PHSX815 HW2 - First Derivative

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The goal here is to study a few different ways to numerically solve ordinary differential equations (ODEs). In particular, this is done by solving the ODE for a pendulum

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta, \tag{1}$$

using Euler, RK2, and RK4 stepping algorithms after converting it to a system of two coupled first-order ODEs in (θ, ω).

Time Period & Error

Under the small angle approximation $\sin\theta \approx \theta$, the system reduces to a simple harmonic oscillator and thus has a time period of $T_0 = 2\pi\sqrt{\frac{l}{g}}$, where l and g are the length of the pendulum and the acceleration due to gravity.

However, if a pendulum (here, with length $l = 1.5\ m$) is released from an angle of $50°$ with respect to the vertical, the small angle approximation does not provide a very good approximation. To see this, we use the functions defined in pendulumODE.py and obtain the following results, as seen in Pendulum.ipynb. The error caused due to the small-angle approximation is shown below.

```
Approximation for small amplitude T0 = 2.458173 s
Exact period T  = 2.580548 s
Difference: 0.122375 s
Ratio T/T0: 1.049783
```

Fig. 1: The difference between the exact time period and that computed using the small-angle approximation

Fourth-Order Range-Kutta

The RK4 method improves upon the Euler and RK2 methods for computing the first derivative by defining it as

$$
\begin{aligned}
k_1 &= hf(x_n, y_n), \\
k_2 &= hf\left(x_n + \frac{h}{2}, y_n + \frac{1}{2}k_1\right), \\
k_3 &= hf\left(x_n + \frac{h}{2}, y_n + \frac{1}{2}k_2\right), \\
k_4 &= hf(x_n + h, y_n + k_3), \\
&\text{and}
\end{aligned}
$$

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) + \mathcal{O}(h^5), \tag{2}$$

by modifying stepper.py appropriately. This gives us the following results for the pendulum.

```
End state  20.0 [1.62296606e-03 2.16045319e+00]
End time:  20.0 s  angle: 0.0016229660593336248 rad
End angular speed is  2.160453192981916  rad/s
End energy ratio  0.9999999945487286
t,istep 20.0 2000
```

Fig. 2: The output on using the RK4 method

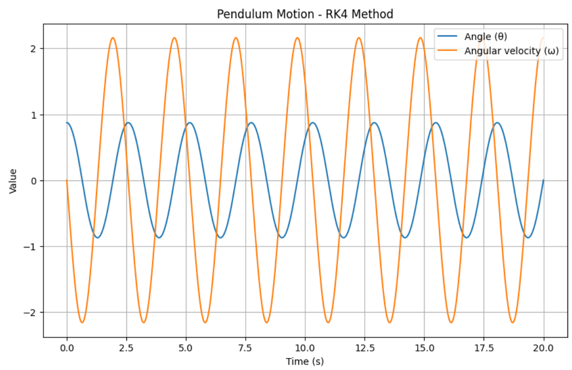

Fig. 3: The evolution of the angular position and velocity with time for the RK4 method

Comparing Different Methods

To see why RK4 is a better method than the Euler or even second-order Runge-Kutta methods, we look at the solutions of our ODE using the latter two methods.

```
For Euler Method:

End state  20.0 [-1.25655306 -2.03336526]
End time:  20.0 s  angle: -1.2565530643033849 rad
End angular speed is  -2.0333652582071564  rad/s
End energy ratio  2.8199599132535966
t,istep 20.0 2000
```

Fig. 4: We can see that the Euler method does not conserve energy, as the ratio of the final energy to the initial one is $\simeq 2.8$

```
For RK2 Method:

End state  20.0 [0.00538034 2.16060202]
End time:  20.0 s  angle: 0.005380337871804794 rad
End angular speed is  2.1606020226836575  rad/s
End energy ratio  1.0001746075134332
t,istep 20.0 2000
```

Fig. 5: The RK2 method significantly improves the ratio of the final energy and gives a much more physical result

Time Evolution of Position and Velocity

We can get a better idea by plotting the evolution of the angular position and momenta with time for all three methods - Euler, RK2, and RK4.

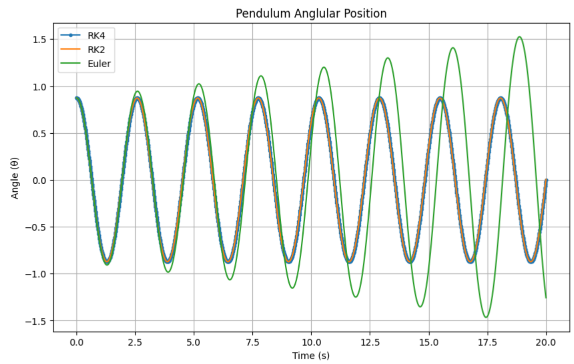

Fig. 6: The amplitude of oscillations increases for the Euler method

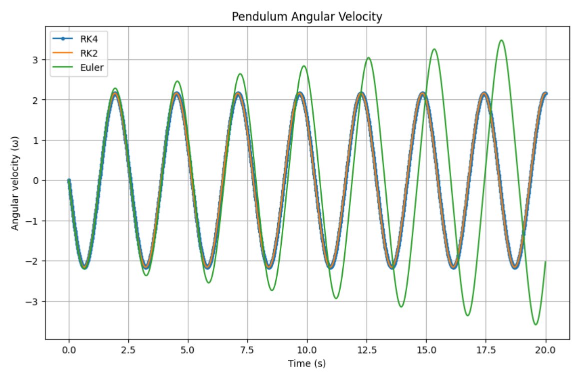

Fig. 7: Similarly, the amplitude of the angular velocity also increases

Energy Ratios

We can also see similar results by plotting the ratio of the energy at time t and the initial energy with time.

To better understand the improvement caused by the RK4 method over the RK2 method, we plot just those two energy ratios.

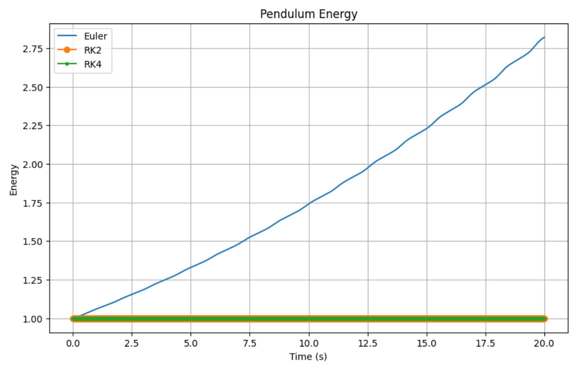

Fig. 8: The total mechanical energy of the system keeps increasing with the Euler method

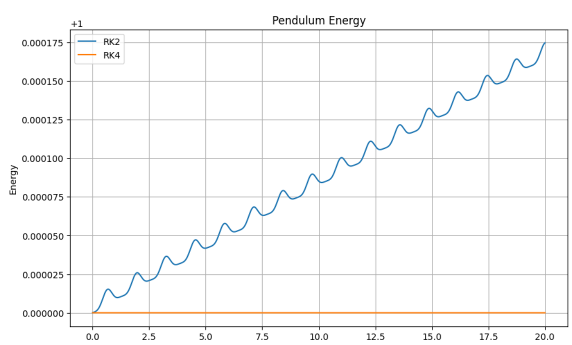

Fig. 9: Although RK2 is much better than Euler, we still see an increase in energy, albeit on a much smaller scale