UNIT 2

Image Enhancement in Frequency Domain

Course: Digital Image Processing(EC662)

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The 2-D Continuous Fourier Transform Pair

- Let f(t,z) be a continuous function of two continuous variables, t and z.
- The two-dimensional, continuous Fourier transform pair is given by the expressions

$$F(\mu, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t, z) e^{-j2\pi(\mu t + vz)}\, dt\, dz \tag{4-59}$$

and

$$f(t, z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(\mu, v)\, e^{j2\pi(\mu t + vz)}\, d\mu\, dv \tag{4-60}$$

- Where µ and v are the frequency variables. When referring to images, t and z are interpreted to be continuous spatial variables.

2-D Discrete Fourier Transform and its inverse

- 2-D discrete Fourier transform(DFT)

$$F(u, v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) e^{-j2\pi(ux/M + vy/N)} \tag{4-67}$$

- Where f(x,y) is a digital image of size M x N.
- Discreet variables u and v in the ranges u=0,1,2.....M-1 and v = 0,1....N-1.
- Given the transform F(u, v) we can obtain f(x, y) by using the inverse Fourier transform(IDFT).

$$f(x, y) = \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u, v) e^{j2\pi(ux/M + vy/N)} \tag{4-68}$$

Where x=0,1,2...M-1 and y = 0,1,2...N-1

Some properties of the 2-D DFT and IDFT

Relationships between Spatial and Frequency Intervals

- A Continuous function f(t,z) is a sampled to form a digital image,
- f(x, y) , consisting of M x N samples taken in the t- and z- directions.
- Let ΔT and ΔZ denote the separations between samples.
- Then, the separations between the corresponding discrete , frequency domain variables are given by

$$\Delta u = \frac{1}{M \Delta T} \tag{4-69}$$

and

$$\Delta v = \frac{1}{N \Delta Z} \tag{4-70}$$

- Note the important property that the separation between samples in the frequency domain are inversely proportional both to the spacing between spatial samples and to the number of samples.

- **Translation and Rotation**

The validity of the following Fourier transform pairs can be demonstrated by direct substitution into eqs. (4.67) and (4.68)

$$f(x,y)e^{j2\pi(u_0x/M+v_0y/N)} \Leftrightarrow F(u-u_0, v-v_0) \qquad (4\text{-}71)$$

and

$$f(x-x_0, y-y_0) \Leftrightarrow F(u,v)\, e^{-j2\pi(x_0u/M+y_0v/N)} \qquad (4\text{-}72)$$

I. E, multiplying f(x,y) by the exponential shown shifts the origin of the DFT to (U_0,V_0) and conversely, multiplying F(u,v) by the negative of that exponential shifts the origin of f(x,y) to (X_0,Y_0).

Translation has no effect on the magnitude(spectrum) of F(u,v).

Using the polar coordinates

$x = r\cos\theta$, $y=r\sin\theta$, $u=\omega\cos\psi$, $v= \omega\sin\psi$

- Results in the following transform pair:

$$f(r, \theta + \theta_0) \Leftrightarrow F(\omega, \varphi + \theta_0) \qquad (4\text{-}73)$$

- Which indicates that rotating f(x,y) by an angle θ_0 rotates F(u,v) by the same angle.
- Conversely, rotating F(u,v) rotates f(x,y) by the same angle.

Periodicity

The 2-D Fourier transform and its inverse are infinitely periodic in the u and v; i.e

$$F(u, v) = F(u + k_1 M, v) = F(u, v + k_2 N) = F(u + k_1 M, v + k_2 N) \qquad (4\text{-}74)$$

and

$$f(x, y) = f(x + k_1 M, y) = f(x, y + k_2 N) = f(x + k_1 M, y + k_2 N) \qquad (4\text{-}75)$$

K_1 and K_2 are integers.

a
b
c d

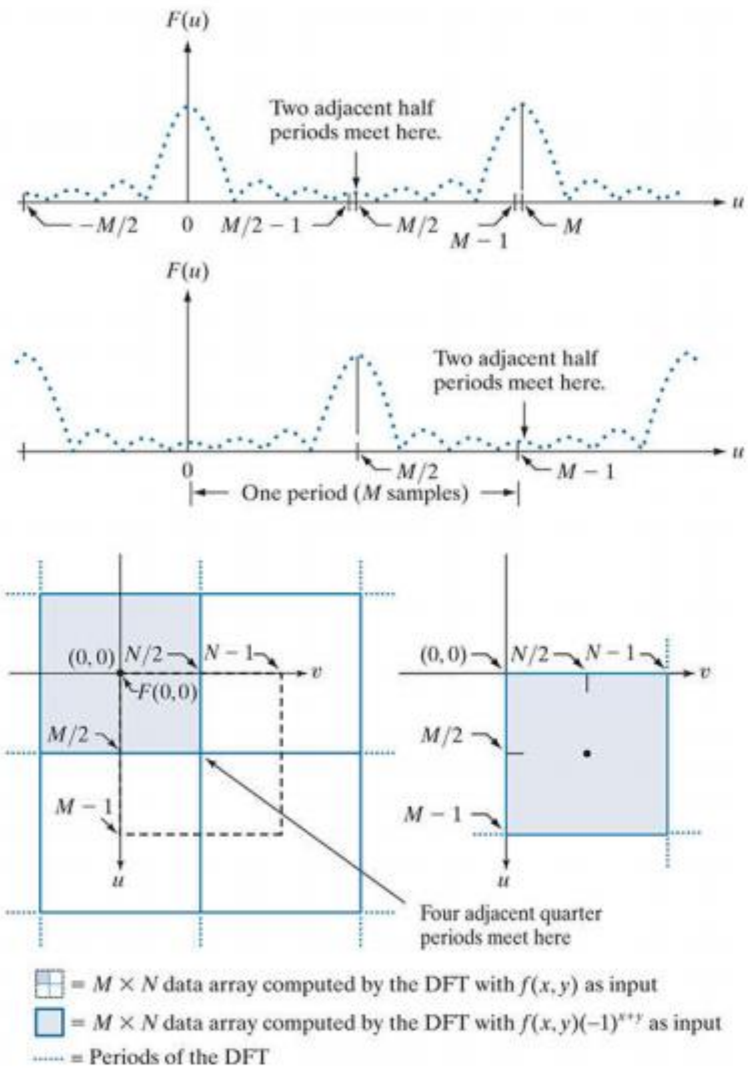


FIGURE 4.22

Centering the Fourier transform. (a) A 1-D DFT showing an infinite number of periods. (b) Shifted DFT obtained by multiplying $f(x)$ by $(-1)^x$ before computing $F(u)$. (c) A 2-D DFT showing an infinite number of periods. The area within the dashed rectangle is the data array, $F(u,v)$, obtained with Eq. (4-67) with an image $f(x, y)$ as the input. This array consists of four quarter periods. (d) Shifted array obtained by multiplying $f(x, y)$ by $(-1)^{x+y}$ before computing $F(u,v)$. The data now contains one complete, centered period, as in (b).

$$f(x)e^{j2\pi(u_0x/M)} \Leftrightarrow F(u-u_0)$$

- In other words, multiplying f(x,y)by the exponential term shown shifts the transform data so that the origin, F(0) is moved to u_0.
- If we let u_0= M/2, the exponential term becomes $e^{j\pi x}$, which is equal to -1^x because x is an integer.

$$f(x)(-1)^x \Leftrightarrow F(u-M/2)$$

- That is, multiplying f(x) $(-1)^x$ shifts the data so that F(u) is centered on the interval [0, M-1], which corresponds to Fig.4.22(b), as desired.
- In 2-D the situation is more difficult to graph, but the principle is the same as Fig.4.22(c) shows.
- Instead of two half periods, there are now four quarter periods meeting at the point (M/2,N/2).

- As in the 1-D case, we want to shift the data so that F(0,0) Is at (M/2,N/2).
- Letting (u_0,v_0)= (M/2,N/2) in Eq.(4-71) results in the expression

$$f(x, y)(-1)^{x+y} \Leftrightarrow F(u - M/2, v - N/2) \qquad (4\text{-}76)$$

- Using this equation shifts the data so that F(0,0) is moved to the center of the frequency rectangle(i.e. the rectangle defined by the intervals [0,M-1] and [0,N-1] in the frequency domain). Fig.4.22(d) shows the result.

Symmetry Properties

An important result from functional analysis is that any real or complex function, w(x,y), can be expressed as the sum of an even and an odd part, each of which can be real or complex:

$$W(x, y) = W_e(x, y) + W_o(x, y) \qquad (4\text{-}77)$$

where the even and odd parts are defined as

$$w_e(x, y) \triangleq \frac{w(x, y) + w(-x, -y)}{2} \quad (4\text{-}78)$$

and

$$w_o(x, y) \triangleq \frac{w(x, y) - w(-x, -y)}{2} \quad (4\text{-}79)$$

- For all valid values of x and y. substituting Eqs (4-78) and (4-79) into Eq.(4-77) gives the identity w(x, y) Ξ w(x,y), thus proving the validity of the latter equation. It follows from the preceding definitions that

$$w_e(x, y) = w_e(-x, -y) \quad (4\text{-}80)$$

and

$$w_o(x, y) = -w_o(-x, -y) \quad (4\text{-}81)$$

- Even functions are said to be symmetric and odd functions antisymmetric.
- Because all indices in the DFT and IDFT are nonnegative integers, when we talk about symmetry(antisymmetry) we are referring to symmetry(antisymmetry) about the center point of a sequence, in which case the definitions of even and odd become.

$$w_e(x, y) = w_e(M - x, N - y) \quad (4\text{-}82)$$

and

$$w_o(x, y) = -w_o(M - x, N - y) \quad (4\text{-}83)$$

- For x = 0,1,2….M-1 and y = 0,1,2,…N-1. M and N are the number of rows and columns of a 2-D array.
- We know from elementary mathematical analysis that the product of two even or two odd functions is even, and that the product of an even and an odd function is odd. In addition, the only way that a discrete function can be odd is if all its samples sum to zero.
- These properties lead to the important result that

$$\sum_{x=0}^{M-1} \sum_{y=0}^{N-1} w_e(x, y) w_o(x, y) = 0 \quad (4\text{-}84$$

- For any two discrete even and odd functions We and Wo. In other words, because the argument of Eq.(4-84) is odd, the result of the summations is 0. The functions can be real or complex.

Fourier Spectrum and Phase Angle

- Because the 2-D DFT is complex in general, it can be expressed in polar form:

$$\begin{aligned} F(u,v) &= R(u,v) + jI(u,v) \\ &= |F(u,v)|e^{j\phi(u,v)} \end{aligned} \quad (4\text{-}86)$$

When the magnitude

$$|F(u,v)| = \left[R^2(u,v) + I^2(u,v)\right]^{1/2} \quad (4\text{-}87)$$

- Is called the Fourier (or frequency)spectrum and

$$\phi(u,v) = \arctan\left[\frac{I(u,v)}{R(u,v)}\right] \quad (4\text{-}88)$$

- Is the phase angle or phase spectrum.
- Finally, the power spectrum is defined as

$$\begin{aligned} P(u,v) &= |F(u,v)|^2 \\ &= R^2(u,v) + I^2(u,v) \end{aligned} \quad (4\text{-}89)$$

- R and I are the real and imaginary parts of F(u, v), and all computations are carried out for the discrete variables u = 0,1,2….M-1 and v = 0,1,2…N-1.
- Therefore [F(u, v)],Φ(u, v) and P(u, v) are arrays of size M x N.
- The Fourier transform of a real function is conjugate symmetric, which implies that the spectrum has even symmetry about the origin.

$$|F(u,v)| = |F(-u,-v)| \tag{4-90}$$

- The phase angle exhibits odd symmetry about the origin:

$$\phi(u,v) = -\phi(-u,-v) \tag{4-91}$$

- It follows from Eq.(4-67) that

$$F(0,0) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y)$$

- Which indicates that zero-frequency term of the DFT is proportional to the average of f(x,y). That is

$$F(0,0) = MN\frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y) \tag{4-92}$$
$$= MN\overline{f}$$

where $\overline{f}$ (a scalar) denotes the average value of *f*(*x*, *y*). Then,

$$|F(0,0)| = MN|\overline{f}| \tag{4-93}$$

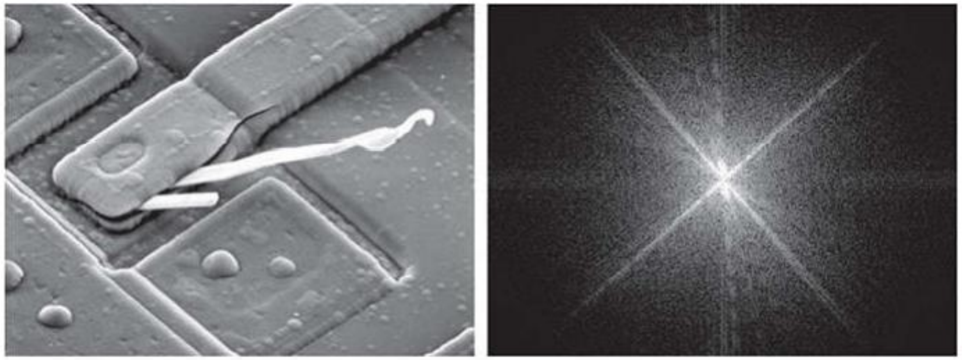
Because the proportionality constant *MN* usually is large, $|F(0,0)|$ typically is the largest component of the spectrum by a factor that can be several orders of magnitude larger than other terms. Because frequency components *u* and *v* are zero at the origin, *F*(0,0) sometimes is called the *dc component* of the transform. This terminology is from electrical engineering, where “dc” signifies direct current (i.e., current of zero frequency).

Frequency Domain Filtering Fundamentals

Filtering in the frequency domain consists of modifying the Fourier transform of an image, then computing the inverse transform to obtain the spatial domain representation of the processed result. Thus, given (a padded) digital image, $f(x, y)$, of size $P \times Q$ pixels, the basic filtering equation in which we are interested has the form:

$$g(x, y) = \text{Real}\{\mathfrak{J}^{-1}[H(u, v)F(u, v)]\} \tag{4-104}$$

where $\mathfrak{J}^{-1}$ is the IDFT, $F(u,v)$ is the DFT of the input image, $f(x, y)$, $H(u,v)$ is a *filter transfer function* (which we often call just a *filter* or *filter function*), and $g(x, y)$ is the *filtered* (*output*) *image*. Functions F, H, and g are arrays of size $P \times Q$, the same as the padded input image. The product $H(u,v)F(u,v)$ is formed using elementwise multiplication, as defined in Section 2.6 . The filter transfer function modifies the transform of the input image to yield the processed output, $g(x, y)$. The task of specifying $H(u,v)$ is simplified considerably by using functions that are symmetric about their center, which requires that $F(u,v)$ be centered also. As explained in Section 4.6 , this is accomplished by multiplying the input image by $(-1)^{x+y}$ prior to computing its transform.†

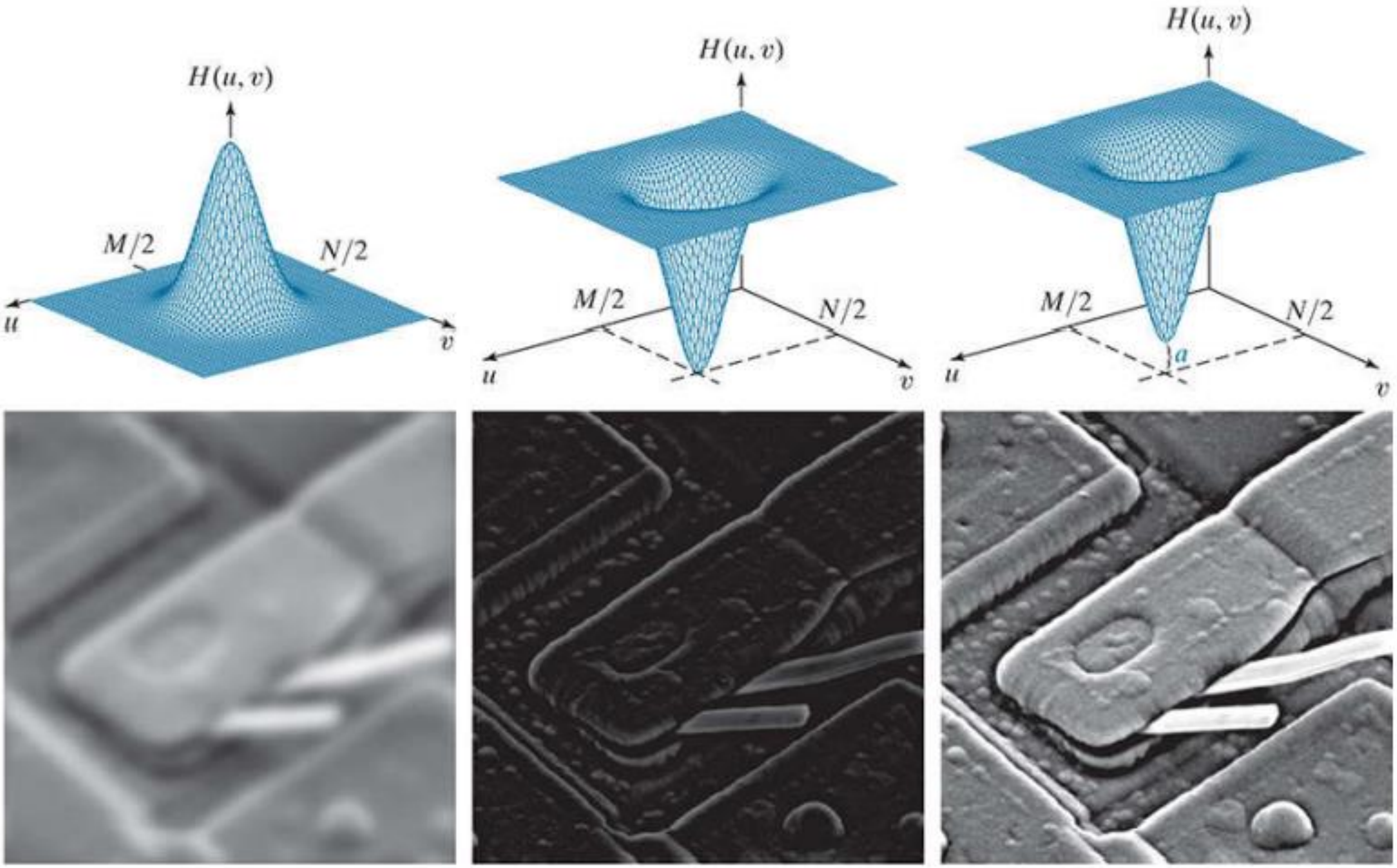


a b

FIGURE 4.28

(a) SEM image of a damaged integrated circuit. (b) Fourier spectrum of (a).

(Original image courtesy of Dr. J. M. Hudak, Brockhouse Institute for Materials Research, McMaster University, Hamilton, Ontario, Canada.)



a b c
d e f

FIGURE 4.30

Top row: Frequency domain filter transfer functions of (a) a lowpass filter, (b) a highpass filter, and (c) an offset highpass filter. Bottom row: Corresponding filtered images obtained using **Eq. (4-104)** . The offset in (c) is $a = 0.85$, and the height of $H(u,v)$ is 1. Compare (f) with **Fig. 4.28(a)** .

a b c

FIGURE 4.31

(a) A simple image. (b) Result of blurring with a Gaussian lowpass filter without padding. (c) Result of lowpass filtering with zero padding. Compare the vertical edges in (b) and (c).

a c
b d

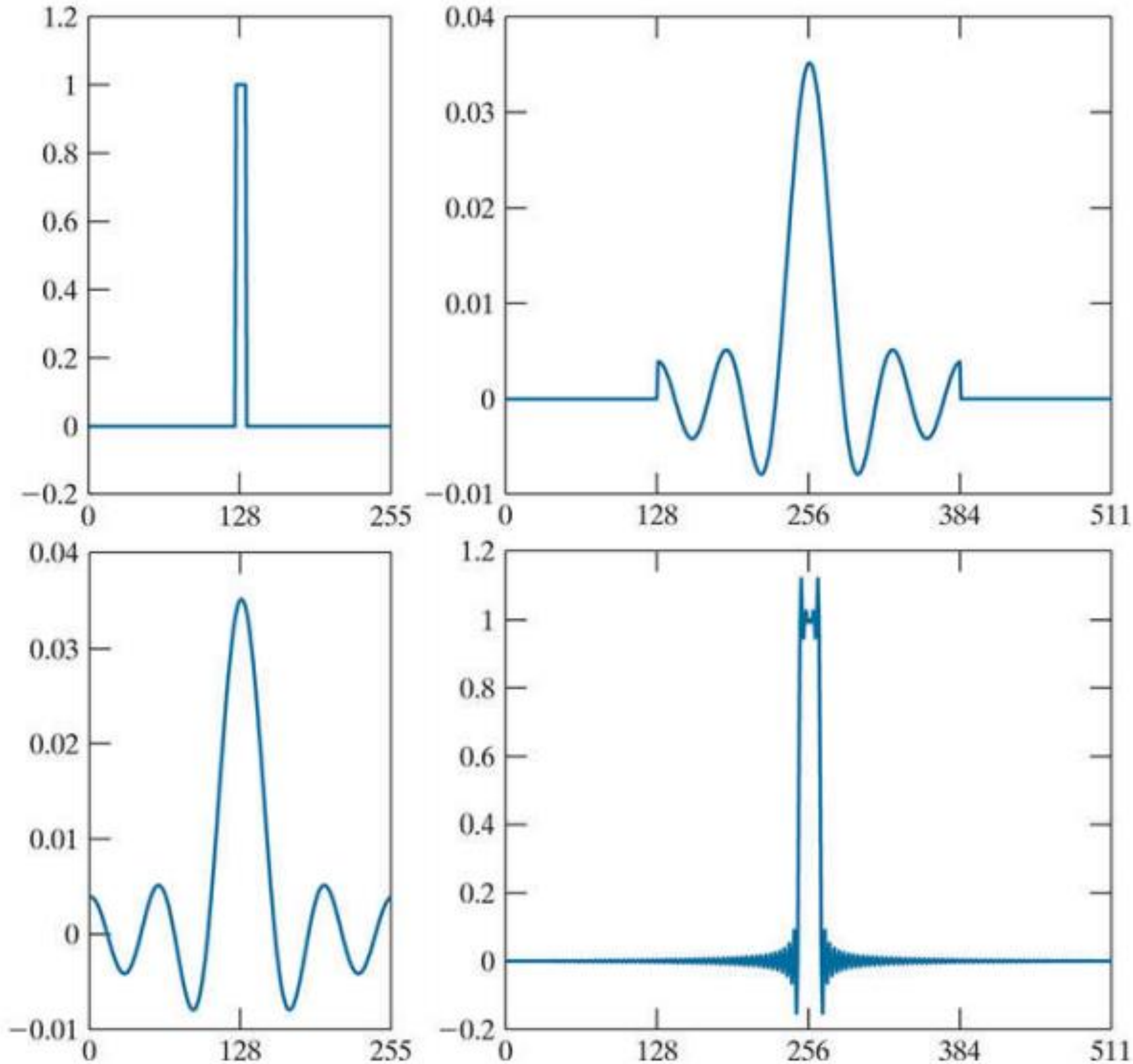


FIGURE 4.33

(a) Filter transfer function specified in the (centered) frequency domain. (b) Spatial representation (filter kernel) obtained by computing the IDFT of (a). (c) Result of padding (b) to twice its length (note the discontinuities). (d) Corresponding filter in the frequency domain obtained by computing the DFT of (c). Note the ringing caused by the discontinuities in (c). Part (b) of the figure is below (a), and (d) is below (c).

Summary of steps for filtering in the frequency domain

The process of filtering in the frequency domain can be summarized as follows:

1. Given an input image *f*(*x*, *y*) of size $M \times N$, obtain the padding sizes *P* and *Q* using Eqs. (4-102) and (4-103) ; that is, $P = 2M$ and $Q = 2N$.
2. Form a padded† image $f_p(x, y)$ of size $P \times Q$ using zero-, mirror-, or replicate padding (see Fig. 3.45 for a comparison of padding methods).

   † Sometimes we omit padding when doing "quick" experiments to get an idea of filter performance, or when trying to determine quantitative relationships between spatial features and their effect on frequency domain components, particularly in band and notch filtering, as explained later in Section 4.10 and in Chapter 5 .

3. Multiply $f_p(x, y)$ by $(-1)^{x+y}$ to center the Fourier transform on the $P \times Q$ frequency rectangle.
4. Compute the DFT, *F*(*u*, *v*), of the image from Step 3.
5. Construct a real, symmetric filter transfer function, *H*(*u*, *v*), of size $P \times Q$ with center at (*P*/2,*Q*/2).
6. Form the product $G(u, v) = H(u, v)F(u, v)$ using elementwise multiplication; that is, $G(i, k) = H(i, k)F(i, k)$ for $i = 0, 1, 2, \ldots, M - 1$ and $k = 0, 1, 2, \ldots, N - 1$.

   > See Section 2.6 for a definition of elementwise operations.

7. Obtain the filtered image (of size $P \times Q$ by computing the IDFT of *G*(*u*,*v*):

$$g_p(x, y) = \left(\text{real}\left[\mathfrak{J}^{-1}\{G(u, v)\}\right]\right)(-1)^{x+y}$$

8. Obtain the final filtered result, *g*(*x*, *y*), of the same size as the input image, by extracting the $M \times N$ region from the top, left quadrant of $g_p(x, y)$.

We will discuss the construction of filter transfer functions (Step 5) in the following sections of this chapter. In theory, the IDFT in Step 7 should be real because *f*(*x*, *y*) is real and *H*(*u*,*v*) is real and symmetric. However, parasitic complex terms in the IDFT resulting from computational inaccuracies are not uncommon. Taking the real part of the result takes care of that. Multiplication by $(-1)^{x+y}$ cancels out the multiplication by this factor in Step 3.

Correspondence Between Filtering in the Spatial and Frequency Domains

- we defined filtering in the frequency domain as the element wise product of a filter transfer function, H(u, v), and F(u, v), the Fourier transform of the input image.
- Given H(u, v), suppose that we want to find its equivalent kernel in the spatial domain.
- If we let f(x, y) = ∂(x, y), and F(u, v)=1the filtered output is $T^{-1}\{H(u,v)\}$.
- This expression as the inverse transform of the frequency domain filter transfer function, which is the corresponding kernel in the spatial domain. Conversely, it follows from a similar analysis and the convolution theorem that, given a spatial filter kernel, we obtain its frequency domain representation by taking the forward Fourier transform of the kernel. Therefore, the two filters form a Fourier transform pair:

$$h(x, y) \Leftrightarrow H(u, v) \quad (4\text{-}106)$$

- Where h(x, y) is the spatial kernel. Because this kernel can be obtained from the response of a frequency domain filter to an impulse, h(x, y) sometimes is referred to as the impulse response of H(u, v).
- Also, because all quantities in a discrete implementation of Eq. (4-106) are finite, such filters are called finite impulse response (FIR) filters.
- Let H(u) denote the 1-D frequency domain Gaussian transfer function

$$H(u) = Ae^{-u^2/2\sigma^2} \quad (4\text{-}107)$$

where σ is the standard deviation of the Gaussian curve. The kernel in the spatial domain is obtained by taking the inverse DFT of *H*(*u*)

$$h(x) = \sqrt{2\pi}\sigma A\, e^{-2\pi^2\sigma^2x^2} \quad (4\text{-}108)$$

These two equations are important for two reasons:

(1) They are a Fourier transform pair, both components of which are Gaussian and real. This facilitates analysis because we do not have to be concerned with complex numbers. In addition, Gaussian curves are intuitive and easy to manipulate.

(2) The functions behave reciprocally. When H(u) has a broad profile (large value of), h(x) has a narrow profile, and vice versa. In fact, as approaches infinity, H(u) tends toward a constant function and h(x) tends toward an impulse, which implies no filtering in either domain.

Figures 4.36(a) and **(b)** show plots of a Gaussian lowpass filter transfer function in the frequency domain and the corresponding function in the spatial domain. Suppose that we want to use the shape of $h(x)$ in **Fig. 4.36(b)** as a guide for specifying the coefficients of a small kernel in the spatial domain. The key characteristic of the function in **Fig. 4.36(b)** is that all its values are positive. Thus, we conclude that we can implement lowpass filtering in the spatial domain by using a kernel with all positive coefficients (as we did in **Section 3.5**). For reference, **Fig. 4.36(b)** also shows two of the kernels discussed in that section. Note the reciprocal relationship between the width of the Gaussian functions, as discussed in the previous paragraph. The narrower the frequency domain function, the more it will attenuate the low frequencies, resulting in increased blurring. In the spatial domain, this means that a larger kernel must be used to increase blurring, as we illustrated in **Example 3.14** .

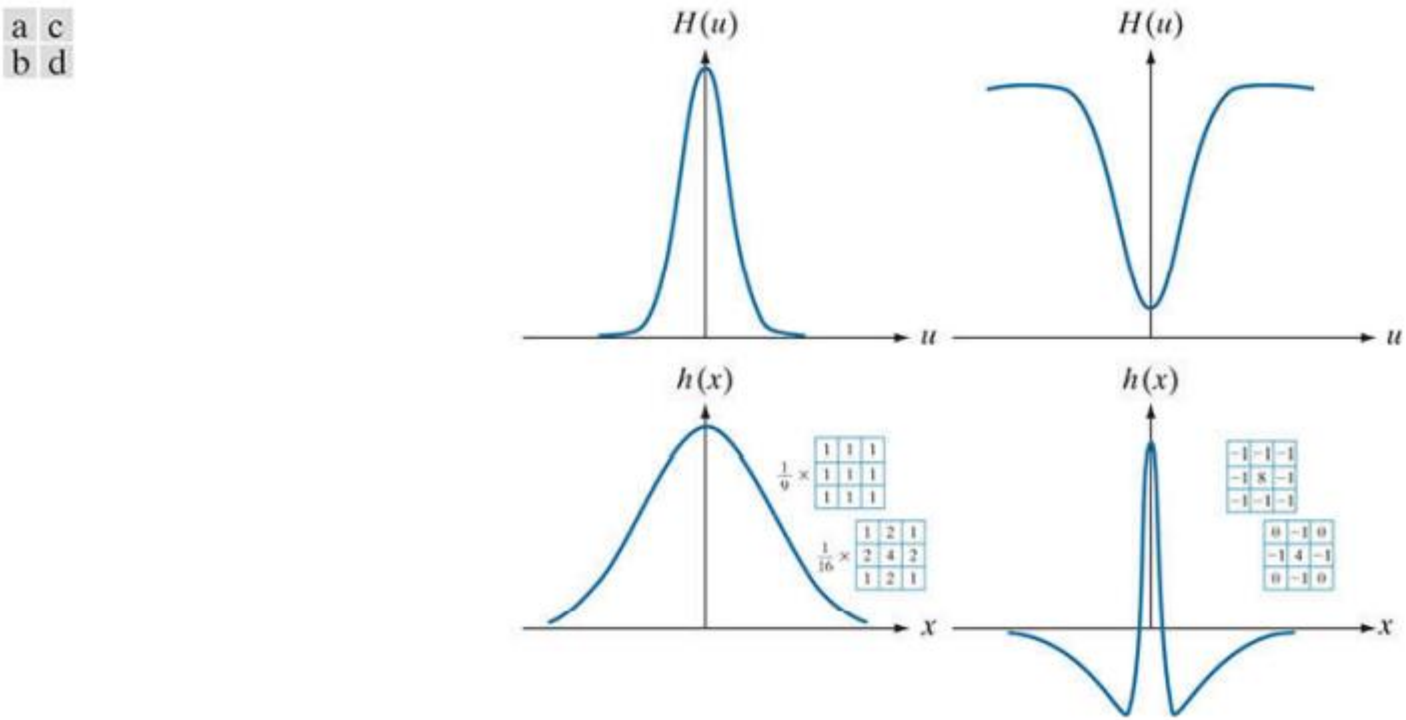


FIGURE 4.36

(a) A 1-D Gaussian lowpass transfer function in the frequency domain. (b) Corresponding kernel in the spatial domain. (c) Gaussian highpass transfer function in the frequency domain. (d) Corresponding kernel. The small 2-D kernels shown are kernels we used in

EXAMPLE 4.15: Obtaining a frequency domain transfer function from a spatial kernel.

a b

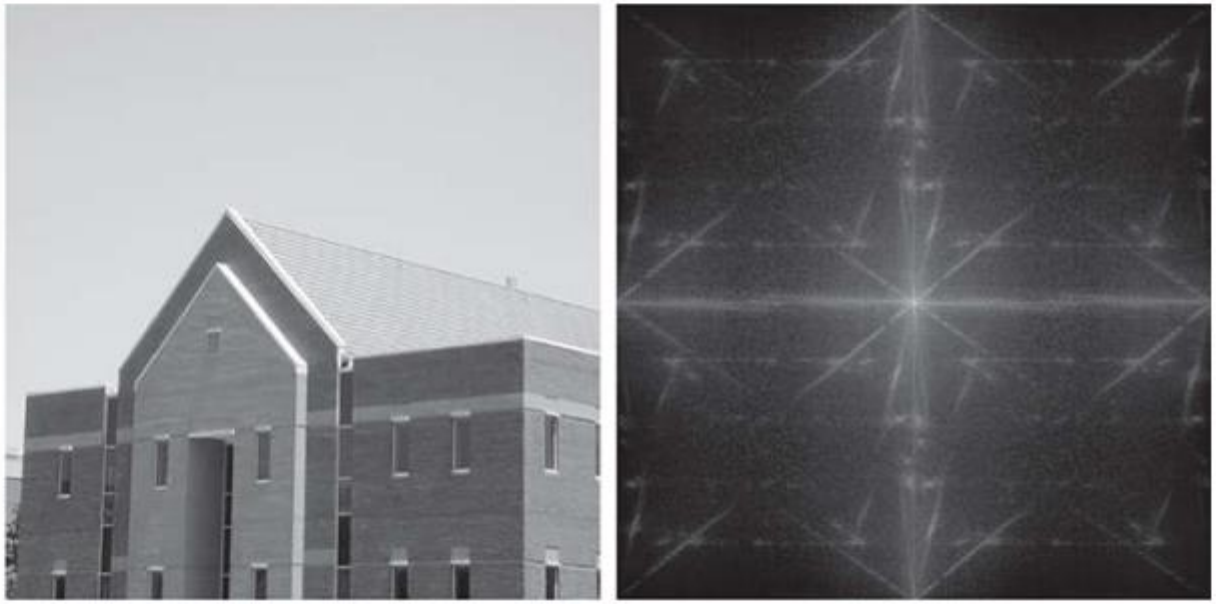


FIGURE 4.37
(a) Image of a building, and (b) its Fourier spectrum.

a b
c d

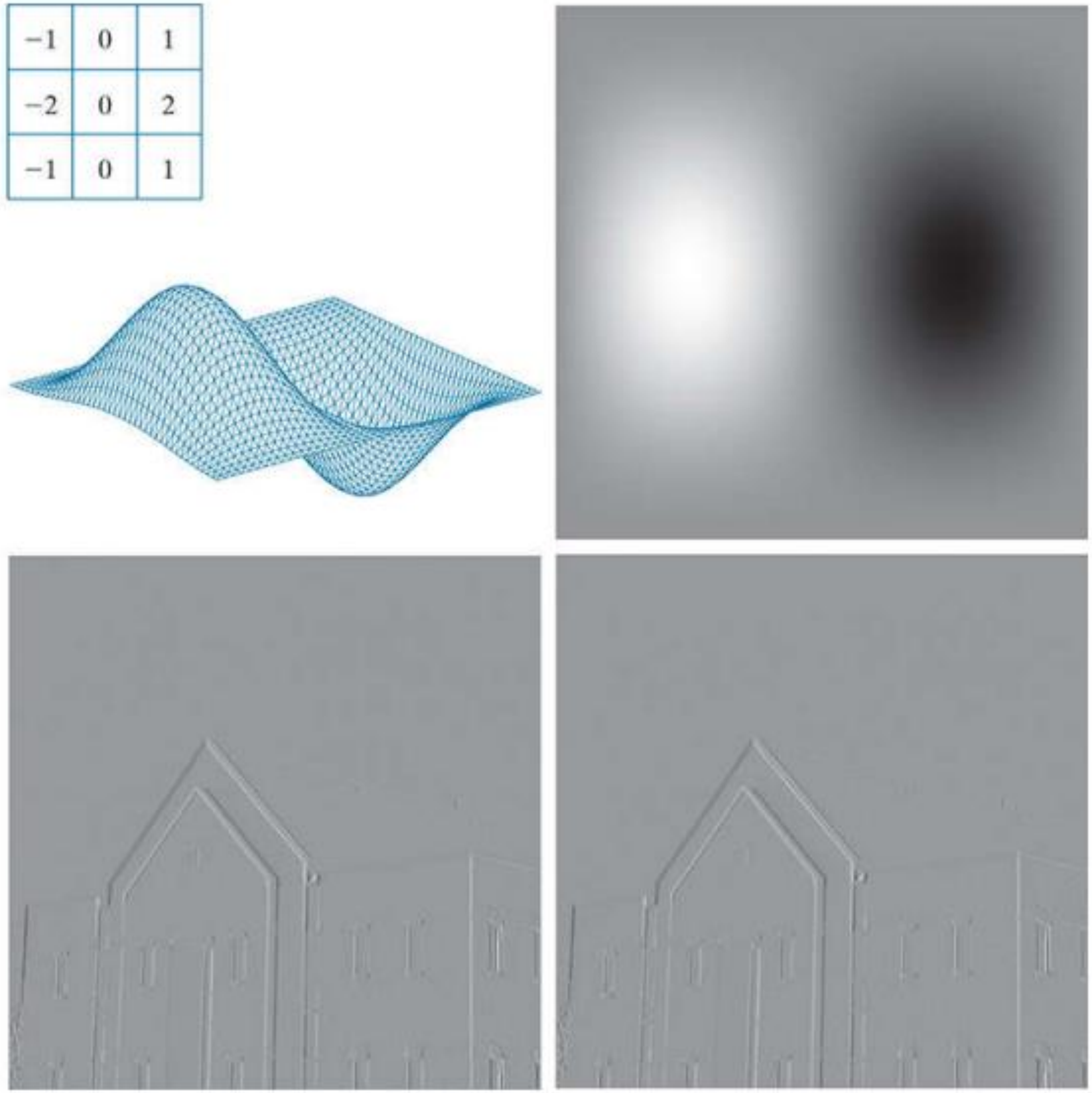


FIGURE 4.38

(a) A spatial kernel and perspective plot of its corresponding frequency domain filter transfer function. (b) Transfer function shown as an image. (c) Result of filtering Fig. 4.37(a) in the frequency domain with the transfer function in (b). (d) Result of filtering the same image in the spatial domain with the kernel in (a). The results are identical.

Image Smoothing Using Lowpass Frequency Domain Filters

Three types of lowpass filters:

1. Ideal lowpass filter
2. Butterworth lowpass filter
3. Gaussian lowpass filter

- These three categories cover the range from very sharp(ideal) to very smooth(Gaussian) filtering.
- The shape of a Butterworth filter is controlled by a parameter called the filter order.
- For lower values, the butterworth filter is more like a Gaussian filter.

- **Ideal Lowpass Filters**

➢ A 2-D lowpass filter that passes without attenuation all frequencies within a circle of radius from the origin, and "cut off" all frequencies outside this, circle is called an ideal low pass filter(ILPF).

➢ It is specified by the transfer function

$$H(u,v) = \begin{cases} 1 & \text{if } D(u,v) \le D_0 \\ 0 & \text{if } D(u,v) > D_0 \end{cases} \qquad (4\text{-}111)$$

➢ Where D_0 is a positive constant and D(u,v) is the distance between a point(u,v) in the frequency domain and the center of the PxQ frequency rectangle; that is

$$D(u,v) = \left[(u - P/2)^2 + (v - Q/2)^2\right]^{1/2} \qquad (4\text{-}112)$$

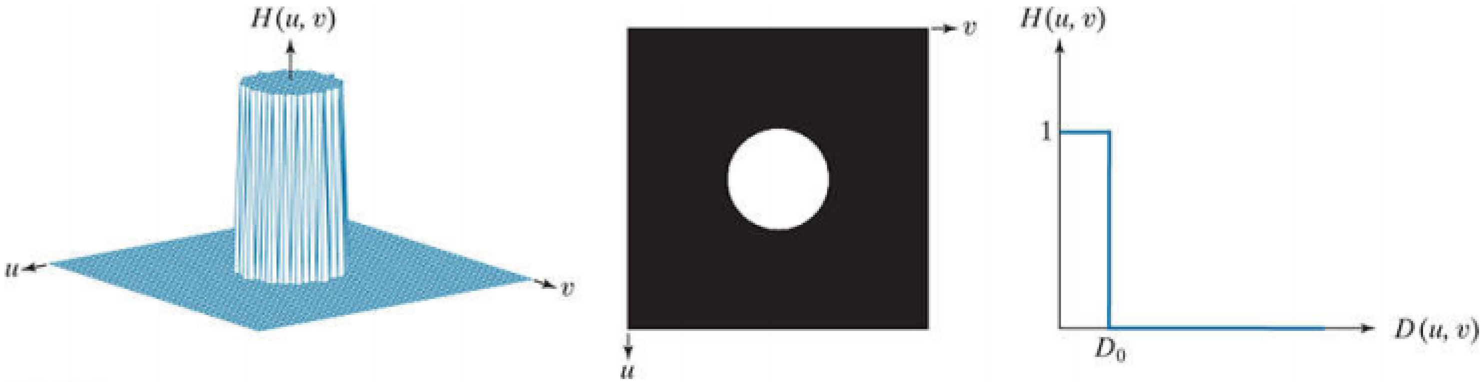


FIGURE 4.39

(a) Perspective plot of an ideal lowpass-filter transfer function. (b) Function displayed as an image. (c) Radial cross section.

- Figure 4.39(a) shows a perspective plot of transfer function H(u,v) and
- Fig.4.39(b) shows it displayed as an image.
- Fig 4.39(c) indicates that all frequencies on or inside a circle of radius D_0 are passed without attenuation, whereas all frequencies outside the circle are completely attenuated (flitted out).
- For an ILPF cross section, the point of transition between the values H(u,v)=1 and H(u,v)=0 is called the cutoff frequency.
- In fig 4.39 the cutoff frequency is D_0.

- One way to establish standard cutoff frequency loci using circles that enclose specified amounts of total image power PT, which we obtain by summing the components of the power spectrum of the padded images at each point(u,v) for u = 0,1,2….P-1 and v = 0,1,2…..Q-1.

$$P_T = \sum_{u=0}^{P-1} \sum_{v=0}^{Q-1} P(u,v) \qquad (4\text{-}113)$$

- If the DFT has been centered, a circle of radius D_0 with origin at the center of the frequency rectangle encloses α percent of the power, where

$$\alpha = 100\left[\sum_{u} \sum_{v} P(u,v)/P_T\right] \qquad (4\text{-}114)$$

and the summation is over values of (u,v) that lie inside the circle or on its boundary.

- Fig 4.42 (a) shows an image of a frequency domain ILPF transfer function of radius 15 and size 1000x1000 pixels.
- Fig 4.42(b) is the spatial representation h(x,y) of the ILPF, obtained by taking the IDFT of (a)
- Fig 4.42(c) shows the intensity profile of a line passing through the center of (b). This profile resembles a sinc function.

a b c

FIGURE 4.42
(a) Frequency domain ILPF transfer function. (b) Corresponding spatial domain kernel function. (c) Intensity profile of a horizontal line through the center of (b).

Gaussian Lowpass Filters

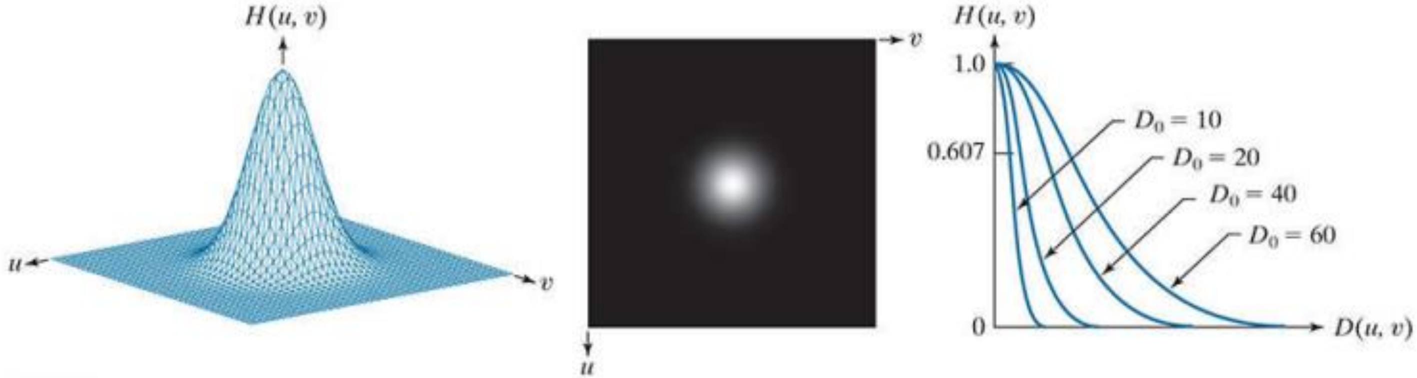
Gaussian lowpass filter (GLPF) transfer functions have the form

$$H(u,v) = e^{-D^2(u,v)/2\sigma^2} \tag{4-115}$$

where, as in Eq. (4-112), $D(u,v)$ is the distance from the center of the $P \times Q$ frequency rectangle to any point, (u, v), contained by the rectangle. Unlike our earlier expressions for Gaussian functions, we do not use a multiplying constant here in order to be consistent with the filters discussed in this and later sections, whose highest value is 1. As before, σ is a measure of spread about the center. By letting $\sigma = D_0$, we can express the Gaussian transfer function in the same notation as other functions in this section:

$$H(u,v) = e^{-D^2(u,v)/2D_0^2} \tag{4-116}$$

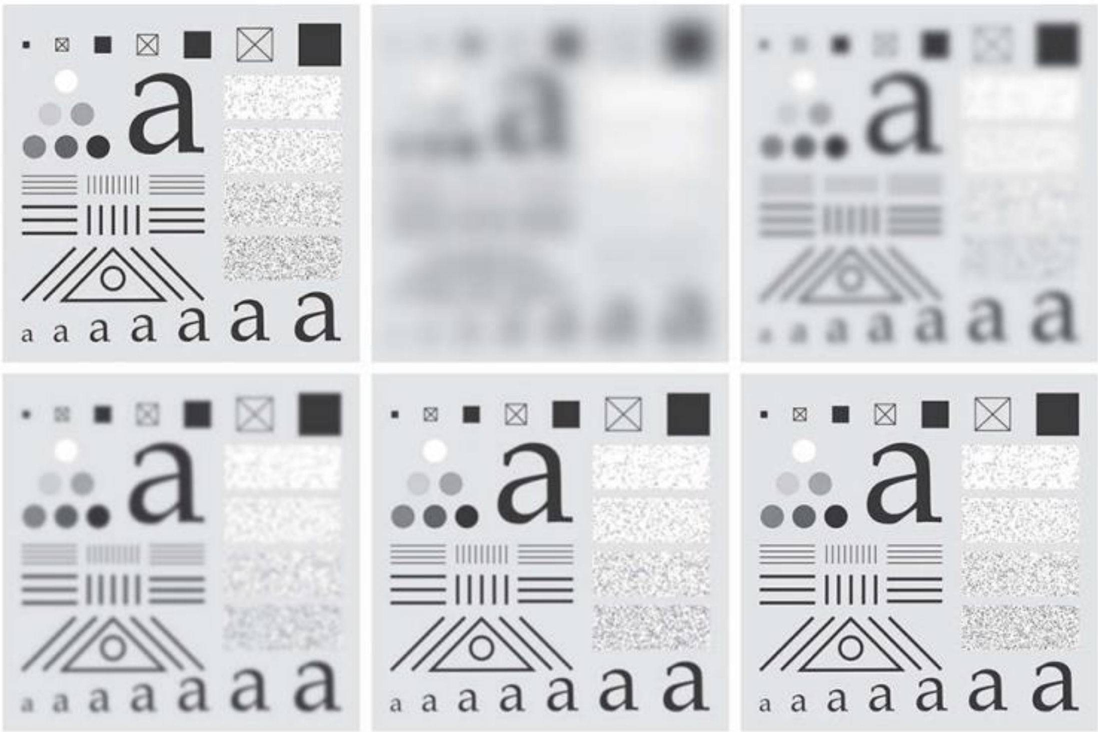
where D_0 is the cutoff frequency. When $D(u,v) = D_0$, the GLPF transfer function is down to 0.607 of its maximum value of 1.0.



a b c

FIGURE 4.43
(a) Perspective plot of a GLPF transfer function. (b) Function displayed as an image. (c) Radial cross sections for various values of D_0.

EXAMPLE 4.17: Image smoothing in the frequency domain using Gaussian lowpass filters.



a b c
d e f

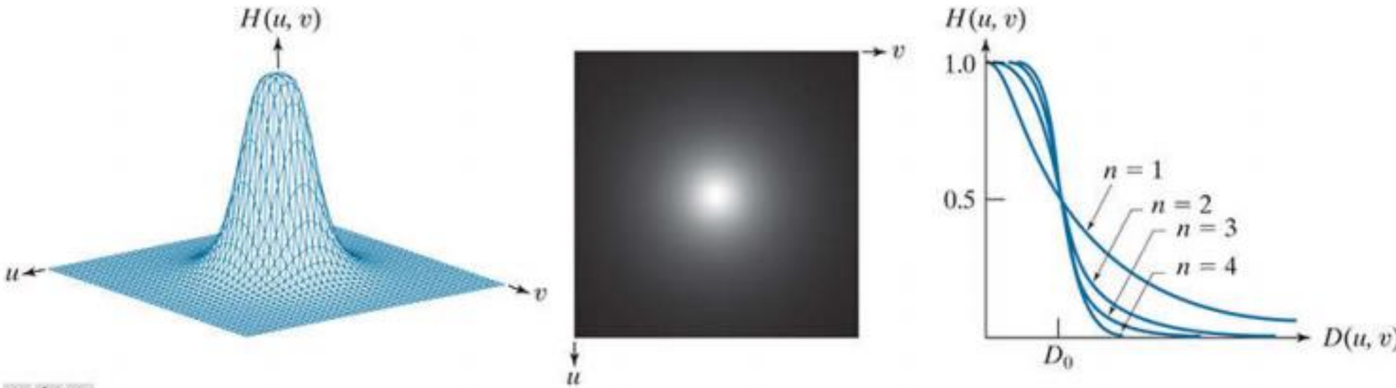
FIGURE 4.44
(a) Original image of size 688×688 pixels. (b)–(f) Results of filtering using GLPFs with cutoff frequencies at the radii shown in Fig. 4.40 . Compare with Fig. 4.41 . We used mirror padding to avoid the black borders characteristic of zero padding.

Butterworth Lowpass Filters

The transfer function of a Butterworth lowpass filter (BLPF) of order *n*, with cutoff frequency at a distance D_0 from the center of the frequency rectangle, is defined as

$$H(u,v) = \frac{1}{1 + [D(u,v)/D_0]^{2n}} \tag{4-117}$$

where $D(u,v)$ is given by Eq. (4-112) . Figure 4.45 shows a perspective plot, image display, and radial cross sections of the BLPF function. Comparing the cross section plots in Figs. 4.39 , 4.43 , and 4.45 , we see that the BLPF function can be controlled to approach the characteristics of the ILPF using higher values of *n*, and the GLPF for lower values of *n*, while providing a smooth transition in from low to high frequencies. Thus, we can use a BLPF to approach the sharpness of an ILPF function with considerably less ringing.



a b c

FIGURE 4.45

(a) Perspective plot of a Butterworth lowpass-filter transfer function. (b) Function displayed as an image. (c) Radial cross sections of BLPFs of orders 1 through 4.

EXAMPLE 4.18: Image smoothing using a Butterworth lowpass filter.

Figures 4.46(b) -(f) show the results of applying the BLPF of Eq. (4-117) to Fig. 4.46(a), with cutoff frequencies equal to the five radii in Fig. 4.40(b), and with $n = 2.25$. The results in terms of blurring are between the results obtained with using ILPFs and GLPFs. For example, compare Fig. 4.46(b), with Figs. 4.41(b) and 4.44(b). The degree of blurring with the BLPF was less than with the ILPF, but more than with the GLPF.

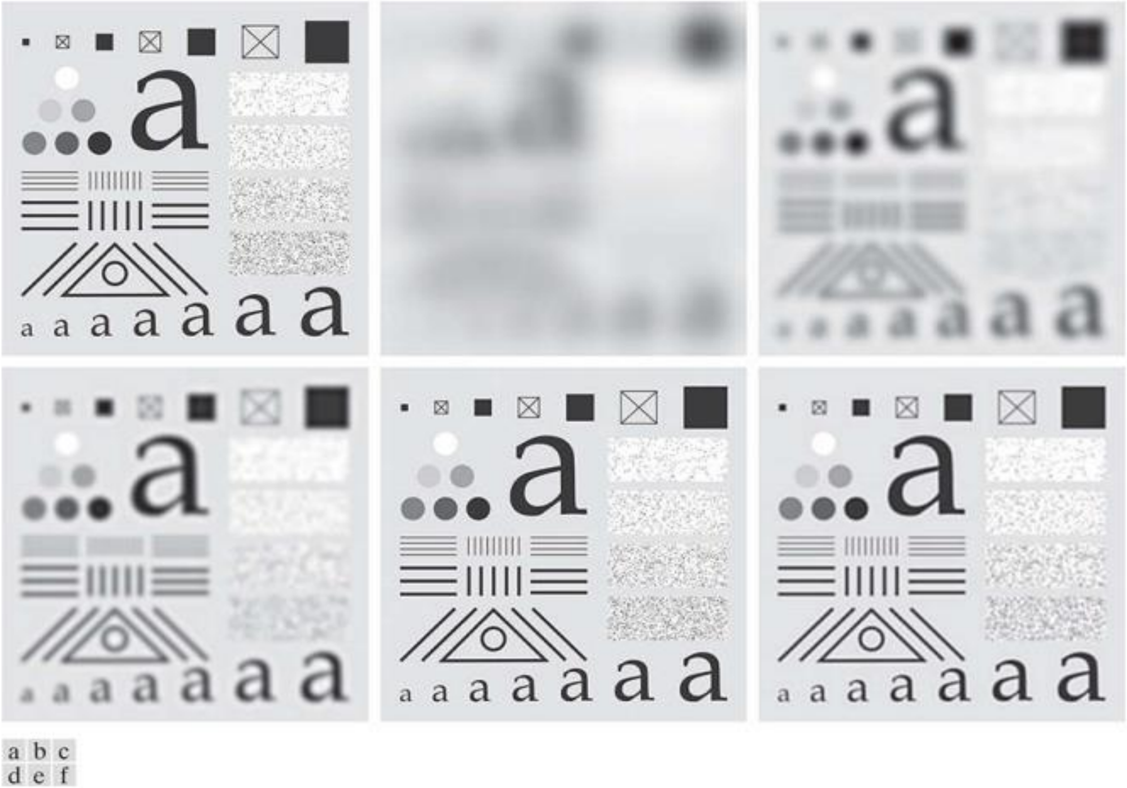


FIGURE 4.46

(a) Original image of size 688×688 pixels. (b)–(f) Results of filtering using BLPFs with cutoff frequencies at the radii shown in Fig. 4.40 and $n = 2.25$. Compare with Figs. 4.41 and 4.44. We used mirror padding to avoid the black borders characteristic of zero padding.

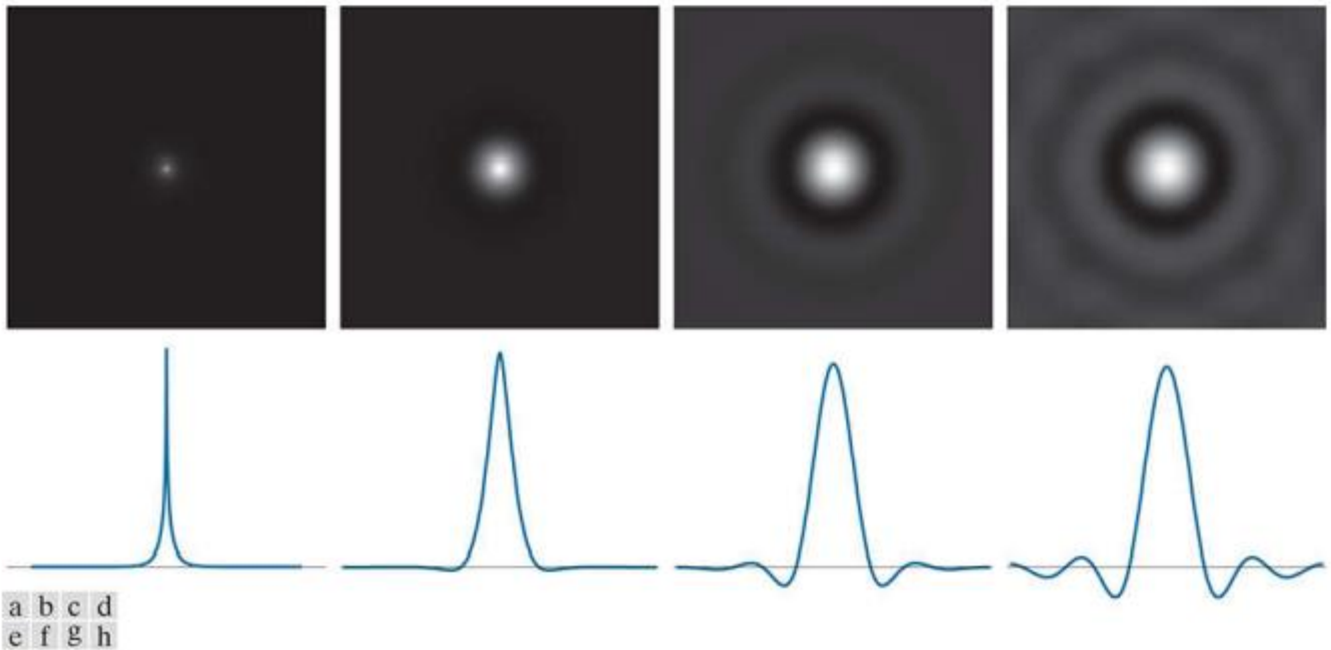
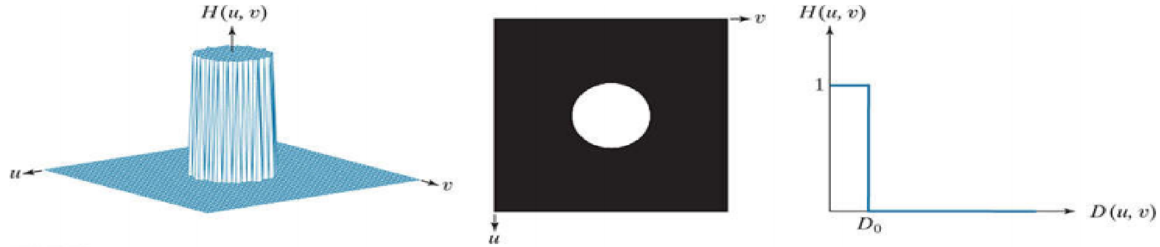


FIGURE 4.47
(a)–(d) Spatial representations (i.e., spatial kernels) corresponding to BLPF transfer functions of size 1000×1000 pixels, cut-off frequency of 5, and order 1, 2, 5, and 20, respectively. (e)–(h) Corresponding intensity profiles through the center of the filter functions.

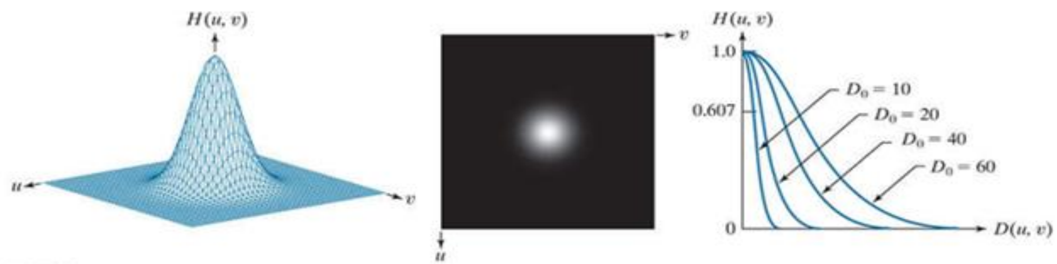
TABLE 4.5
Lowpass filter transfer functions. D_0 is the cutoff frequency, and *n* is the order of the Butterworth filter.

Ideal	Gaussian	Butterworth
$H(u,v) = \begin{cases} 1 & \text{if } D(u,v) \le D_0 \\ 0 & \text{if } D(u,v) > D_0 \end{cases}$	$H(u,v) = e^{-D^2(u,v)/2D_0^2}$	$H(u,v) = \frac{1}{1+[D(u,v)/D_0]^{2n}}$



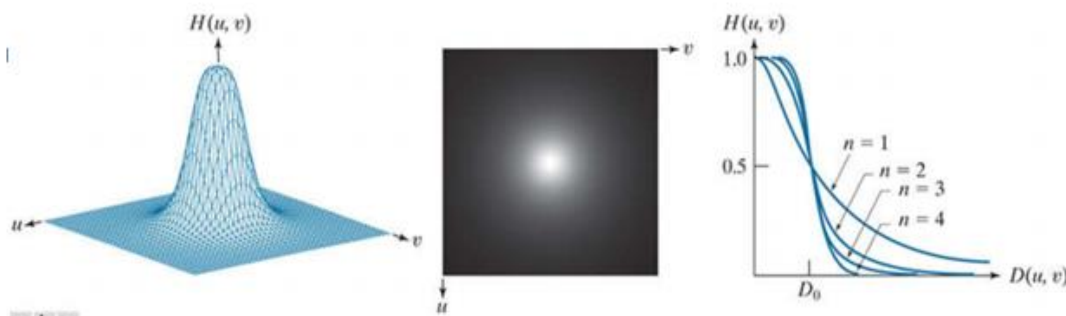
a b c

FIGURE 4.39
(a) Perspective plot of an ideal lowpass-filter transfer function. (b) Function displayed as an image. (c) Radial cross section.



a b c

FIGURE 4.43
(a) Perspective plot of a GLPF transfer function. (b) Function displayed as an image. (c) Radial cross sections for various values of D_0.



a b c

FIGURE 4.45
(a) Perspective plot of a Butterworth lowpass-filter transfer function. (b) Function displayed as an image. (c) Radial cross sections of BLPFs of orders 1 through 4.

Examples of Lowpass Filtering

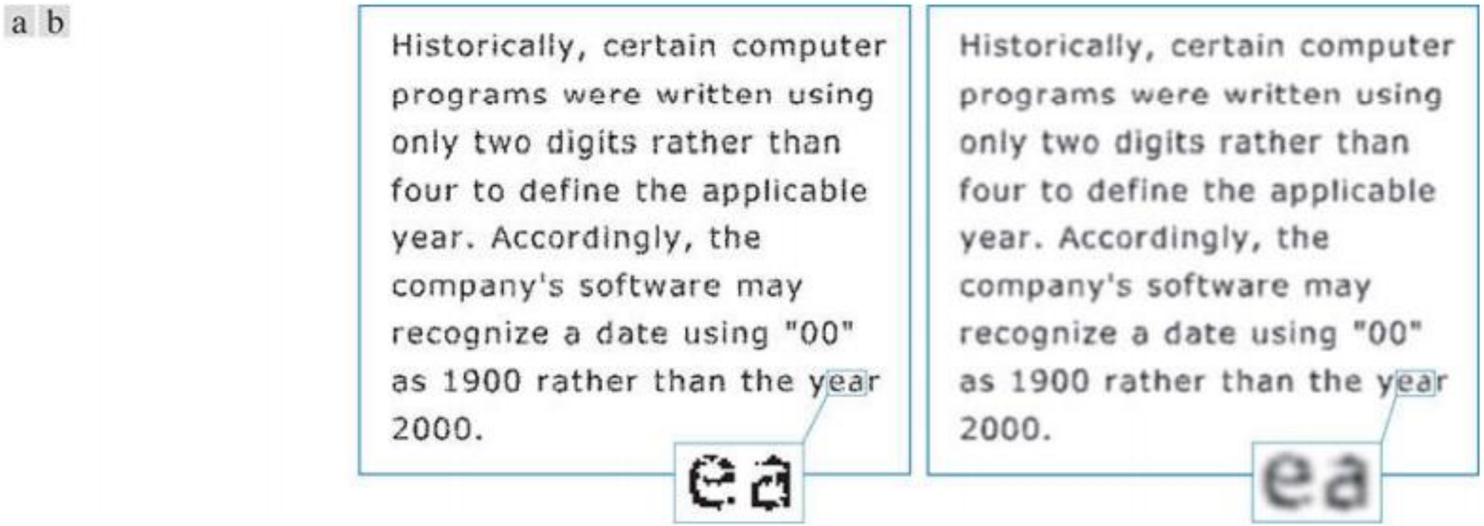


FIGURE 4.48
(a) Sample text of low resolution (note the broken characters in the magnified view). (b) Result of filtering with a GLPF, showing that gaps in the broken characters were joined.

a b c

FIGURE 4.49

(a) Original 785×732 image. (b) Result of filtering using a GLPF with $D_0 = 150$. (c) Result of filtering using a GLPF with $D_0 = 130$. Note the reduction in fine skin lines in the magnified sections in (b) and (c).

Image Sharpening Using Highpass Filters

- Highpass filter transfer function in the frequency domain.

$$H_{\mathrm{HP}}(u,v) = 1 - H_{\mathrm{LP}}(u,v) \tag{4-118}$$

where $H_{\mathrm{LP}}(u,v)$ is the transfer function of a lowpass filter. Thus, it follows from Eq. (4-111) that an ideal highpass filter (IHPF) transfer function is given by

$$H(u,v) = \begin{cases} 0 & \text{if } D(u,v) \le D_0 \\ 1 & \text{if } D(u,v) > D_0 \end{cases} \tag{4-119}$$

where, as before, $D(u,v)$ is the distance from the center of the $P \times Q$ frequency rectangle, as given in Eq. (4-112). Similarly, it follows from Eq. (4-116) that the transfer function of a Gaussian highpass filter (GHPF) transfer function is given by

$$H(u,v) = 1 - e^{-D^2(u,v)/2D_0^2} \tag{4-120}$$

and, from Eq. (4-117), that the transfer function of a Butterworth highpass filter (BHPF) is

$$H(u,v) = \frac{1}{1 + [D_0/D(u,v)]^{2n}} \tag{4-121}$$

a b c
d e f
g h i

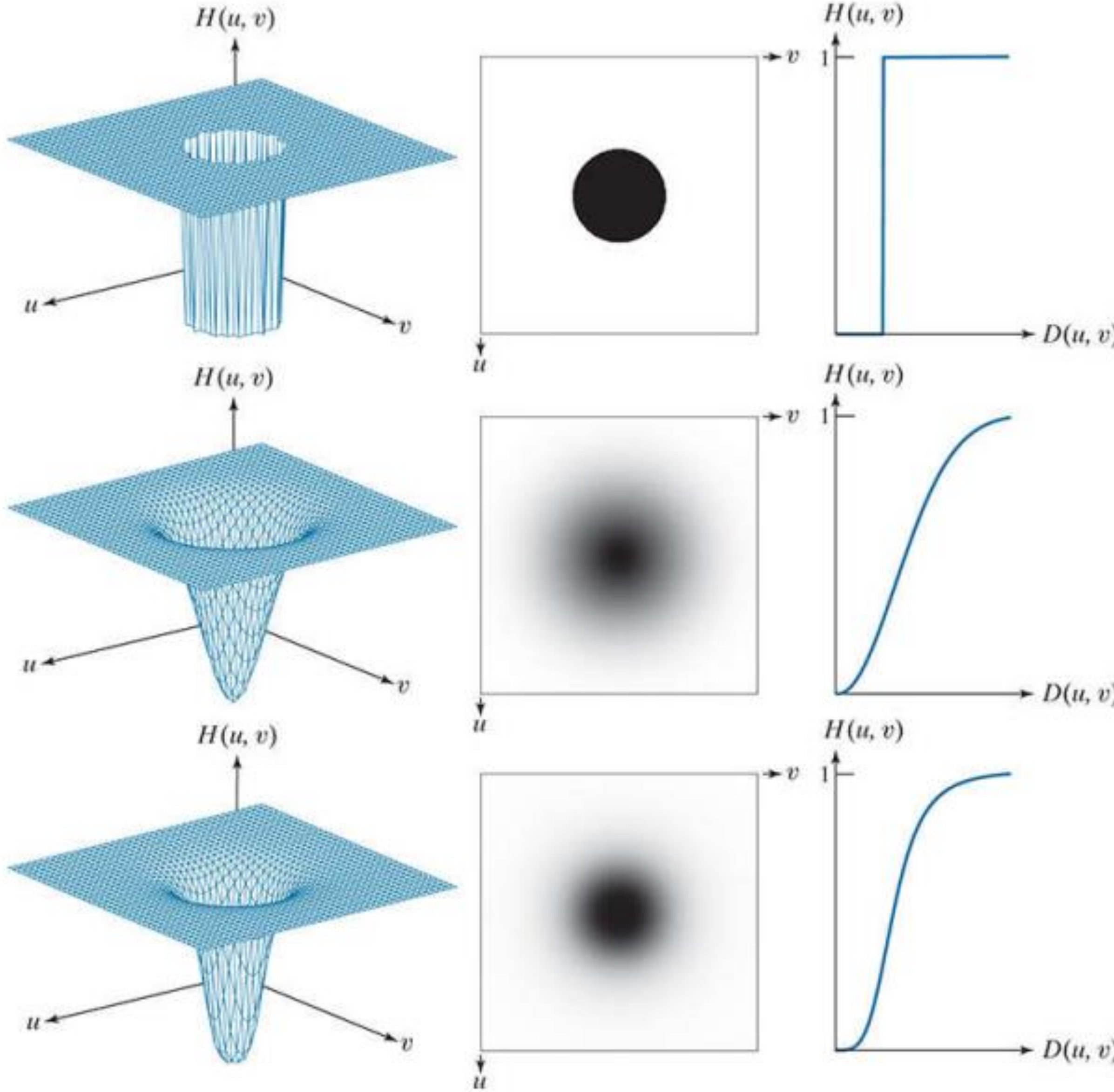


FIGURE 4.51
Top row: Perspective plot, image, and, radial cross section of an IHPF transfer function. Middle and bottom rows: The same sequence for GHPF and BHPF transfer functions. (The thin image borders were added for clarity. They are not part of the data.)

- Figure 4.51 shows 3-D plots, image representations, and radial cross sections for the preceding transfer functions. As before, we see that the BHPF transfer function in the third row of the figure represents a transition between the sharpness of the IHPF and the broad smoothness of the GHPF transfer function.

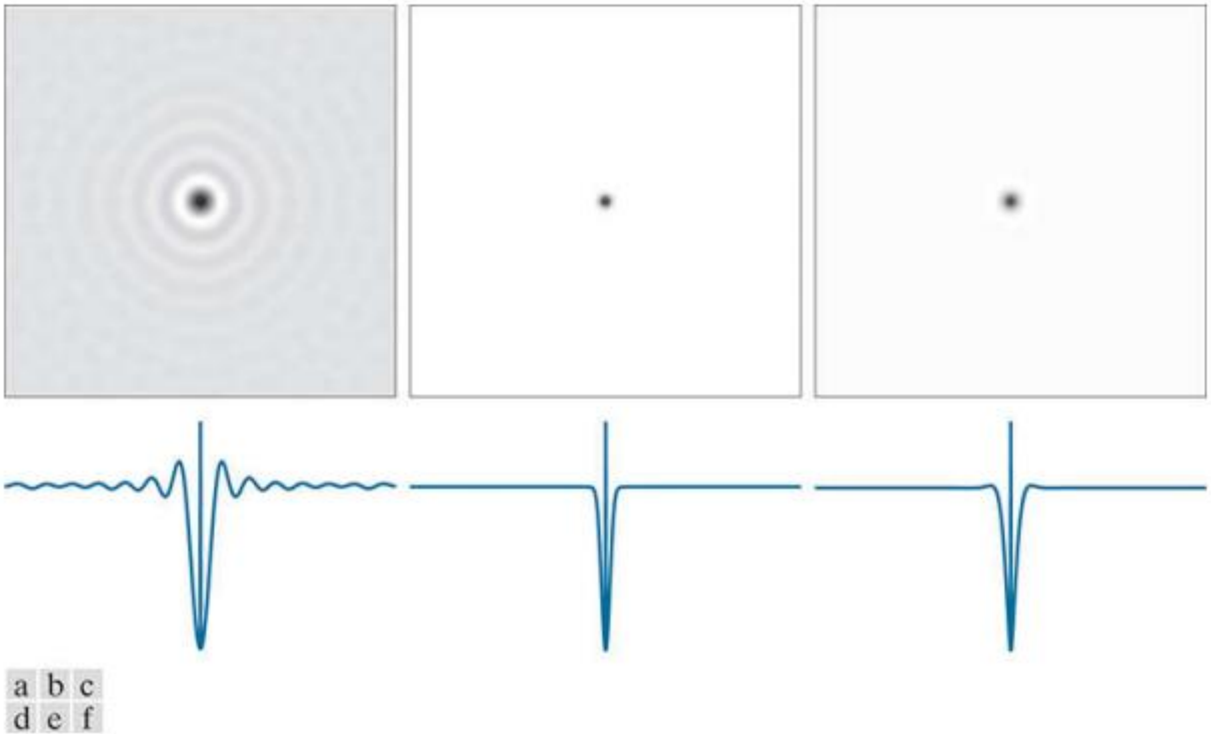


FIGURE 4.52
(a)–(c): Ideal, Gaussian, and Butterworth highpass spatial filters obtained from IHPF, GHPF, and BHPF frequency-domain transfer functions. (The thin image borders are not part of the data.) (d)–(f): Horizontal intensity profiles through the centers of the kernels.

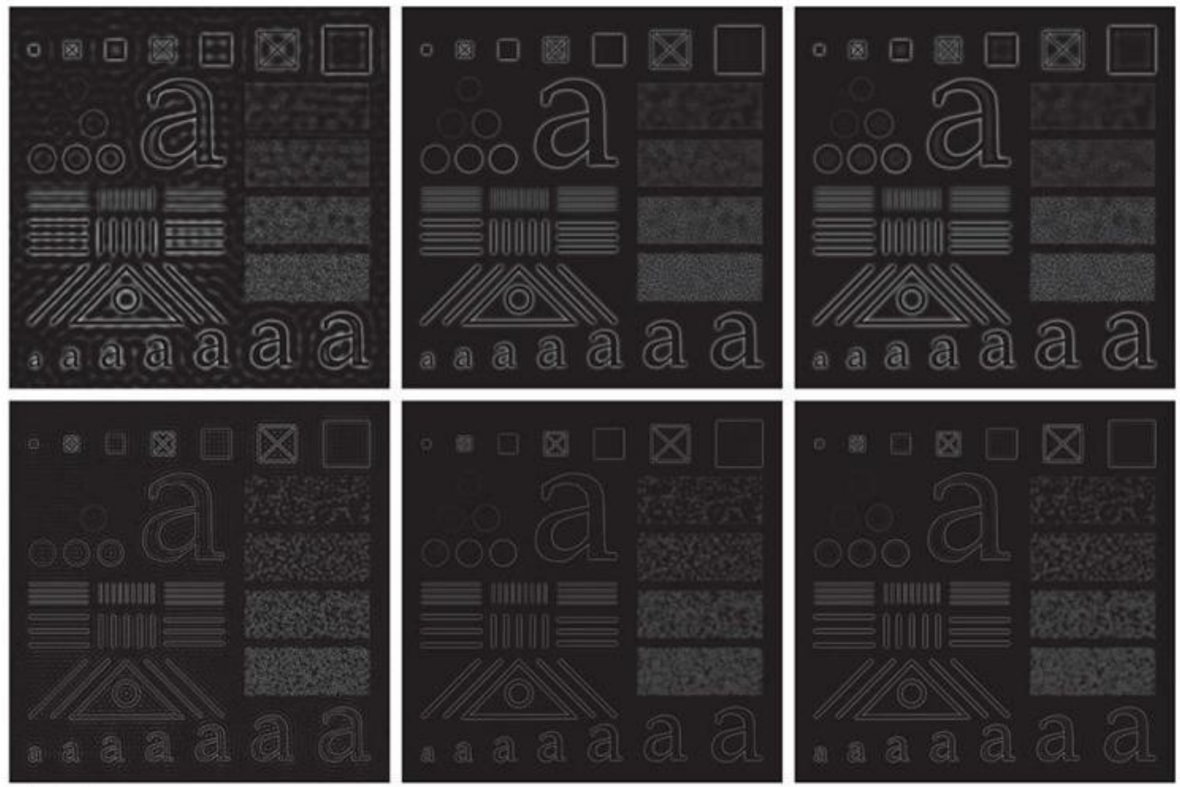
TABLE 4.6

Highpass filter transfer functions. D_0 is the cutoff frequency and n is the order of the Butterworth transfer function.

Ideal	Gaussian	Butterworth
$H(u,v) = \begin{cases} 0 & \text{if } D(u,v) \le D_0 \\ 1 & \text{if } D(u,v) > D_0 \end{cases}$	$H(u,v) = 1 - e^{-D^2(u,v)/2D_0^2}$	$H(u,v) = \dfrac{1}{1+[D_0/D(u,v)]^{2n}}$

EXAMPLE 4.19:

Highpass filtering of the character test pattern.



a b c
d e f

FIGURE 4.53

Top row: The image from Fig. 4.40 (a) filtered with IHPF, GHPF, and BHPF transfer functions using $D_0 = 60$ in all cases ($n = 2$ for the BHPF). Second row: Same sequence, but using $D_0 = 160$.

EXAMPLE 4.20: Using highpass filtering and thresholding for image enhancement.

a b c

FIGURE 4.55
(a) Smudged thumbprint. (b) Result of highpass filtering (a). (c) Result of thresholding (b).

The Laplacian in the Frequency Domain

- We used the Laplacian for image sharpening in the spatial domain.
- In this section, we revisit the Laplacian and show that it yields equivalent results using frequency domain techniques.

- The Laplacian can be implemented in the frequency domain using the filter transfer function.

$$H(u,v) = -4\pi^2(u^2 + v^2) \tag{4-123}$$

- Or, with respect to the center of the frequency rectangle, using the transfer function.

$$\begin{aligned} H(u,v) &= -4\pi^2\left[(u - P/2)^2 + (v - Q/2)^2\right] \\ &= -4\pi^2 D^2(u,v) \end{aligned} \tag{4-124}$$

- D(u,v) is the distance function

- Using this transfer function, the Laplacian of an image f(x,y), is obtained in the familiar manner.

$$\nabla^2 f(x, y) = \mathfrak{J}^{-1}[H(u,v)F(u,v)] \qquad (4\text{-}125)$$

- Where F(u, v) is the DFT of f(x, y).enhancement is implemented using the equation

$$g(x, y) = f(x, y) + c\nabla^2 f(x, y) \qquad (4\text{-}126)$$

- Here , C = -1 because H(u, v) is negative. f(x, y) and $\nabla^2 f(x, y)$ had comparable values.
- However, computing $\nabla^2 f(x, y)$ with Eq.(4-125) introduce DFT scaling factors that can be several orders of magnitude larger than the maximum value of f.
- Thus, the difference between f and its Laplacian must be brought into comparable ranges.
- The easiest way to handle this problem is to normalize the values of f(x, y) to the range (0,1)(before computing its DFT) and divid $\nabla^2 f(x, y)$ by its maximum value, which will bring it to the approximate range[-1, 1].

- We can write Eq.(4-126) directly in the frequency domain as

$$
\begin{aligned}
g(x, y) &= \mathfrak{J}^{-1}\{F(u,v) - H(u,v)F(u,v)\} \\
&= \mathfrak{J}^{-1}\{[1 - H(u,v)]F(u,v)\} \\
&= \mathfrak{J}^{-1}\{[1 + 4\pi^2 D^2(u,v)]F(u,v)\}
\end{aligned}
\tag{4-127}
$$

- EXAMPLE 4.21: Image sharpening in the frequency domain using the Laplacian.

a b

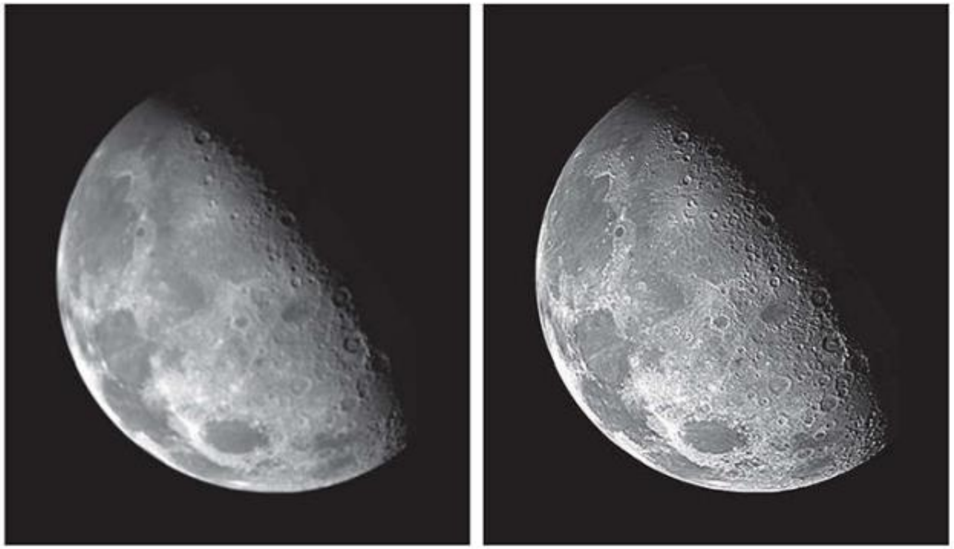


FIGURE 4.56

(a) Original, blurry image. (b) Image enhanced using the Laplacian in the frequency domain. Compare with Fig. 3.52(d) .

(Original image courtesy of NASA.)

Unsharp Masking, High-Boost Filtering, and High-Frequency-Emphasis Filtering

- In This section, we discuss frequency domain formulations of the unsharp masking and high- boost filtering image sharpening techniques introduced.
- Using frequency domain methods, the mask defined in Eq.(3.64) is given by

$$g_{\text{mask}}(x, y) = f(x, y) - f_{\text{LP}}(x, y) \tag{4-128}$$

with

$$f_{\text{LP}}(x, y) = \mathfrak{J}^{-1}[H_{\text{LP}}(u, v)F(u, v)] \tag{4-129}$$

Where H_{LP} is a lowpass filter transfer function and f(u, v) is the DFT of f(x, y) is a smoothed image analogous to f(x, y).

$$g(x, y) = f(x, y) + kg_{mask}(x, y) \qquad (4.130)$$

This expression defines unsharp masking when k =1 and high-boost filtering when k > 1.

- Using the preceding results, we can express Eq.(4-130) entirely in terms of frequency domain computations involving a lowpass filter.

$$g(x,y) = \mathcal{J}^{-1}\{(1 + k[1 - H_{\text{LP}}(u,v)])F(u,v)\} \qquad (4\text{-}131)$$

- We can express this result in terms of a highpass filter using Eq.(4-118)

$$g(x,y) = \mathcal{J}^{-1}\{[1 + kH_{\text{HP}}(u,v)]F(u,v)\} \qquad (4\text{-}132)$$

- The expression contained within the square brackets is called a high-frequency –emphasis filter function.
- As noted earlier, highpass filters set the dc term to zero, thus reducing the average intensity in the filtered image to 0.
- The high-frequency-emphasis filter does not have this problem because of the 1 that is added to the highpass filter transfer function.
- Constant k gives control over the proportional of high frequencies that influences the final result.
- A slightly more general formulation of high-frequency-emphasis filtering is the expression

$$g(x,y) = \mathcal{J}^{-1}\{[k_1 + k_2 H_{\text{HP}}(u,v)]F(u,v)\} \qquad (4\text{-}133)$$

- Where $K_1 \geq 0$ offsets the value the transfer function so as not to zero-out the dc term and $K_2 > 0$ controls the contribution of high frequencies.

EXAMPLE 4.22: Image enhancement using high-frequency-emphasis filtering.

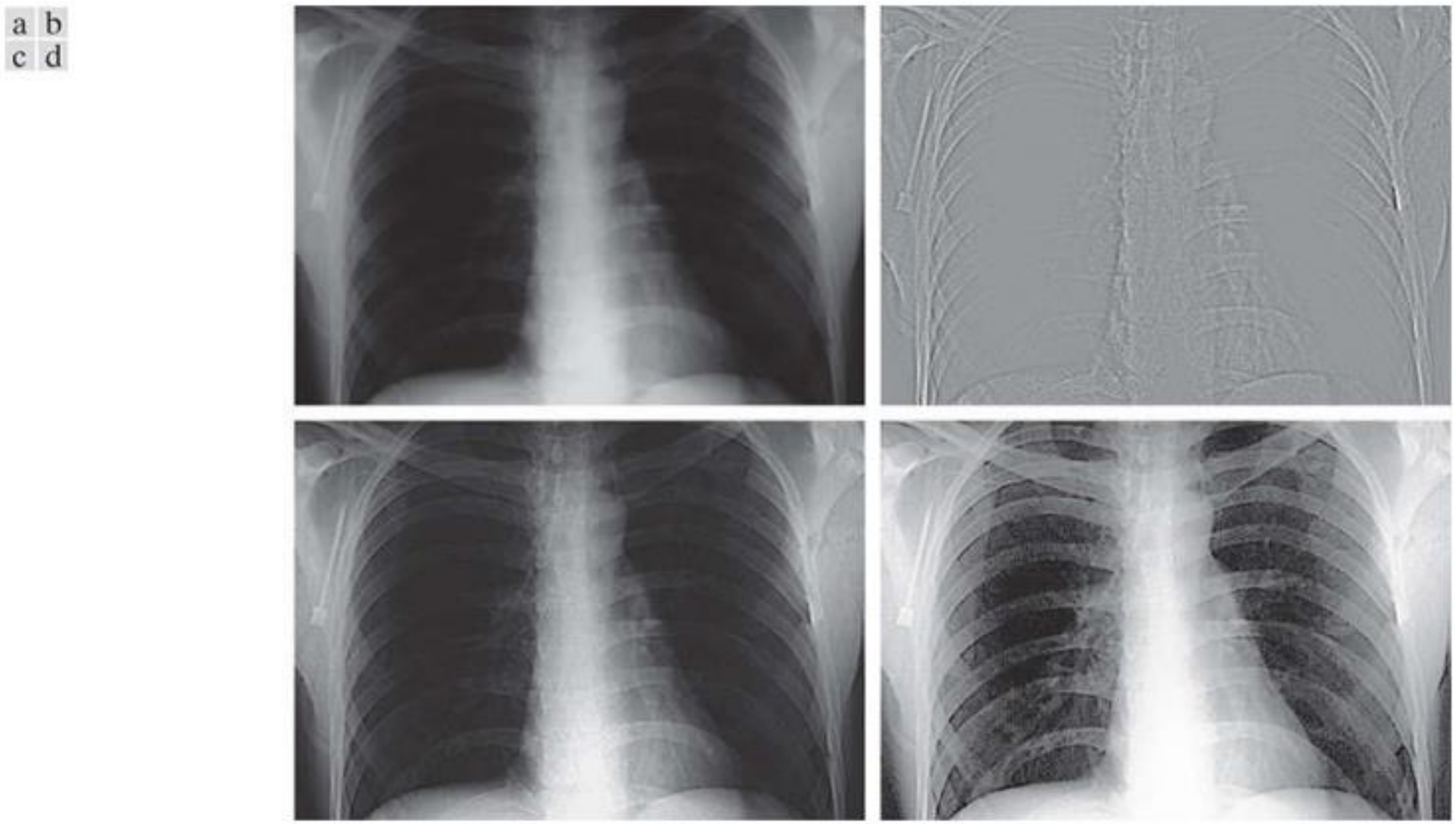


FIGURE 4.57
(a) A chest X-ray. (b) Result of filtering with a GHPF function. (c) Result of high-frequency-emphasis filtering using the same GHPF. (d) Result of performing histogram equalization on (c).

(Original image courtesy of Dr. Thomas R. Gest, Division of Anatomical Sciences, University of Michigan Medical School.)

Homomorphic Filtering

$$f(x, y) = i(x, y)r(x, y) \tag{4-134}$$

This equation cannot be used directly to operate on the frequency components of illumination and reflectance because the Fourier transform of a product is not the product of the transforms:

$$\mathfrak{J}[f(x, y)] \neq \mathfrak{J}[i(x, y)]\mathfrak{J}[r(x, y)] \tag{4-135}$$

However, suppose that we define

$$\begin{aligned} z(x, y) &= \ln f(x, y) \\ &= \ln i(x, y) + \ln r(x, y) \end{aligned} \tag{4-136}$$

If *f*(*x*, *y*) has any zero values, a 1 must be added to the image to avoid having to deal with ln(0). The 1 is then subtracted from the final result.

Then,

$$\mathcal{J}[z(x,\ y)] = \mathcal{J}[\ln f(x,\ y)] \\ = \mathcal{J}[\ln i(x,\ y)] + \mathcal{J}[\ln r(x,\ y)] \tag{4-137}$$

or

$$Z(u,v) = F_i(u,v) + F_r(u,v) \tag{4-138}$$

where $F_i(u,v)$ and $F_r(u,v)$ are the Fourier transforms of $\ln i(x,\ y)$ and $\ln r(x,\ y)$, respectively.

We can filter *Z*(*u*,*v*) using a filter transfer function *H*(*u*,*v*) so that

$$S(u,v) = H(u,v)Z(u,v) \\ = H(u,v)F_i(u,v) + H(u,v)F_r(u,v) \tag{4-139}$$

The filtered image in the spatial domain is then

$$s(x,\ y) = \mathcal{J}^{-1}[S(u,v)] \\ = \mathcal{J}^{-1}[H(u,v)F_i(u,v)] + \mathcal{J}^{-1}[H(u,v)F_r(u,v)] \tag{4-140}$$

By defining

$$i'(x,\ y) = \mathcal{J}^{-1}[H(u,v)F_i(u,v)] \tag{4-141}$$

and

$$r'(x,\ y) = \mathcal{J}^{-1}[H(u,v)F_r(u,v)] \tag{4-142}$$

we can express Eq. (4-140) in the form

$$s(x, y) = i'(x, y) + r'(x, y) \tag{4-143}$$

Finally, because $z(x, y)$ was formed by taking the natural logarithm of the input image, we reverse the process by taking the exponential of the filtered result to form the output image:

$$\begin{aligned} g(x, y) &= e^{s(x, y)} \\ &= e^{i'(x, y)} e^{r'(x, y)} \\ &= i_0(x, y)\, r_0(x, y) \end{aligned} \tag{4-144}$$

where

$$i_0(x, y) = e^{i'(x, y)} \tag{4-145}$$

and

$$r_0(x, y) = e^{r'(x, y)} \tag{4-146}$$

are the illumination and reflectance components of the output (processed) image.

Figure 4.58 is a summary of the filtering approach just derived. This method is based on a special case of a class of systems known as *homomorphic systems*. In this particular application, the key to the approach is the separation of the illumination and reflectance components achieved in the form shown in Eq. (4-138) . The *homomorphic filter transfer function*, $H(u,v)$, then can operate on these components separately, as indicated by Eq. (4-139) .

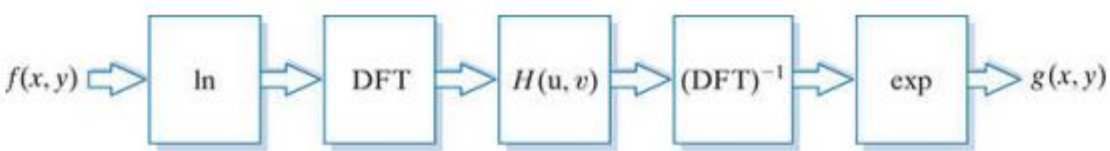


FIGURE 4.58
Summary of steps in homomorphic filtering.

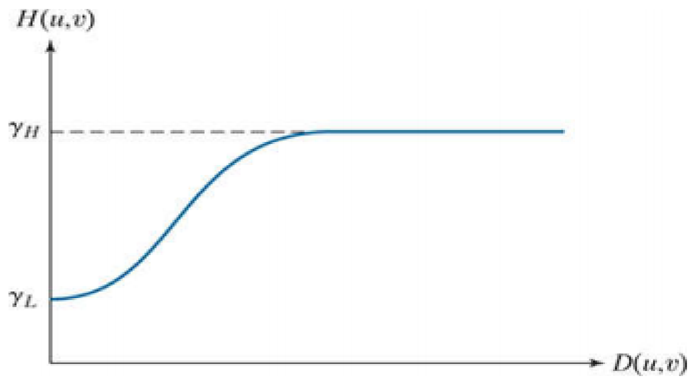


FIGURE 4.59
Radial cross section of a homomorphic filter transfer function.

A good deal of control can be gained over the illumination and reflectance components with a homomorphic filter. This control requires specification of a filter transfer function $H(u,v)$ that affects the low- and high-frequency components of the Fourier transform in different, controllable ways. Figure 4.59 shows a cross section of such a function. If the parameters γ_L and γ_H are chosen so that $\gamma_L < 1$ and $\gamma_H \geq 1$, the filter function in Fig. 4.59 will attenuate the contribution made by the low frequencies (illumination) and amplify the contribution made by high frequencies (reflectance). The net result is simultaneous dynamic range compression and contrast enhancement.

- The shape of the function in Fig.4.59 can be approximated using a highpass filter transfer function. For example, using a slightly modified form of the GHPF function yields the homomorphic function

$$H(u,v) = (\gamma_H - \gamma_L)\left[1 - e^{-cD^2(u,v/D_0^2)}\right] + \gamma_L \qquad (4\text{-}147)$$

EXAMPLE 4.23: Homomorphic filtering.

a b

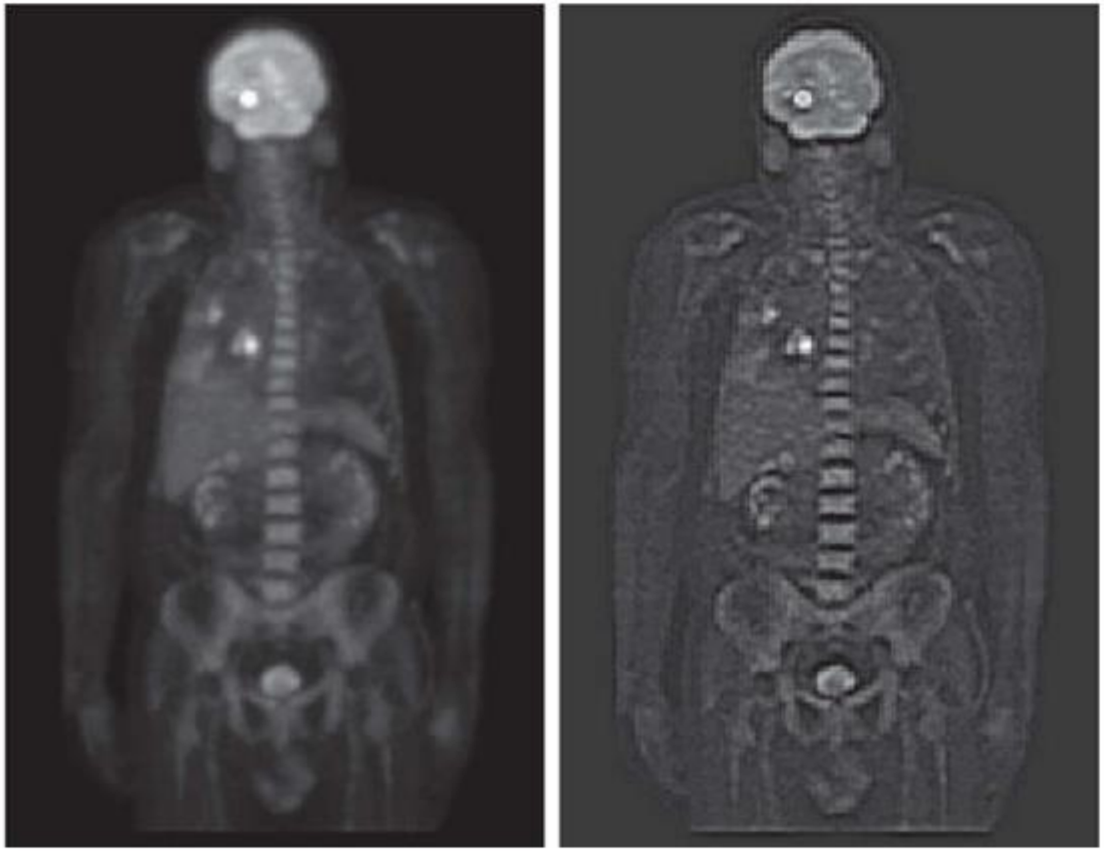


FIGURE 4.60

(a) Full body PET scan. (b) Image enhanced using homomorphic filtering.

(Original image courtesy of Dr. Michael E. Casey, CTI Pet Systems.)