta v, \text{ where } \beta = \frac{v}{\|v\|}$$

Theorem: The Cauchy - Schwarz inequality

If u and v are any two vectors in an inner product space V, then

$$|\langle u, v \rangle| \leq \|u\| \|v\|$$

Equality hold iff u and v are linearly dependent.

Pf: If $p = \text{proj}_v(u)$, then $u-p$ and p are orthogonal.

$\therefore$ by Pythagorean law

$$\|u-p\|^2 + \|p\|^2 = \|u\|^2$$

$$\Rightarrow \|p\|^2 = \|u\|^2 - \|u-p\|^2$$

But $\|p\| = \dfrac{\langle u, v \rangle}{\|v\|}$

$$\Rightarrow \frac{(\langle u, v \rangle)^2}{\|v\|^2} = \|u\|^2 - \|u-p\|^2$$

$$\Rightarrow (\langle u, v \rangle)^2 = \|u\|^2 \|v\|^2 - \|u-p\|^2 \|v\|^2 \quad (*)$$

$$\leq \|u\|^2 \|v\|^2$$

Equality holds in $(*)$ iff $u = p$ or $v = 0$

or simplify u and v are dependent.

Orthonormal Sets

Defn: Non-zero vectors $v_1, v_2, \cdots, v_n$ in an inner product space V is said to be an orthogonal set if

$$\langle v_i, v_j \rangle = 0 \quad \text{whenever } i \neq j$$

Ex: The set $\{(1,1,1)^T, (2,1,-3)^T, (4,-5,1)^T\}$ is an orthogonal set in $\mathbb{R}^3$, since

$$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ -5 \\ 1 \end{bmatrix} = 0$$

$$\begin{bmatrix} 2, 1, -3 \end{bmatrix} \begin{bmatrix} 4 \\ -5 \\ 1 \end{bmatrix} = 0$$

Theorem: If $\{v_1, v_2, \cdots, v_n\}$ is an orthogonal set of nonzero vectors in an inner product space V, then $v_1, v_2, \cdots, v_n$ are linearly independent.

Pf: Let $v_1, v_2, \cdots, v_n$ be orthogonal set.

Let $$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0 \qquad \text{——— } (1)$$

By taking the inner product of v_j with both sides of (1), we see that $(1 \leq j \leq n)$

$$c_1 \langle v_j, v_1 \rangle + c_2 \langle v_j, v_2 \rangle + \cdots + c_n \langle v_j, v_n \rangle = 0$$

$$\Rightarrow \quad c_j \|v_j\|^2 = 0$$

and hence all the scalars $c_1, c_2, \cdots, c_n$ must be 0.

Defn: An orthonormal set of vectors is an orthogonal set of unit vectors.

Ex: Let v_1, v_2 and v_3 be orthogonal set in $\mathbb{R}^3$, where

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 4 \\ -5 \\ 1 \end{bmatrix}.$$

Form an orthonormal set.

Soln: let $u_1 = \dfrac{v_1}{\|v_1\|}$ | let $u_2 = \dfrac{v_2}{\|v_2\|}$ | let $u_3 = \dfrac{v_3}{\|v_3\|}$

$\|v_1\| = \sqrt{1^2+1^2+1^2} = \sqrt{3}$ | $\|v_2\| = \sqrt{4+1+9} = \sqrt{14}$ | $\|v_3\| = \sqrt{16+25+1} = \sqrt{42}$

$\therefore$ orthonormal set is

$$u_1 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}, \quad u_2 = \begin{bmatrix} 2/\sqrt{14} \\ 1/\sqrt{14} \\ -3/\sqrt{14} \end{bmatrix}, \quad u_3 = \begin{bmatrix} 4/\sqrt{42} \\ -5/\sqrt{42} \\ 1/\sqrt{42} \end{bmatrix}$$

Ex: Let $C[-\pi,\pi]$ be an inner product space, where inner product is defined by

$$\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, g(x)\, dx$$

S.T the set $\{1, \cos x, \cos 2x, \dots, \cos nx\}$ is an orthogonal set. and form its orthonormal set.

Soln: Consider

$$\langle 1, \cos kx \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos kx\, dx = 0$$

$$\langle \cos jx, \cos kx \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos jx \cdot \cos kx\, dx \qquad (j \neq k)$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1}{2}\left(\cos(j+k)x + \cos(j-k)x\right) dx$$

$$= \frac{1}{2\pi} \left[\frac{\sin(j+k)x}{j+k} + \frac{\sin(j-k)x}{j-k}\right]_{-\pi}^{\pi}$$

$$= 0$$

Thus, the set is orthogonal.

Consider $\|1\|^2 = \langle 1, 1\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} 1\, dx = 2 \implies \|1\| = \sqrt{2}$

and $\|\cos kx\|^2 = \langle \cos kx, \cos kx\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos^2 kx\, dx$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1+\cos 2x}{2}\, dx$$

$$= \frac{1}{2\pi}\left[x + \frac{\sin 2x}{2}\right]_{-\pi}^{\pi}$$

$$= 1$$

$\therefore$ The orthonormal set is

$$\left\{\frac{1}{\sqrt{2}}, \cos x, \cos 2x, \ldots, \cos nx\right\}$$

Orthonormal basis

A set $S = \{u_1, u_2, \ldots, u_n\}$ is said to be an orthonormal basis of an inner product space V if

i) $S = \{u_1, u_2, \ldots, u_n\}$ is orthonormal set.

ii) $\text{Span}(S) = V$

Theorem: Let $S=\{u_1,u_2,\dots,u_n\}$ be an orthonormal basis for an inner product space V. For any vector $v\in V$ Suppose $v=c_1u_1+c_2u_2+\dots+c_nu_n$. Then $\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} \langle v,u_1\rangle \\ \langle v,u_2\rangle \\ \vdots \\ \langle v,u_n\rangle \end{bmatrix}$

i.e coordinate vector of v wrt S is

$$[v]_S = \begin{bmatrix} \langle v,u_1\rangle \\ \langle v,u_2\rangle \\ \vdots \\ \langle v,u_n\rangle \end{bmatrix}.$$

pf: Let $v = c_1u_1+c_2u_2+\dots+c_nu_n = \sum_{i=1}^{n} c_i u_i$

By taking inner product with u_j on both side,

$$\langle v,u_j\rangle = \left\langle \sum_{i=1}^{n} c_i u_i,\ u_j \right\rangle$$

$$= \sum_{i=1}^{n} c_i \langle u_i,u_j\rangle = \sum_{i=1}^{n} c_i \delta_{ij} = c_j$$

Thus, $[v]_S = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} \langle v,u_1\rangle \\ \langle v,u_2\rangle \\ \vdots \\ \langle v,u_n\rangle \end{bmatrix}$

Corollary: Let $S=\{u_1,u_2,\dots,u_n\}$ be orthonormal basis for an inner product space V. If $u=\sum_{i=1}^{n} a_i u_i$ and $v=\sum_{i=1}^{n} b_i u_i$, then

i) $\langle u,v\rangle = \sum_{i=1}^{n} a_i b_i \quad \left(= \langle [u]_S, [v]_S\rangle\right)$

ii) $\|v\|^2 = \sum_{i=1}^{n} c_i^2 \quad \left(= \|[v]_S\|^2\right)$ [Parseval's formula]

Pf: Consider

i) $\langle u,v\rangle = \langle \sum_{i=1}^{n} a_i u_i , v\rangle = \sum_{i=1}^{n} a_i \langle u_i, v\rangle$

$= \sum_{i=1}^{n} a_i \langle v, u_i\rangle$

$= \sum_{i=1}^{n} a_i b_i \qquad (\because b_i = \langle v, u_i\rangle)$

ii) $\|v\|^2 = \langle v,v\rangle = \sum_{i=1}^{n} c_i^2$

Ex: Let $S=\{u_1,u_2,u_3\}$ be orthonormal basis for $\mathbb{R}^3$,

where $u_1 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$, $u_2 = \begin{bmatrix} 2/\sqrt{14} \\ 1/\sqrt{14} \\ -3/\sqrt{14} \end{bmatrix}$, $u_3 = \begin{bmatrix} 4/\sqrt{42} \\ -5/\sqrt{42} \\ 1/\sqrt{42} \end{bmatrix}$

Let $x = \begin{bmatrix} 2 \\ 4 \\ -7 \end{bmatrix}$ and $y = \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}$ be vectors in $\mathbb{R}^3$.

i) Determine coordinate vectors of x and y wrt S.

(That is, $[x]_S$ and $[y]_S$)

ii) Verify that $\langle x,y\rangle = \langle [x]_S, [y]_S\rangle$

iii) Verify that $\|x\| = \|[x]_S\|$.

Soln i) $[x]_S = \begin{bmatrix} \langle x,u_1\rangle \\ \langle x,u_2\rangle \\ \langle x,u_3\rangle \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{3}} + \frac{4}{\sqrt{3}} - \frac{7}{\sqrt{3}} \\ 4/\sqrt{14} + 4/\sqrt{14} + 21/\sqrt{14} \\ 8/\sqrt{42} - 20/\sqrt{42} - 7/\sqrt{42} \end{bmatrix} = \begin{bmatrix} -1/\sqrt{3} \\ 29/\sqrt{14} \\ -19/\sqrt{42} \end{bmatrix}$

$[y]_S = \begin{bmatrix} \langle y,u_1\rangle \\ \langle y,u_2\rangle \\ \langle y,u_3\rangle \end{bmatrix} = \begin{bmatrix} 2/\sqrt{3} \\ -11/\sqrt{14} \\ -1/\sqrt{42} \end{bmatrix}$

ii) $\langle x, y\rangle = -2+0-21 = -23$

$$\langle [x]_S, [y]_S\rangle = -\tfrac{2}{3} - \frac{319}{14} + \frac{19}{42} = \frac{-28-1157+19}{42}$$

$$= -23$$

iii) $\|x\| = \sqrt{4+16+49} = \sqrt{69}$

$$\|[x]_S\| = \sqrt{\frac{1}{3} + \frac{29^2}{14} + \frac{19^2}{42}} = \sqrt{\frac{14+3\times 29^2+19^2}{42}} = \sqrt{69}$$

Orthogonal Matrices

Defn: An $n\times n$ matrix Q is said to be orthogonal matrix if the column vectors of Q form an orthonormal set in $\mathbb{R}^n$.

Theorem: An $n\times n$ matrix is orthogonal iff $Q^TQ = I$.

pf: It follows from the defn.

This implies that $Q^T = Q^{-1}$.

Examples: 1) For a fixed θ

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \text{ is orthogonal.}$$

Because $Q^TQ = I$

2) permutation matrix,

$$Q = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \text{ is orthogonal.}$$

Since $Q^TQ = I$

3) Let $Q = \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 & -1/2 \\ 1/2 & 1/2 & -1/2 & -1/2 \\ 1/2 & -1/2 & -1/2 & 1/2 \end{bmatrix}$ is orthogonal too.

Properties of Orthogonal matrix:

If Q is an $n \times n$ orthogonal matrix, then

a) The column vectors of Q form an orthonormal basis for $\mathbb{R}^n$.

b) $Q^T Q = I$

c) $Q^T = Q^{-1}$

d) Inner product are preserved under multiplication by an orthogonal matrix.

i.e $\quad \langle x, y \rangle = \langle Qx, Qy \rangle$

Because, $\langle Qx, Qy \rangle = (Qx)^T Q y$

$= (x^T Q^T) Q y$

$= x^T (Q^T Q) y$

$= x^T I y \quad = x^T y = \langle x, y \rangle$

e) Multiplication by an orthogonal matrix preserves the length of vectors.

i.e, $\quad \|Qx\| = \|x\|$

Since from d) $\quad \langle Qx, Qx \rangle = \|Qx\|^2 = \langle x, x \rangle = \|x\|^2$.

Note: If Q is any $m \times n$ rectangular matrix such that column vectors are orthonormal vectors, then

$Q^T Q = I$, but $Q Q^T$ need not be I.

Ex:

Let $\quad Q = \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \\ 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}$.

Clearly column vectors of Q are orthonormal vectors.

$$Q^T Q = \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ -1/2 & 1/2 & -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \\ 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Theorem (Projection onto subspaces with orthonormal bases.)

Let V be a subspace of $\mathbb{R}^n$ and $B = \{u_1, u_2, u_3, \cdots, u_k\}$ be orthonormal basis of V.
Then projection of $x \in \mathbb{R}^n$ onto the subspace V is

$$\boxed{\begin{aligned} \mathrm{proj}_V(x) &= \mathrm{proj}_{u_1}(x) + \mathrm{proj}_{u_2}(x) + \cdots + \mathrm{proj}_{u_k}(x) \\ &= \langle u_1, x\rangle u_1 + \langle u_2, x\rangle u_2 + \cdots + \langle u_k, x\rangle u_k \end{aligned}}$$

Moreover, if

$$Q = \begin{bmatrix} u_1 & u_2 & \cdots & u_k \end{bmatrix}_{n\times k},$$

then $\boxed{\mathrm{Proj}_V(x) = QQ^T(x)}$

Where, QQ^T is called the projection matrix onto V with orthonormal basis $\{u_1, u_2, \cdots, u_k\}$.

Pf: Let $x \in \mathbb{R}^n$ be any vector. Then $x = v + w$, where $v \in V$ and $w \in V^\perp$. Thus, $v = \mathrm{proj}_V(x)$ and $w = \mathrm{Proj}_{V^\perp}(x)$.

Since B is orthonormal basis of V

$v = c_1u_1 + c_2u_2 + \cdots + c_ku_k$ — ① for some $c_i \in \mathbb{R}$

and $x = c_1u_1 + c_2u_2 + \cdots + c_ku_k + w$ — ②

$V^\perp = \{ w \in \mathbb{R}^n \mid \langle w, v\rangle = 0$ for every $v \in V \}$
Called orthogonal Complement of V.
$V^\perp$ is also subspace of $\mathbb{R}^n$

$$\Rightarrow \langle u_1, x\rangle = c_1 \cancelto{1}{\langle u_1, u_1\rangle} + c_2 \cancelto{0}{\langle u_1, u_2\rangle} + \cdots + c_k \cancelto{0}{\langle u_1, u_k\rangle} + \cancelto{0}{\langle u_1, w\rangle} = c_1$$

In general, $\langle u_i, x\rangle = c_i$, $i = 1, 2, \cdots, k$

Thus,

$$v = \mathrm{proj}_V(x) = \langle u_1, x\rangle u_1 + \langle u_2, x\rangle u_2 + \cdots + \langle u_k, x\rangle u_k \quad (\text{from } ①)$$

$$= \mathrm{Proj}_{u_1}(x) + \mathrm{Proj}_{u_2}(x) + \cdots + \mathrm{Proj}_{u_k}(x)$$

OR

We have

$$\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix} = \begin{bmatrix} \langle u_1, x \rangle \\ \langle u_2, x \rangle \\ \vdots \\ \langle u_k, x \rangle \end{bmatrix} \quad \text{—} \quad (3)$$

From (2),

$$x = \begin{bmatrix} u_1 & u_2 & \cdots & u_k \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix} + w$$

$$= Q \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix} + w$$

Taking Q^T on BS.

$$Q^T x = Q^T Q \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix} + Q^T w$$

(Since $Q^T Q = I$

$Q^T w = \begin{bmatrix} u_1^T w \\ u_2^T w \\ \vdots \\ u_k^T w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$)

$$\Rightarrow \quad Q^T x = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix} \quad \text{—} \quad (4)$$

$$\therefore \ \text{proj}_V(x) = \langle u_1, x \rangle u_1 + \langle u_2, x \rangle u_2 + \cdots + \langle u_k, x \rangle u_k$$

$$= \begin{bmatrix} u_1 & u_2 & \cdots & u_k \end{bmatrix} \begin{bmatrix} \langle u_1, x \rangle \\ \langle u_2, x \rangle \\ \vdots \\ \langle u_k, x \rangle \end{bmatrix} = (Q Q^T) x$$

(from (3) and (4))

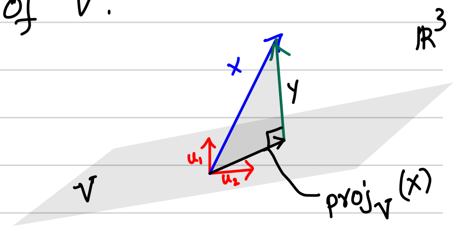
Ex: Let $V = \text{span}(u_1, u_2)$ be subspace in $\mathbb{R}^3$, let $x \in \mathbb{R}^3$

where $u_1 = \begin{bmatrix} 2/3 \\ 2/3 \\ 1/3 \end{bmatrix}$, $u_2 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix}$, $x = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$

Obtain i) Projection of x onto the plane V.

ii) the shortest distance between the point x and the plane V.

Soln: Clearly, $\{u_1, u_2\}$ is an orthonormal basis of V.



Let $$Q = \begin{bmatrix} 2/3 & 1/\sqrt{2} \\ 2/3 & -1/\sqrt{2} \\ 1/3 & 0 \end{bmatrix}, \quad Q^T = \begin{bmatrix} 2/3 & 2/3 & 1/3 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{bmatrix}$$

i) Projection of x onto V is

$$\begin{aligned} \text{Proj}_V x &= \text{proj}_{u_1} x + \text{proj}_{u_2} x \\ &= \langle u_1, x\rangle u_1 + \langle u_2, x\rangle u_2 \\ &= 2\begin{bmatrix} 2/3 \\ 2/3 \\ 1/3 \end{bmatrix} + \frac{3}{\sqrt{2}}\begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix} = \begin{bmatrix} 17/6 \\ -1/6 \\ 2/3 \end{bmatrix} \end{aligned}$$

$\langle u_1, x\rangle$
$= 4/3 - 2/3 + 4/3$
$= 2$

$\langle u_2, x\rangle$
$= \frac{2}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{3}{\sqrt{2}}$

Or

projection matrix onto V $= QQ^T = \begin{bmatrix} \frac{17}{18} & \frac{-1}{18} & 2/9 \\ -1/18 & \frac{17}{18} & 2/9 \\ 2/9 & 2/9 & 1/9 \end{bmatrix}$

$\therefore$ proj of x onto V is

$$\text{proj}_V(x) = QQ^T x = \begin{bmatrix} \frac{17}{18} & \frac{-1}{18} & 2/9 \\ -1/18 & \frac{17}{18} & 2/9 \\ 2/9 & 2/9 & 1/9 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 17/6 \\ -1/6 \\ 2/3 \end{bmatrix}$$

ii) $$y = x - \text{proj}_V(x) = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} - \begin{bmatrix} 51/18 \\ -1/6 \\ 2/3 \end{bmatrix} = \begin{bmatrix} -5/6 \\ -5/6 \\ 10/3 \end{bmatrix}$$

$\therefore$ Shortest distance is

$$\|y\| = \|x - \text{proj}_V(x)\| = \left\|\begin{bmatrix} -5/6 \\ -5/6 \\ 10/3 \end{bmatrix}\right\| = \sqrt{\frac{25}{36} + \frac{25}{36} + \frac{100}{9}} = \frac{5\sqrt{2}}{2}$$

The Gram-Schmidt Orthogonalization Process

We discuss a process of contructing orthonormal basis for an n-dimensional inner product space V.

In particular, we learn to transform an ordinary basis $\{w_1, w_2, w_3, \dots w_n\}$ into an orthonormal basis $\{u_1, u_2, u_3, \dots, u_n\}$

Construction is such that

$$\operatorname{span}(w_1, w_2, w_3, \dots, w_n) = \operatorname{span}(u_1, u_2, u_3, \dots u_n)$$

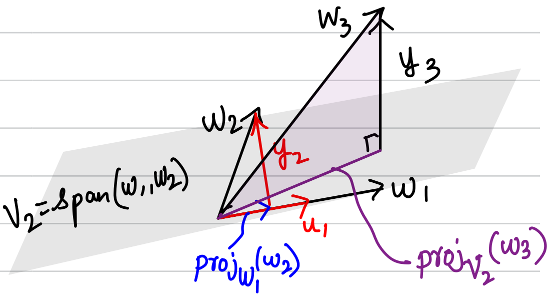
Let $V_1 = \operatorname{span}(w_1)$, $V_2 = \operatorname{span}(w_1, w_2)$.

In general

$$V_i = \operatorname{span}(w_1, w_2, \dots, w_i), \quad i = 1, 2, 3, \dots, n$$

and $V_n = V$.

Let $\boxed{u_1 = \frac{w_1}{\|w_1\|}}$. Then $V_1 = \operatorname{span}(u_1)$ and $V_2 = \operatorname{span}(u_1, w_2)$



From fig,

$$y_2 = w_2 - \operatorname{proj}_{V_1}(w_2)$$

$$\Rightarrow \boxed{y_2 = w_2 - \langle u_1, w_2 \rangle u_1, \quad u_2 = \frac{y_2}{\|y_2\|}}$$

$\therefore$ $V_2 = \operatorname{span}(u_1, u_2)$ and $V_3 = \operatorname{span}(u_1, u_2, w_3)$

Now,

$$y_3 = w_3 - \operatorname{proj}_{V_2}(w_3)$$

$$= w_3 - \operatorname{proj}_{u_1}(w_3) - \operatorname{proj}_{u_2}(w_3)$$

$$\Rightarrow \boxed{y_3 = w_3 - \langle u_1, w_3 \rangle u_1 - \langle u_2, w_3 \rangle u_2, \quad u_3 = \frac{y_3}{\|y_3\|}}$$

$\therefore$ $V_3 = \operatorname{span}(u_1, u_2, u_3)$

In general,

$$y_i = w_i - \text{proj}_{V_i}(w_i)$$

$$= w_i - \text{proj}_{u_1}(w_i) - \text{proj}_{u_2}(w_i) - \cdots - \text{proj}_{u_{i-1}}(w_i)$$

$$\Rightarrow \quad y_i = w_i - \langle u_1, w_i \rangle u_1 - \langle u_2, w_i \rangle u_2 - \cdots - \langle u_{i-1}, w_i \rangle u_{i-1}$$

$$\text{and} \quad u_i = \frac{y_i}{\|y_i\|}, \quad i = 1, 2, \cdots, n$$

Ex1: Let

$$A = \begin{bmatrix} 1 & -1 & 4 \\ 1 & 4 & -2 \\ 1 & 4 & 2 \\ 1 & -1 & 0 \end{bmatrix}.$$

Find orthonormal basis for the column space of A.

Soln: Let

$$w_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad w_2 = \begin{bmatrix} -1 \\ 4 \\ 4 \\ -1 \end{bmatrix}, \quad w_3 = \begin{bmatrix} 4 \\ -2 \\ 2 \\ 0 \end{bmatrix}$$

w_1, w_2 and w_3 are LI, hence form a basis for 3-dimensional subspace of $\mathbb{R}^4$.

We use Gram-Schmidt process to construct orthonormal basis $\{u_1, u_2, u_3\}$

$$u_1 = \frac{w_1}{\|w_1\|} = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$$

Let $y_2 = w_2 - \langle u_1, w_2 \rangle u_1$, $\quad u_2 = \dfrac{y_2}{\|y_2\|}$

where $\langle u_1, w_2 \rangle = u_1^T w_2 = -\frac{1}{2} + \frac{4}{2} + \frac{4}{2} - \frac{1}{2} = 3$

$\therefore \quad y_2 = w_2 - 3u_1$

$$= \begin{bmatrix} -1 - 3/2 \\ 4 - 3/2 \\ 4 - 3/2 \\ -1 - 3/2 \end{bmatrix} = \begin{bmatrix} -5/2 \\ 5/2 \\ 5/2 \\ -5/2 \end{bmatrix}$$

$$\|y_2\| = \sqrt{\frac{25}{4} + \frac{25}{4} + \frac{25}{4} + \frac{25}{4}} = \sqrt{\frac{100}{4}} = 5$$

$$\therefore \quad u_2 = \frac{y_2}{\|y_2\|}$$

$$\Rightarrow \quad \boxed{u_2 = \begin{bmatrix} -1/2 \\ 1/2 \\ 1/2 \\ -1/2 \end{bmatrix}}$$

Let $\quad y_3 = w_3 - \langle u_1, w_3 \rangle u_1 - \langle u_2, w_3 \rangle u_2 \quad$ and $\quad u_3 = \dfrac{y_3}{\|y_3\|}$

where $\langle u_1, w_3 \rangle = u_1^T w_3 = \frac{4}{2} - \frac{2}{2} + \frac{2}{2} + 0 = 2$

$$\langle u_2, w_3 \rangle = u_2^T w_3 = -4/2 - 2/2 + 2/2 + 0 = -2$$

$$\therefore \quad y_3 = w_3 - 2u_1 + 2u_2$$

$$= \begin{bmatrix} 4 \\ -2 \\ 2 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 2 \\ -2 \end{bmatrix}$$

$$\|y_3\| = \sqrt{4+4+4+4} = \sqrt{16} = 4$$

$$\therefore \quad u_3 = \frac{y_3}{\|y_3\|} \quad \Rightarrow \quad \boxed{u_3 = \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix}}$$

Thus orthonormal basis is $\quad u_1 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}, \quad u_2 = \begin{bmatrix} -1/2 \\ 1/2 \\ 1/2 \\ -1/2 \end{bmatrix}, \quad u_3 = \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix}$

Gram - Schmidt QR factorization

If A is an $m \times n$ matrix of rank n, then A can be factored into QR,

where Q is $m \times n$ matrix with orthonormal column vectors

R is $n \times n$ upper triangular matrix (diagonal entries are positive)

Pf: Let

$$A = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$$

Given that $\{a_1, a_2, \ldots, a_n\}$ are L.I ($\because$ rank$(A) = n$)

Let $\{u_1, u_2, \ldots, u_n\}$ be orthonormal basis for $C(A)$.

Let $\quad r_{11} = \|a_1\|, \quad r_{KK} = \left\| a_K - \sum_{i=1}^{K-1} \langle a_K, u_i \rangle u_i \right\|, \quad K = 2, 3, \ldots, n.$

Then by Gram - Schmidt process,

$$u_1 = \frac{a_1}{r_{11}}$$

$$\Rightarrow \quad a_1 = r_{11} \, u_1$$

$$u_2 = \frac{a_2 - \langle u_1, a_2 \rangle u_1}{r_{22}}$$

$$\Rightarrow \quad a_2 = \langle u_1, a_2 \rangle u_1 + r_{22} u_2$$

$$u_3 = \frac{a_3 - \langle u_1, a_3 \rangle u_1 - \langle u_2, a_3 \rangle u_2}{r_{33}}$$

$$\Rightarrow \quad a_3 = \langle u_1, a_3 \rangle u_1 + \langle u_2, a_3 \rangle u_2 + r_{33} u_3$$

$\vdots$

$$u_n = \frac{a_n - \langle u_1, a_n \rangle u_1 - \cdots - \langle u_{n-1}, a_n \rangle u_{n-1}}{r_{nn}}$$

$$\Rightarrow \quad a_n = \langle u_1, a_n \rangle u_1 + \langle u_2, a_n \rangle u_2 + \cdots + \langle u_{n-1}, a_n \rangle u_{n-1} + r_{nn} a_n$$

We have $r_{kk} = \langle u_k, a_k \rangle$

Consider

$$r_{kk} u_k = a_k - \langle u_1, a_k \rangle u_1 - \cdots - \langle u_{k-1}, a_k \rangle u_{k-1}$$

$$\Rightarrow \langle u_k, r_{kk} u_k \rangle = \langle u_k, a_k - \langle u_1, a_k \rangle u_1 - \cdots - \langle u_{k-1}, a_k \rangle u_{k-1} \rangle$$

$$\Rightarrow r_{kk} \langle u_k, u_k \rangle$$

$$= \langle u_k, a_k \rangle - \langle u_1, a_k \rangle \langle u_k, u_1 \rangle - \cdots - \langle u_{k-1}, a_k \rangle \langle u_k, u_{k-1} \rangle$$

But $\langle u_k, u_k \rangle = 1$ and $\langle u_k, u_j \rangle = 0$, $j = \{1, 2, \cdots, k-1\}$

$\therefore$ $r_{kk} = \langle u_k, a_k \rangle$

Therefore,

$$a_1 = \langle u_1, a_1 \rangle u_1$$

$$a_2 = \langle u_1, a_2 \rangle u_1 + \langle u_2, a_2 \rangle u_2$$

$$a_3 = \langle u_1, a_3 \rangle u_1 + \langle u_2, a_3 \rangle u_2 + \langle u_3, a_3 \rangle u_3$$

$$\vdots$$

$$a_n = \langle u_1, a_n \rangle u_1 + \langle u_2, a_n \rangle u_2 + \cdots + \langle u_n, a_n \rangle u_n$$

$$\Rightarrow \underbrace{\begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} \langle u_1, a_1 \rangle & \langle u_1, a_2 \rangle & \cdots & \langle u_1, a_n \rangle \\ 0 & \langle u_2, a_2 \rangle & \cdots & \langle u_2, a_n \rangle \\ 0 & 0 & \cdots & \langle u_3, a_n \rangle \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & \langle u_n, a_n \rangle \end{bmatrix}}_{R}$$

$$\Rightarrow A = QR.$$

or

From Gram-Schmidt process,

if $\{u_1, u_2, \dots, u_n\}$ are orthonormal set of $\{a_1, a_2, \dots, a_n\}$,

Then

$$a_1 \in \text{span}(u_1) \qquad (\because \text{span}(a_1) = \text{span}(u_1))$$

$$\Rightarrow \quad a_1 = \langle u_1, a_1 \rangle u_1$$

$$a_2 \in \text{span}(u_1, u_2) \qquad (\because \text{span}(a_1, a_2) = \text{span}(u_1, u_2))$$

$$\Rightarrow \quad a_2 = \langle u_1, a_2 \rangle u_1 + \langle u_2, a_2 \rangle u_2$$

$$a_3 \in \text{span}(u_1, u_2, u_3) \qquad (\because \text{span}(a_1, a_2, a_3) = \text{span}(u_1, u_2, u_3))$$

$$\Rightarrow \quad a_3 = \langle u_1, a_3 \rangle u_1 + \langle u_2, a_3 \rangle u_2 + \langle u_3, a_3 \rangle u_3$$

$\vdots$

$$a_n \in \text{span}(u_1, u_2, \dots, u_n)$$

$$\Rightarrow \quad a_n = \langle u_1, a_n \rangle u_1 + \langle u_2, a_n \rangle u_2 + \cdots + \langle u_n, a_n \rangle u_n$$

If we set

$$Q = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix}, \qquad R = \begin{bmatrix} \langle u_1, a_1 \rangle & \langle u_1, a_2 \rangle & \langle u_1, a_3 \rangle & \cdots & \langle u_1, a_n \rangle \\ 0 & \langle u_2, a_2 \rangle & \langle u_2, a_3 \rangle & \cdots & \langle u_2, a_n \rangle \\ 0 & 0 & \langle u_3, a_3 \rangle & \cdots & \langle u_3, a_n \rangle \\ 0 & 0 & 0 & \cdots & \langle u_4, a_n \rangle \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & & \langle u_n, a_n \rangle \end{bmatrix}$$

Then

$$A = QR$$

Ex2: Obtain QR factorization for

$$A = \begin{bmatrix} 1 & -1 & 4 \\ 1 & 4 & -2 \\ 1 & 4 & 2 \\ 1 & -1 & 0 \end{bmatrix}.$$

Soln: Let

$$a_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad a_2 = \begin{bmatrix} -1 \\ 4 \\ 4 \\ -1 \end{bmatrix}, \quad a_3 = \begin{bmatrix} 4 \\ -2 \\ 2 \\ 0 \end{bmatrix}$$

If u_1, u_2 and u_3 are orthonormal sets, then from previous example 1,

$$u_1 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}, \quad u_2 = \begin{bmatrix} -1/2 \\ 1/2 \\ 1/2 \\ -1/2 \end{bmatrix}, \quad u_3 = \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix}$$

$$\therefore \quad Q = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \\ 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & -1/2 \end{bmatrix}$$

$$R = \begin{bmatrix} \langle u_1, a_1 \rangle & \langle u_1, a_2 \rangle & \langle u_1, a_3 \rangle \\ 0 & \langle u_2, a_2 \rangle & \langle u_2, a_3 \rangle \\ 0 & 0 & \langle u_3, a_3 \rangle \end{bmatrix} = \begin{bmatrix} 2 & 3 & 2 \\ 0 & 5 & -2 \\ 0 & 0 & 4 \end{bmatrix}$$

Thus

$$\begin{bmatrix} 1 & -1 & 4 \\ 1 & 4 & -2 \\ 1 & 4 & 2 \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \\ 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & -1/2 \end{bmatrix} \cdot \begin{bmatrix} 2 & 3 & 2 \\ 0 & 5 & -2 \\ 0 & 0 & 4 \end{bmatrix}$$

is required QR factorization of A.

Ex3: Compute the Gram-Schmidt QR factorization of the matrix

$$A = \begin{bmatrix} 1 & -2 & -1 \\ 2 & 0 & 1 \\ 2 & -4 & 2 \\ 4 & 0 & 0 \end{bmatrix}$$

Soln: Column vectors of A are linearly independent. Since $\text{rank}(A) = 3$ (verify). Hence we can obtain QR factorization.

Let
$$a_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 4 \end{bmatrix}, \quad a_2 = \begin{bmatrix} -2 \\ 0 \\ -4 \\ 0 \end{bmatrix}, \quad a_3 = \begin{bmatrix} -1 \\ 1 \\ 2 \\ 0 \end{bmatrix}.$$

Let $\{u_1, u_2, u_3\}$ be corresponding orthonormal set.

Then

$$u_1 = \frac{a_1}{\|a_1\|}$$

where $\|a_1\| = (1^2 + 2^2 + 2^2 + 4^2)^{1/2} = 5$

$$\Rightarrow \quad u_1 = \begin{bmatrix} 1/5 \\ 2/5 \\ 2/5 \\ 4/5 \end{bmatrix}$$

$$y_2 = a_2 - \text{proj}_{u_1} a_2$$
$$= a_2 - \langle u_1, a_2 \rangle u_1$$
$$= \begin{bmatrix} -2 \\ 0 \\ -4 \\ 0 \end{bmatrix} - \underbrace{\left(-2/5 - 8/5\right)}_{-2} \begin{bmatrix} 1/5 \\ 2/5 \\ 2/5 \\ 4/5 \end{bmatrix} = \begin{bmatrix} -8/5 \\ 4/5 \\ -16/5 \\ 8/5 \end{bmatrix}$$

$$\|y_2\| = \left(\frac{64 + 16 + 256 + 64}{25}\right)^{1/2} = 4$$

$$\therefore \quad u_2 = \frac{y_2}{\|y_2\|} = \begin{bmatrix} -2/5 \\ 1/5 \\ -4/5 \\ 2/5 \end{bmatrix}$$

$$y_3 = a_3 - \langle u_1, a_3\rangle u_1 - \langle u_2, a_3\rangle u_2$$

$$= \begin{bmatrix} -1 \\ 1 \\ 2 \\ 0 \end{bmatrix} - \underbrace{\left(-\tfrac{1}{5} + \tfrac{2}{5} + \tfrac{4}{5}\right)}_{1} \begin{bmatrix} 1/5 \\ 2/5 \\ 2/5 \\ 4/5 \end{bmatrix} - \underbrace{\left(\tfrac{2}{5} + \tfrac{1}{5} - \tfrac{8}{5}\right)}_{-1} \begin{bmatrix} -2/5 \\ 1/5 \\ -4/5 \\ 2/5 \end{bmatrix}$$

$$= \begin{bmatrix} -8/5 \\ 4/5 \\ 4/5 \\ -2/5 \end{bmatrix}$$

$$\|y_3\| = \left(\frac{64+16+16+4}{25}\right)^{1/2} = 2$$

$$\therefore \quad u_3 = \frac{y_3}{\|y_3\|} = \begin{bmatrix} -4/5 \\ 2/5 \\ 2/5 \\ -1/5 \end{bmatrix}$$

Thus

$$A = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} \begin{bmatrix} \langle u_1, a_1\rangle & \langle u_1, a_2\rangle & \langle u_1, a_3\rangle \\ 0 & \langle u_2, a_2\rangle & \langle u_2, a_3\rangle \\ 0 & 0 & \langle u_3\, a_3\rangle \end{bmatrix}$$

$$= \begin{bmatrix} 1/5 & -2/5 & -4/5 \\ 2/5 & 1/5 & 2/5 \\ 2/5 & -4/5 & 2/5 \\ 4/5 & 2/5 & -1/5 \end{bmatrix} \begin{bmatrix} 5 & -2 & 1 \\ 0 & 4 & -1 \\ 0 & 0 & 2 \end{bmatrix} = Q \quad R$$

Recall : If A is an $m \times n$ matrix, then

i) Every vector of $R(A)$ is orthogonal to every vector of $N(A)$.

ii) $N(A) \cap R(A) = \{0\}$

iii) $\dim(R(A)) + \dim(N(A)) = \dim(\mathbb{R}^n)$

iv) Basis of $R(A)$ $\cup$ Basis of $N(A)$ = Basis of $\mathbb{R}^n$.

Ex 4: Let $a_1 = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$.

i) Extend this to an orthonormal basis of $\mathbb{R}^3$

ii) Suppose $\{u_1, u_2, u_3\}$ is orthonormal basis, where $u_1 = \frac{a_1}{\|a_1\|}$

- Find the projection matrix P_1 that projects vectors in $\mathbb{R}^3$ onto Span (u_2, u_3).

- Find the projection matrix P_2 that projects vectors in $\mathbb{R}^3$ on Span (u_1).

Soln: Let A be a matrix given by

$$A = [3 \quad 1 \quad -1]$$

Clearly, $R(A) = \text{span}(a_1) \Rightarrow$ Basis of $R(A) = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$

We know that $N(A)$ and $R(A)$ are orthogonal Complements.

i.e every vector in $N(A)$ is orthogonal to every vector in $R(A)$

and $N(A) \cap R(A) = \{0\}$

$\therefore$ Basis of $N(A)$ $\cup$ Basis of $R(A)$ $=$ Basis of $\mathbb{R}^3$.

Null space of A:

Consider $AX = 0$

$$\Rightarrow [3 \quad 1 \quad -1] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\Rightarrow 3x_1 + x_2 - x_3 = 0$$

Let $x_2 = k_1$, $x_3 = k_2$ be free variables

$$\Rightarrow x_1 = \frac{-k_1 + k_2}{3}$$

$$\therefore \quad X = \begin{bmatrix} -\frac{k_1+k_2}{3} \\ k_1 \\ k_2 \end{bmatrix}$$

$$N(A) = \left\{ k_1 \begin{bmatrix} -1/3 \\ 1 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} 1/3 \\ 0 \\ 1 \end{bmatrix}, \quad k_1, k_2 \in \mathbb{R} \right\}$$

$$\text{Basis of } N(A) = \left\{ \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \right\}$$

$$\text{Basis of } \mathbb{R}^3 = \left\{ \underbrace{\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}}_{a_1}, \underbrace{\begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}}_{a_2}, \underbrace{\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}}_{a_3} \right\}$$

Clearly a_1 is orthogonal to a_2 and a_3.

($\therefore$ we use GS process to orthonormal set of $\{a_2, a_3\}$)

Let $u_1 = \dfrac{a_1}{\|a_1\|} = \begin{bmatrix} 3/\sqrt{11} \\ 1/\sqrt{11} \\ -1/\sqrt{11} \end{bmatrix}$

$$u_2 = \frac{a_2}{\|a_2\|} = \begin{bmatrix} -1/\sqrt{10} \\ 3/\sqrt{10} \\ 0 \end{bmatrix}$$

$$y_3 = a_3 - \text{proj}_{u_2} a_3$$

$$= a_3 - \langle u_2, a_3 \rangle u_2$$

$$= \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} - \left(\frac{-1}{\sqrt{10}}\right) \begin{bmatrix} -1/\sqrt{10} \\ 3/\sqrt{10} \\ 0 \end{bmatrix} = \begin{bmatrix} 9/10 \\ 3/10 \\ 3 \end{bmatrix}$$

$$u_3 = \frac{y_3}{\|y_3\|}, \quad \|y_3\| = \sqrt{\frac{99}{10}}$$

$$u_3 = \begin{bmatrix} \frac{9}{\sqrt{99}\cdot\sqrt{10}} \\ 3/\sqrt{99}\sqrt{10} \\ 30/\sqrt{99}\sqrt{10} \end{bmatrix}$$

Orthonormal basis of $\mathbb{R}^3$.

$$\left\{ \begin{bmatrix} 3/\sqrt{11} \\ 1/\sqrt{11} \\ -1/\sqrt{11} \end{bmatrix}, \begin{bmatrix} -1/\sqrt{10} \\ 3/\sqrt{10} \\ 0 \end{bmatrix}, \begin{bmatrix} 9/\sqrt{99}\sqrt{10} \\ 3/\sqrt{99}\sqrt{10} \\ 30/\sqrt{99}\sqrt{10} \end{bmatrix} \right\}$$

ii) Let $Q = [u_2 \;\; u_3]$

Projection matrix onto span(u_2, u_3) is

$$QQ^T = \begin{bmatrix} -1/\sqrt{10} & 9/\sqrt{99}\sqrt{10} \\ 3/\sqrt{10} & 3/\sqrt{99}\sqrt{10} \\ 0 & 30/\sqrt{99}\sqrt{10} \end{bmatrix} \begin{bmatrix} -1/\sqrt{10} & 3/\sqrt{10} & 0 \\ 9/\sqrt{99}\sqrt{10} & \frac{3}{\sqrt{99}\sqrt{10}} & \frac{30}{\sqrt{99}\sqrt{10}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{11} & -3/11 & \frac{3}{11} \\ -3/11 & \frac{10}{11} & \frac{1}{11} \\ \frac{3}{11} & 1/11 & 10/11 \end{bmatrix}$$

iii) projection matrix onto span (u_1) is $u_1 u_1^T$

$$= \begin{bmatrix} 3/\sqrt{11} \\ 1/\sqrt{11} \\ -1/\sqrt{11} \end{bmatrix} \begin{bmatrix} 3/\sqrt{11} & \frac{1}{\sqrt{11}} & -1/\sqrt{11} \end{bmatrix} = \begin{bmatrix} \frac{9}{11} & \frac{3}{11} & \frac{-3}{11} \\ \frac{3}{11} & \frac{1}{11} & \frac{-1}{11} \\ \frac{-3}{11} & \frac{-1}{11} & \frac{1}{11} \end{bmatrix}$$

Ex5: Let A be an $m\times n$ matrix. If $Ax=b$ is consistent, the projection of b onto $C(A)$ is _______.

Ans : b itself

Since $Ax=b$ is consistent, b is in span of col. vectors of A $\Rightarrow$ $b \in C(A)$.

$\therefore \ \text{proj}_{C(A)} b = b$

Ex6: Obtain the orthogonal projection of the vector $x=\begin{bmatrix}1\\1\\-1\\-1\end{bmatrix}$ onto the column space of the matrix A. Given that the columns of A are LI.

$$A=\begin{bmatrix}1 & 6 & 6\\ 3 & -8 & 3\\ 1 & -2 & 6\\ 1 & -4 & -3\end{bmatrix}$$

Soln: Let col. vectors of A be

$$a_1=\begin{bmatrix}1\\3\\1\\1\end{bmatrix}, \quad a_2=\begin{bmatrix}6\\-8\\-2\\-4\end{bmatrix}, \quad a_3=\begin{bmatrix}6\\3\\6\\-3\end{bmatrix}$$

First we obtain orthonormal basis of $C(A)$ using Gram-Schmidt process, then we obtain proj. of given vector onto $C(A)$.

Let $\{u_1, u_2, u_3\}$ be orthonormal basis of $C(A)$. Then

$$u_1 = \frac{a_1}{\|a_1\|}$$

$$\Rightarrow \quad u_1 = \begin{bmatrix} 1/\sqrt{12} \\ 3/\sqrt{12} \\ 1/\sqrt{12} \\ 1/\sqrt{12} \end{bmatrix}$$

$$u_2 = \frac{y_2}{\|y_2\|}, \qquad \text{where} \qquad y_2 = a_2 - \langle u_1, a_2 \rangle u_1$$

$$= \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix} + 2\sqrt{12} \begin{bmatrix} 1/\sqrt{12} \\ 3/\sqrt{12} \\ 1/\sqrt{12} \\ 1/\sqrt{12} \end{bmatrix}$$

$$= \begin{bmatrix} 8 \\ -2 \\ 0 \\ -2 \end{bmatrix}$$

$$\|y_2\| = (64+4+0+4)^{1/2} = \sqrt{72}$$

$$\therefore \quad u_2 = \begin{bmatrix} 8/\sqrt{72} \\ -2/\sqrt{72} \\ 0 \\ -2/\sqrt{72} \end{bmatrix}$$

$$y_3 = \frac{y_3}{\|y_3\|}, \qquad y_3 = a_3 - \langle u_1, a_3 \rangle u_1 - \langle u_2, a_3 \rangle u_2$$

$$= \begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix} - \frac{18}{\sqrt{12}} \begin{bmatrix} 1/\sqrt{12} \\ 3/\sqrt{12} \\ 1/\sqrt{12} \\ 1/\sqrt{12} \end{bmatrix} - \frac{48}{\sqrt{72}} \begin{bmatrix} 8/\sqrt{72} \\ -2/\sqrt{72} \\ 0 \\ -2/\sqrt{72} \end{bmatrix}$$

$$= \begin{bmatrix} -5/6 \\ -1/6 \\ 9/2 \\ -19/6 \end{bmatrix}$$

$$\|y_3\| = \left(\frac{25}{36} + \frac{1}{36} + \frac{27^2}{36} + \frac{19^2}{36}\right)^{1/2}$$

$$= \sqrt{31}$$

$$u_3 = \begin{bmatrix} -5/6\sqrt{31} \\ -1/6\sqrt{31} \\ 9/2\sqrt{31} \\ -19/6\sqrt{31} \end{bmatrix}$$

Orthonormal basis of $C(A)$ is

$$\left\{ \begin{bmatrix} 1/\sqrt{12} \\ 3/\sqrt{12} \\ 1/\sqrt{12} \\ 1/\sqrt{12} \end{bmatrix}, \begin{bmatrix} 8/\sqrt{72} \\ -2/\sqrt{72} \\ 0 \\ -2/\sqrt{72} \end{bmatrix}, \begin{bmatrix} -5/6\sqrt{31} \\ -1/6\sqrt{31} \\ 9/2\sqrt{31} \\ -19/6\sqrt{31} \end{bmatrix} \right\}$$

$$\text{proj}_{C(A)} x = \text{proj}_{u_1} x + \text{proj}_{u_2} x + \text{proj}_{u_3} x$$

$$= \langle u_1, x\rangle u_1 + \langle u_2, x\rangle u_2 + \langle u_3, x\rangle u_3$$

$$= \frac{2}{\sqrt{12}} \begin{bmatrix} 1/\sqrt{12} \\ 3/\sqrt{12} \\ 1/\sqrt{12} \\ 1/\sqrt{12} \end{bmatrix} + \frac{4}{\sqrt{72}} \begin{bmatrix} 8/\sqrt{72} \\ -2/\sqrt{72} \\ 0 \\ -2/\sqrt{72} \end{bmatrix} - \frac{7}{3\sqrt{31}} \begin{bmatrix} -5/6\sqrt{31} \\ -1/6\sqrt{31} \\ 9/2\sqrt{31} \\ -19/6\sqrt{31} \end{bmatrix}$$

$$= \begin{bmatrix} 104/93 \\ 9/31 \\ -16/93 \\ 17/93 \end{bmatrix}$$

Eigenvalues and Eigenvectors

Let A be a square matrix of order $n \times n$. Then a non-zero vector X is said to be an eigenvector of A if there exist a scalar λ such that

$$AX = \lambda X$$

here λ is called corresponding eigen value or Characteristic value.

That is,

Suppose $A: \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation, a non-zero vector $X \in \mathbb{R}^n$ is said to be an eigen vector if the image of X under A is a vector lies in the same line where X lies. The scaling factor is called the eigenvalue.

Ex0: If $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, then $IX = X$ for every $X \in \mathbb{R}^3$

$\therefore$ Every vector is an eigenvector with eigenvalue $\lambda = 1$

It has three LI eigenvectors.

Ex1: Let P be a projection matrix onto a subspace V. Then eigenvalues of P are

i) $\lambda_1 = 1$ corresponding eigenvectors are basis of V.

ii) $\lambda_2 = 0$ corresponding eigenvectors are basis of $V^\perp$

where $V^\perp = \{ v \mid \langle v, w \rangle = 0 \text{ for all } w \in V \}$

a) Let $P = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$.

It is a projection matrix onto the line $y = x$.

i) $Px = x$ when x is on the line $y = x$

$\therefore$ eigenvector is $x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ with eigenvalue $\lambda_1 = 1$.

ii) $Px = 0$ when x is orthogonal to the line $y = x$

$\therefore$ eigenvector is $x_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ with eigenvalue $\lambda_2 = 0$.

b) $P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. It is a projection matrix onto e_1-axis.

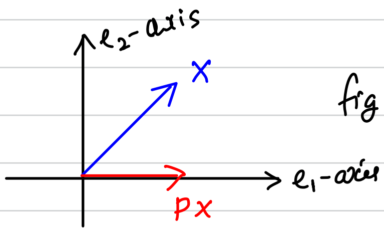


fig 1

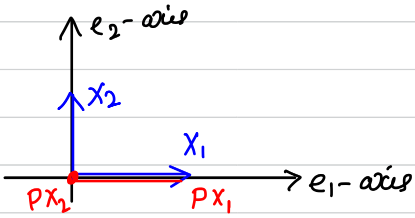


fig 2

From fig 2.

i) $Px_1 = x_1 \Rightarrow x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigenvector with $\lambda_1 = 1$

ii) $Px_2 = 0 \Rightarrow x_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is another eigenvector with $\lambda_2 = 0$

c) Let $V = \text{span}(u_1, u_2)$ be subspace in $\mathbb{R}^3$,

where $u_1 = \begin{bmatrix} 2/3 \\ 2/3 \\ 1/3 \end{bmatrix}$, $u_2 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix}$.

Clealy V is a subspace with orthonormal basis $\{u_1, u_2\}$

If $Q = [u_1 \ \ u_2]$, then

projection matrix P onto V $= QQ^T = \begin{bmatrix} \frac{17}{18} & \frac{-1}{18} & 2/9 \\ -1/18 & \frac{17}{18} & 2/9 \\ 2/9 & 2/9 & 1/9 \end{bmatrix}$

Eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 0$.

Eigenvectors when $\lambda_1 = 1$ are u_1 and u_2, and eigenvector vector when $\lambda_2 = 0$ is a vector u_3 $\perp^{le}$ to both u_1 and u_2.

To find u_3 consider

$$A = \begin{bmatrix} u_1^T \\ u_2^T \end{bmatrix} \Rightarrow A = \begin{bmatrix} 2/3 & 2/3 & 1/3 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{bmatrix}$$

we find Null space of A:

$$Ax = 0$$

$$\Rightarrow \begin{bmatrix} 2/3 & 2/3 & 1/3 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & 2 & 1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & 2 & 1 \\ 0 & -4 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$4x_2 + x_3$

$$\Rightarrow \begin{aligned} 2x_1 + 2x_2 + x_3 &= 0 \\ 4x_2 + x_3 &= 0 \end{aligned}$$

Let $x_3 = k \Rightarrow x_2 = -k/4$ and $x_1 = \dfrac{k/2 - k}{2} = -k/4$

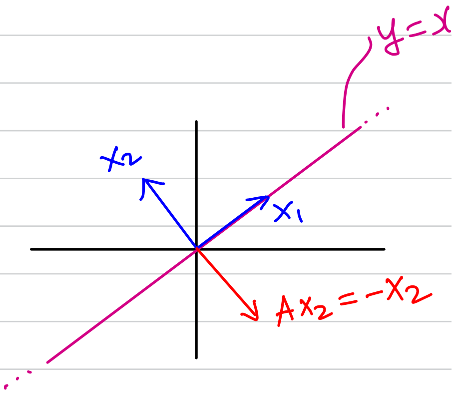
$$X = k \begin{bmatrix} -1/4 \\ -1/4 \\ 1 \end{bmatrix}$$

$\therefore$ eigen vector $u_3 = \begin{bmatrix} -1/4 \\ -1/4 \\ 1 \end{bmatrix}$

Note: Clearly, projection matrix is symmetric and if P is a projection matrix of order n, then it has n $\perp$ eigenvectors.

Ex2: Reflection matrix about a line $y=x$

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$



From the fig:

i) $AX_1 = X_1$, when X_1 is on the line $y=x$. $\therefore$ eigen vector $X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ with $\lambda_1 = 1$

ii) $AX_2 = -X_2$, when X_2 is $\perp^{le}$ to X_1

$\therefore$ eigenvector $X_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ with $\lambda_2 = -1$

Ex3: Rotation matrix by an angle $90°$ in counterclockwise direction.

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$ No real eigenvalues and eigenvectors

Method to find eigenvalues and eigenvectors

Let A be an $n\times n$ matrix. If X is an eigenvector and λ is the eigenvalue, then

$$AX = \lambda X$$

$$\Rightarrow AX - \lambda X = 0$$

$$\Rightarrow \boxed{[A-\lambda I]\, X = 0} \quad \text{——— } (1)$$

It has non-trivial soln iff

$$\boxed{|A-\lambda I| = 0} \quad \text{—— } (2)$$

(2) is called characteristic Equation, Solving we get eigenvalues and a corresponding eigenvectors are found using (1).

Property 1:

If A is an $n\times n$ matrix, then it has atmost n linearly independent eigenvectors.

Property 2:

A symmetric matrix of order n has n linearly independent eigenvectors

Ex. Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

Characteristic Eqn :

$$|A-\lambda I| = 0 \quad \Rightarrow \begin{vmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^3 - (a_{11}+a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21}) = 0$$

$$\Rightarrow \lambda^3 - \text{trace}(A)\lambda + |A| = 0.$$

Now, let
$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Characteristic eqn :

$|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^3 - (a_{11}+a_{22}+a_{33})\lambda^2 + \left(\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}\right)\lambda$$
$$- |A| = 0$$

$$\Rightarrow \lambda^3 - \text{trace}(A)\lambda^2 + \left(\begin{array}{c}\text{sum of minors of} \\ \text{diagonal enteries}\end{array}\right)\lambda - |A| = 0$$

property 3 :

Sum of eigenvalues of A = trace (A)

product of eigenvalues of A = det (A)

property 4 :

A matrix B is singular iff 0 is an eigenvalue.

If B is singular $\Leftrightarrow |B| = 0$

$\Leftrightarrow |B - 0I| = 0$

$\Leftrightarrow 0$ is the eigenvalue.

property 5 :

Matrices A and A^T have same eigenvalues

Suppose λ is eigenvalue of A, then

$$|A-\lambda I| = 0$$

$$\Rightarrow \left|(A-\lambda I)^T\right| = 0 \qquad (\because \text{ A matrix and its transpose have same determinant})$$

$$\Rightarrow |A^T - \lambda I| = 0$$

$\Rightarrow$ λ is eigenvalue of A^T.

Property 6:

If λ is eigenvalue of A, then eigenvalue of $A+KI$ is $\lambda+K$

Consider
$$[A+KI]X = AX + KIX$$
$$= AX + KX$$
$$= \lambda X + KX \qquad (\because \lambda \text{ is eigenvalue of } A)$$
$$= (\lambda+K)X$$

$\Rightarrow$ $\lambda+K$ is eigenvalue of $A+KI$

Eigenvectors of A and $A+KI$ are same

Property 7:

If A is triangular or diagonal matrix, then its eigenvalues are diagonal entries of A.

For instance, $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$

Char Eqn: $|A-\lambda I| = 0$

$$\Rightarrow \begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ 0 & a_{22}-\lambda & a_{23} \\ 0 & 0 & a_{33}-\lambda \end{vmatrix} = 0 \Rightarrow (a_{11}-\lambda)(a_{22}-\lambda)(a_{33}-\lambda) = 0$$

$\therefore$ eigenvalues are a_{11}, a_{22} and a_{33}.

property 8

If A is symmetric, then its eigenvalues are real.

property 9.

If A is skew-symmetrix, then its eigenvalues are zero or purely imaginary

property 10.

If λ is eigenvalue of A, then λ^k is eigenvalue of A^k, $k \in \mathbb{Z}^+$.

Let $AX = \lambda X$

$\Rightarrow A(AX) = A(\lambda X)$

$\Rightarrow (AA)X = \lambda AX$

$\Rightarrow A^2 X = \lambda^2 X$

$\Rightarrow \lambda^2$ is eigenvalue of A^2. By induction we can easily prove that for any $k \in \mathbb{Z}^+$ λ^k is eigenvalue of A^k.

Property 11

If A is nonsingular and λ is eigenvalue of A, then $\frac{1}{\lambda}$ is eigenvalue of A^{-1}.

Let $AX = \lambda X$

$\Rightarrow (A^{-1}A)X = A^{-1}(\lambda X)$

$\Rightarrow X = \lambda (A^{-1}X)$

$\Rightarrow A^{-1}X = \frac{1}{\lambda} X$

Property 12:

Eigenvalues of orthogonal matrices are +1 or −1.

If Q is orthogonal matrix, then $Q^{-1} = Q^T$

If λ is eigenvalue of Q, then $\frac{1}{\lambda}$ is eigenvalue of Q^{-1} and λ is eigenvalue of Q^T.

$$\therefore \frac{1}{\lambda} = \lambda \Rightarrow \lambda^2 = 1 \Rightarrow \lambda = \pm 1$$

Property 13:

If $\lambda_1, \lambda_2, \ldots, \lambda_K$ are distinct eigenvalues of A, then its K eigenvectors are linearly independent.

Repeated eigenvalues may or may not have linearly independent eigenvectors.

property 14:

If A is symmetric, then eigenvectors to different eigenvalues are orthogonal.

We can find orthogonal eigenvector for repeated eigenvalue using Gram Schmidt process.

Property 15:

Matrices A and B are said to be similar if there exist non-singular matrix S such that

$$B = S^{-1} A S$$

Two similar matrices have same eigenvalues.

Ex 1: Let $B = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$. Find eigenvalues and eigenvectors of B.

Soln: To find eigenvalues:

Characteristic eqn is $|B - \lambda I| = 0$

$$\Rightarrow \left| \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0$$

$$\Rightarrow \begin{vmatrix} 4-\lambda & 1 \\ 1 & 4-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (4-\lambda)^2 - 1 = 0$$

$$\Rightarrow \lambda^2 - 8\lambda + 15 = 0$$

Solve for λ,

$$\lambda = \frac{8 \pm \sqrt{64-60}}{2}$$

$$\Rightarrow \lambda = 5, 3$$

Eigenvalues are $\lambda_1 = 5$, $\lambda_2 = 3$

To find eigenvectors: Consider $[B - \lambda I] X = 0$

i) $\lambda_1 = 5$,

$$[B - 5I] X = 0$$

$$\Rightarrow \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -x_1 + x_2 = 0$$

Let $x_2 = k \Rightarrow x_1 = k$ $\quad (k \neq 0)$

Eigenvector for $\lambda = 5$ is

$$X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad (k = 1)$$

Now, $\lambda = 3$,

$$[B - 3\lambda] X = 0$$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x_1 + x_2 = 0$$

Let $x_2 = k$, $x_1 = -k$

Eigenvector for $\lambda = 3$ is

$$X_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad (k=1)$$

Diagonalisation:

Let A be an $n \times n$ matrix which has n linearly independent eigenvectors. Then we can factor A into

$A = XDX^{-1}$, is called diagonalization.

where columns of X are eigenvectors of A, it is called a diagonalising matrix (or eigenvector matrix).

D is a diagonal matrix, diagonal entries are eigenvalues.

pf: Let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be eigenvalues of A and let $x_1, x_2, \cdots, x_n$ be corresponding eigenvectors. Then

$$A x_i = \lambda_i x_i \quad , \quad i = 1, 2, \cdots, n$$

$$\Rightarrow AX = A \begin{bmatrix} x_1 & x_2 & x_3 & \cdots & x_n \end{bmatrix}$$

$$= \begin{bmatrix} Ax_1 & Ax_2 & Ax_3 & \cdots & Ax_n \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 x_1 & \lambda_2 A_2 & \lambda_3 x_3 & \cdots & \lambda_n x_n \end{bmatrix}$$

$$= \begin{bmatrix} X_1 & X_2 & \cdots & X_n \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_1 & 0 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

$$= X D$$

$$\Rightarrow \quad A = X D X^{-1}$$

or

$$D = X^{-1} A X$$

Remark 1: If the matrix A has no repeated eigenvalues, then it has n LI eigenvectors. Therefore it is diagonalizable.

Remark 2: Diagonalizing matrix X is not Unique. Any eigenvector can be multiplied by a constant, and remains an eigenvector.

Remark 3: Not all matrices possess n linearly independent eigenvectors, so not all matrices are diagonalizable.

Remark 4: If $A = X D X^{-1}$, then $A^k = X D^k X^{-1}$ for any $k \in \mathbb{Z}^+$.

$$A = X D X^{-1}$$

$$\Rightarrow A^2 = (X D X^{-1})(X D X^{-1})$$

$$\Rightarrow A^2 = X D (X^{-1} X) D X^{-1}$$

$$\Rightarrow A^2 = X D^2 X^{-1} \qquad (\because X^{-1} X = I)$$

Ex 2: Let $A = \begin{bmatrix} 2 & -3 \\ 2 & -5 \end{bmatrix}$.

i) Diagonalize the matrix A

ii) Find eigenvalues and eigenvectors of $(A^{-1} + K)^3$

Soln: Given $A = \begin{bmatrix} 2 & -3 \\ 2 & -5 \end{bmatrix}$

Characteristic eqn : $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 2-\lambda & -3 \\ 2 & -5-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 + 3\lambda - 4 = 0$$

Solving, $\lambda = -4, 1$

eigenvalues $\lambda_1 = -4$, $\lambda_2 = 1$.

To find eigenvectors, consider

$[A - \lambda I] X = 0$

i) $\lambda_1 = -4$

$$[A + 4I] X = 0$$

$$\Rightarrow \begin{bmatrix} 6 & -3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 2x_1 - x_2 = 0$$

put $x_2 = k \Rightarrow x_1 = k/2$

$\therefore X_1 = \begin{bmatrix} k/2 \\ k \end{bmatrix}$, $k \neq 0$

Eigenvector for $\lambda_1 = -4$ is $X_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $(k = 2)$

ii) $\lambda = 1$

$[A - I]X = 0$

$$\Rightarrow \begin{bmatrix} 1 & -3 \\ 2 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad x_1 - 3x_2 = 0$

put $x_2 = k_1 \Rightarrow x_1 = 3k_1$

$\therefore \quad X_2 = \begin{bmatrix} 3k_1 \\ k_1 \end{bmatrix}$

eigenvector for $\lambda_2 = 1$ is $X_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ $(k_1 = 1)$

Diagonalizing matrix $\quad X = [X_1 \;\; X_2] = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$

Diagonal matrix $\quad D = \begin{bmatrix} -4 & 0 \\ 0 & 1 \end{bmatrix}$

and $\quad X^{-1} = \frac{1}{-5} \begin{bmatrix} 1 & -3 \\ -2 & 1 \end{bmatrix}$

$\therefore$ Diagonalization : $\quad A = X D X^{-1}$

$$\Rightarrow \begin{bmatrix} 2 & -3 \\ 2 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -4 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1/5 & 3/5 \\ 2/5 & -1/5 \end{bmatrix}$$

Ex 3 :