PHSX815 HW2 - Parameter Estimation

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The goal here is to study different ways of finding the best-fit parameter to given data and the effect our model and the background have on the predictions and likelihood of the parameters, along with their uncertainties, using

$$\sigma(E_b; m_\tau) = \frac{\pi\alpha^2}{6E_b^2}\frac{\sqrt{E_b^2 - m_\tau^2}}{E_b}\left(3 - \frac{E_b^2 - m_\tau^2}{E_b^2}\right). \tag{1}$$

Problem 1

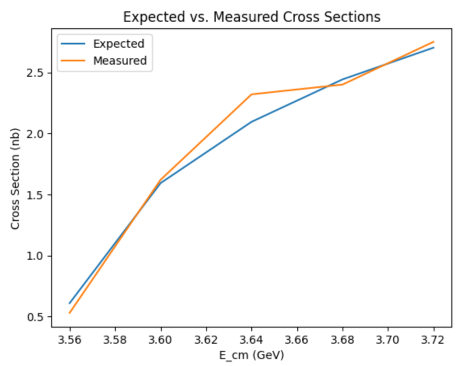

Fig. 1: The measured and expected cross-sections with the Tau mass m_τ = 1776.860.12 MeV and the fine-structure constant $\alpha = 1/137.036$

Simple Hypothesis

Here, we check if our known value of the tau mass reproduces the observed data based on our relation. We do this using the chi-squared value, assuming the uncertainties are 5% of the measured cross-section.

```
Chi-Squared: 11.797337135153589
```

Fig. 2: The chi-squared value for the given PDG Tau lepton mass and the observed data

Since there are no free parameters here, there are $5 = 5 - 0$ degrees of freedom. Using this, we can find the p-value of this hypothesis.

```
(pvalueArgs.getArgs     ) Found argument list:  Namespace(which=0, value=11.797337135153589, ndof=5)
(pvalueArgs.getArguments) Assigning arguments to program variables
(pvalueArgs.ShowArgs    ) Program has set
which:    0
value:    11.797337135153589
ndof:    5
chisq =  11.797337135153589

Observed chi-squared p-value of 3.7672765319651913 %
Normalized deviation =  2.149506737112912
```

Fig. 3: The p-value is only $\simeq 3.8\%$, implying that the given tau mass is unlikely to lead to our observations using this simple relation. Better uncertainties, detector effects, or higher-order corrections

Chi-Squared Minimization

We can try to obtain a better estimate for the tau mass by minimizing the chi-squared using our given data.

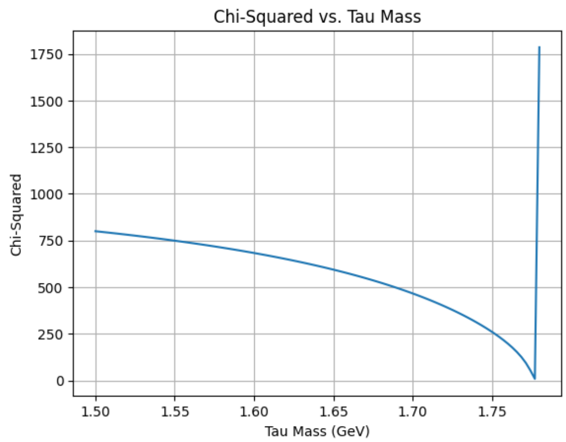

Fig. 4: The variation of chi-squared value with tau mass

This can be carried out using golden section minimization, giving us the minimum chi-squared value and the mass corresponding to that value.

```
Best Fit Mass: 1.777577698765262
Minimum Chi-Squared: 6.134018699746366
```

Fig. 5: The minima is obtained for a mass greater than the PDG value

```
(pvalueArgs.getArgs     ) Found argument list:  Namespace(which=0, value=6.134018699746366, ndof=4)
(pvalueArgs.getArguments) Assigning arguments to program variables
(pvalueArgs.ShowArgs    ) Program has set
which:    0
value:    6.134018699746366
ndof:    4
chisq =  6.134018699746366

Observed chi-squared p-value of 18.936073657138042 %
Normalized deviation =  0.7544895468847771
```

Fig. 6: The Tau mass obtained by minimizing the chi-squared value has a much better p-value of 18.9%

Uncertainties

To find the $\pm 1\sigma$ uncertainty interval, we find the values of m_τ corresponding to the roots of $\chi^2 = \chi^2_{min} + 1$. Since the difference of the two changes signs around the root, we use the hybrid Brent's method to find these bounds on the tau mass.

```
Lower bound of 68% CI (±1σ): 1.777307767842892 GeV
Best Fit Mass: 1.777577698765262
Upper bound of 68% CI (±1σ): 1.7778050337287203 GeV
```

Fig. 7: Uncertainties on the Tau mass

Problem 2

If we are given the observed number of events, we can obtain the cross-section from that using the relation

$$\mu = \sigma \mathcal{L}. \tag{2}$$

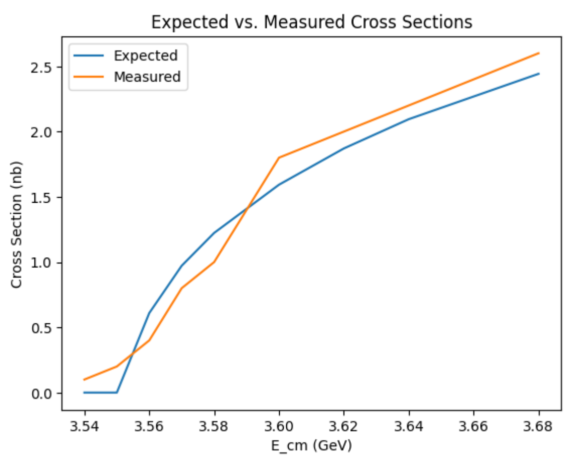

Fig. 8: The observed cross-sections for a luminosity of 10 nb $^{-1}$

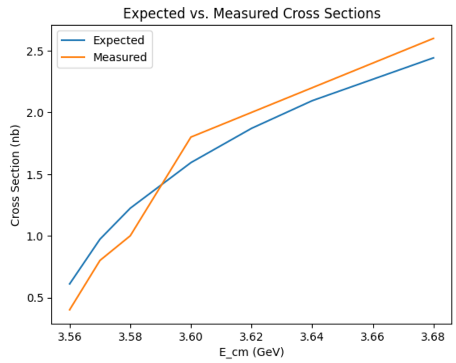

Fig. 9: Looking only at the non-zero expected cross-sections where the beam energy is greater than the tau mass

To better model the uncertainties, assuming the deviations in the observed number of events arise from Poisson statistics, we can calculate the modified negative log likelihood as

$$L' = -2 \ln L = -2 \sum_{i=1}^{N} \ln p(x_i; m_\tau), \tag{3}$$

where

$$p(N_{obs} = n; \mu) = e^{-\mu} \frac{\mu^n}{n!}. \tag{4}$$

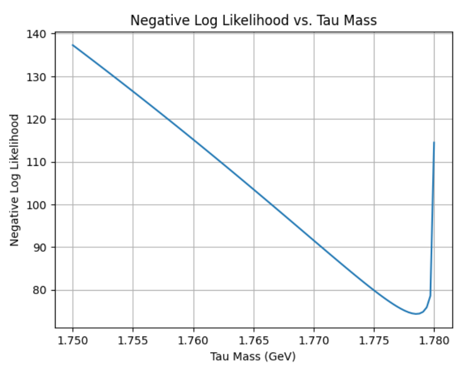

Fig. 10: The variation of the modified negative log likelihood with Tau mass

We can find the best-fit value for m_τ by minimizing the modified negative log likelihood.

```
Most likely mass: 1.778541005590813
Minimized negative log likelihood: 74.33418580286866
Negative log likelihood for given tau mass: 76.18477675807978
Negative log likelihood for tau mass with minimum chi squared: 75.08092884233477
```

Fig. 11: The minimum occurs at a mass even higher than that obtained by minimizing the chi-squared. Henceforth, we appropriately modify the range of the plot to discount unphysical scenarios where a Tau mass greater than the beam energies leads to undefined logarithms

The modified negative log likelihood allows us to once again calculate the uncertainties on the tau mass with a 68% central confidence interval.

```
Lower bound of 68% CI (±1σ): 1.7773939532131129 GeV
Best Fit Mass: 1.778541005590813
Upper bound of 68% CI (±1σ): 1.7792661570827175 GeV
```

Fig. 12: The uncertainties on the Tau mass

Problem 3

We can assume a constant background that does not depend on the Tau mass to better account for detector effects.

Effects of a Constant Background

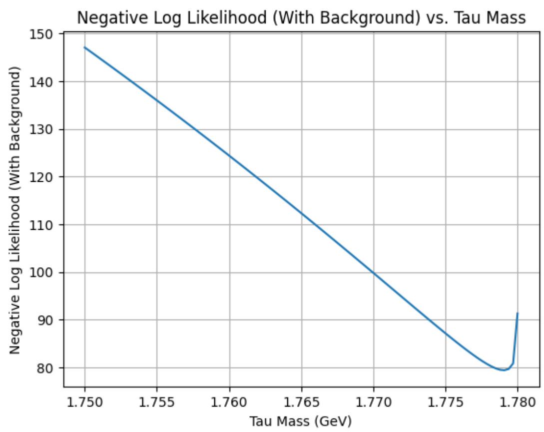

Fig. 13: The minima with background

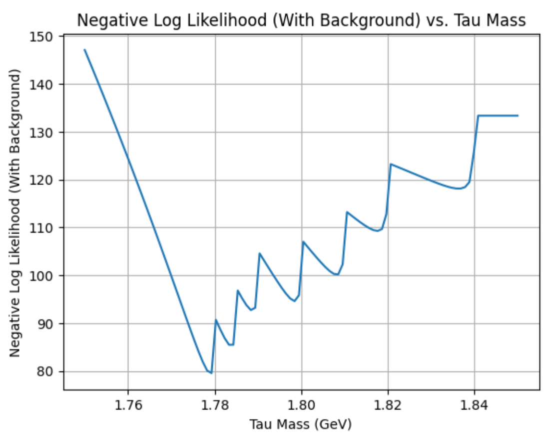

Fig. 14: If we plot L' for a larger range, we get some unexpected behaviour, but the absolute minima in this region remains unchanged

```
Most likely mass with constant background: 1.7790441123007859
Minimized negative log likelihood with constant background: 79.42586959386267
Negative log likelihood with constant background for given tau mass: 82.79157544247741
Negative log likelihood with constant background for tau mass with minimum chi squared: 81.2986256672132
```

Fig. 15: The minimized modified negative log likelihood

Uncertainties

Once again, we can find the bounds for the Tau mass that give the 68% central confidence interval.

```
Lower bound of 68% CI (±1σ): 1.7780620637869098 GeV
Best Fit Mass: 1.7790441123007859
Upper bound of 68% CI (±1σ): 1.7796224644358725 GeV
```

Fig. 16: The uncertainties on the Tau mass considering background effects