Milestone 2: Trajectories and Jacobians

Robot Kinematics and Dynamics

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Contents

1 Overview

In this capstone, you will form a team of 3 students to program a Franka Emika Panda robot to draw with lights to create a long-exposure image.

2 Introduction

2.1 Notation and Terminology

Throughout this document we will use the following terms consistently:

Configuration (or config for short) specifies the settable degrees of freedom of a Robot. For Franka robots, this is a 7-element vector where the first coefficient is the angle of the first joint, etc.

Cartesian coordinates refers to the x, y, z position in workspace coordinates for translation, and a rotation relative to a fixed workspace identity rotation.

Position refers to the Cartesian translation of a an object.

Orientation refers to the rotation of an object.

Pose refers the the position and orientation of an object.

Linear interpolation given a start and end vector, this is an interpolation that starts at the start vector at $t = 0$ and ends at the end vector at $t = 1$. Mathematically $f(x_0, x_1, t) = x_0(1 - t) + x_1 t$, for $x_0, x_1 \in \mathbb{R}^n$.

2.2 Picking / Grasping

The picking/grasping task in the capstone is the same as in the hands-on section of Assignment 6: moving the robot to a config where the target object is at the center of the gripper and closing the gripper.

2.3 Trajectories

For the capstone, we recommend switching, as appropriate, between running joint-space or Cartesian-space interpolations. In almost all cases, the start configuration (q_0) or pose ($x_0 = \mathrm{fk}(q_0)$) will be given from the robot's current configuration or forward kinematics. The end configuration or pose will be task-based. You can also compute a sequence of interpolations by chaining the end-configuration of one interpolation into the start of the next.

2.3.1 Joint-Space Trajectories

Joint-space **l**inear int**erp**olation (LERP) is straight-forward. Compute

$$\mathrm{lerp}(q_0, q_1, t) = q_0(1 - t) + q_1 t,$$

where $q_0 \in \mathbb{R}^7$ is the start configuration, $q_1 \in \mathbb{R}^7$ is the end configuration, and $t \in [0, 1]$. This hopefully seems simple, as it is just standard vector times scalar and a vector addition.

2.3.2 End-Effector Trajectories

Cartesian-space interpolation is a bit more complicated. If considering just the translation components, you can hold the rotation fixed and linearly interpolate from start to end translation, then use inverse kinematics to get the joint angles. If you want to include rotation too, you will need to interpolate both translation and rotation. Fortunately, there is a really good way to interpolation rotations: **s**pherical **l**inear int**erp**olation or SLERP. SLERP requires converting the rotations to quaternions, interpolating a quaternion, then converting the quaternion back.

$$\mathrm{slerp}(p_0, p_1, t) = \frac{\sin((1-t)\phi)}{\sin\phi} p_0 + \frac{\sin(t\phi)}{\sin\phi} p_1,$$

where $p_0, p_1 \in \mathbb{R}^4$ are the coefficients of unit quaternions stacked into vectors, and $\cos\phi = p_0 \cdot p_1$ (dot product). SLERP can trace the short way or the long way. To ensure taking the shortest path, test if $\cos\phi < 0$, and if so, negate one of the quaternions (remember p_0 and $-p_0$ are the same rotation), and recompute $\cos\phi$. You can take a shortcut to compute $\sin\phi$ using the identity $\sin^2\phi = 1 - \cos^2\phi$. Note also that ϕ is the angular distance between rotations—you can use this to decide how many steps to take along the interpolation.

Putting together, interpolating in Cartesian space, given $x_0 = [d_0, p_0]$ and $x_1 = [d_1, p_1]$ are the translations (displacements) (d_0, d_1) and rotations (p_0, p_1). Note that $x_0, x_1 \in SE(3)$ and can be expressed as homogeneous matrices and converted to and from a [translation, quaternion] tuple (we leave these conversions as implicit here). Compute interpolation of a configuration q_t at time $t \in [0, 1]$ as:

$$q_t = \mathrm{ik}(x_t) = \mathrm{ik}([d_t, p_t]) = \mathrm{ik}([\mathrm{lerp}(d_0, d_1, t), \mathrm{slerp}(p_0, p_1, t)]).$$

When computing an interpolation, the most sensible setting for q_{init} in the inverse kinematics computation is the previous configuration computed. Thus, switching the subscript i to reflect the i^{th} waypoint along the interpolation:

$$q_{i+1} = \mathrm{ik}(x_{i+1}, q_i),$$

where q_0 is the start of the interpolation.

2.3.3 Trajectory Timing and Execution

When computing an interpolation-based trajectory, the next thing to consider is velocity and acceleration, thus timing. Robots cannot instantaneously accelerate, and the Franka robot has velocity and acceleration limits, both in joint space (Table. 1) and end-effector/Cartesian space (Table. 2). You should thus smoothly vary the time that you pass to the interpolation function to smoothly accelerate and avoid instantaneous changes. You can usually stay within the documented limits in the tables by trial-and-error or rough estimations.

We recommend starting with a trapezoidal profile, but you are free to try other profiles. Example: suppose you want to interpolate a line in joint space from q_0 to q_1, where $q_0, q_1 \in \mathbb{R}^7$. You want to interpolate over 2 seconds, accelerating for 0.5 seconds and decelerating for 0.5 seconds. First, define the number of waypoints you want to compute. We'll say 20 waypoints, thus 5 for accelerating, then 10 at constant velocity, and 5 for decelerating.

Table 1: **Franka Panda Joint Limits**—Documented limits for the robot joints. Configuration limits have an upper and lower bound that cannot be exceeded due to hardware limits. Velocity, acceleration, and jerk limits list the upper bound in any direction—these may be enforced by software limits in the robot, and exceeding them may result in unexpected operation. Torque limits are limited by the capabilities of the motors.

Joint	$q_{\min}$ [rad]	$q_{\max}$ [rad]	$\dot{q}_{\max}$ $[\frac{\text{rad}}{\text{s}}]$	$\ddot{q}_{\max}$ $[\frac{\text{rad}}{\text{s}^2}]$	$\dddot{q}_{\max}$ $[\frac{\text{rad}}{\text{s}^3}]$	$\tau_{j\max}$ [N m]	$\dot{\tau}_{j\max}$ $[\frac{\text{N m}}{\text{s}}]$
1	-2.8973	2.8973	2.1750	15	7500	87	1000
2	-1.7628	1.7628	2.1750	7.5	3750	87	1000
3	-2.8973	2.8973	2.1750	10	5000	87	1000
4	-3.0718	-0.0698	2.1750	12.5	6250	87	1000
5	-2.8973	2.8973	2.6100	15	7500	12	1000
6	-0.0175	3.7525	2.6100	20	10 000	12	1000
7	-2.8973	2.8973	2.6100	20	10 000	12	1000

Source: https://frankaemika.github.io/docs/control_parameters.html#limits-for-panda

Table 2: **Franka Panda Cartesian Limits**—Documented limits of the robot's motion. Trajectories that exceed these limits may not run properly, so try to keep the trajectories within these limits.

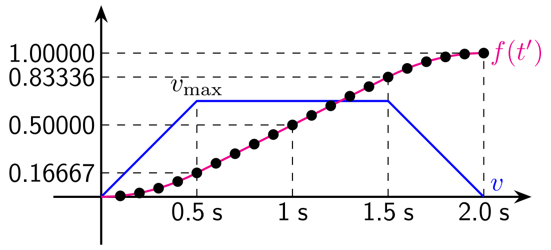
	Translation	Rotation	Elbow
$\dot{p}_{\max}$	$1.7\ \frac{\text{m}}{\text{s}}$	$2.5\ \frac{\text{rad}}{\text{s}}$	$2.1750\ \frac{\text{rad}}{\text{s}}$
$\ddot{p}_{\max}$	$13.0\ \frac{\text{m}}{\text{s}^2}$	$25.0\ \frac{\text{rad}}{\text{s}^2}$	$10.0\ \frac{\text{rad}}{\text{s}^2}$
$\dddot{p}_{\max}$	$6500.0\ \frac{\text{m}}{\text{s}^3}$	$12\,500.0\ \frac{\text{rad}}{\text{s}^3}$	$5000.0\ \frac{\text{rad}}{\text{s}^3}$

Source: https://frankaemika.github.io/docs/control_parameters.html#limits-for-panda

Our goal is to modify our interpolation from:

$$\text{lerp}(q_0, q_1, t) \qquad \text{to} \qquad \text{lerp}(q_0, q_1, f(t')),$$

where $f(t')$ maps from $t' \in [0, 2]$ to $f(t') \in [0, 1]$. The interpolation will look something like the following:



With $f(\cdot)$ defined, you can then interpolate t' from 0 to 2 seconds and send the waypoints to the robot to run. You can apply the same method to interpolate in Cartesian space.

Note: The robot controller receives new waypoints every 0.02 seconds, so your interpolation function should ensure that all waypoints are spaced 0.02 seconds apart.

Each successive waypoint must be reachable within 0.02 seconds from the previous one-the robot should not exceed its joint limits during the 0.02-second motion to avoid triggering errors. The final output of your interpolation function must be an $n \times 7$ array, where each row represents a waypoint defined by the 7 joint angles.

2.4 Angular and Linear Velocity Jacobians

As a reminder, each column of the Jacobian is influenced by each joint. The Jacobian for each joint varies based on the type of the joint as shown in the table below.

*i*th joint	Revolute	Prismatic
Linear	$z_{i-1}^0 \times (o_n - o_{i-1})$,	z_{i-1}^0
Angular	z_{i-1}^0	0

2.5 Milestone 2: Trajectories and Jacobians

Due: **Thurs, Nov. 20, 2025**

In this milestone, you will get the robot to interpolate between two configurations smoothly.

In code, write a function `compute_joint_trajectory(q1, q2, num_steps)` expanding your `Robot` class in `milestone-2.py` , where q1 is the starting configuration, q2 is the ending configuration, and `time_step (which is fixed to be 0.02s) * num_steps` is the duration (or end time `T`) of the trajectory. The trajectory it returns should be a list of waypoints where each waypoint is separated by `time_step` seconds (for the robot, this should usually be 0.02 (i.e., 2 ms), matching the robot's control frequency), though larger values may be useful for testing and debugging.

You can also use `plot_end_effector_trajectory(self, traj)` in the `Robot` class to visualize your interpolated trajectory.

Once the code works, test it on the real robot. For the robot, use teach mode to move the robot to a pose and record the joint configuration, and repeat 3 more times. Generate trajectories that move between successive pairs of recorded configurations in sequence and then close the loop. Get the robot to complete a loop motion within 10 seconds. You will need to call the `follow_trajectory(traj)` method 4 times to close the loop. Use `set_config(q1)` to get to the starting configuration, but do NOT use it for anything else.

Note: be careful about the motions between points when collecting them—if you do too much motion, the trajectory you generate could cause a collision.

- You should also finish writing a script to pick and place the flashlights in the workspace. The idea is similar: use `guide_mode` to record the joint configurations for the pick and place actions, then use interpolation to generate smooth trajectories, and use the gripper functions to pick up and release objects. Remember to break the trajectory to small reachable chunks.
  - When picking the flashlight, it should be standing up (facing down) rather than towards the camera.
  - When placing, drop it gently in the same orientation—do NOT drop it from a height as it may be damaged.
  - The goal here is functionality: to have pieces ready and focus on trajectory generation and inverse kinematics for the rest of the semester.
- Implement a recovery / standby mode. See Section 4.2 “Recovery policies” and “Sensors” in the Capstone PDF. An example of how to incorporate user inputs is given in `demo.py`. The robot should wait in standby until the user signals to continue, either to retry the current action or to move on to the next action.
- Implement `compute_jacobian_analytical` and `compute_jacobian_numerical` in `milestone-2.py`. Validate that the analytic and numeric jacobian computations compute approximately the same result and experiment with your numeric finite difference being `1e1, 1e3, 1e6, 1e10`, what approximation error do you get for each?

Codebase Files

The codebase contains the following five files:

1. `guide_mode.py` This script allows you to interactively control the robot:
   - Press 1 to get the end-effector pose.
   - Press 2 to get the joint angles.
   - Press 3 to stop the current skill.
   - Press 4 to move the robot to the home position.

2. `FrankaRobot16384.py` This is a wrapper class that you will use for most of the capstone to access different functions of the robot.

3. `example.py` This file contains examples demonstrating how to use the wrapper class, including following trajectories and controlling the gripper.

4. `demo.py` This file demonstrates how to incorporate user inputs to put the robot in a standby or recovery mode. The robot waits for a user signal to either retry the current action or move on to the next one.

5. `milestone-2.py` This file contains two 4 new functions for you to implement:
   - `compute_joint_trajectory(q1, q2, num_steps)` – generates a joint-space trajectory with a fixed time step of 0.02 s.
   - `plot_end_effector_trajectory(traj)` – uses your forward kinematics to compute the end-effector pose for all waypoints in a trajectory and visualizes the trajectory in a graph.
   - `compute_jacobian_analytical` – takes in thetas to calculate analytical jacobian
   - `compute_jacobian_numerical` – takes in thetas to calculate numerical jacobian

3 Rubric

(1) Autograder Your code will be tested automatically on `compute_joint_trajectory` (**Note: We will test it on a trapezoidal profile, You are free to experiment with other profiles but keep it trapezoidal for the autograder**) ,`compute_jacobian_analytical`, `compute_jacobian_numerical`. Points will be assigned based on the number of test cases passed. Partial credit may be given for small numerical errors within tolerance.

(2) Loop Motion Validation Complete a loop of four joint configurations within 10 seconds. Include in your submission:

- A single PDF report linking a video showing the robot completing the loop. You may include all demonstrations in one video if you prefer.
- A plot of the end-effector trajectory generated from your recorded joint waypoints using your `plot_end_effector_trajectory()` function.
- Any relevant observations or notes about the motion.

(3) Pick-and-Place Demonstration Demonstrate the robot safely picking and placing a flash light in the workspace. Include in your submission:

- A PDF report linking a video showing the robot picking up the flash light standing upright (facing down, not toward the camera) and placing it gently in the target location.
- A description of the motion and trajectory used.
- Note: The flash lights will be provided later. Focus on functionality, safe handling, and trajectory generation. Do **not** drop the object from height to avoid damage.

(4) Standby / Recovery Mode Demonstration Demonstrate the robot waiting for user input between actions, allowing retries or moving to the next action. Include in your submission:

- A PDF report linking a video showing the robot entering standby or recovery mode.
- An explanation of your design for recovery/standby.
- Note: Since long exposure photography may be involved, waiting between actions ensures undesired motion does not ruin the experiment. The design of this mode is left completely up to you. An example of this is given in `demo.py`

(5) Writeup

- Submit a single PDF report compiling all video links, plots, and explanations for the above demonstrations. A single video covering all parts is allowed.

4 Submission Checklist

☐ Upload `all files` to Gradescope.