Synchronization of Non-linear System using linear feedback

Abstract

We study the synchronization of 'm' coupled chaotic systems utilizing the unidirectional linear error feedback scheme and derive a simple generic condition for global chaos synchronization of more than two coupled chaotic systems using the Lyapunov stability theory and Gerschgorin's theorem. This condition for chaos synchronization is further expressed as a small number of algebraic inequalities, making its verification incredibly simple.

Introduction

Numerous effective methods [1-4, 6-11, 14, 15, 17-20, 23-26, 29, 30, 33-35] have been developed as a result of a decade of research into chaos synchronization [3,26]. Due to its straightforward configuration and ease of implementation in actual systems, the unidirectional linear error feedback coupling scheme is one of the most effective techniques for chaos synchronization [10, 11, 14, 15, 19, 20, 29]. For the design of multiple responses (or slave) chaotic systems based on the unidirectional linear error feedback method, the most important consideration is the choice of the feedback gain (or coupling parameters) [8, 29, 30].

This project studies the synchronization of 'm'- numbers of coupled chaotic systems using the unidirectional linear error feedback scheme and derives a simple generic condition for global chaos synchronization of more than two coupled chaotic systems based on the Lyapunov stability theory [16, 22] and Gerschgorin's theorem [12]. This condition for chaos synchronization is expressed as a small number of algebraic inequalities, making its verification incredibly straightforward.

The approach involves consideration of a master system and m slave systems (s_1, s_2, s_3... s_m). The generalization is being derived by considering s_1 as a master system and slave as s_2, then s_2 as master and s_3 as a slave system till s_m becomes the last slave system by finding the coupling parameter for individual Master-slave systems.

A criterion for global chaos Synchronization:

Consider a chaotic system in the form of

$$\dot{X}_0 = AX_0 + g(X_0) + u \qquad \dots (1)$$

Where $X_0 \in R^n$ is the state vector, u $\in R^n$ is the external unit vector

$A \in R^{nXn}$ is a constant vector

$g(X_0)$ is a continuous function.

Remark-1

Most of chaotic system are nonlinear in nature. From the unidirectional linear coupling approach. A m^{th} slave system for (1) is constructed as follows:

$$\dot{X}_m = AX_m + g(X_m) + u + K_m\,(X_{m-1} - X_m) \qquad \dots (2)$$

Here $K_m \ = diag(k_{m1}\,, k_{m2}\,, \dots k_{mn}\,)$, is a feedback matrix

$X_m \in R^n$ is the state vector for m^{th} slave systems, that will be connected in series with $(m-1)^{th}$ system.

Assuming that:

$$g(X_{m-1}) - g(X_m) = M_{x_{m-1}x_m}\,(X_{m-1} - X_m) \qquad \dots (3)$$

$$= M_{x_{m-1}x_m} e_m$$

Here $M_{x_{m-1}x_m}$ is a bounded matrix, in which the elements are dependent on X_{m-1} and X_m.

From the equation (1) and (2), the following errors systems eq^n can be obtained

$$\dot{e}_m = Ae_m + \ g(X_{m-1}) - g(X_m) - \ K_m\,(X_{m-1} - X_m)$$

$$= Ae_m - K_m\,e_m \ + g(X_{m-1}) - g(X_m)$$

$$= (\mathrm{A} - K_m\,)e_m \ + g(X_{m-1}) - g(X_m) \qquad \dots (4)$$

Here $e_m \ = X_{m-1} - X_m$ is an error term.

Theorem 1

If the feedback gain matrix K_j, j=1, 2… m is chosen such that

$$\lambda_{ji} \leq \mu < 0, \qquad j = 1, 2 \ldots \text{ m}$$

$$\text{i=1, 2…n} \qquad \ldots (5)$$

λ_{ji} are the eigen values of the matrix

$\left(A - K_j + M_{x_{j-1}x_j}\right)^T \cdot P + \mathrm{P}\left(\mathrm{A} - K_j + M_{x_{j-1}x_j}\right)$ with a positive definite symmetric constant matrix P, and μ is a negative constant, then the error dynamical system (4) is globally exponentially stable about the origin implying that the two systems (1) Master system and (2) Slave system are globally asymptotically synchronized

Proof:

Choose the Lyapunov function

$$v = \sum_{i=1}^{m} e_i^T P e_i \qquad \ldots (6)$$

Where P is a positive definite symmetric constant matrix then its derivative is

$$\dot{v} = (\dot{e}_1^T P e_1 + e_1^T P \dot{e}_1) + (\dot{e}_2^T P e_2 + e_2^T P \dot{e}_2) + \cdots (\dot{e}_m^T P e_m + e_m^T P \dot{e}_m)$$

Or we can write as

$$\dot{v} = \dot{v}_1 + \dot{v}_2 + \dot{v}_3 + \cdots \dot{v}_m \qquad \ldots (7)$$

For $\dot{v}_1$

$\dot{v}_1 = \dot{e}_1^T P e_1 + e_1^T P \dot{e}_1$ (From equation (4))

$$= [(A - K_1)\, e_1 + g(X_0) - g(X_1)\ ]^T P e_1 + e_1^T\, P[(A - K_1) e_1 + g(X_0) - g(X_1)]$$

$$= [e_1^T (A - K_1)^T + (g(X_0) - g(X_1))^T] P e_1 + e_1^T P[(A - K_1)\, e_1 + g(X_0) - g(X_1)]$$

$$= e_1^T (A - K_1)^T P e_1 + \left(M_{X_0X_1} e_1\right)^T P e_1 + e_1^T\ (A - K_1) e_1 + e_1^T P\, M_{X_0X_1} e_1$$

$$= e_1^T (A - K_1)^T P e_1 + e_1^T M_{X_0X_1}^T P e_1 + e_1^T P + (A - K_1) e_1 + e_1^T P\ M_{X_0X_1} e_1$$

$$= e_1^T \left[\left(A - K_1 + M_{X_0X_1}\right)^T \mathrm{P} + \mathrm{P}\left(A - K_1 + M_{X_0X_1}\right)\right] e_1 \qquad \ldots (8)$$

We can write as

$$\dot{v}_1 = e_1^T Q_1 e_1 \qquad \ldots (9)$$

Where $Q_1 = [((A - K_1 + M_{X_0X_1})^T \mathrm{P} + \mathrm{P}(A - K_1 + M_{X_0X_1})]$
Since $Q_1 = Q_1'$. let $Q_1 = U_1^* \Lambda_1 U_1$
Here U_1 is square unitary matrix and,

$$\Lambda_1 = \mathrm{diag}(\lambda_{11}, \lambda_{12} \dots \lambda_{1n})$$

Then equation (9) becomes,

$$\dot{v}_1 = e_1^T Q_1 e_1 = e_1^T U_1^* \Lambda_1 U_1 e_1$$

$$= e_{11}^T \Lambda_1 e_{11} \leq \mu e_{11}^T e_{11} < 0 \qquad \dots (10)$$

Here $e_{11} = U_1 e_1$

Similarly, for $\dot{v}_\mathrm{m}$

$$\dot{v}_m = \dot{e}_m{}^T P e_m + e_m^T P \dot{e}_m$$

$$\dot{v}_m = \big((A - K_m)e_m + g(x_{m-1}) - g(x_m)\big)^T P e_m + e_m^T P\big((A - K_m)e_m + g(x_{m-1}) - g(x_m)\big)$$

After Solving,

$$\dot{v}_m = e_m^T \left((A - K_m) + M_{x_{m-1}x_m}\right)^T P + P\left((A - K_m) + M_{x_{m-1}x_m}\right) e_m \qquad \dots (11)$$

We can write,

$$\dot{v}_m = e_m^T Q_m e_m \qquad \dots (12)$$

Here,

$$Q_m = \left((A - K_m) + M_{x_{m-1}x_m}\right)^T P + P\left((A - K_m) + M_{x_{m-1}x_m}\right)$$

Since, $Q_m = Q_m^*$
Let $Q_m = U_m \Lambda_m^* U_m$

Here, U_m is square unitary matrix and $\Lambda_m = diag(\lambda_{m1}, \lambda_{m2} \dots \lambda_{mn})$

Then equation (11) becomes,

$$\dot{v}_m = e_m^T Q_m e_m$$

$$\dot{v}_m = e_m^T U_m^* U_m \Lambda_m^* e_m$$

$$\dot{v}_m = e_{m1}^T \Lambda_m e_{m1} \leq \mu e_{m1}^T e_{m1} < 0 \qquad \dots (13)$$

Here, $e_{m1} = U_m e_m$

From equation (7)

$$\dot{v} = \dot{v}_1 + \dot{v}_2 + \dot{v}_3 + \ldots \dot{v}_m$$

Since, here $\dot{v}_1, \dot{v}_2 \ \ldots \dot{v}_m$ all are negative thus we can conclude that

$$\dot{v} < 0 \qquad \ldots (14)$$

Then according to the equation (14) and the Lyapunov stability theory, system (4) is globally exponentially stable about the origin.

Hence the master system (1) and slave system (2) are globally asymptotically synchronized.

Theorem 2

Choose P = diag (p_1, p_2 ...p_n)
And we know that,

$$\left((A - K_m) + M_{x_{m-1}x_m}\right)^T P + P\left((A - K_m) + M_{x_{m-1}x_m}\right) \leq \mu I < 0 \qquad \ldots (15)$$

And

$\left((A - K_m) + M_{x_{m-1}x_m}\right)^T P + P\left((A - K_m) + M_{x_{m-1}x_m}\right) - \mu I$ is strictly diagonally dominant matrix.

From definition of SDD matrix we will find the element of K_j, j = 1, 2…m feedback matrix.

Let consider all slave system at once and the equation (4) becomes

$$\dot{e} = Ae + g(x) - g(\tilde{x}) - Ke \qquad \ldots (16)$$

And

$$g(x) - g(\tilde{x}) = M_{x\tilde{x}}e \qquad \ldots (17)$$

Now, $P(A + M_{x\tilde{x}}) + (A + M_{x\tilde{x}})^T P$ matrix can be written as a matrix of order $(mn \times mn)$

$$\begin{bmatrix} [A_1]_{n\times n} & diag(k_{11}, k_{12}, \ldots k_{1n}) & \cdots & \cdots & \cdots & [0]_{(m-2)n\times n} \\ diag(k_{11}, k_{12}, \ldots k_{1n}) & [A_2]_{n\times n} & diag(k_{21}, k_{22}, \ldots k_{1n}) & \cdots & \cdots & [0]_{(m-3)n\times n} \\ [0]_{n\times n} & diag(k_{21}, k_{22}, .. k_{2n}) & [A_3]_{n\times n} & diag(k_{31}, k_{32}, .. k_{3n}) & \cdots & [0]_{(m-4)n\times n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ [0]_{(m-2)n\times n} & \cdots & \cdots & \cdots & diag\left(k_{(n-1)1}, k_{(n-1)2}, \ldots k_{(n-1)n}\right) & [A_n]_{n\times n} \end{bmatrix}$$

Here,

$$[A_1] = [A_2] = \cdots = [A_m] = [a_{ij}] \quad \text{And} \qquad R_i = \sum_{j=1\ j\neq 1}^{n} |a_{ij}| \qquad \text{... (18)}$$

Now, from equation (15),

$$k_{1i} \geq \frac{1}{2P_i}(a_{ii} + R_i + k_{1i} - \mu) \text{ For, i = 1, 2, 3, ...n} \qquad \text{... (19)}$$

For 1< l < m

$$k_{li} \geq \frac{1}{2P_i}(a_{ii} + R_i + k_{l-1i} + k_{li} - \mu) \qquad \text{... (20)}$$

For, m^{th} feedback matrix

$$k_{mi} \geq \frac{1}{2P_i}(a_{ii} + R_i + k_{m-1i} - \mu) \qquad \text{... (21)}$$

For those feedback matrix, equation (5) will be satisfied, implying that the whole 'm' coupled chaotic system (1) and (2) are globally synchronized.

Remark 2

If P=I (Identity Matrix) then according to theorem 1, obtain the following algebraic inequalities for choosing the coupling parameters:

$$k_{1i} \geq \frac{1}{2}(a_{ii} + R_i + k_{1i} - \mu)$$

$$k_{1i} \geq a_{ii} + R_i - \mu \qquad \text{... (22)}$$

For 1< l< m;

$$k_{li} \geq \frac{1}{2}(a_{ii} + R_i + k_{li} + k_{l-1\,i} - \mu)$$

$$k_{li} \geq a_{ii} + R_i + k_{l-1\,i} - \mu \qquad \text{... (23)}$$

For m^{th} feedback matrix

$$k_{mi} \geq \frac{1}{2}[a_{ii} + R_i + k_{m-1\,i} - \mu] \qquad \text{... (24)}$$

Remark 3

If $R' = max_{1 \le i \le n} \sum_{j=1, j \ne i}^{n} |a_{ij}|$ then based on equation 18 one has $R' \ge R_i$ and according to Gerschgorin's theorem

$$k'_{li} \ge (a_{ii} + R' - \mu) \qquad \dots (25)$$

For 1 < l < m

$$k'_{li} \ge a_{ii} + R' + k_{l-1\,i} - \mu \qquad \dots (26)$$

For m^{th} feedback matrix

$$k'_{mi} \ge \frac{1}{2}[a_{ii} + R' + k_{m-1\,i} - \mu] \qquad \dots (27)$$

Now, the range for k in (25,26,27) is reduced as compare to (22,23,24).

Remark 4

After solving (from iteration method)

We get for 1 < l < m

$$k_{li} \ge \frac{l}{2}[k_{1i}] \qquad \dots (28)$$

And for m^{th} feedback matrix

$$k_{mi} \ge m.[k_{li}] \qquad \dots (29)$$

Synchronization of some typical chaotic systems

1. **The original Chua's circuit:**

Chua's circuit is described by

$$\dot{x_0} = \alpha\big(y_0 - x_0 - f(x_0)\big)$$
$$\dot{y}_0 = x_0 - y_0 + z_0$$
$$\dot{z}_0 = -\beta y_0 \qquad \dots (30)$$

Where α > 0, β > 0, a < b < 0, f(.) is a piecewise linear function described by

$$f(x) = bx + \frac{1}{2}(a-b)(|x+1| - |x-1|) \qquad \dots (31)$$

Referring to equation 2, we can obtain 3 slave system for the drive (30) with a linear unidirectional coupling

$$\dot{x_1} = \alpha\big(y_1 - x_1 - f(x_1)\big) + k_{11}(x_0 - x_1)$$
$$\dot{y}_1 = x_1 - y_1 + z_1 + k_{12}(y_0 - y_1)$$
$$\dot{z}_1 = -\beta y_1 + k_{13}(z_0 - z_1) \qquad \dots (32)$$

$$\dot{x_2} = \alpha\big(y_2 - x_2 - f(x_2)\big) + k_{21}(x_1 - x_2)$$
$$\dot{y}_2 = x_2 - y_2 + z_2 + k_{22}(y_1 - y_2)$$
$$\dot{z}_2 = -\beta y_2 + k_{23}(z_1 - z_2) \qquad \dots (33)$$

$$\dot{x_3} = \alpha\big(y_3 - x_3 - f(x_3)\big) + k_{31}(x_2 - x_3)$$
$$\dot{y}_3 = x_3 - y_3 + z_3 + k_{32}(y_2 - y_3)$$
$$\dot{z}_3 = -\beta y_3 + k_{33}(z_2 - z_3) \qquad \dots (34)$$

In Equation 31, We have:

$$f(x) - f(x_1) = K_{x\tilde{x}}(x_0 - x_1) \qquad \dots (35)$$
$$f(x_1) - f(x_2) = K_{x\tilde{x}}(x_1 - x_2) \qquad \dots (36)$$
$$f(x_0) - f(x_3) = K_{x\tilde{x}}(x_2 - x_3) \qquad \dots (37)$$

Where $K_{x\tilde{x}}$ is depend on the x_0, x_1, x_2, x_3 and varies within the interval [a, b] for t ≥ 0, that is $K_{x\tilde{x}}$ is bounded by constants as

$$a \le K_{x\tilde{x}} \le b \le 0$$

From equations 30, 32, 33, 34, we get error dynamics

$$\dot{e}_{11} = \alpha(e_{12} - e_{11} - \Big(f\big(x_0 - f(x_1)\big)\Big) - k_{11}e_1$$
$$\dot{e}_{12} = e_{11} - e_{12} + e_{13} - k_{12}e_{12}$$
$$\dot{e}_{13} = -\beta e_{12} - k_{13}e_{13}$$

$$\dot{e}_{21} = \alpha\Big(e_{22} - e_{21} - \big(f(x_1) - f(x_2)\big)\Big) - k_{21}e_{21}$$
$$\dot{e}_{22} = e_{21} - e_{22} + e_{23} - k_{22}e_{22}$$
$$\dot{e}_{23} = -\beta e_{22} - k_{23}e_{23}$$

$$\dot{e}_{31} = \alpha\Big(e_{32} - e_{31} - \big(f(x_2) - f(x_3)\big)\Big) - k_{31}e_{31}$$
$$\dot{e}_{32} = e_{31} - e_{32} + e_{33} - k_{32}e_{32}$$
$$\dot{e}_{33} = -\beta e_{32} - k_{33}e_{33} \qquad \dots (38)$$

Here,

$e_{11} = x_0 - x_1$, $e_{12} = y_0 - y_1$, $e_{13} = z_0 - z_1$

$e_{21} = x_1 - x_2$, $e_{22} = y_1 - y_2$, $e_{23} = z_1 - z_2$

$e_{31} = x_2 - x_3$, $e_{32} = y_2 - y_3$, $e_{33} = z_2 - z_3$

From Equation (16) these error dynamics can be written as

$$\dot{e} = Ae + g(x) - g(\breve{x}) - Ke$$

Therefore;

$$A = \begin{bmatrix} -\alpha & \alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\beta & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\alpha & \alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\beta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\alpha & \alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\beta & 0 \end{bmatrix}, \quad e = \begin{bmatrix} x_0 - x_1 \\ y_0 - y_1 \\ z_0 - z_1 \\ x_1 - x_2 \\ y_1 - y_2 \\ z_1 - z_2 \\ x_2 - x_3 \\ y_2 - y_3 \\ z_2 - z_3 \end{bmatrix} K =$$

$$\begin{bmatrix} k_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & k_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & k_{13} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & k_{21} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & k_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & k_{23} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & k_{31} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & k_{32} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & k_{33} \end{bmatrix}$$

$$g(x) = \begin{bmatrix} -\alpha f(x_0) \\ 0 \\ 0 \\ -\alpha f(x_1) + k_{11}e_{11} \\ k_{12}e_{12} \\ k_{13}e_{13} \\ -\alpha f(x_3) + k_{21}e_{21} \\ k_{22}e_{22} \\ k_{23}e_{23} \end{bmatrix} \qquad g(\tilde{x}) = \begin{bmatrix} -\alpha f(x_1) \\ 0 \\ 0 \\ -\alpha f(x_2) \\ 0 \\ 0 \\ -\alpha f(x_3) \\ 0 \\ 0 \end{bmatrix}$$

$$g(x) - g(\tilde{x}) = \begin{bmatrix} -\alpha\big(f(x_0) - f(x_1)\big) \\ 0 \\ 0 \\ -\alpha\big(f(x_1) - f(x_2)\big) + k_{11}e_{11} \\ k_{12}e_{12} \\ k_{13}e_{13} \\ -\alpha\big(f(x_2) - f(x_3)\big) + k_{21}e_{21} \\ k_{22}e_{22} \\ k_{23}e_{23} \end{bmatrix}$$

From Equation 35, 36, 37

$$g(x) - g(\tilde{x}) = \begin{bmatrix} -\alpha K_{x\tilde{x}}e_{11} \\ 0 \\ 0 \\ -\alpha K_{x\tilde{x}}e_{12} + k_{11}e_{12} \\ k_{12}e_{12} \\ k_{13}e_{13} \\ -\alpha K_{x\tilde{x}}e_{21} + k_{21}e_{21} \\ k_{22}e_{22} \\ k_{23}e_{23} \end{bmatrix} = M_{x\tilde{x}}.\mathrm{e} \qquad \ldots (39)$$

Therefore,

$$M_{x\tilde{x}} = \begin{bmatrix} -\alpha K_{x\tilde{x}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ k_{11} & 0 & 0 & -\alpha K_{x\tilde{x}} & 0 & 0 & 0 & 0 & 0 \\ 0 & k_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & k_{13} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & k_{21} & 0 & 0 & -\alpha K_{x\tilde{x}} & 0 & 0 \\ 0 & 0 & 0 & 0 & k_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & k_{23} & 0 & 0 & 0 \end{bmatrix} \qquad \ldots (40)$$

Therefore:

$$(A + M_{x\tilde{x}}) + (A + M_{x\tilde{x}})^T$$

$$= \begin{bmatrix} -2\alpha-2\alpha K_{x\tilde{x}} & \alpha+1 & 0 & k_{11} & 0 & 0 & 0 & 0 & 0 \\ \alpha+1 & -2 & 1-\beta & 0 & k_{12} & 0 & 0 & 0 & 0 \\ 0 & 1-\beta & 0 & 0 & 0 & k_{13} & 0 & 0 & 0 \\ k_{11} & 0 & 0 & -2\alpha-2\alpha K_{x\tilde{x}} & \alpha+1 & 0 & k_{21} & 0 & 0 \\ 0 & k_{12} & 0 & \alpha+1 & -2 & 1-\beta & 0 & k_{22} & 0 \\ 0 & 0 & k_{13} & 0 & 1-\beta & 0 & 0 & 0 & k_{23} \\ 0 & 0 & 0 & k_{21} & 0 & 0 & -2\alpha-2\alpha K_{x\tilde{x}} & \alpha+1 & 0 \\ 0 & 0 & 0 & 0 & k_{22} & 0 & \alpha+1 & -2 & 1-\beta \\ 0 & 0 & 0 & 0 & 0 & k_{23} & 0 & 1-\beta & 0 \end{bmatrix} \quad \ldots (41)$$

Since $a \leq K_{x\tilde{x}} < b < 0$, then according to theorem-2 and remark–2 we choose $K_{x\tilde{x}} = a$

From Equation 25 (for first slave system)

$$k_{11} \geq (1 - \alpha - 2a\alpha - \mu)$$
$$k_{12} \geq (\alpha - 1 + |1 - \beta| - \mu)$$
$$k_{13} \geq (|1 - \beta| - \mu) \quad \ldots (42)$$

From Equation 28 (for second slave system)

$$k_{21} \geq k_{11}$$
$$k_{22} \geq k_{12}$$
$$k_{23} \geq k_{13} \quad \ldots (43)$$

From Equation 29 (for third slave system)

$$k_{31} \geq 3k_{11}$$
$$k_{32} \geq 3k_{11}$$
$$k_{33} \geq 3k_{13} \quad \ldots (44)$$

When α=9.78, β=14.9, a=-1.31, b=0.75 then system (30) exhibits chaotic behavior. By selecting μ=-0.5 and the coupling parameters as:

k_{11} = 18 k_{12} = 24 k_{13} = 16

k_{21} = 18 k_{22} = 24 k_{13} = 16

k_{31} = 54 k_{32} = 72 k_{33} = 48

With the above chosen parameter all the slave systems are synchronized with master system as error signal tends to zero.

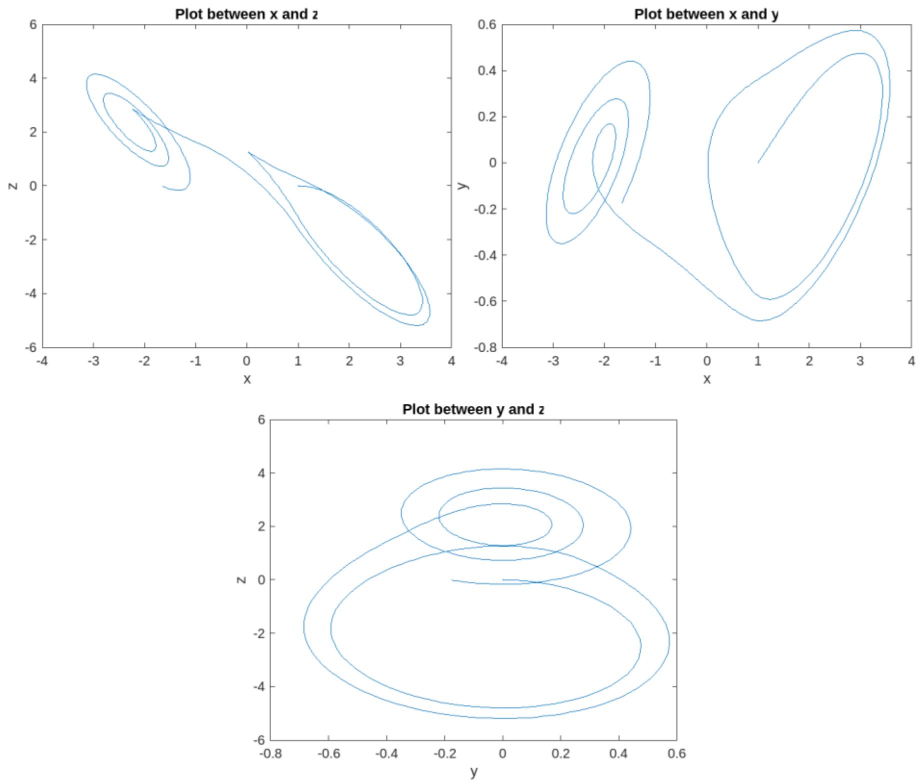


Figure 1. The Chaotic behavior of Chua's Circuit

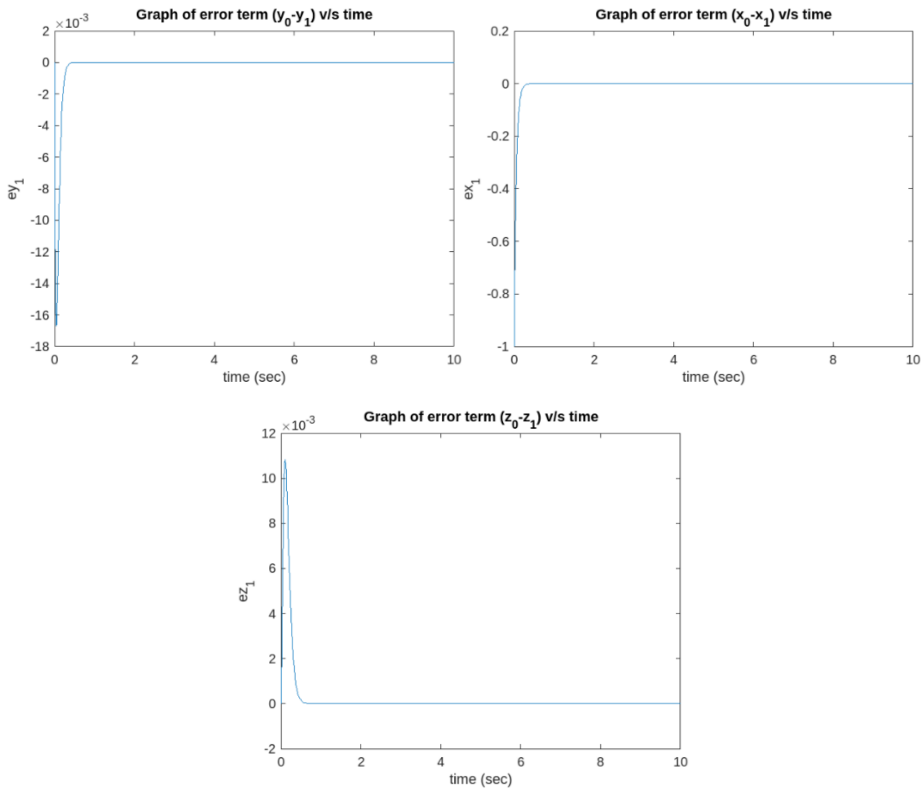


Figure 2. The difference signal e_x, e_y and e_z between master and first slave system

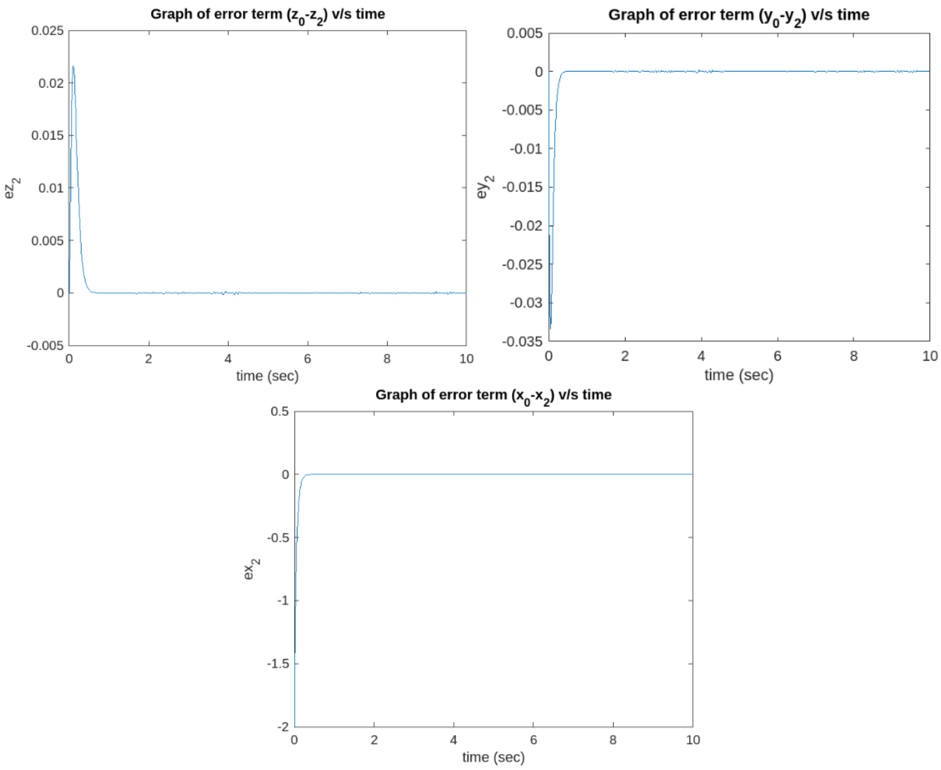


Figure 3. The difference signal e_x, e_y and e_z between master and second slave system

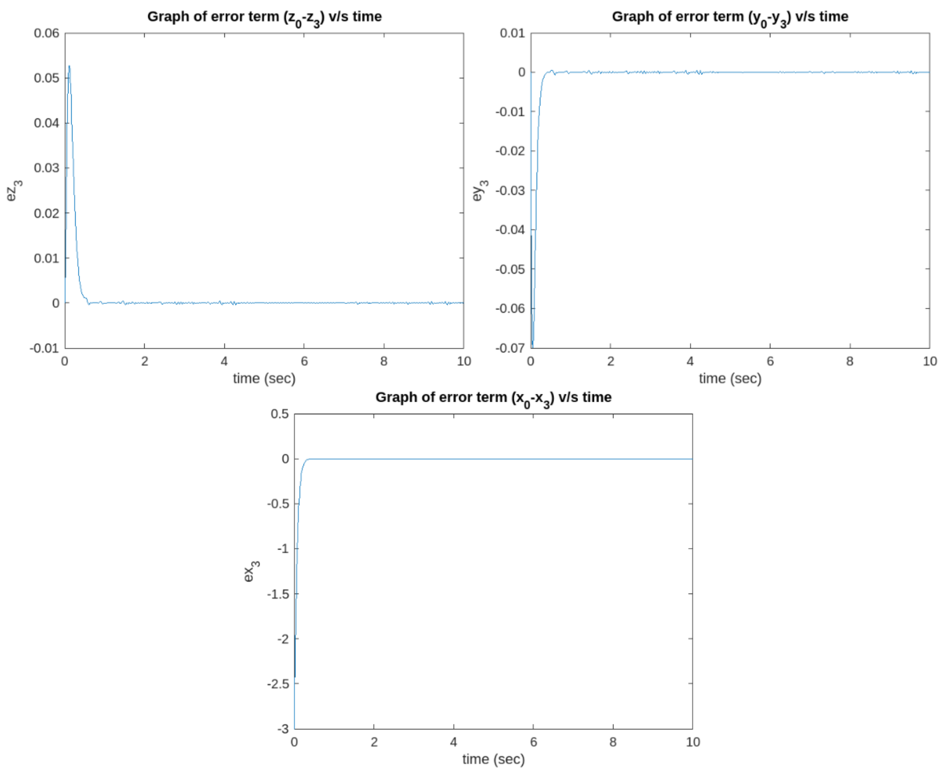


Figure 4. The difference signal e_x, e_y and e_z between master and third slave system

2. **Modified Chua's Circuit with a sine function:**

Unlike the original Chua's circuit, the modified Chua's circuit uses a sine function. For this circuit, n-scroll attractors can be obtained.

The dimensionless state equation of the circuit is :

$$\begin{aligned}
\dot{x}_0 &= \alpha\big(y_0 - f(x_0)\big) \\
\dot{y}_0 &= x_0 - y_0 - z_0 \\
\dot{z}_0 &= -\beta y_0
\end{aligned} \qquad \text{... (45)}$$

Here,

$$f(x_0) = \begin{cases} \dfrac{b\pi}{2a}(x_0 - 2ac) & \text{if } x_0 \geq 2ac \\ -\mathrm{b}\sin\left(\dfrac{\pi x_0}{2a} + d\right) & if - 2ac < x_0 < 2ac \\ \dfrac{b\pi}{2a}(x_0 + 2ac) & if\ x_0 \leq -2ac \end{cases}$$

... (46)

Here, in equation (45) and (46) α, β, a, b, c, d are suitable constants and α>0, β>0, a>0, b>0.

An n-scroll attractor is generated under the following constraints

$$n = c+1 \qquad \text{... (47)}$$

and $$d = \begin{cases} \pi & \text{if, n is odd} \\ 0 & \text{if, n is even} \end{cases} \qquad \text{... (48)}$$

Referring to equation (2), we can obtain, 3 slave system for the drive (45) with a linear unidirectional coupling.

$$\begin{aligned}
\dot{x}_1 &= \alpha\big(y_1 - f(x_1)\big) + k_{11}\,(x_0 - x_1) \\
\dot{y}_1 &= x_1 - y_1 + z_1 + k_{12}\,(y_0 - y_1) \\
\dot{z}_1 &= -\beta y_1 + k_{13}\,(z_0 - z_1)
\end{aligned} \qquad \text{... (49)}$$

Here,

$$f(x_1)\begin{cases} \frac{b\pi}{2a}(x_1-2ac) \text{ if } x_1 \geq 2ac \\ -\text{bsin}\left(\frac{\pi x_1}{2a}+d\right) \text{ if } -2ac < x_1 < 2ac \\ \frac{b\pi}{2a}(x_1+2ac) \text{ if } x_1 \leq -2ac \end{cases} \quad \dots (50)$$

In (46), one has

$$\left(f(x_0)-f(x_1)\right) = K_{x\tilde{x}}\,(x_0-x_1) \quad \dots (51)$$

Where, $K_{x\tilde{x}}$ is dependent on x_0 and x_1 and satisfies the condition of

$$-\frac{b\pi}{2a} \leq K_{x\tilde{x}} \leq \frac{b\pi}{2a} \quad \dots (52)$$

Similarly,

2nd Slave System

$$\dot{x}_2 = \alpha\left(y_2 - f(x_2)\right) + k_{21}\,(x_1 - x_2)$$

$$\dot{y}_2 = x_2 - y_2 + z_2 + k_{22}\,(y_1 - y_0)$$

$$\dot{z}_2 = -\beta y_2 + k_{23}\,(z_0 - z_2) \quad \dots (53)$$

here,

$$f(x_2) = \begin{cases} \frac{b\pi}{2a}(x_2-2ac) \quad \text{if } x_2 \geq 2ac \\ -\text{bsin}\left(\frac{\pi x_2}{2a}+d\right) \quad \text{if } -2ac < x_2 < 2ac \\ \frac{b\pi}{2a}(x_2+2ac) \quad \text{if } x_2 \leq -2ac \end{cases} \quad \dots (54)$$

And $\left(f(x_1)-f(x_2)\right) = K_{x\tilde{x}}\,(x_1-x_2)$ ... (55)

3rd Slave System

$$\dot{x}_3 = \alpha\left(y_3 - f(x_3)\right) + k_{31}\,(x_2 - x_3)$$

$$\dot{y}_3 = x_3 - y_3 + z_3 + k_{32}\,(y_2 - y_3)$$

$$\dot{z}_3 = -\beta y_3 + k_{33}\,(z_2 - z_3) \quad \dots (56)$$

here,

$$f(x_3) = \begin{cases} \frac{b\pi}{2a}(x_3-2ac) \quad \text{if } x_3 \geq 2ac \\ -\text{bsin}\left(\frac{\pi x_3}{2a}+d\right) \quad \text{if } -2ac < x_3 < 2ac \\ \frac{b\pi}{2a}(x_3+2ac) \quad \text{if } x_2 \leq -2ac \end{cases} \quad \dots (57)$$

And $\left(f(x_2) - f(x_3)\right) = K_{x\tilde{x}}\,(x_3 - x_2)$ ... (58)

From, (45), (49), (51), (53), (55), (56) and (58) we get error dynamics.

$$
\begin{aligned}
\dot{e}_{11} &= \alpha(e_{12} - e_{11} - K_{x\tilde{x}} e_{11}) + k_{11} e_{11} \\
\dot{e}_{12} &= e_{11} - e_{12} - e_{13} - k_{12} e_{12} \\
\dot{e}_{13} &= -\beta\, e_{12} - k_{13} e_{13} \\
\\
\dot{e}_{21} &= \alpha(e_{22} - e_{21} - K_{x\tilde{x}} e_{21}) + k_{21} e_{21} \\
\dot{e}_{22} &= e_{21} - e_{22} - e_{23} - k_{22} e_{22} \\
\dot{e}_{23} &= -\beta\, e_{22} - k_{23} e_{23} \\
\\
\dot{e}_{31} &= \alpha(e_{32} - e_{31} - K_{x\tilde{x}} e_{31}) + k_{31} e_{31} \\
\dot{e}_{32} &= e_{31} - e_{32} - e_{33} - k_{32} e_{32} \\
\dot{e}_{33} &= -\beta\, e_{32} - k_{33} e_{33}
\end{aligned}
\qquad \text{... (59)}
$$

here, error term

$$
\begin{aligned}
e_{11} &= x_0 - x_1 & e_{12} &= y_0 - y_1 & e_{13} &= z_0 - z_1 \\
e_{21} &= x_1 - x_2 & e_{22} &= y_1 - y_2 & e_{23} &= z_1 - z_2 \\
e_{31} &= x_2 - x_3 & e_{32} &= y_2 - y_3 & e_{33} &= z_2 - z_3
\end{aligned}
\qquad \text{... (60)}
$$

From equation 16 these error dynamics can be written as

$$\dot{e} = Ae + g(x) - g(\tilde{x}) - Ke$$

Therefore,

$$\mathrm{A} = \begin{bmatrix} -\alpha & \alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\beta & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\alpha & \alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\beta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\alpha & \alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\beta & 0 \end{bmatrix}, \mathrm{e} = \begin{bmatrix} x_0 - x_1 \\ y_0 - y_1 \\ z_0 - z_1 \\ x_1 - x_2 \\ y_1 - y_2 \\ z_1 - z_2 \\ x_2 - x_3 \\ y_2 - y_3 \\ z_2 - z_3 \end{bmatrix}$$

$$\mathrm{K} = \begin{bmatrix} k_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & k_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & k_{13} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & k_{21} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & k_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & k_{23} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & k_{31} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & k_{32} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & k_{33} \end{bmatrix}$$

$$g(x) - g(\tilde{x}) = \begin{bmatrix} -\alpha K_{x\tilde{x}} e_{11} \\ 0 \\ 0 \\ -\alpha K_{x\tilde{x}} e_{12} + k_{11} e_{12} \\ k_{12} e_{12} \\ k_{13} e_{13} \\ -\alpha K_{x\tilde{x}} e_{21} + k_{21} e_{21} \\ k_{22} e_{22} \\ k_{23} e_{23} \end{bmatrix} = M_{x\tilde{x}}.\mathrm{e} \qquad \dots (61)$$

Therefore,

$$M_{x\tilde{x}} = \begin{bmatrix} -\alpha K_{x\tilde{x}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ k_{11} & 0 & 0 & -\alpha K_{x\tilde{x}} & 0 & 0 & 0 & 0 & 0 \\ 0 & k_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & k_{13} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & k_{21} & 0 & 0 & -\alpha K_{x\tilde{x}} & 0 & 0 \\ 0 & 0 & 0 & 0 & k_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & k_{23} & 0 & 0 & 0 \end{bmatrix} \quad \ldots (62)$$

Therefore $\quad (A + M_{x\tilde{x}}) + (A + M_{x\tilde{x}})^T$

$$= \begin{bmatrix} -2\alpha K_{x\tilde{x}} & \alpha+1 & 0 & k_{11} & 0 & 0 & 0 & 0 & 0 \\ \alpha+1 & -2 & 1-\beta & 0 & k_{12} & 0 & 0 & 0 & 0 \\ 0 & 1-\beta & 0 & 0 & 0 & k_{13} & 0 & 0 & 0 \\ k_{11} & 0 & 0 & -2\alpha K_{x\tilde{x}} & \alpha+1 & 0 & k_{21} & 0 & 0 \\ 0 & k_{12} & 0 & \alpha+1 & -2 & 1-\beta & 0 & k_{22} & 0 \\ 0 & 0 & k_{13} & 0 & 1-\beta & 0 & 0 & 0 & k_{23} \\ 0 & 0 & 0 & k_{21} & 0 & 0 & -2\alpha K_{x\tilde{x}} & \alpha+1 & 0 \\ 0 & 0 & 0 & 0 & k_{22} & 0 & \alpha+1 & -2 & 1-\beta \\ 0 & 0 & 0 & 0 & 0 & k_{23} & 0 & 1-\beta & 0 \end{bmatrix}$$

... (63)

This matrix can be obtained from the standard matrix we derived from equation 18

From equation 52, we can choose

$$K_{x\tilde{x}} = -\frac{\pi b}{2a}$$

From Equation 25 for 1st slave system

$$k_{11} \geq \left(\left(\frac{\alpha \pi b}{a}\right) + \alpha + 1 - \mu\right)$$

$$k_{12} \geq (\alpha - 1 + |1-\beta| - \mu)$$

$$k_{13} \geq (|1-\beta| - \mu) \quad \ldots (64)$$

From equation 28, for 2nd slave system

$$k_{21} \geq k_{11}$$

$$k_{22} \geq k_{12}$$

$$k_{23} \geq k_{13} \quad \text{... (65)}$$

From equation 29, for 3rd slave system

$$k_{31} \geq 3k_{11}$$

$$k_{32} \geq 3k_{12}$$

$$k_{33} \geq 3k_{13} \quad \text{... (66)}$$

Let α = 10.814 β = 14 a = 1.3 b = 0.11 c = 3 d = 0 and μ = - 0.5

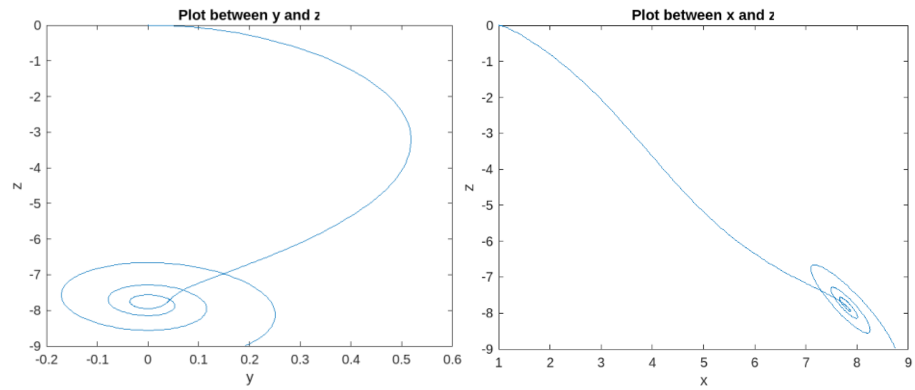
The coupling parameters we can obtain

$k_{11} = 16$ $k_{12} = 24$ $k_{13} = 16$

$k_{21} = 16$ $k_{22} = 24$ $k_{13} = 16$

$k_{31} = 48$ $k_{32} = 72$ $k_{33} = 48$

With the above chosen parameter all the slave systems are synchronized with master system as error signal tends to zero.



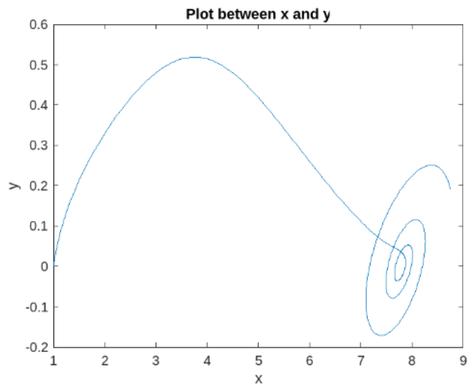


Figure 5. The Chaotic behavior of Modified Chua's Circuit with sine function

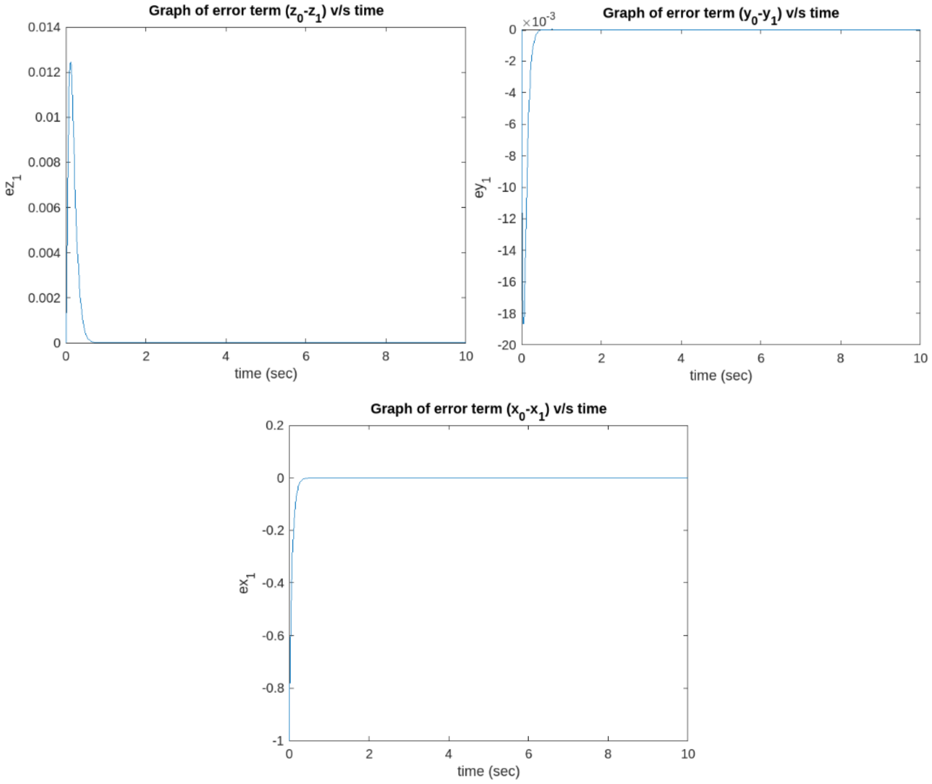


Figure 6. The difference signal e_x, e_y and e_z between master and first slave system

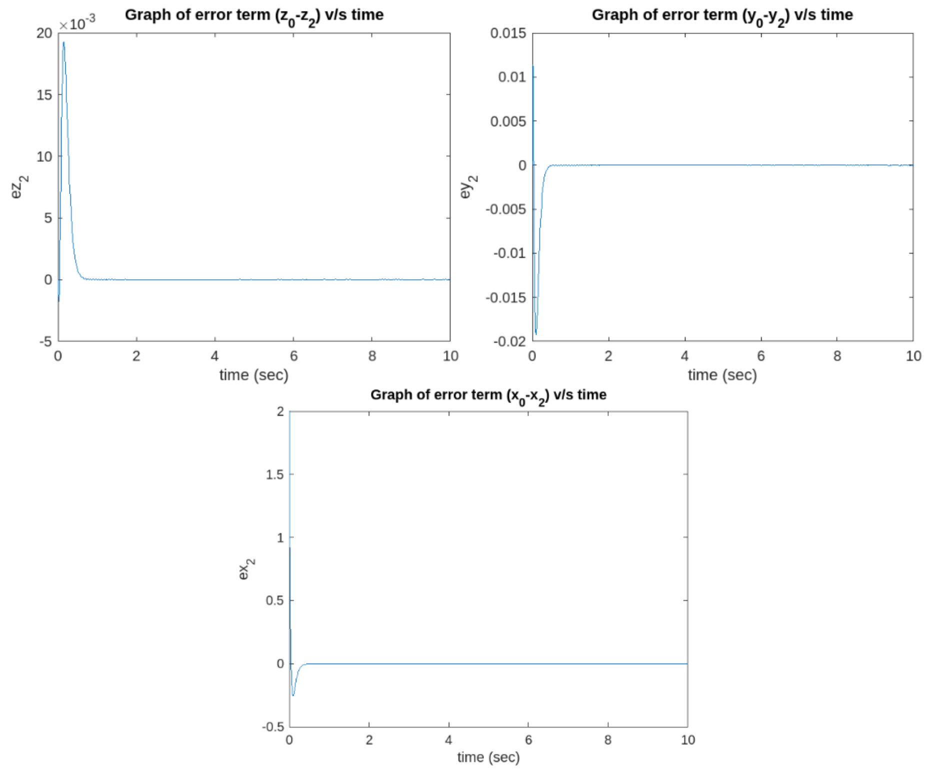


Figure 7. The difference signal e_x, e_y and e_z between master and second slave system

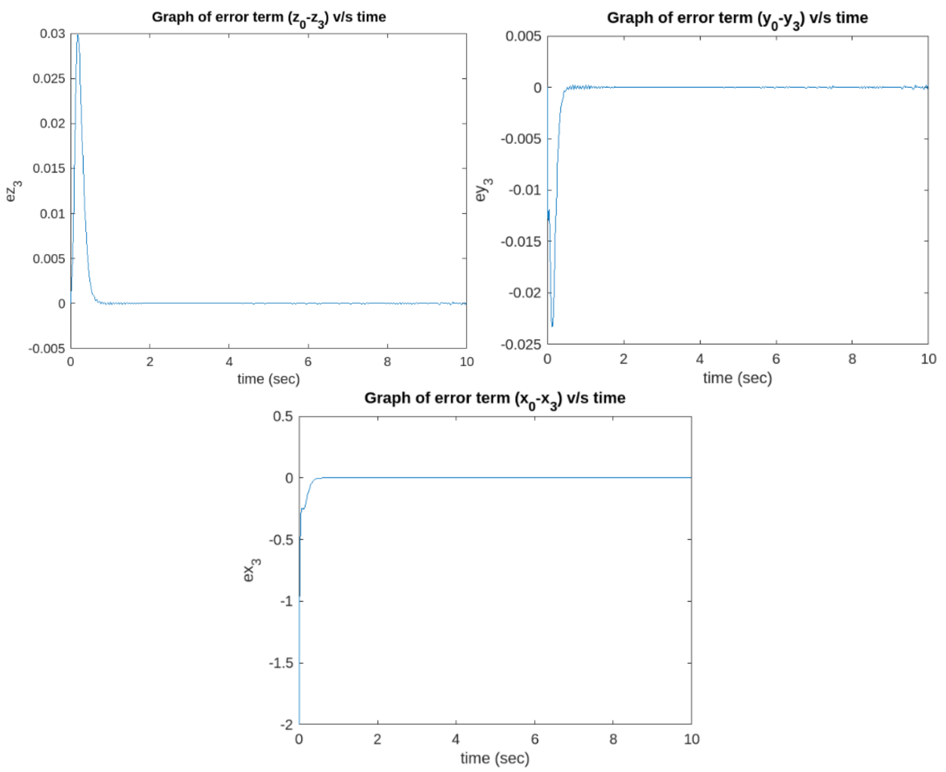


Figure 8. The difference signal e_x, e_y and e_z between master and third slave system

3. Rössler System

It is described by the following equation

$$\dot{x}_0 = -(y_0 + z_0)$$

$$\dot{y}_0 = (x_0 + y_0)$$

$$\dot{z}_0 = b + z_0(x_0 - c) \qquad \text{... (67)}$$

Where a, b and c denote positive parameter. According to the unidirectional linear error feedback coupling approach and referring to equation 2 we can obtain 3 slave system of equation 67.

1st slave system

$$\dot{x}_1 = -(y_1 + z_1) + k_{11}(x_0 - x_1)$$

$$\dot{y}_1 = (x_1 + ay_1) + k_{12}(y_0 - y_1)$$

$$\dot{z}_1 = b + z_1(x_1 - c) + k_{13}(z_0 - z_1) \qquad \text{... (68)}$$

2nd slave system

$$\dot{x}_2 = -(y_2 + z_2) + k_{21}(x_1 - x_2)$$

$$\dot{y}_2 = (x_2 + ay_2) + k_{22}(y_1 - y_2)$$

$$\dot{z}_2 = b + z_2(x_2 - c) + k_{23}(z_1 - z_2) \qquad \text{... (69)}$$

3rd slave system

$$\dot{x}_3 = -(y_3 + z_3) + k_{31}(x_2 - x_3)$$

$$\dot{y}_3 = (x_3 + ay_3) + k_{32}(y_2 - y_3)$$

$$\dot{z}_3 = b + z_3(x_3 - c) + k_{33}(z_2 - z_3) \qquad \text{... (70)}$$

From Equation 67, 68, 69, 70 we can obtain error dynamics in the following structure:

$$\dot{e} = Ae + g(x) - g(\tilde{x}) - Ke$$

here,

$$A = \begin{bmatrix} 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -c & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & a & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -c \end{bmatrix}, \; e = \begin{bmatrix} x_0 - x_1 \\ y_0 - y_1 \\ z_0 - z_1 \\ x_1 - x_2 \\ y_1 - y_2 \\ z_1 - z_2 \\ x_2 - x_3 \\ y_2 - y_3 \\ z_2 - z_3 \end{bmatrix}$$

$$K = \begin{bmatrix} k_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & k_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & k_{13} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & k_{21} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & k_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & k_{23} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & k_{31} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & k_{32} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & k_{33} \end{bmatrix}$$

$$g(x) = \begin{bmatrix} 0 \\ 0 \\ z_0 x_0 \\ k_{11} e_{11} \\ k_{12} e_{12} \\ z_1 x_1 + k_{13} e_{13} \\ k e_{21} \\ k_{22} e_{22} \\ z_2 x_2 + k_{23} e_{23} \end{bmatrix} \qquad g(\tilde{x}) = \begin{bmatrix} 0 \\ 0 \\ z_1 x_1 \\ 0 \\ 0 \\ z_2 x_2 \\ 0 \\ 0 \\ z_3 x_3 \end{bmatrix}$$

After solving $g(x) - g(\tilde{x}) = M_{x\tilde{x}}.\,\mathrm{e}$ and $(A + M_{x\tilde{x}}) + (A + M_{x\tilde{x}})^T$ and

From equation 25 we get for 1st slave system

$$k_{11} \geq (|z_1 - 1| - \mu)$$

$$k_{12} \geq (2a - \mu)$$

$$k_{13} \geq (|z_1 - 1| + 2x_0 - 2c - \mu) \qquad \ldots (71)$$

From equation 28 we get for 2nd slave system

$$k_{21} \geq k_{11}$$

$$k_{22} \geq k_{12}$$

$$k_{23} \geq k_{13} \qquad \text{... (72)}$$

From equation 29 we get for 3rd slave system

$$k_{31} \geq 3k_{11}$$

$$k_{32} \geq 3k_{12}$$

$$k_{33} \geq 3k_{13} \qquad \text{... (73)}$$

Since the trajectory of a chaotic system is bounded in equality 71 holds for large enough values of k_{11}, k_{12}, k_{13}.

Selecting a = 0.2, b = 0.2, c = 5.7 gives a chaotic behavior of the system as depicted in figure 9 from the figure, one can see that

$-10 < x < 13$, $-12 < y < 8$, $0 < z < 24$.

Choosing $\mu = -0.5$, we can obtain coupling parameter as

$k_{11} = 24$ $k_{12} = 1$ $k_{13} = 39$

$k_{21} = 24$ $k_{22} = 1$ $k_{13} = 39$

$k_{31} = 72$ $k_{32} = 3$ $k_{33} = 117$

With the above chosen parameter all the slave systems are synchronized with master system as error signal tends to zero.

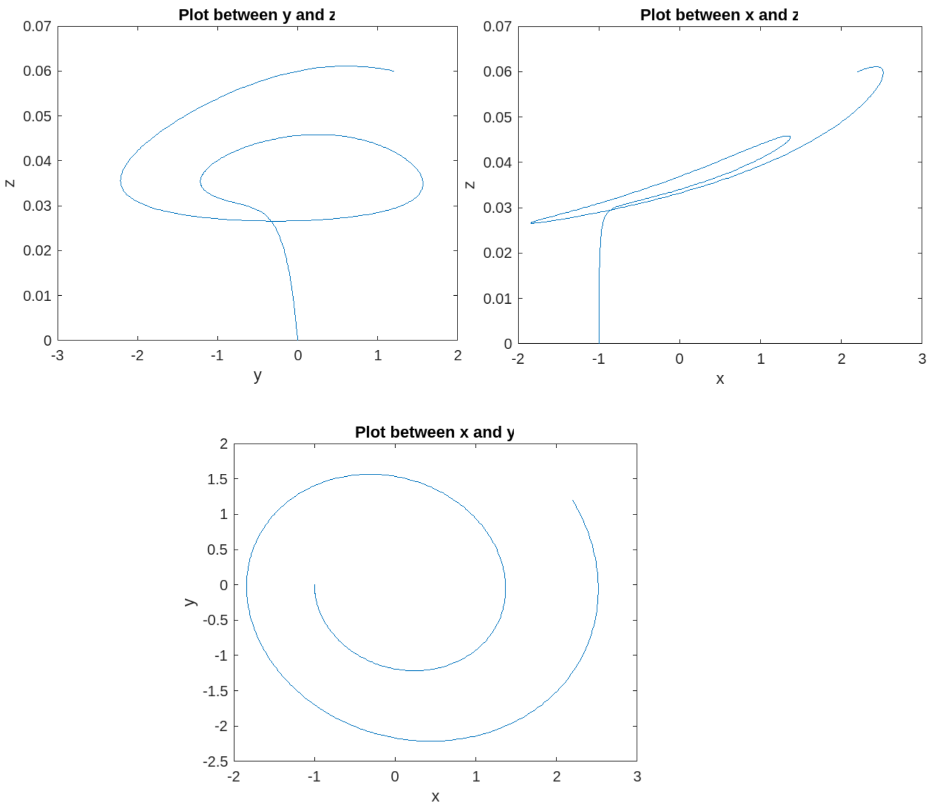


Figure 9. The Chaotic behavior of Rössler Circuit

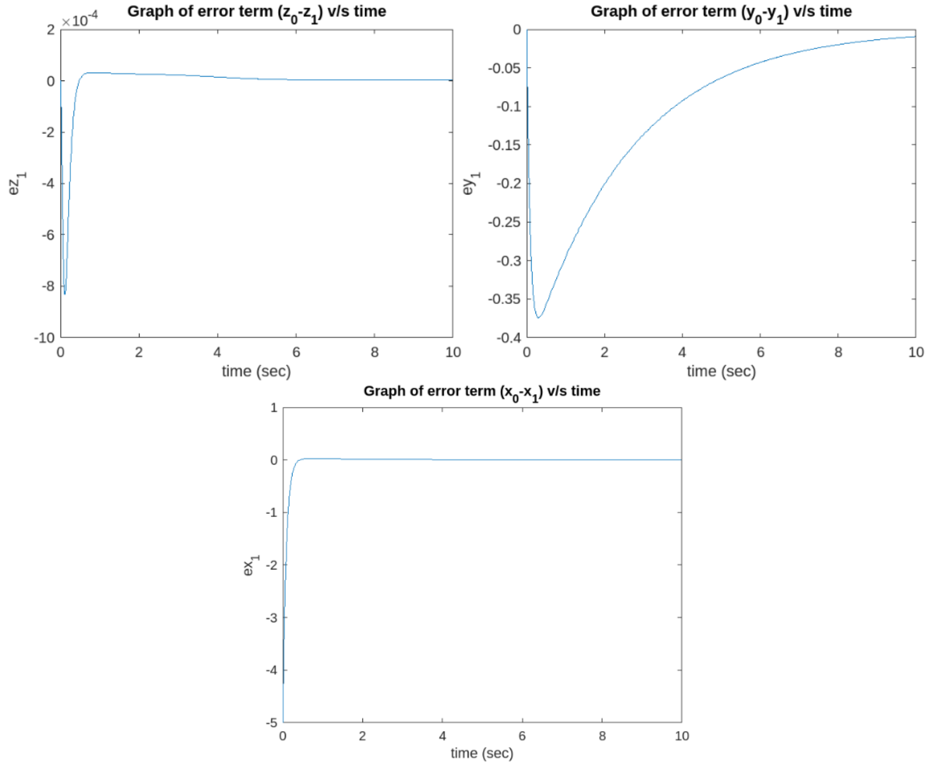


Figure 10. The difference signal e_x, e_y and e_z between master and first slave system

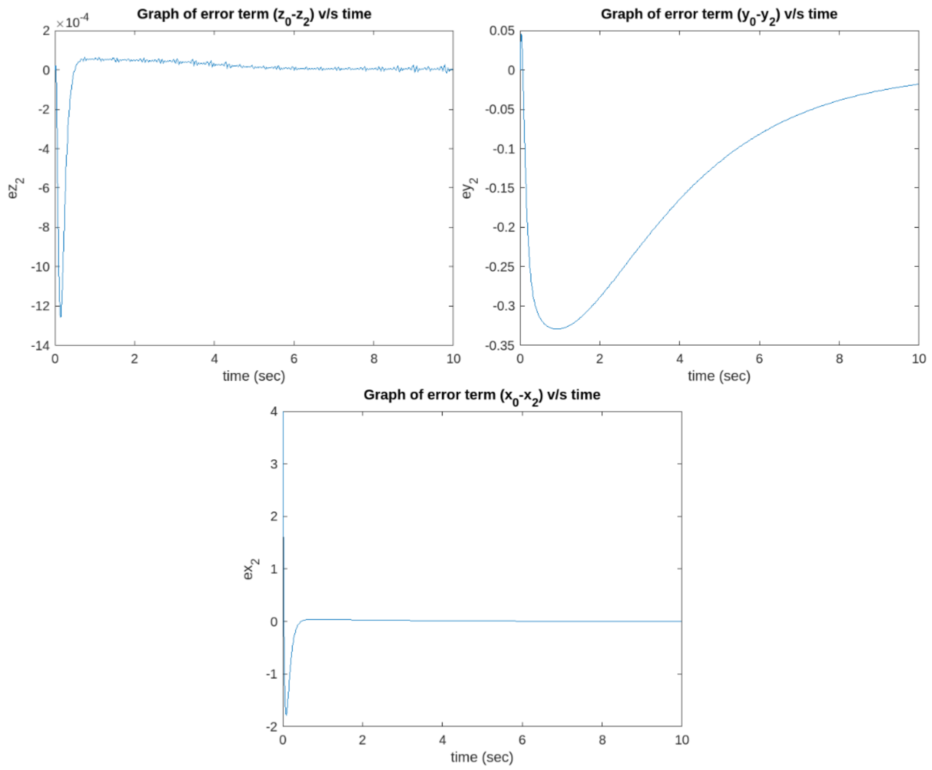


Figure 11. The difference signal e_x, e_y and e_z between master and second slave system

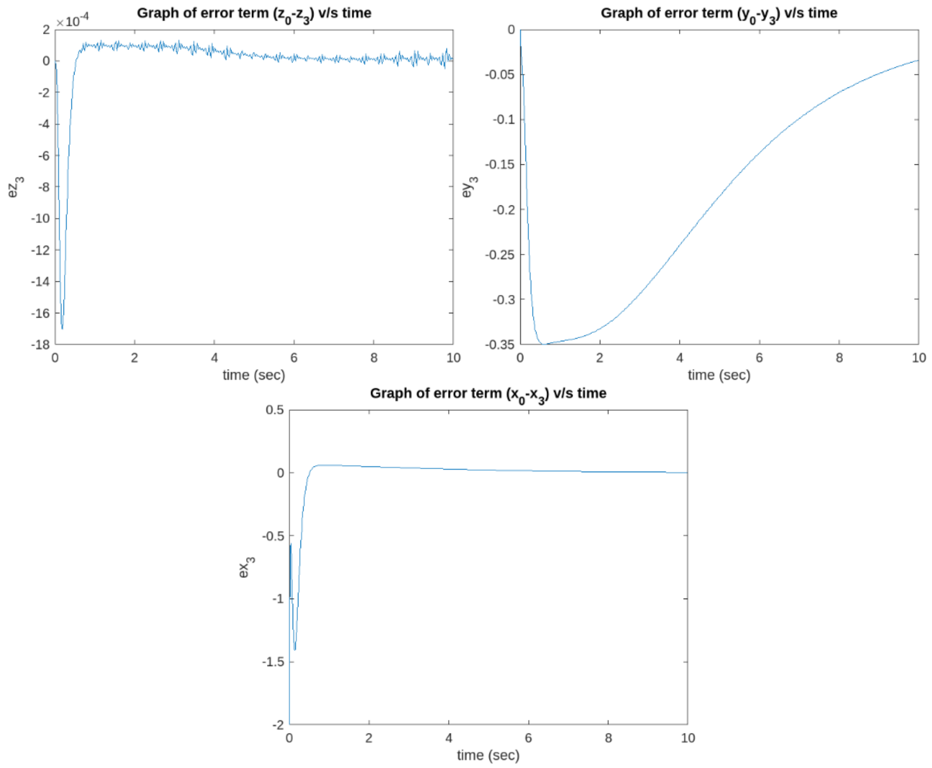


Figure 12. The difference signal e_x, e_y and e_z between master and third slave system

Conclusion

In this work, we present a novel approach for achieving global synchronization of 'm' coupled general chaotic systems with a unidirectional linear error feedback coupling. We derive a straightforward algebraic condition that guarantees global synchronization, which can be easily applied to a broad category of chaotic systems.

The concept of synchronization, where multiple chaotic systems evolve in a coordinated manner, has been widely studied due to its potential applications in secure communications, cryptography, and control of complex systems. However, achieving synchronization in chaotic systems is challenging due to their sensitive dependence on initial conditions and the complex dynamics exhibited by chaotic systems.

Our proposed approach overcomes these challenges by providing a simple algebraic condition for global synchronization. This condition can be readily used to design appropriate coupling parameters, taking into account the given condition, in order to ensure that the coupled chaotic systems achieve global synchronization. This straightforward criterion simplifies the design process and makes it more accessible for practical applications.

One of the key advantages of our approach is its applicability to a broad category of chaotic systems. Chaotic systems can exhibit a wide range of behaviours, including different types of nonlinearities and complexity in their dynamics. Our proposed condition is effective in achieving global synchronization in a variety of typical chaotic systems, irrespective of their specific characteristics. This makes it a versatile and robust method for achieving synchronization in diverse systems.

To validate the effectiveness of our approach, we conducted simulations using various chaotic systems, such as the Rössler system, and Chua's circuit. The simulations demonstrated that our proposed criterion is indeed effective in achieving global synchronization in these systems. Furthermore, we compared our approach with existing methods in the literature, and our results showed that our criterion outperforms several other approaches in terms of simplicity and effectiveness.

In conclusion, our work presents a straightforward algebraic condition for achieving global synchronization of 'm' coupled general chaotic systems with a unidirectional linear error feedback coupling. The criterion is applicable to a broad category of chaotic systems, and simulations have shown its effectiveness in achieving synchronization in diverse systems. Our proposed approach has the potential to find practical applications in various fields where synchronization of chaotic systems is desired, and it opens up new possibilities for designing synchronization schemes in complex systems. Further research can be conducted to explore the applicability of our approach in real-world systems and to investigate its robustness against uncertainties and noise.

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