Practical No. 1

Topic: Limits & Continuity

1) $\lim_{x \to a} \left[\dfrac{\sqrt{a+2x} - \sqrt{3x}}{\sqrt{3a+x} - 2\sqrt{x}} \right]$

2) $\lim_{y \to 0} \left[\dfrac{\sqrt{a+y} - \sqrt{a}}{y\sqrt{a+y}} \right]$

3) $\lim_{x \to \pi/6} \left[\dfrac{\cos x - \sqrt{3}\sin x}{\pi - 6x} \right]$

4) $\lim_{x \to \infty} \left[\dfrac{\sqrt{x^2+5} - \sqrt{x^2-3}}{\sqrt{x^2+3} - \sqrt{x^2+1}} \right]$

5) Examine the continuity of the following function at given points.

(i) $f(x) = \dfrac{\sin 2x}{\sqrt{1-\cos 2x}}$ for $0 < x \le \dfrac{\pi}{2}$
$= \dfrac{\cos x}{\pi - 2x}$, for $\dfrac{\pi}{2} < x < \pi$ } at $x = \dfrac{\pi}{2}$

(ii) $f(x) = \dfrac{x^2-9}{x-3}$ $0 < x < 3$
$= x+3$ $3 \le x < 6$
$= \dfrac{x^2-9}{x+3}$ $6 \le x < 9$ } at $x = 3$ & $x = 6$

(6) Find value of K, so that the function f(x) is cts at the indicated point.

(i) $f(x) = \frac{1-\cos 4x}{x^2} \quad x < 0$
$\quad = K \quad x = 0$ } at $x = 0$

(ii) $f(x) = (\sec^2 x)^{\cot^2 x} \quad x \neq 0$
$\quad = K \quad x = 0$ } at $x = 0$

(iii) $f(x) = \frac{\sqrt{3} - \tan x}{\pi - 3x} \quad x \neq \frac{\pi}{3}$
$\quad = K \quad x = \frac{\pi}{3}$ } at $x = \frac{\pi}{3}$

(6) Discuss the continuity of the following functions which of these functions have a removable discontinuity? Redefine the function so as to remove the discontinuity.

(i) $f(x) = \frac{1-\cos 3x}{x \tan x} \quad x \neq 0$
$\quad = 9 \quad x \neq 0$ } at $x = 0$

(ii) $f(x) = \frac{(e^{3x} - 1)\sin x^\circ}{x^2} \quad x \neq 0$
$\quad = \frac{\pi}{60}, \quad x = 0$ } at $x = 0$.

(7) If $f(x) = \dfrac{\cos^2 x (e^{x^2} - 1) \sin x^\circ}{x^2}$ for $x \neq 0$ is 32 cts at $x = 0$ find $f(0)$

(8) If $f(x) = \dfrac{\sqrt{2} - \sqrt{1 + \sin x}}{\cos^2 x}$ for $x \neq \dfrac{\pi}{2}$ is cts at $x = \pi/2$

Find $f(\pi/2)$

Solution:

(1) $\lim\limits_{x \to a} \left[\dfrac{\sqrt{a + 2x} - \sqrt{3x}}{\sqrt{3a + x} - 2\sqrt{x}} \right]$

$\lim\limits_{x \to a} \left[\dfrac{\sqrt{a+2x} - \sqrt{3x}}{\sqrt{3a+x} - 2\sqrt{x}} \times \dfrac{\sqrt{a+2x} + \sqrt{3x}}{\sqrt{a+2x} + \sqrt{3x}} \times \dfrac{\sqrt{3a+x} + 2\sqrt{x}}{\sqrt{3a+x} + 2\sqrt{x}} \right]$

$\lim\limits_{x \to a} \left[\dfrac{(a + 2x - 3x) \cdot (\sqrt{3a+x} + 2\sqrt{x})}{(3a + x - 4x) \cdot (\sqrt{a+2x} + \sqrt{3x})} \right]$

$\lim\limits_{x \to a} \dfrac{(a - x)(\sqrt{3a+x} + 2\sqrt{x})}{(3a - 3x)(\sqrt{a+2x} + \sqrt{3x})}$

$\dfrac{1}{3} \lim\limits_{x \to a} \dfrac{(a-x)(\sqrt{3a+x} + 2\sqrt{x})}{(a-x)(\sqrt{a+2x} + \sqrt{3x})}$

$\dfrac{1}{3} \times \dfrac{\sqrt{3a+a} + 2\sqrt{a}}{\sqrt{a+2a} + \sqrt{3a}}$

$\dfrac{1}{3} \times \dfrac{\sqrt{4a} + 2\sqrt{a}}{\sqrt{3a} + \sqrt{3a}}$

$\dfrac{1}{3} \times \dfrac{\cancel{4}^{2}\sqrt{a}}{\cancel{2}\sqrt{3a}}$

$= \dfrac{2}{3\sqrt{3}}$

(2) $\lim\limits_{y \to 0} \left[\dfrac{\sqrt{a+y} - \sqrt{a}}{y\sqrt{a+y}} \right]$

$= \lim\limits_{y \to 0} \left[\dfrac{\sqrt{a+y} - \sqrt{a}}{y\sqrt{a+y}} \times \dfrac{\sqrt{a+y} + \sqrt{a}}{\sqrt{a+y} + \sqrt{a}} \right]$

$= \lim\limits_{y \to 0} \dfrac{a+y-a}{y\sqrt{a+y}\,(\sqrt{a+y} + \sqrt{a})}$

$= \lim\limits_{y \to 0} \dfrac{\cancel{y}}{\cancel{y}\sqrt{a+y}\,(\sqrt{a+y} + \sqrt{a})}$

$= \dfrac{1}{\sqrt{a+0}\,(\sqrt{a+0} + \sqrt{a})}$

$= \dfrac{1}{\sqrt{a}\,(2\sqrt{a})}$

$= \dfrac{1}{2a}$

(3) $\lim\limits_{x \to \pi/6} \dfrac{\cos x - \sqrt{3}\sin x}{\pi - 6x}$

By substituting $x - \pi/6 = h$

$x = h + \pi/6$

where $h \to 0$

$\lim\limits_{h \to 0} \dfrac{\cos(h+\pi/6) - \sqrt{3}\sin(h+\pi/6)}{\pi - 6(h+\pi/6)}$

using

$\cos(A+B) = \cos A \cdot \cos B - \sin A \cdot \sin B$

$\sin(A+B) = \sin A \cdot \cos B + \cos A \cdot \sin B$

$\lim\limits_{h \to 0} \dfrac{\cos h \cdot \cos \pi/6 - \sin h \cdot \sin \pi/6 - \sqrt{3}\sin h \cos \pi/6 + \cos h \cdot \sin \pi/6}{\pi - 6\left(\dfrac{6h+\pi}{6}\right)}$

$\cos \pi/6 = \cos 30° = \sqrt{3}/2$

$\sin \pi/6 = \sin 30° = 1/2$

$$\lim_{h\to 0} \frac{\cos h \cdot \sqrt{3}/2 \cdot \sin h \; 1/\sqrt{2} - \sqrt{3}\left(\sin h \sqrt{3}/2 + \cos h \cdot 1/\sqrt{2}\right)}{\cancel{\pi} - 6h + \cancel{\pi}}$$

$$\lim_{h\to 0} \frac{\cos \cdot \sqrt{3}\,h/2 - \sin h/2 - \sin 3h/2 - \cos \sqrt{3}\,h/2}{-6h}$$

$$\lim_{h\to 0} \frac{-\sin 4h/2}{-6h}$$

$$\frac{1}{3} \lim_{h\to 0} \frac{\sin h}{h} = \frac{1}{3} \times 1 = \frac{1}{3}$$

(4) $$\lim_{x\to\infty} \left[\frac{\sqrt{x^2+5} - \sqrt{x^2-3}}{\sqrt{x^2+3} - \sqrt{x^2+1}} \right]$$

By rationalising numerator & Denominator both.

$$\lim_{x\to\infty} \left[\frac{\sqrt{x^2+5} - \sqrt{x^2-3}}{\sqrt{x^2+3} - \sqrt{x^2+1}} \times \frac{\sqrt{x^2+5} + \sqrt{x^2-3}}{\sqrt{x^2+5} + \sqrt{x^2-3}} \times \frac{\sqrt{x^2+3} + \sqrt{x^2+1}}{\sqrt{x^2+3} + \sqrt{x^2+1}} \right]$$

$$\lim_{x\to\infty} \left[\frac{(\cancel{x^2} + 5 - \cancel{x^2} + 3)\left(\sqrt{x^2+3} + \sqrt{x^2+1}\right)}{(\cancel{x^2} + 3 - \cancel{x^2} - 1)\left(\sqrt{x^2+5} + \sqrt{x^2-3}\right)} \right]$$

$$\lim_{x\to\infty} \frac{\cancel{8}^{\,4}}{\cancel{2}} \frac{\left(\sqrt{x^2+3} + \sqrt{x^2+1}\right)}{\left(\sqrt{x^2+5} + \sqrt{x^2-3}\right)}$$

$$4 \lim_{x\to\infty} \frac{\sqrt{x^2(1+3/x^2)} + \sqrt{x^2(1+1/x^2)}}{\sqrt{x^2(1+5/x^2)} + \sqrt{x^2(1-3/x^2)}}$$

After applying limit, we get $= 4$

(5) $f(x) = \dfrac{\sin 2x}{\sqrt{1-\cos 2x}}$, for $0 \leq x \leq \dfrac{\pi}{2}$

$= \dfrac{\cos x}{\pi - 2x}$, for $\dfrac{\pi}{2} < x < \pi$ } at $x = \pi/2$

$f(\pi/2) = \dfrac{\sin 2(\pi/2)}{\sqrt{1-\cos 2(\pi/2)}}$ $\therefore f(\pi/2) = 0$

f at $x = \pi/2$ define

(ii) $\lim\limits_{x \to \pi/2} f(x) = \lim\limits_{x \to \pi/2} + \dfrac{\cos x}{\pi - 2x}$

By substituting method,

$x - \dfrac{\pi}{2} = h$

$x = h + \pi/2$

where $h \to 0$.

$\lim\limits_{h \to 0} \dfrac{\cos(h + \pi/2)}{\pi - 2(h + \pi/2)}$

$\lim\limits_{h \to 0} \dfrac{\cos(h + \pi/2)}{\pi - 2(2h + \pi/2)}$

$\lim\limits_{h \to 0} \dfrac{\cos(h + \pi/2)}{-2h}$

$\lim\limits_{h \to 0} \dfrac{\cos h \cdot \cos \pi/2 - \sin h \cdot \sin \pi/2}{-2h}$ using $\cos(A+B) = \cos A \cdot \cos B - \sin A \sin$

$\lim\limits_{h \to 0} \dfrac{\cos h \cdot 0 - \sin h}{-2h}$

$\lim\limits_{h \to 0} \dfrac{-\sin h}{-2h}$

$\dfrac{1}{2} \lim\limits_{h \to 0} \dfrac{\sin h}{h} = 1/2$

b) $\lim_{x\to\pi/2} f(x) = \lim_{x\to\pi/2} \dfrac{\sin 2x}{\sqrt{1-\cos 2x}}$ using $\sin 2x = 2\sin x \cdot \cos x$

$\lim_{x\to\pi/2} \dfrac{2\sin x \cdot \cos x}{\sqrt{2\sin^2 x}}$

$\lim_{x\to\pi/2} \dfrac{2\cancel{\sin x} \cdot \cos x}{\sqrt{2}\,\cancel{\sin x}}$

$\lim_{x\to\pi/2} \dfrac{2\cos x}{\sqrt{2}}$

$\dfrac{2}{\sqrt{2}} \lim_{x\to\pi/2} \cdot \cos x.$

$\therefore$ LHL $\neq$ RHL

$\therefore f$ is not continuous at $x = \pi/2$

(ii) $f(x) = \dfrac{x^2-9}{x-3}$ $\quad 0 < x \leq 3$

$= x+3$ $\quad 3 \leq x \leq 6$

$= \dfrac{x^2-9}{x+3}$ $\quad 6 \leq x < 9$

at $x=3$ & $x=6$

at $x=3$

(i) $f(3) = \dfrac{x^2-9}{x-3} = 0$

f at $x=3$ define

(ii) $\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} x+3$

$f(3) = x+3,\ 3+3 = 6$

f is define at $x=3$

$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} (x+3 = 6$

$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} \dfrac{x^2-9}{x-3} = \dfrac{\cancel{(x-3)}(x+3)}{\cancel{(x-3)}}$

$\therefore$ LHL = RHL

$\therefore f$ is continuous at $x=3$

for $x=6$

$$f(6) = \frac{x^2-9}{x+3} = \frac{36-9}{6+3} = \frac{27}{9} = 3$$

(7) $\lim_{x\to 6^+} \frac{x^2-9}{x+3}$

$\lim_{x\to 6^+} \frac{(x-3)\cancel{(x+3)}}{\cancel{(x+3)}}$

$\lim_{x\to 6^+} (x-3) = 6-3 = 3$

$\lim_{x\to 6^-} x+3 = 3+6 = 9$

$\therefore$ LHL $\neq$ RHL

function is not continuous.

(6) (1) $f(x) = \frac{1-\cos 4x}{x^2} \quad x<0$
$\quad = k \quad x=0$ } at $x=0$

$\Rightarrow f(\cdot)$ continuous at $x=0$

$\lim_{x\to 0} f(x) = f(0)$

$\lim_{x\to 0} \frac{1-\cos 4x}{x^2} = k$

$\lim_{x\to 0} \frac{2\sin^2 2x}{x^2} = k$

$2\lim_{x\to 0} \frac{\sin^2 x}{x^2} = k$

$2(2)^2 = k$

$\therefore k = 8$

(ii) $f(x) = (\sec^2 x)^{\cot^2 x} \quad x \neq 0$
$= k \quad x = 0$ } at $x = 0$

$\rightarrow f(x) = (\sec^2 x)^{\cot^2 x}$

using
$\tan^2 x - \sec^2 x = 1$
$\therefore \sec^2 x = 1 + \tan^2 x$
$\cot^2 x = \frac{1}{\tan^2 x}$

$\therefore \lim_{x \to 0} (\sec^2 x)^{\cot^2 x}$

$\lim_{x \to 0} (1 + \tan^2 x)^{1/\tan^2 x}$

we know that.
$\lim_{x \to 0} (1 + px)^{1/px} = e$

$= e.$

$\therefore k = e$

(iii) $f(x) = \frac{\sqrt{3} - \tan x}{\pi - 3x} \quad x \neq \pi/3$
$= k \quad x = \pi/3$ } at $x = \pi/3$

$x - \frac{\pi}{3} = h$

$\therefore x = h + \pi/3$

when $h \to 0$
$f(\pi/3 + h) = \frac{\sqrt{3} - \tan(\pi/3 + h)}{\pi - 3(\pi/3 + h)}$

$\lim_{h \to 0} \frac{\sqrt{3} - \tan(\pi/3 + h)}{\pi - 3(\pi/3 + h)}$

using

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \cdot \tan B}$$

$$\lim_{h \to 0} \frac{\sqrt{3} - \dfrac{\tan \pi/3 + \tan h}{1 - \tan \pi/3 \cdot \tan h}}{\pi - \pi - 3h}$$

$$\lim_{h \to 0} \frac{\dfrac{\sqrt{3}\,(1 - \tan \pi/3 \cdot \tan h) - (\tan \pi/3 + \tan h)}{1 - \tan \pi/3 \cdot \tan h}}{-3h}$$

$$\lim_{h \to 0} \frac{\dfrac{(\sqrt{3} - \sqrt{3} \times \sqrt{3} \cdot \tan h) - (\sqrt{3} + \tan h)}{1 - \tan \pi/3 \cdot \tan h}}{-3h}$$

$$\lim_{h \to 0} \frac{\dfrac{(\sqrt{3} - 3 \tan h) - (\sqrt{3} + \tan h)}{1 - \sqrt{3} \cdot \tan h}}{-3h}$$

$$\lim_{h \to 0} \frac{\dfrac{(\cancel{\sqrt{3}} - 3\tan h) - \cancel{\sqrt{3}} - \tan h}{1 - \sqrt{3}\tan h}}{-3h}$$

$$\lim_{h \to 0} \frac{-4 \tan h}{-3h\,(1 - \sqrt{3}\tan h)}$$

$$\lim_{h \to 0} \frac{4 \tan h}{3h\,(1 - \sqrt{3} \tan h)}$$

$$\frac{4}{3} \lim_{h \to 0} \frac{\tan h}{h} \quad \lim_{h \to 0} \frac{1}{(1 - \sqrt{3} \tan h)}$$

$$= \frac{4}{3} \cdot \frac{1}{1 - \sqrt{3}(0)}$$

$$= \frac{4}{3}\left(\frac{1}{1}\right) = \frac{4}{3}$$

(4) $f(x) = \dfrac{1-\cos 3x}{x\tan x}$ $\quad x \neq 0$ $\left.\begin{matrix} \\ \\ \end{matrix}\right\}$ at $x=0$

$= 9$ $\quad x = 0$

$f(x) = \dfrac{1-\cos 3x}{x\tan x}$

$\lim_{x\to 0} = \dfrac{2\sin^2 3/2\, x}{x\tan x}$

$\lim_{x\to 0} = \dfrac{\dfrac{2\sin^2 3x/2}{x^2} \times x^2}{\dfrac{x\cdot\tan x}{x^2} \times x^2}$

$= \dfrac{2\lim_{x\to 0} \left(\frac{3}{2}\right)^2}{1}$

$= 2\times\dfrac{9}{4} = 9/2$

$\lim_{x\to 0} f(x) = 9/2 \neq f(x_0)$

$\therefore f$ is not continuous at $x=0$.

Redefine function

$f(x) = \dfrac{(1-\cos 3x)}{x\tan x}$ $\quad x\neq 0$

$9/2.$

Now, $\lim_{x\to 0} f(x) = f(0)$

f has removable discontinuity at $x=0$

(7)

(ii) $f(x) = \frac{(e^{3x}-1)\sin x^\circ}{x^2}$ $\quad x \neq 0$ $\Big\}$ at $x=0$

$= \pi/6 \quad x=0$

$$\lim_{x\to 0} \frac{(e^{3x}-1)\sin\left(\frac{\pi x}{180}\right)}{x^2}$$

$$\lim_{x\to 0} \frac{(e^{3x}-1)\sin(\pi x/180)}{x^2}$$

$$\lim_{x\to 0} \frac{e^{3x}-1}{x} \lim_{x\to 0} \frac{\sin(\pi x/180)}{x}$$

$$\lim_{x\to 0} \frac{3\cdot e^{3x}-1}{3x} \lim_{x\to 0} \frac{\sin(\pi x/180)}{x}$$

$$3\lim_{x\to 0} \frac{e^{3x}-1}{3x} \lim_{x\to 0} \frac{\sin(\pi x/180)}{x}$$

$$3\cdot \log_e \frac{\pi}{\cancel{180}_{60}} = \pi/60 = f(0)$$

f is continuous at $x=0$.

(8) $f(x) = \frac{e^{x^2} - \cos x}{x^2}$ $\quad x=0$

is continuous at $x=0$

$$\lim_{x\to 0} f(x) = f(0)$$

$$= \lim_{x\to 0} \frac{e^{x^2} - \cos x}{x^2} = f(0)$$

$$= \lim_{x\to 0} \frac{e^{x^2} - \cos x - 1 + 1}{x^2}$$

$$= \lim_{x\to 0} \frac{(e^{x^2}-1) + (1-\cos x)}{x^2}$$

$$= \lim_{x\to 0} \frac{(e^{x^2}-1) + (1-\cos x)}{x^2}$$

$$= \lim_{x\to 0} \frac{e^{x^2}-1}{x^2} + \lim \frac{1-\cos x}{x^2}$$

$$\log e + \lim_{x\to 0} \frac{2\sin^2 x/2}{x}$$

$$\log e + 2\lim_{x\to 0} \left(\frac{\sin x/2}{x}\right)^2$$

Multiply with 2 in Num & Denominator

$$= 1 + 2 \times \frac{1}{\cancel{4}\,2} = \frac{3}{2} = f(0)$$

(9) $f(x) = \frac{\sqrt{2} - \sqrt{1+\sin x}}{\cos^2 x} \quad x \neq \pi/2.$

$f(x)$ is continuous at $x = \pi/2$.

$$\lim_{x\to\pi/2} \frac{\sqrt{2} - \sqrt{1+\sin x}}{\cos^2 x} \times \frac{\sqrt{2} + \sqrt{1+\sin x}}{\sqrt{2} + \sqrt{1+\sin x}}$$

$$\lim_{x\to\pi/2} \frac{2 - 1 + \sin x}{\cos^2 x(\sqrt{2} + \sqrt{1+\sin x})}$$

$$\lim_{x\to\pi/2.} \frac{1 + \sin x}{1 - \sin^2 x(\sqrt{2} + \sqrt{1+\sin x})}$$

$$\lim_{x\to\pi/2} \frac{\cancel{1+\sin x}}{(1-\sin x)\cancel{(1+\sin x)}(\sqrt{2} + \sqrt{1+\sin x})}$$

$$\lim_{x\to\pi/2} \frac{1}{(1-\sin x)(\sqrt{2} + \sqrt{1+\sin x})}$$

$$\frac{1}{2(\sqrt{2}+\sqrt{2})}$$

$$= \frac{1}{2(2\sqrt{2})}$$

$$= \frac{1}{4\sqrt{2}}$$

$$\therefore f(\pi/2) = \frac{1}{4\sqrt{2}}$$

Practical - 2

Topic : Derivative

Q1] Show that the following function defined from $\mathbb{R}$ to $\mathbb{R}$ are differentiable.

(i) $\cot x$

$f(x) = \cot x$

$$Df(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$= \lim_{x \to a} \frac{\cot x - \cot a}{x - a}$$

$$= \lim_{x \to a} \frac{1/\tan x - 1/\tan(a)}{x - a}$$

$$= \lim_{x \to 0} \frac{\tan a - \tan x}{(x - a) \tan x \tan a}$$

put $x - a = h$

$x = a + h$

as $x \to a$, $h \to 0$

$$Df(h) = \lim_{h \to 0} \frac{\tan a - \tan(a+h)}{(x + h - a) \tan(a+h) \tan a}$$

$$= \lim_{h \to 0} \frac{\tan a - \tan(a+h)}{h \times \tan(a+h) \tan a}$$

formula: $\tan(A - B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

$\tan(A - B)(1 + \tan A \cdot \tan B) = \tan A - \tan B)$

$$= \lim_{h \to 0} \frac{\tan(a - a - h) - (1 + \tan a + \tan(a+h))}{h \times \tan(a+h) \tan a}$$

$$= \lim_{h \to 0} \frac{2\cos\left(\frac{a+a+h}{2}\right) \cdot \sin\left(\frac{a-a-h}{2}\right)}{h \times \sin a \cdot \sin(a+h)}$$

$$= \lim_{h \to 0} \frac{-\sin h/2}{h/2} \times 1/2 \times \frac{2\cos\left(\frac{2a+h}{2}\right)}{\sin a \cdot \sin(a+h)}$$

$$= -1/2 \times \frac{2\cos\left(\frac{2a+0}{2}\right)}{\sin(a+0)}$$

$$= \frac{-\cos a}{\sin^2 a} = -\cot a \operatorname{cosec} a$$

(iii) $\sec x$

$f(x) = \sec x$

$$Df(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x-a}$$

$$= \lim_{x \to a} \frac{\sec x - \sec a}{x-a}$$

$$= \lim_{x \to a} \frac{1/\cos x - 1/\cos a}{x-a}$$

$$= \lim_{x \to a} \frac{\cos a - \cos x}{(x-a)\cos a \cdot \cos x}$$

put $x - a = h$

$x = a+h$

as $x \to a$, $h \to 0$

$$Df(h) = \lim_{h \to 0} \frac{\cos a - \cos(a+h)}{h \times \cos a \cdot \cos(a+h)}$$

Formula: $-2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$

$$= \lim_{h \to 0} \frac{-2\sin\left(\frac{a+a+h}{2}\right)\sin\left(\frac{a-a-h}{2}\right)}{h \times \cos a \cos(a+h)}$$

$$= \lim_{h \to 0} \frac{-2\sin\left(\frac{2a+h}{2}\right)\sin\frac{-h}{2}}{\cos a \cdot \cos(a+h) \times \frac{-h}{2}} \times \frac{-1}{2}$$

$$= \lim \frac{-1}{2} \times \frac{-2\sin\left(\frac{2a+0}{2}\right)}{\cos a \cos(a+0)}$$

$$= \frac{-1}{2} \times -2 \frac{\sin a}{\cos a \times \cos a}$$

$= \tan a \sec a$

Q2) If $f(x) = 4x+1, \quad x \leq 5$
$\qquad = x^2+5 \quad x > 0$ at $x=2$, then find function is differentiable or not.

→ LHD:

$$Lf'(2^-) = \lim_{x \to 2^-} \frac{f(x) - f(2)}{x-2}$$

$$= \lim_{x \to 2^-} \frac{4x+1-(4\times 2+1)}{x-2}$$

$$= \lim_{x \to 2^-} \frac{4x+1-9}{x-2}$$

$$= \lim_{x \to 2^-} \frac{4x-8}{x-2}$$

$$= \lim_{x \to 2^-} \frac{4(x-2)}{x-2} = 4$$

$Lf'(2^-) = 4$

RHD:

$$Rf'(2^+) = \lim_{x \to 2^+} \frac{x^2+5-9}{x-2}$$

$$= \lim_{x \to 2^+} \frac{x^2 - 4}{x - 2}$$

$$= \lim_{x \to 2^+} \frac{(x+2)\cancel{(x-2)}}{\cancel{x-2}}$$

$$= 2 + 2 = 4.$$

$f'(2^+) = 4.$

RHD = LHD

$\therefore f$ is diffrentiable at $x = 2$

Q3) If $f(x) = 4x + 7, \quad x < 3$

$= x^2 + 3x + 1, \quad x \geq 3$ at $x = 3$, then find f is diffrentiable or not?

→ RHD:

$$f'(3^+) = \lim_{x \to 3^+} \frac{f(x) - f(3)}{x - 3}$$

$$= \lim_{x \to 3^+} \frac{x^2 + 3x + 1 - (3^2 + 3 \times 3 + 1)}{x - 3}$$

$$= \lim_{x \to 3^+} \frac{x^2 + 3x - 18}{x - 3}$$

$$= \lim_{x \to 3^+} \frac{x^2 + 6x - 3x - 18}{x - 3}$$

$$= \lim_{x \to 3^+} \frac{(x+6)(x-3)}{(x-3)}$$

$$= \lim_{x \to 3^+} \frac{(x+6)\cancel{(x-3)}}{\cancel{(x-3)}}$$

$$= 3 + 6$$

$\therefore f'(3^+) = 9$

LHD = $f'(3^-)$

$= \lim_{x \to 3^-} \frac{f(x) - f(3)}{x - 3}$

$= \lim_{x \to 3^-} \frac{4x + 7 - 19}{x - 3}$

$= \lim_{x \to 3^-} \frac{4x - 12}{x - 3}$

$= \lim_{x \to 3} \frac{4\cancel{(x-3)}}{\cancel{(x-3)}}$

$f'(3^-) = 4$

RHD $\neq$ LHD

f is not differentiable at $x = 3$

Q4) If $f(x) = 8x - 5$, $x \leq 2$
$= 3x^2 - 4x + 7$, $x > 2$ at $x = 2$. Then find f is differentiable or not.

→ $f(2) = 8 \times 2 - 5 = 16 - 5 = 11$

RHD:
$f'(2^+) = \lim_{x \to 2^+} \frac{f(x) - f(2)}{x - 2}$

$= \lim_{x \to 2^+} \frac{3x^2 - 4x + 7 - 11}{x - 2}$

$= \lim_{x \to 2^+} \frac{3x^2 - 4x - 4}{x - 2}$

$= \lim_{x \to 2^+} \frac{3x^2 - 6x + 2x - 4}{x - 2}$

$= \lim_{x \to 2^+} \frac{3x(x-2) + 2(x-2)}{x - 2}$

$= \lim_{x \to 2^+} \frac{(3x+2)(x-2)}{(x-2)}$

$= 3 \times 2 + 2 = 8$

$f'(2^+) = 8$

LHD:

$$of(2^-) = \lim_{x \to 2^-} \frac{f(x) - f(2)}{x - 2}$$

$$= \lim_{x \to 2^-} \frac{8x - 5 - 11}{x - 2}$$

$$= \lim_{x \to 2^-} \frac{8x - 16}{x - 2}.$$

$$= \lim_{x \to 2^-} \frac{8\cancel{(x-2)}}{\cancel{(x-2)}}$$

$$= 8$$

$$of(2^-) = 8$$

LHD = RHD

$\therefore$ f is differentiable at $x = 3$.

Practical No: 3

Topic: Application of ~~Depri~~ Derivatives

① Find the intervals in which function is increasing or decreasing.

(i) $f(x) = x^3 - 5x - 11$

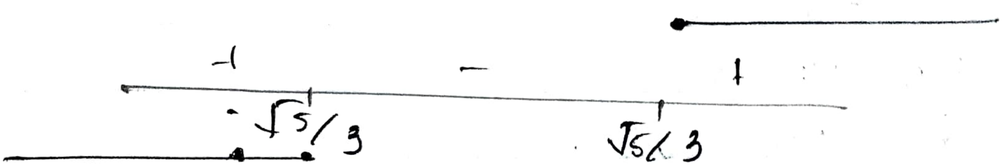
$\therefore f(x) = 3x^2 - 5$

$\therefore$ f is increasing iff $f'(x) > 0$

$3x^2 - 5 > 0$

$3(x^2 - 5/3) > 0$

$(x - \sqrt{5/3})(x - \sqrt{5/3}) > 0$



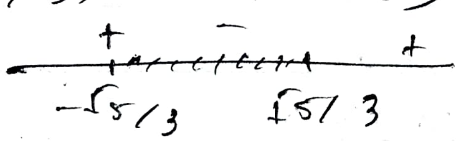
$x \in (-\infty, -\sqrt{5/3}) \cup (\sqrt{5/3}, \infty)$

and f is decreasing i.e. $f(x) < 0$

$\therefore 3x^2 - 5 < 0$

$3(x^2 - 5/3) < 0$

$(x - \sqrt{5/3})(x + \sqrt{5/3}) < 0$



$x \in (-5/3, \sqrt{5/3})$

(2) $f(x) = x^2 - 4x$.

$f'(x) = 2x - 4$

$\therefore$ f is increasing iff $f'(x) > 0$

$\therefore 2x - 4 > 0$

$2(x - 2) > 0$

$x - 2 > 0$

$x \in (2, \infty)$

and it is decreasing iff $f'(x) < 0$

$\therefore 2x - 4 < 0$

$\therefore 2(x-2) < 0$

$\therefore x - 2 < 0$

$x \in (-\infty, 2)$

2) $f(x) = 2x^3 + x^2 - 20x + 4$

$\therefore f'(x) = 6x^2 + 2x - 20$

$\because f$ is increasing its $f'(x) > 0$

$\therefore 6x^2 - 2x - 20 > 0$

$\therefore 2(3x^2 + x - 10) > 0$

$\therefore 3x^2 + x - 10 > 0$

$\therefore 3x^2 + 6x - 5x - 10 > 0$

$3x(x+2) - 5(x+2) > 0$

$(x+2)(3x-5) > 0$

$+ \quad - \quad +$

$-2 \quad 5/3$

$x \in (-\infty, -2) \cup (5/3, \infty)$ and

f is decreasing iff $f'(x) < 0$

$\therefore 6x^2 + 2x - 20 < 0$

$\therefore 2(3x^2 + x - 10) < 0$

$\therefore 3x^2 + x - 10 < 0$

$\therefore 3x^2 + 6x - 5x - 10 < 0$

$\therefore 3x(x+2) - 5(x+2) < 0$

$\therefore (x+2)(3x-5) < 0$

$x = -2, 5/3$

$+ \quad - \quad +$

$-2 \quad 5/3$

$x \in (-2, 5/3)$

(4) $f(x) = x^3 - 27x + 5$

$f'(x) = 3x^2 - 27$

$\therefore f$ is increasing iff $f'(x) > 0$

$3(x^2 - 9) > 0$

$(x-3)(x+3) > 0$

$x = 3, -3$

$x \in (-\infty, -3) \cup (3, \infty)$

and f is decreasing iff $f'(x) < 0$

$\therefore 3(x^2 - 9) < 0$

$\therefore \cancel{3}(x^2 - 9) < 0$

$\therefore (x+3)(x-3) < 0$

$\therefore x \in (-3, 3)$

(5) $f(x) = 2x^3 - 9x^2 - 24x + 69$

$f'(x) = 6x^2 - 18x - 24$

$\therefore f$ is increasing iff $f'(x) > 0$

$\therefore 6x^2 - 18x - 24 > 0$

$6(x^2 - 3x - 4) > 0$

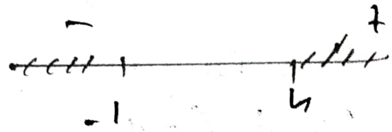
$x^2 - 3x - 4 > 0$

$x^2 - 4x + x - 4 > 0$

$\therefore x(x-4) + 1(x-4) > 0$

$\therefore (x+1)(x-4)$

$x = 4, -1$



$x \in (-\infty, -1) \cup (4, \infty)$.

and f is decreasing iff $f'(x) < 0$

$\therefore 6x^2 - 18x - 24 < 0$

$6(x^2 - 3x - 4) < 0$

$\therefore x^2 - 4x + x - 4 < 0$

$\therefore x(x-4) + 1(x-4) < 0$

$\therefore (x-4)(x+1)$

$x \in (-1, 4)$.

Q2 a) $y = 3x^2 - 2x^3$

$\therefore f(x) = 3x^2 - 2x^3.$

$\therefore f'(x) = 6(x) - 6x^2$

$\therefore f''(x) = 6 - 12x$

f is concave upwards if $f''(x) > 0$

$\therefore (6 - 12x) > 0$

$\therefore 12(6/12 - x) > 0$

$\therefore x - 1/2 > 0$

$\therefore x > 1/2$

$\therefore f''(x) > 0$

$x \in (1/2, \infty)$

(b) $y = x^4 - 6x^3 + 12x^2 + 5x - 7$

$f'(x) = 4x^3 - 18x^2 + 24x + 5$

$f''(x) = 12x^2 - 36x + 24$

f is concave upward if $f''(x) > 0$

$\therefore 12x^2 - 36x + 24 > 0$

$\therefore 12(x^2 - 3x + 2) > 0$

$\therefore x^2 - 3x + 2 > 0$

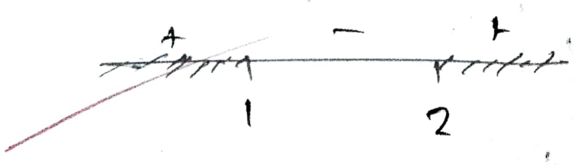
$\therefore x^2 - 2x - x + 2 > 0$

$\therefore x(x-2) - 1(x-2) > 0$

$(x-2)(x-1) > 0$

$x = 1, 2.$

$x \in (-\infty, 1) \cup (2, \infty$



(c) $y = x^3 - 27x + 5$

$f'(x) = 3x^2 - 27$

$f''(x) = 6x$

f is concave upward iff $f''(x) > 0$

$\therefore 6x > 0$

$\therefore x > 0$

$\therefore x \in (0, \infty)$

(d) $y = 69 - 24x - 9x^2 + 2x^3$

$f'(x) = 6x^2 - 18x - 24$

$f''(x) = 12x - 18$

f is concave upward iff $f''(x) > 0$

$\therefore 12x - 18 > 0$

$\therefore 12(x - 18/12) > 0$

$\therefore x - 3/2 > 0$ $\quad x > 3/2.$

$x \in (3/2; \infty)$

(e) $y = 2x^3 + x^2 - 20x + 4$

$f(x) = 2x^3 + x^2 - 20x + 4$

$f'(x) = 6x^2 + 2x - 20$

$f''(x) = 12x + 2$

f is concave upward iff $f''(x) > 0$

$\therefore f''(x) > 0$

$\therefore 12x + 2 > 0$

$\therefore 12(x + 2/12) > 6$

$\therefore x + 1/6 > 0$ $\quad \therefore x \in (-1/6, \infty)$

$x < -1/6$

21/12/19

$f''(x) \not> 0$

$\therefore$ There exist no interval

Practical- 4

Topic: Application of Derivative & Newton's Method.

Q1] Find maximum & minimum values of following functions.

(i) $f(x) = x^2 + \frac{16}{x^2}$

(ii) $f(x) = 3 - 5x^3 + 3x^5$

(iii) $f(x) = x^3 - 3x^2 + 1$ in $[-1/2, 4]$

(iv) $f(x) = 2x^3 - 3x^2 - 12x + 1$ in $[-2, 3]$

Q2] Find the root of following equation by Newton's method. (Take 4 iteration only) correct upto 4 decimal.

(i) $f(x) = x^3 - 3x^2 - 55x + 9.5$ (take $x_0 = 0$)

(ii) $f(x) = x^3 - 4x - 9$ in $[2, 3]$

(iii) $f(x) = x^3 - 1.8x^2 - 10x + 17$ in $[1, 2]$

Solution:

Q1)

(1) $f(x) = x^2 + \frac{16}{x^2}$.

$f'(x) = 2x - 32/x^3$

Now consider, $f'(x) = 0$

$\therefore 2x - 32/x^3 = 0$

$\therefore 2x = 32/x^3$

$\therefore x^4 = 32/2$

$\therefore x^4 = 16$

$\therefore x = \pm 16$

$f''(x) = 2 + 96/x^4$

$f''(2) = 2 + 96/\cancel{2}6$

$= 2 + 96/16$

$= 2 + 6$

$= 8 > 0$

$\therefore$ f has minimum value at $x = 2$

$\therefore f(2) = 2^2 + 16/22$

$= 4 + 16/4$

$= 4 + 4$

$= 8$

$\therefore f''(-2) = 2 + 96/(-2)4$

$= 2 + 96/16$

$= 2 + 6$

$= 8 > 0$

$\therefore$ f has minimum value at $x = -2$.

$\therefore$ Function reaches minimum value at

$x = 2$, & $x = -2$

(ii) $f(x) = 3-5x^3+3x^5$

$\therefore f'(x) = -15x^2+15x^4$

Consider, $f'(x) = 0$

$\therefore -15x^2+15x^4 = 0$

$\therefore 15x^4 = 15x^2$

$\therefore x^2 = 1$

$\therefore x = \pm 1$

$\therefore f''(x) = -30x+60x^3$

$f(1) = -30+60$

$= 30 > 0$ $\therefore$ f has minimum value at $x=1$

$f(1) = 3-5(1)^3+3(1)^5$

$= 6-5$

$= 1$

$\therefore f(-1) = -30(-1)+60(-1)^3$

$= 30-60$

$= -30 < 0$ $\therefore$ f has maximum value at $x=-1$

$\therefore f(-1) = 3-5(-1)^3+3(-1)^5$

$= 3+5-3 = 5$

$\therefore$ f has maximum value at 5 at $x=-1$ & has the minimum value 1 at $x=1$

(iii) $f(x) = x^3-3x^2+1$

$\therefore f'(x) = 3x^2-6x$

Consider, $f'(x) = 0$

$\therefore 3x^2-6x = 0$

$\therefore 3x(x-2) = 0$

$\therefore 3x = 0$ or $x-2 = 0$.

$\therefore x = 0$ or $x = 2$

$\therefore f''(x) = 6x-6$

$\therefore f''(0) = 6(0)-6$

$= -6 < 0$ $\therefore$ f has maximum value at $x=0$

$\therefore f(0) = (0)^3-3(0)^2+1 = 1$

$\therefore f''(2) = 6(2) - 6$

$= 12 - 6$

$= 6 > 0$

$\therefore f$ has minimum value at $x = 2$

$\therefore f(2) = (2)^3 - 3(2)^2 + 1$

$= 8 - 3(4) + 1$

$= 9 - 12$

$= -3$

$\therefore f$ has maximum value 1 at $x = 0$ & f has minimum value at -3 at $x = 2$

(iv) $f(x) = 2x^3 - 3x^2 - 12x + 1$

$\therefore f'(x) = 6x^2 - 6x - 12$

Consider, $f'(x) = 0$

$\therefore 6x^2 - 6x - 12 = 0$

$\therefore 6(x^2 - x - 2) = 0$

$\therefore x^2 - x - 2 = 0$

$\therefore x^2 + x - 2x - 2 = 0$

$\therefore x(x+1) - 2(x+1) = 0$

$\therefore (x-2)(x+1) = 0$

$\therefore x = 2$ or $x = -1$

$\therefore f''(x) = 12x - 6$

$\therefore f''(2) = 12(2) - 6$

$= 24 - 6$

$= 18 > 0$

$\therefore f$ has minimum value at $x = 2$.

$\therefore f(x) = 2(2)^3 - 3(2)^2 - 12(2) + 1$

$= 16 - 12 - 24 + 1$

$= -19$

$\therefore f''(-1) = 12(-1) - 6$

$= -12 - 6$

$= -18 < 0$

$\therefore f$ has min. value at $x = -1$

$\therefore f(-1) = 2(-1)^3 - 3(-1)^2 - 12(1) + 1$

$= -2 - 3 + 12 + 1$

$= 8$

$\therefore f$ has maximum value 8 at $x = -1$ & f has minimum value -19 at $x = 2$.

Q2] (1) $f(x) = x^3 - 3x^2 - 55x + 9.5$ $\quad \underline{x_0 = 0} \rightarrow$ given

$f'(x) = 3x^2 - 6x - 55$

By Newton's Method,

$x_{n+1} = x_n - f(x_n) / f'(x_n)$

$\therefore x_1 = x_0 - f(x_0) / f'(x_0)$

$\therefore x_1 = 0 + 9.5 / 55$

$\therefore x_1 = \underline{0.1727}$

$\therefore f(x_1) = (0.1727)^3 - 3(0.1727)^2 - 55(0.1727) + 9.5$

$= 0.0051 - 0.0895 - 9.4985 + 9.5$

$= \underline{-0.0829}$

$\therefore f'(x_1) = 3(0.1727)^2 - 6(0.1727) - 55$

$= 0.0898 - 1.0362 - 55$

$= \underline{-55.9467}$

$\therefore x_2 = x_1 - f(x_1) / f'(x_1)$

$= 0.1727 - 0.0829 / 55.9467$

$= \underline{0.1712}$

$f(x_2) = (0.1712)^3 - 3(0.1712)^2 - 55(0.1712) + 9.5$

$= 0.0050 - 0.0879 - 9.416 + 9.5$

$= \underline{0.011}$

$f'(x_2) = 3(0.1712)^2 - 6(0.1712) - 55$

$= 0.0879 - 1.0272 - 55$

$= \underline{-55.9393}$

$\therefore x_3 = x_2 - f(x_2) / f'(x_2)$

$= 0.1712 + 0.0011 / 55.9393$

$= \underline{0.1712}$

$\therefore$ The root of the equation is 0.1712.

$f(x) = x^3 - 4x - 9$ $\quad [2, 3]$

$f'(x) = 3x^2 - 4$

$f(2) = 2^3 - 4(2) - 9$

$= 8 - 8 - 9$

$= -9$

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$f(3) = 3^3 - 4(3) - 9$

$= 27 - 12 - 9$

$= 6$

Let $x_0 = 3$, be the initial approximation.

$\therefore$ By Newton's Method,

$x_{n+1} = x_n - f(x_n) / f'(x_n)$

$x_1 = x_0 - f(x_0) / f'(x_0)$

$= 3 - 6/23$

$= 2.7392$

$f(x_1) = (2.7392)^3 - 4(2.7392) - 9$

$= 20.5528 - 10.9568 - 9$

$= 0.596$

$f'(x_1) = 3(2.7392)^2 - 4$

$= 22.5096 - 4$

$= 18.5096$

$x_2 = x_1 - f(x_1) / f'(x_1)$

$= 2.7392 - 0.596 / 18.5096$

$= 2.7071$

$f(x_2) = (2.7071)^3 - 4(2.7071)$

$= 19.8386 - 10.8284$

$= 0.0102$

$f'(x_2) = 3(2.7071)^2 - 4$

$= 21.9851 - 4$

$= 17.9851$

$= 2.7071 - \dfrac{0.0102}{17.9851}$

$= 2.7071 - 0.0056$

$= \underline{2.7015}$

$f(x_3) = (2.7015)^3 - 4(2.7015) - 9$

$= 19.7152 - 10.806 - 9 = -0.0901$

$f'(3) = 3(2.7015)^2 - 4 = 21.8943 - 4 = 17.8943$

$x_4 = 2.7015 + 0.0901 / 17.8943 = 2.7015 + 0.0050$

$= 2.7065$

(8) $f(x) = x^3 - 1.8x^2 - 10x + 17$ $[1, 2]$

$f'(x) = 3x^2 - 3.6x - 10$

$f(1) = (1)^3 - 1.8(1)^2 - 10(1) + 17$

$= -1.8 - 10 + 17$

$= 6.2$

$f(2) = (2)^3 - 1.8(2)^2 - 10(2) + 17$

$= \cancel{-7.8 - 10 + 17}$

$= 8 - 7.2 - 20 + 17 = 2.2$

Let $x_0 = 2$ be initial approximation.

By Newton's Method,

$x_{n+1} = x_n - f(x_n) / f'(x_n)$

$x_1 = x_0 - f(x_2) / f'(x_2)$

$= 2 - 2.2 / 5.2$

$= 2 - 0.4230 = 1.577$

$f(x_1) = (1.577)^3 - 1.8(1.577)^2 - 10(1.577) + 17$

$= 3.9219 - 4.4764 - 15.77 + 17$

$= \underline{0.6755}$

$f'(x) = 3(1.577)^2 - 3.6(1.577) - 10$

$= 7.4608 - 5.6772 - 10$

$= \underline{-8.2164}$

$\therefore x_2 = x_1 - f(x_1) / f'(x_1)$

$= 1.577 + 0.6755 / 8.2164$

$= 1.577 + 0.0822$

$= \underline{1.6592}$

$f(x_2) = (1.6592)^3 - 1.8(1.6592)^2 - 10(1.6592) + 17$

$= 4.5677 - 4.9553 - 16.592 + 17$

$= 0.0204$

$f'(x_2) = 3(1.6592)^2 - 3.6(1.6592) - 10$

$= 8.2588 - 5.97312 - 10$

$= -7.7143$

$x_3 = x_2 - f(x_2) / f'(x_2)$

$= 1.6592 + 0.0204 / 7.7143$

$= 1.6592 + 0.0026$

$= 1.6618$

$f(x_3) = (1.6618)^3 - 1.8(1.6618)^2 - 10(1.6618) + 17$

$= 4.8892 - 4.9708 - 16.618 + 17$

$= 0.0004$

$f'(x_3) = 3(1.6618)^2 - 3.6(1.6618) - 10$

$= 8.2847 - 5.9824 - 10$

$= \underline{-7.6977}$

$x_4 = x_3 - f(x_3) / f'(x_3)$

$= 1.8618 + \dfrac{0.0004}{7.6977}$

$= \underline{\underline{1.6618}}$

Practical - 5

Topic: Integration.

Q] Solve the following:

(i) $\int \frac{dx}{\sqrt{x^2+2x-3}}$

(ii) $\int (4e^{3x}+1)\, dx$

(iii) $\int (2x^2 - 3\sin x + 5\sqrt{x})\, dx$

(iv) $\int \frac{x^3+3x+4}{\sqrt{x}} \cdot dx$

(v) $\int t^7 \sin(2t^4)\, dt$

(vi) $\int \sqrt{x}\,(x^2-1)\, dx$

(vii) $\int \frac{1}{x^3} \sin\left(\frac{1}{x^2}\right) dx$

(viii) $\int \frac{\cos x}{\sqrt[3]{\sin^2 x}}\, dx$

(ix) $\int e^{\cos^2 x} \sin 2x\, dx$

(x) $\int \left(\frac{x^2-2x}{(x^3-3x^2+1)}\right) dx$

Solution:

(1) $\int \frac{1}{\sqrt{x^2+2x-3}} \cdot dx$

$= \int \frac{1}{\sqrt{x^2+2x+1-4}} \cdot dx$

$a^2+2ab+b^2 = (a+b)^2$

$= \int \frac{1}{\sqrt{(x+1)^2-4}} \cdot dx$

Substitute

put $x+1 = t$

$dx = \frac{1}{t} \times dt$ where $t = 1$, $t = x+1$

$\int \frac{1}{\sqrt{t^2-4}} dt.$

using.

$\int \frac{1}{\sqrt{x^2-a^2}} dx = \ln(|x+\sqrt{x^2-a^2}|)$

$= \ln(|t+\sqrt{t^2-4}|)$

$t = x+1.$

$= \ln(|x+1+\sqrt{(x+1)^2-4}|)$

$= \ln(|x+1+\sqrt{x^2+2x-3}|)$

$= \ln(|x+1+\sqrt{x^2+2x-3}|) + c.$

(2) $\int (4e^{3x}+1) dx$

$= \int 4e^{3x} \cdot dx + \int 1 \cdot dx$

$= 4\int e^{3x} dx + \int 1 \cdot dx$ # $\int e^{ax} = \frac{1}{a} \times e^{ax}.$

$= \frac{4e^{3x}}{3} + x.$

$= \frac{4e^{3x}}{3} + x + c.$

③ $\int 2x^2 - 3\sin(x) + 5\sqrt{x}\, dx.$

$= \int 2x^2 - 3\sin(x) + 5^{1/2}\, dx.$ # $\sqrt[n]{a^m} = a^{m/n}$

$= \int 2x^2\, dx - \int 3\sin(x)\, dx + \int 5x^{1/2}\, dx.$

$= \frac{2x^3}{3} + 3\cos x + \frac{10\sqrt{x}}{3} + C$

$= \frac{2x^3 + 10x\sqrt{x}}{3} + 3\cos x + C$

④ $\int \frac{x^3 + 3x + 4}{\sqrt{x}}\, dx.$

$= \int \frac{x^3 + 3x + 4}{x^{1/2}}\, dx.$

split the denominator.

$= \int \frac{x^3}{x^{1/2}} + \frac{3x}{x^{1/2}} + \frac{4}{x^{1/2}}\, dx$

$= \int x^{5/2} + 3x^{1/2} + 4/x^{1/2}\, dx.$

$= \int x^{5/2}\, dx + \int 3x^{1/2}\, dx + \int 4/x^{1/2}\, dx.$

$= \frac{2x^3\sqrt{x}}{7} + 2x\sqrt{x} + 8\sqrt{x} + C.$

⑤ $\int t^7 \times \sin(2t^4)\, dx.$

put $u = 2t^4.$

$du = 2 \times 4t^3$

$= \int t^7 \times \sin(2t^4) \times \frac{1}{2 \times 4t^3}\, dx.$

$= \int t^4 \sin(2t^4) \times \frac{1}{2 \times 4}\, dx.$

$= \int t^4 \cdot \sin(2t^4) \times 1/8 \, du$

$= \frac{t^4 \times \sin(2t^4)}{8} \cdot du$

Substitute t^4 with $u/2$.

$= \int \frac{u/2 \times \sin(u)}{8} \, du$

$= \int \frac{u \times \sin(u)}{2} / 8 \, du$

$= \int \frac{u \times \sin(u)}{6} \, du$

$= \frac{1}{16} \int u \times \sin(u) \, du$

$\# \int u \, dv = uv - \int v \, du$.

where $u = u$

$dv = \sin(u) \times du$.

$du = 1 \, du \qquad v = -\cos(u)$

$= \frac{1}{16} \left(u \times (-\cos(u)) \right) - \int -\cos(u) \, du$.

$= 1/16 \times (u \times (-\cos u)) + \int \cos(u) \, du$

$\# \int \cos x \cdot dx = \sin x$.

$= \frac{1}{16} \times \left(u \times (-\cos(u)) + \sin(u) \right)$

$=$ Return the substitution $u = 2t^4$

$= \frac{1}{16} \times \left(2t^4 \times (-\cos(2t^4)) + \sin(2t^4) \right)$

$= -\frac{t^4 \times \cos(2t^4)}{8} + \frac{\sin(2t^4)}{16} + C$

(vi) $\int \sqrt{x}\,(x^2-1)\,dx$

$= \int \sqrt{x}\; x^2 - \sqrt{x}\; dx$

$= \int x^{1/2} \times x^2 - x^{1/2}\, dx$

$= \int x^{5/2} - x^{1/2}\, dx$

$= \int x^{5/2}\, dx - \int x^{1/2}\, dx$

$$I_1 = \frac{x^{5/2+1}}{5/2+1} = \frac{x^{7/2}}{7/2} = \frac{2x^{7/2}}{7} = \frac{2\sqrt{x^7}}{7} = \frac{2x^3\sqrt{x}}{7}$$

$$I_2 = \frac{x^{1/2+1}}{1/2+1} = \frac{x^{3/2}}{3/2} = \frac{2x^{3/2}}{3/2} = \frac{2\sqrt{x^3}}{3}$$

$$= \frac{2x^3\sqrt{x}}{7} + \frac{2\sqrt{x^3}}{3} + C$$

(viii) $\int \frac{\cos x}{\sqrt[3]{\sin(x)^2}}\, dx$

$= \int \frac{\cos x}{\sin x^{2/3}}\, dx$

put $t = \sin(x)$

$t = \cos x$

$= \int \frac{\cos(x)}{\sin(x)^{3/2}} \times \frac{1}{\cos(x)}\, dx$

$= \frac{1}{\sin x^{3/2}}\, dt$

$= \int \frac{1}{t^{2/3}}\, dt = \frac{-1}{(2/3-1)t^{2/3-1}}$

$= \frac{-1}{-1/3\, t^{2/3-1}} = \frac{1}{1/3\, t^{-1/3}} = \frac{t^{1/3}}{1/3} = 3t^{1/3} = 3\sqrt[3]{t}$

Return substitution $t = \sin(x)$

$= 3\sqrt[3]{\sin(x)} + C$

(x) $\int \frac{x^2 - 2x}{x^3 - 3x^2 + 1} dx$.

put $x^3 - 3x^2 + 1 = dt$

$I = \int \frac{x^2 - 2x}{x^3 - 3x^2 + 1} \times \frac{1}{3x^2 - 3 \times 2x} dt$

$= \int \frac{x^2 - 2x}{x^3 - 3x^2 + 1} \times \frac{1}{3x^2 - 6x} dx$

$= \int \frac{x^2 - 2x}{x^3 - 3x^2 + 1} \times \frac{1}{3(x^2 - 2x)} dt$

$= \int \frac{1}{x^3 - 3x^2 + 1} \times 1/3 \, dt$

$= \int \frac{1}{3(x^3 - 3x^2 + 1)} dt \qquad = \int \frac{1}{3t} dt$

$= 1/3 \int 1/t \, dt \qquad \int 1/x \, dx = \ln|x|$

$= 1/3 \times \ln(t) + C$

$= 1/3 \times \ln(|x^3 - 3x^2 + 1|) + C$

(vii) $\int \frac{1}{x^3} \sin\left(\frac{1}{x^2}\right) dx$

$I = \int \frac{1}{x^3} \sin\left(\frac{1}{x^2}\right) dx$

let $\frac{1}{x^2} = t$.

$x^{-2} = t$.

$-\frac{2}{x^3} dx = dt$

$I = -\frac{1}{2}\int \frac{-2}{x^3} \sin\left(\frac{1}{x^2}\right) dx$

$= -\frac{1}{2}\int \sin t.$

$= -\frac{1}{2}(-\cos t) + c$

$= \frac{1}{2}\cos t + c$

Resubstitution, $t = 1/x^2$

$I = \frac{1}{2}\cos\left(\frac{1}{x^2}\right) + c$

(9x) $\int e^{\cos^2 x} \cdot \sin 2x \, dx.$

$I = \int e^{\cos^2 x} \cdot \sin 2x \, dx.$

Let $\cos^2 x = t.$

$-2\cos x \cdot \sin x \, dx = dt.$

$-\sin 2x \, dx = dt$

$I = \int -\sin 2x \cdot e^{\cos^2 x} \, dx.$

$= -\int e^t . dt$

$= -e^t + c$

Resubstituting $t = \cos^2 x$

$I = -e^{\cos^2 x} + c$

AK
04/01/2020

Practical - 6

Aim: Application of integration & numerical integrations.

(a) $y = \sqrt{4-x^2} \qquad x \in [-2, 2]$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{4-x^2}} (-2x)$$

$$= \frac{-2x}{2\sqrt{4-x^2}}$$

$$= \int_{-2}^{2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \cdot dx$$

$$= \int_{-2}^{2} \sqrt{1 + \left(\frac{-x}{\sqrt{4-x^2}}\right)^2} \, dx$$

$$= \int_{-2}^{2} \sqrt{\frac{4 - x^2 + x^2}{4 - x^2}} \cdot dx$$

$$= \int_{-2}^{2} \sqrt{\frac{4}{4-x^2}} \cdot dx$$

$$= \int_{-2}^{2} \frac{2}{\sqrt{4-x^2}} \cdot dx$$

$$= \int_{-2}^{2} \frac{2}{\sqrt{(2)^2 - x^2}} \cdot dx$$

$$= 2\left[\sin^{-1}(x/2)\right]_{-2}^{2}$$

$$= \sin^{-1}(1) - \sin^{-1}(-1)$$

$$= 2\left[(\pi/2) - (-\pi/2)\right]$$

$$= 2\left[\pi/2 + \pi/2\right]$$

$$= 2\pi$$

3) $y = x^{3/2}$ in $[0, 4]$

$$\frac{dy}{dx} = \frac{3}{2} \cdot x^{3/2 - 1}$$

$$I = \int_0^4 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \cdot dx$$

$$= \int_0^4 \sqrt{1 + \left(\frac{3\sqrt{x}}{2}\right)^2} \cdot dx$$

$$= \int_0^4 \sqrt{\left(1 + \frac{9x}{4}\right)} \cdot dx$$

$$= \int_0^4 \sqrt{\frac{4 + 9x}{4}} \cdot dx.$$

$$= \frac{1}{2} \int_0^4 \sqrt{(4 + 9x)} \cdot dx$$

$$= \frac{1}{2} \left[\frac{(4 + 9x)^{1/2 + 1}}{1/2 + 1} \right]_0^4 dx.$$

$$= \frac{1}{2} \left[\frac{(4 + 9x)^{3/2}}{3/2} \times \frac{1}{9} \right]_0^4$$

$$= 1/27 \left[(4 + 9x)^{3/2} \right]_0^4$$

$$= 1/27 \left[(4 + 0)^{3/2} - (4 + 36)^{3/2} \right]$$

$$= 1/27 \left[(4)^{3/2} - (40)^{3/2} \right]$$

$$= 1/27 \left[(4)^{3/2} - 8 \right]$$

Q2] $y = \sqrt{4 - x^2}$ $x \in [-2, 2]$.

$$\frac{dy}{dx} = \frac{-\cancel{2}x}{\cancel{2}\sqrt{4 - x^2}} = \frac{-x}{\sqrt{4 - x^2}}$$

$$I = 2 \int_0^2 \sqrt{1 + \left(\frac{-x}{\sqrt{4 - x^2}}\right)^2} \, dx.$$

$= 2\int_0^2 \sqrt{\frac{1+x^2}{\sqrt{4-x^2}}} \, dx$

$= 4\left(\sin^{-1}(x/2)\right)_0^2$

$= 2\pi$

(4) $x = 3\sin t$, $y = 3\cos t$, $t \in [0, 2\pi]$

$\rightarrow \frac{dy}{dx} = 3\cos t$ dy $\frac{dy}{dt} = 3\sin t$

$I = \int_0^{2\pi} \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2} \cdot dt$

$= \int_0^{2\pi} \sqrt{9\sin^2 + 9\cos^2 t} \cdot dt$

$= 3\int_0^{2\pi} \sqrt{3 x} \cdot dt$

$= 3\int_0^{2\pi} x \cdot dt$

$= 3[x]_0^{2\pi}$

$= 3(2\pi - 0)$

$I = 6\pi$ units.

(5) $x = \frac{1}{6}y^3 + 1/2y$ on $y \in [1, 2]$

$\rightarrow \frac{dx}{dy} = y^3/2 - 1/2y^2$

$\frac{dx}{dy} = \frac{y^4 - 1}{2y^2}$

$I = \int_1^2 \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \cdot dy = \int_1^4 \sqrt{1 + \frac{(y^4 - 1)}{4y^2}} \cdot dy$

$= \int_1^2 \sqrt{\frac{(y^4+1)^2}{(2y^2)^2}} \cdot dy$

$= \int_1^2 \frac{y^4 + 1}{2y^2} \cdot dy$

$= 1/2 \int_1^2 y^2 \cdot dy + 1/2 \int_1^2 y^{-2} \, dy$

$= 1/2 \left[y^3/3 - y/1^2 \right]_1^2$

$= 1/2 \left[\frac{8}{3} \cdot 1/2 - 1/3 + 1 \right]$

$= 1/2 \left[7/3 + 1/3 \right]$

$= 1/2 \left[17/6 \right]$

$= 17/12$ units

Q2] Solve the following simpson's rule.

(1) $\int_0^2 e^{x^2} \cdot dx$ with $n = 4$

x	0	0.5	1	1.5	2
y	1	1.284	2.7183	9.9877	54.5982
	y_0	y_1	y_2	y_3	y_4

$\rightarrow \int_0^2 e^{x^2} \cdot dx = 16.4526$

$h = \frac{2-0}{4} = 1/2$

By simpson's rule,

$\int_0^2 dx = 1/2/3 \left[(y_0 + y_1 \, 1/2 + 2 \, 1/2 + y \, 1/3 + 1/4 \right]$

$= 1/2/3 \left(e^{x^2} + 4e^{(0.5)^2} + 2e^{(1)^2} + 4e^{(1.5)^2} + e^{(2)^2} \right)$

≈ 17.3536

(2) $\int_0^4 x^2 \cdot dx$; $n = 4$.

$h = 4/4 - 0 = 1$

$\int_0^c f(x) dx = \frac{h}{3} \left[y_0 + 4^1/1 + 2 + 2 \, 1/2 + 4^1/3 + 1/4 \right)$

$= 1/3 \left[1/10 \right] + 4 f(1) + 2 f(2) + 4(3) + (4) \right)$

$= 1/3 \left[0^2 + 4(1)^2 + 2(2)^2 + 4(3)^2 + 4^2 \right)$

$= \frac{64}{3} \approx 21.333.$

x	0	1	2	3	4
y	0	1	4	9	16
	y_0	y_1	y_2	y_3	y_4

3) $\int_0^{\pi/3} \sqrt{\sin x}\cdot dx$, $n=6$

$\Delta x = \frac{b-a}{n} = \frac{\pi/3 - 0}{6} = \frac{\pi}{18}$

x	0	$\pi/8$	$2\pi/18$	$3\pi/18$	$4\pi/18$	$5\pi/18$
y	0	0.1467	0.584	0.707	0.801	0.875
	y_0	y_1	y_2	y_3	y_4	y_5

~~$\int_0^{\pi/3} \sqrt{\sin x}\cdot dx \approx \frac{h}{3}\left(y_0 + 4(y_1+y_2+y_3) + 2(y_2+y_4) + y_4\right)$~~

~~$\approx \frac{\pi/18}{3}\left(0 + 4(0.4167 + 0.707 + 0.8715) +\right.$~~

~~$\approx 2(0.584 + 0.801) + 0.930$~~

~~≈ 0.681~~

$\int_0^{\pi/3} \sqrt{\sin x}\cdot dx = \frac{h}{3}\left[(y_0+y_6) + 4(y_1+y_3+y_5) + 2(y_2+y_4)\right]$

$= \frac{\pi/18}{3}\left[0.4167 + 0.9306 + 4(0.4167 + 0.7071 + 0.9752) + 2(0.5848 + 0.8017)\right]$

$= \pi/54\left[1.3473 + 4(1.999) + 2(1.3865)\right]$

$= \pi/54\left[1.3473 + 7.996 + 2.773\right]$

$= \frac{\pi}{54} \times 12.1163$

$= 0.7049.$

Practical - 7

Topic : Diffrential Equation

Q.1)

(1) $x \frac{dy}{dx} + y = e^x$

$\frac{dy}{dx} + \frac{1}{x} y = \frac{e^x}{x}$

$P(x) = \frac{1}{x}$ $\quad$ $Q(x) = \frac{e^x}{x}$.

$IF = e^{\int 1/(x)} \cdot dx$

$= e^{\int 1/(x)\, dx}$

$= e^{\ln x}$

$IF = x$

$y(IF) = \int Q(x)\,(IF)\, dx + c$

$= \int \frac{e^x}{x} \cdot x \cdot dx + c$

$= \int e^x\, dx + 1$

$xy = e^x + c$

(2) $e^x \frac{dy}{dx} + 2e^x y = 1$

$\frac{dy}{dx} + \frac{2e^x y}{e^x} = \frac{1}{e^x}$

$\frac{dy}{dx} + 2y = \frac{1}{e^x}$

$\frac{dy}{dx} + 2y = e^{-x}$

$P(x) = 2$ $\quad$ $Q(x) = e^{-x}$

$IF = e^{\int 2dx} = e^{2x}$

$y \cdot IF = \int Q(x) \cdot (IF)\, dx + c$

$y \cdot e^{2x} = \int e^{-x} \cdot e^{2x} \cdot dx + c$

$= \int e^{x} \cdot dx + c$

$y \cdot e^{2x} = e^{x} + c$

(3) $x \frac{dy}{dx} + \frac{\cos x}{x} = 2y$

$x \frac{dy}{dx} = \frac{\cos x}{x} - 2y$

$\therefore \frac{dy}{dx} + \frac{2y}{x} = \frac{\cos x}{x^2}$

$P(x) = 2/x \qquad Q(x) = \cos x / x^2$

$IF = e^{\int P dx}$

$= e^{\int 2/x \cdot dx}$

$= e^{\ln|x^2|}$

$= x^2$

$y(IF) = \int Q(x) \cdot (IF)\, dx + c$

$= \int \frac{\cos x}{\cancel{x^2}} \cdot \cancel{x^2} \cdot dx + c$

$= \int \cos x \cdot dx + c$

$x^2 y = \sin x + c$

(4) $x \frac{dy}{dx} + 3y = \frac{\sin x}{x^2}$

$\frac{dy}{dx} + \frac{3y}{x} = \frac{\sin x}{x^3}$

$P(x) = 3/x \qquad Q(x) = \sin x / x^3$

$IF = e^{\int P dx}$

$= e^{\int 3/x \cdot dx}$

$= e^{\ln|x^3|}$

$= x^3$

$y \cdot IF = \int Q(x) \cdot (IF) \cdot dx + c \qquad \therefore x^3 y = -\cos x + c$

$= \int \frac{\sin x}{\cancel{x^3}} \cdot \cancel{x^3}\, dx + c.$

(5) $e^{2x}\frac{dy}{dx} + 2e^{2x}y = 2x$

$\frac{dy}{dx} + 2y = \frac{2x}{e^{2x}}$

$P(x) = 2 \quad Q(x) = 2x/e^{2x}$

$IF = e^{\int P dx}$

$= e^{\int 2 dx}$

$= e^{2x}$

$y(IF) = \int Q(x)(IF)\,dx + c$

$y \cdot e^{2x} = \int \frac{2x}{e^{2x}} \cdot e^{2x}\,dx + c$

$= \int 2x\,dx + c$

$e^{2x} \cdot y = x^2 + c$

(6) $\sec^2 x \tan y\,dx + \sec^2 y \tan x\,dy = 0$

$\sec^2 x \cdot \tan y\,dx = -\sec^2 y \tan x\,dy$

$\frac{\sec^2 x}{\tan x} \cdot dx = -\frac{\sec^2 y}{\tan y} \cdot dy$

On Integrating, we get

$\int \frac{\sec^2 x}{\tan x} \cdot dx = -\int \frac{\sec^2 y}{\tan y} \cdot dy$

$\therefore \log|\tan x| = -\log|\tan y| + c$

$\log|\tan x * \tan y| = c$

$\therefore \tan x \cdot \tan y = e^c$

(7) $\frac{dy}{dx} = \sin^2(x - y + 1)$

put $x - y + 1 = v$

Differentiating on both sides,

$1 - \frac{dy}{dx} = \frac{dv}{dx}$

$1 - \frac{dv}{dx} = \frac{dy}{dx}$

$$1-\frac{dv}{dx} = \sin^2 v$$

$$\frac{dv}{dx} = 1-\sin^2 v \qquad \frac{dv}{dx} = \cos^2 v$$

$$\frac{dv}{\cos^2 v} = dx.$$

$$\int \sec^2 v \cdot dv = \int dx$$

$$\tan v = x+c$$

$$\tan(x+y-1) = x+c.$$

(8) $$\frac{dy}{dx} = \frac{2x+3y-1}{6x+9y+6}$$

put $2x+3y = v$

$$2+3\frac{dy}{dx} = \frac{dv}{dx}$$

$$\frac{dy}{dx} = \frac{1}{3}\left(\frac{dv}{dx}-2\right)$$

$$\frac{1}{3}\left(\frac{dv}{dx}-2\right) = \frac{1}{3}\frac{(v-1)}{(v+2)}$$

$$\frac{dv}{dx} = \frac{v-1}{v+2}+2$$

$$\frac{dv}{dx} = \frac{v-1+2v+4}{v+2}$$

$$= \frac{3v+3}{v+2}$$

$$\frac{dv}{dx} = \frac{3(v+1)}{v+2}$$

$$\frac{v+2}{v+1}\,dv = 3\,dx$$

On integrating, we get

$$\int \frac{v+2}{v+1}\, dv = \int 3\, dx.$$

$$\int \frac{v+1}{v}\, dv + \int \frac{1}{v+1}\, dv = 3x$$

$$v + \log|v+1| = 3x + C$$

$$2x + 3y + \log|2x+3y+1| = 3x + C$$

$$3y = x - \log|2x+3y+1| + C.$$

AK
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Practical - 8

Topic: Euler's method

Using Euler's Method, find the following.

① $\frac{dy}{dx} = y + e^x - 2$, $y(0) = 2$, $h = 0.5$ find $y(2)$

② $\frac{dy}{dx} = 1 + y^2$, $y(0) = 0$, $h = 0.2$, find $y(1)$

③ $\frac{dy}{dx} = \sqrt{\frac{x}{y}}$, $y(0) = 1$, $h = 0.2$ find $y(1)$

④ $\frac{dy}{dx} = 3x^2 + 1$, $y(1) = 2$, find $y(2)$

For $h = 0.5$ & $h = 0.25$

⑤ $\frac{dy}{dx} = \sqrt{xy} + 2$, $y(1) = 1$, find $y(1.2)$ with $h = 0.2$

Solution:

① $\frac{dy}{dx} = y + e^x - 2$, $y(0) = 2$, $h = 0.5$

$f(x) = y + e^x - 2$, $x_0 = 0$

$y(0) = 2$, $h = 0.5$

n	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
0	0	2	1	2.5
1	0.5	2.5	2.1487	3.5743
2	1	3.7543	4.2925	5.7205
3	1.5	5.7205	8.2021	9.8215
4	2	9.8215		

$y(2) = 9.8215$

(2) $\frac{dy}{dx} = 1 + y^2$, $y(0) = 0$, $h = 0.2$.

$x_0 = 0$, $y_0 = 0$

$h = 0.2$

n	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
0	0	0	1	0.2
1	0.2	0.2	1.04	0.408
2	0.4	0.408	1.1664	0.6412
3	0.6	0.6412	1.4111	0.9234
4	0.8	0.9234	1.8526	1.2939
5	1	1.2939		

$y(1) = 1.2939$

(3) $\frac{dy}{dx} = \sqrt{\frac{x}{y}}$, $y(0) = 1$, $h = 0.2$

$x_0 = 0$, $y(0) = 1$, $h = 0.2$

n	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
1	0.	1		
2	0.2	1		
3	0.4	1.0894		
4	0.8	1.2105		
5	1			

n	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
0	0	1	0	1.
1	0.2	1	0.4472	1.089
2	0.4	1.0894	0.6059	1.2105
3	0.6	1.2105	0.7040	1.3513
4	0.8	1.3513	0.76914	1.5051
5	1	1.5051		

$y(1) = 1.5051$

(4) $\frac{dy}{dx} = 3x^2+1$, $y(1) = 2$, $h = 0.5$, & $h = 0.25$

$y_0 = 2$ $x_0 = 1$

n	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
0	1	2	4	4
1	1.5	4	7.75	7.875
2	2	7.875		

$y(2) = 7.875$

n	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
0	1	2	4	3
1	1.25	3	5.6875	4.4218
2	1.5	4.4218	59.6569	19.3360
3	1.75	19.3360	1122.6426	299.9966
4	2	299.996		

$y(2) = 299.9966$

(5) $\frac{dy}{dx} = \sqrt{xy} + 2$, $y(1) = 1$, $h = 0.2$

$y_0 = 1$, $x_0 = 1$, $h = 0.2$.

n	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
0	1	1	3	3.6
1	1.2	3.6		

11/01/2020

Practical - 9

Limits & partial order Derivatives

① $\lim_{(x,y) \to (-4,-1)} \frac{x^3 - 3y + y^2 - 1}{xy + 5}$.

$\lim_{(x,y) \to (-4,-1)} \frac{x^3 - 3y + y^2 - 1}{xy + 5}$.

At $(-4,-1)$ Denominator $\neq 0$

$\therefore$ By applying limit,

$= \frac{(-4)^3 - 3(-4) + (-4)^2 - 1}{-4(-1) + 5}$

$= \frac{-61}{9}$

② $\lim_{(x,y) \to (-2,0)} \frac{(y+1)(x^2 + y^2 - 4x)}{x + 3y}$

$\lim_{(x,y) \to (-2,0)} \frac{(y+1)(x^2 + y^2 - 4x)}{x + 3y}$.

At denominat $(-2,0)$ $\neq 0$

$= \frac{(0+1)(4 + 0 - 8)}{2}$

$= \frac{-4}{2}$

$= -2$

(3) $$\lim_{(x,y,z)\to(1,1,1)} \frac{x^2-y^2z^2}{x^3-x^2y^2}$$

$\rightarrow$ At $(1,1,1)$ Denominator $\neq 0$

$$= \lim_{(x,y,z)\to(1,1,1)} \frac{x^2-y^2z^2}{x^3-x^2y^2}$$

$$= \lim_{(x,y,z)\to(1,1,1)} \frac{(x+yz)}{x^2}$$

on applying limits

$$= \frac{1+1(1)}{1^2}$$

$$= \frac{1}{2}$$

Q2]

(1) $f(x,y) - xye^{x^2}+y^2$

$\therefore f_x = \frac{\partial f}{\partial x}(x,y)$

$$= \frac{\partial(xye^{x^2+y^2})}{\partial x}$$

$$= ye^{x^2+y^2}(2x)$$

$\therefore f_x = 2xye^{x^2+y^2}$

$$f_y = \frac{\partial f(x,y)}{\partial y}$$

$$= \frac{\partial(xye^{x^2+y^2})}{\partial x}$$

$$= xe^{x^2+y^2}(2y)$$

$f_y = 2yxe^{x^2+y^2}$

$f(x,y) = e^x \cos y$

$f_x = \frac{\partial}{\partial x} f(x,y)$

$= \frac{\partial (e^x \cos y)}{\partial x}$

$f_x = e^x \cos y$

$f_y = \frac{\partial f(x,y)}{\partial y}$

$= \frac{\partial (e^x \cos y)}{\partial y}$

$f_y = -e^x \sin y$

⑤ $f(x,y) = x^3y^2 - 3x^2y + y^3 + 1$

$f_x = \frac{\partial f(x,y)}{\partial x}$

$= \frac{\partial f(x^3y^2 - 3x^2y + y^3 + 1)}{\partial x}$

$= 3x^2y^2 - 6xy.$

$f_y = \frac{\partial f(x,y)}{\partial y}$

$= \frac{\partial f(x^3y^2 - 3x^2y + y^3 + 1)}{\partial y}$

$= 2x^3y - 3x^2 + 3y^2$

Q3) (1) $f(x,y) = \frac{2x}{1+y^2}$.

$\rightarrow f_x = \frac{\partial f(x,y)}{\partial x}$

$= \frac{\partial f}{\partial x}(2x/1+y^2)$

$= \frac{1+y^2 \frac{\partial(2x)}{\partial x} - 2x \frac{\partial (1+y^2)}{\partial x}}{(1+y^2)^2}$

$= \frac{2+2y^2}{(1+y^2)^2}$

$= \frac{2}{\underline{(1+y^2}}$.

At $f(0,0) = \frac{2}{1+0} = 2$

$f_y = \frac{\partial f}{\partial y}(2x/(1+y^2))$

$= \frac{1+y^2 \frac{\partial}{\partial x}(2x) - 2x \frac{\partial}{\partial x}(1+y^2)}{(1+y^2)^2}$

$= \frac{1+y^2(0) - 2x(2y)}{(1+y^2)^2}$

$= \frac{-4xy}{(1+y^2)^2}$

At $(0,0) = \frac{-4(0)(0)}{(1+0)^2} = 0$.

(iv) $f(x,y) = \frac{y^2 - xy}{x^2}$ 6n

$f_x = \frac{x^2 \frac{\partial}{\partial x}(y^2 - xy) - (y^2 - xy)\frac{\partial}{\partial x}(x^2)}{(x^2)^2}$

$= \frac{x^2(-y) - (y^2 - xy)(2x)}{x^4}$

$= \frac{-x^2 y - 2x(y^2 - xy)}{x^4}$

$f_y = 2y - \frac{x}{x^2}$

$f_{xx} = \frac{\partial}{\partial x}\left(\frac{-x^2 y - 2x(y^2 - xy)}{x^4}\right)$

$= \frac{x^4\left(\partial/\partial x(-x^2 y - 2xy^2 + 2x^2 y)\right) - (-x^2 y - 2xy + 2x^2 y)\,\partial/\partial x(x^4)}{(x^4)^2}$

$= \frac{x^4(-2xy - 2y^2 + 4xy) - 4x^3(-x^2 y - 2xy + 2x^2 y)}{x^6}$ —— (1)

$f_{yy} = \cancel{\frac{\partial}{\partial y}}\left(\cancel{\frac{-x^2 y - 2xy^2 + 2x^3 y}{x^4}}\right)$

$= \cancel{\frac{x^2 - 4xy + 2x}{x^4}}$ —— (3)

$f_{yy} = \frac{\partial}{\partial y}\left(2y - \frac{x}{x^2}\right)$

$= \frac{2 - 0}{x^2} = \frac{2}{x^2}$ —— (2)

$f_{xy} = \frac{\partial}{\partial y}\left(\frac{-x^2 y - 2xy^2 + 2x^3 y}{x^4}\right)$

$$= x^2 - \frac{4xy + 2x^2}{x^4} \quad \text{——} (3)$$

$$f_{yx} = \frac{\partial}{\partial x}\left(\frac{xy - x}{x^2}\right)$$

$$= \frac{x^2 \frac{\partial}{\partial x}(xy - x) - (xy - x)\frac{\partial}{\partial x}(x^2)}{(x^2)^2}$$

$$= -x^2 - \frac{4xy - 2x^2}{x^4} \quad \text{——} (4)$$

From (3) & (4),

$f_{xy} = f_{yx}$.

(2) $f(x, y) = x^3 + 3x^2y^2 - \log(x^2+1)$

$\rightarrow f_x = \frac{\partial}{\partial x}(x^3 + 3x^2y^2 - \log(x^2+1))$

$$= 3x^2 + 6xy^2 - \frac{2x}{x^2+1}$$

$$f_y = \frac{\partial}{\partial y}(x^3 + 3x^2y^2 - \log(x^2+1))$$

$$= 6x^2y$$

$$f_{xx} = 6x + 6y^2 - \frac{(x^2 + \frac{\partial}{\partial x}(x^2+1))}{(x^2+1)^2}$$

$$= 6x + 6y^2 - \frac{(2(x^2+1) - 4x^2)}{(x^2+1)^2} \quad \text{——} (1)$$

$$f_{yy} = \frac{\partial}{\partial y}(6x^2y)$$

$$= 6x^2 \quad \text{——} (2)$$

$$f_{xy} = \frac{\partial}{\partial y}\left(3x^2 + 6xy^2 - \frac{2}{x^2+1}\right)$$

$= 0 + 12xy - 0$

$= 12xy$ —— ③

$$f_{yx} = \frac{\partial}{\partial x}(6x^2y)$$

$= 12xy$ —— ④

From ③ & ④.

$\underline{f_{xy} = f_{yx}}$

⑧ $f(x,y) = \sin(xy) + e^{x+y}$

$\rightarrow f_x = y\cos(xy) + e^{x+y}$

$= y\cos(xy) + e^{x+y}$

$f_y = x\cos(xy) + e^{x+y}(1)$

$= x\cos(xy) + e^{x+y}$

$$f_{xx} = \frac{\partial}{\partial x}\left(y\cos(xy) + e^{x+y}\right)$$

$= -y\sin(xy)(y) + e^{x+y}(1)$

$= -y^2\sin(xy) + e^{x+y}$ —— ①

$$f_{yy} = \frac{\partial}{\partial y}\left(x\cos(xy) + e^{x+y}\right)$$

$= -x\sin(xy)(x) + e^{x+y}(1)$

$= -x^2\sin(xy) + e^{x+y}$ —— ②

$$f_{xy} = \frac{\partial}{\partial y}\left(y\cos(xy) + e^{x+y}\right)$$

$= -y^2\sin(xy) + \cos(xy) + e^{x+y}$ —— ③

$$f_{yx} = \frac{\partial}{\partial x}\left(x\cos(xy) + e^{x+y}\right)$$

$= -x^2\sin(xy) + \cos(xy) + e^{x+y}$ —— ④

From (3) & (4).

$f_{xy} \neq f_{yx}$

(5)

(i) $f(x,y) = \sqrt{x^2+y^2}$ at $(1,1)$

→ $f(1,1) = \sqrt{1^2+1^2} = \sqrt{2}$

$f_x = \frac{1}{2\sqrt{x^2+y^2}}(2x)$

$= \frac{x}{\sqrt{x^2+y^2}}$

$f_y = \frac{1}{2\sqrt{x^2+y^2}} 2y$

$= \frac{y}{\sqrt{x^2+y^2}}$

f_x at $(1,1) = \frac{1}{\sqrt{2}}$

f_y at $(1,1) = \frac{1}{\sqrt{2}}$

$f(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

$= \sqrt{2} + \frac{1}{\sqrt{2}}(x-1) + \frac{1}{\sqrt{2}}(y-1)$

$= \sqrt{2} + \frac{1}{\sqrt{2}}(x-1+y-1)$

$= \sqrt{2} + \frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y - \frac{2}{\sqrt{2}}$

$= \frac{x+y}{\sqrt{2}}$

$f(x,y) = 1 - x + y \sin x$ at $(\pi/2, 0)$

4) $f(\pi/2, 0) = 1 - \pi/2 + 0 = 1 - \pi/2.$

→ $f_x = 0 - 1 + y\cos x$

f_x at $(\pi/2, 0) = -1$

$f_y = 0 - 0 + \sin x.$

f_y at $(\pi/2, 0) = \sin \pi/2$

$l(y,x) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

$= 1 - \pi/2 + (-1)(x - \pi/2) - 1(y - 0)$

$= 1 - \pi/2 - x + \pi/2 + y$

$= 1 - x + y$

5) $f(x,y) = \log x + \log y$ at $(1,1)$

→ $f(1,1) = \log(1) + \log(1) = 0$

$f_x = 1/x$ $\qquad$ $f_y = 1/y$

f_x at $(1,1) = 1$ $\qquad$ f_y at $(1,1) = 1$

$l(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

$= 0 + 1(x-1) + 1(y-1)$

$= x - 1 + y - 1$

$= x + y - 2$

AK
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Practical-10

Q1] Find the directional directive of the following function at given points & in the direction of given vector.

(i) $f(x,y) = x + 2y - 3$ $a(1,-1)$ $u = 3i - j$

Here, $u = 3i - j$ is not a unit vector.

$|\bar{u}| = \sqrt{(3)^2 + (-1)^2} = \sqrt{9+1} = \sqrt{10}$

unit vector along u is $\frac{u}{|u|} = \frac{1}{\sqrt{10}}(3,-1)$

$= \left(\frac{3}{\sqrt{10}}, \frac{-1}{\sqrt{10}}\right)$

$f(a+hu) = f(1,-1) + h\left(\frac{3}{\sqrt{10}}, \frac{-1}{\sqrt{10}}\right)$

$f(a) = f(1,-1) = (1) + 2(-1) - 3 = 1 - 2 - 3 = -4$

$f(a+hu) = f(1,-1) = (1) + 2(-1) - 3 = 1 - 2 - 3 = -4$

$f(a+hu) = f(1,-1) + h\left(\frac{3}{\sqrt{10}}, \frac{-1}{\sqrt{10}}\right)$

$= f\left(1 + \frac{3}{\sqrt{10}}\right), \left(-1 - \frac{h}{\sqrt{10}}\right)$

$f(a+hu) = \left(1 + \frac{3}{\sqrt{10}}\right) + 2\left(-1 - \frac{h}{\sqrt{10}}\right) - 3$

$= 1 + \frac{3}{\sqrt{10}} - 2 - \frac{2h}{\sqrt{10}} - 3$

$f(a+hu) = -4 + \frac{h}{\sqrt{10}}$

$D_u f(a) = \lim_{h \to 0} \frac{f(a+hu) - f(a)}{h}$

$$= \lim_{h \to a} \frac{\cancel{-4} + h/\sqrt{10} \cancel{+4}}{h}$$

$$D_u f(a) = \frac{1}{\sqrt{10}}$$

$f(x) = y^2 - 4x + 1$, $a = (3, 4)$, $u = i + 5j$

Here $u = i + 5j$ is not a unit vector.

$|\bar{u}| = \sqrt{(1)^2 + (5)^2} = \sqrt{26}$

unit vector along u is $\frac{u}{|u|} = \frac{1}{\sqrt{26}}(1, 5)$

$$= \left(\frac{1}{\sqrt{26}}, \frac{5}{\sqrt{26}}\right)$$

$f(a) = f(3, 4) = (4)^2 - 4(3) + 1 = 5$

$$f(a+hu) = f(3,4) + h\left(\frac{1}{\sqrt{26}}, \frac{5}{\sqrt{26}}\right)$$

$$= f\left(3 + \frac{h}{\sqrt{26}}, 4 + \frac{5h}{\sqrt{26}}\right)$$

$$f_{xy}(a+hu) = \left(4 + \frac{5h}{\sqrt{26}}\right)^2 - 4\left(3 + \frac{h}{\sqrt{26}}\right) + 1$$

$$= 16 + \frac{25h^2}{26} + \frac{40h}{\sqrt{26}} - 12 - \frac{4h}{\sqrt{26}} + 1$$

$$= \frac{25h^2}{26} + \frac{40h}{\sqrt{26}} - \frac{4h}{\sqrt{26}} + 5$$

$$= \frac{25h^2}{26} + \frac{40 - 4h}{\sqrt{26}} + 5$$

$$= \frac{25h^2}{26} + \frac{36h}{\sqrt{26}} + 5$$

2] Find gradient vector for the following functn at given point.

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(i) $f(x,y) = x^y + y^x$, $a = (1,1)$

$f(x) = y \cdot x^{y-1} + y^x \log y$.

$f_y = x^y \log x + x y^{x-1}$.

$\nabla f(x,y) = (f_x, f_y)$

$= (y x^{y-1} + y^x \log y, \; x^y \log x + x y^{x-1})$

$f(1,1) = (1+0, 1+0)$

$= (1,1)$

(ii) $f(x,y) = (\tan^{-1} x) \cdot y^2$ $a = (1,-1)$

$f_x = \frac{1}{1+x^2} \cdot y^2$.

$f_y = 2y \tan^{-1} x$

$\nabla f(x,y) = (f_x, f_y)$

$= \left(\frac{y^2}{1+x^2}, 2y \tan^{-1} x \right)$

$f(1,-1) = \left(1/2, \tan^{-1}(1)(-2) \right)$

$= \left(1/2, \pi/4 (-2) \right)$

$= \left(1/2, -\pi/2 \right)$

(iii) $f(x,y,z) = xyz - e^{x+y+z}$, $a = (1,-1,0)$.

$f_x = yz - e^{x+y+z}$

$f_y = xz - e^{x+y+z}$

$f_z = xy - e^{x+y+z}$.

$\nabla f(x,y,z) = f_x, f_y, f_z$

$x^2+y^2-2x+3y+2=0$ at $(2,-2)$.

$f_x = 2x+0-2+0+0$

$= 2x-2$

$f_y = 0+2y-0+3+0$

$= 2y+3$ $\qquad x_0=2,\ y_0=-2$

$f_x(x_0,y_0) = 2(2)-2 = 2.$

$f_y(x_0,y_0) = 2(-2)+3 = -1$

Eq^n of tangent.

$f_x(x-x_0) + f_y(y-y_0) = 0$

$2(x-2) + (-1(y+2)) = 0$

$2x-2-y-2=0$

$2x-y-4=0$ ⟶ It is required eq^n of tangent.

Eq^n of Normal.

$= ax+by+c=0$

$bx+ay+d=0$

$= -1(x)+2(y)+d=0$

$-x+2y+d=0$ at $(2,-2)$

$-2+2(-2)+d=0$

$-2-4+d=0$

$-6+d=0$

$\therefore \underline{d=6}$

Ex] Find the eq^n of tangent & normal line to each of the following surface

(1) $x^2-2yz+3y+xz=7$ at $(2,1,0)$

$f_x = 2x-0+0+z.$

$f_x = 2x+z$

$f_y = 0 - 2z + 3 + 0$

$= 2x + 3$

$f_z = 0 - 2y + 0 + x$

$= -2y + x$

$(x_0, y_0, z_0) = (2, 1, 0) \therefore x_0 = 2,\ y_0 = 1,\ z_0 = 0$

$f_x(x_0, y_0, z_0) = 2(2) + 0 = 4$

$f_y(x_0, y_0, z_0) = 2(0) + 3 = 3$

$f_z(x_0, y_0, z_0) = -2(1) + 2 = 0.$

eqn of tangent

$f_x(x_0 - x_0) + f_y(y_0 - y_0) + f_z(z_0 - z_0) = 0$

$= 4(x-2) + 3(y-1) + 0(z-0) = 0$

$= 4x - 8 + 3y - 3 = 0.$

$4x + 3y - 11 = 0$ → This is required eqn of tangent

eqn of normal at $(4, 3, -11)$

$$\frac{x - x_0}{f_x} = \frac{y - y_0}{f_y} = \frac{z - z_0}{f_z}$$

$$= \frac{x-2}{4} = \frac{y-1}{3} = \frac{z+11}{0}$$ //

(ii) $3xyz - x - y + z = -4$ at $(1, -1, 2)$

$3xyz - x - y + z + 4 = 0$

$f_x = 3yz - 1 - 0 + 0 + 0$

$= 3yz - 1$

$f_y = 3xz - 0 - 1 + 0 + 0$

$= 3xz - 1.$

$f_2 = 3xy - 0 - 0 + 1 + 0$
$= 3xy + 1$

$(x_0, y_0, z_0) = (1, -1, 2)$ $\therefore x_0 = -1, y_0 = -1, z_0 = 2.$

$f_x(x_0, y_0, z_0) = 3(-1)(2) - 1 = -7$
$f_y(x_0, y_0, z_0) = 3(+1)(2) - 1 = 5$
$f_z(x_0, y_0, z_0) = 3(1)(-1) + 1 = -2$

Eqn of tangent.
$-7(x-1) + 5(y+1) - 2(z-2) = 0$
$-7x + 7 + 5y + 5 - 2z + 4 = 0$
$-7x + 5y - 2z + 16 = 0$ $\to$ Eqn of tangent..

Eqn of Normal

$$\frac{x - x_0}{f_x} = \frac{y - y_0}{f_y} = \frac{z - z_0}{f_z}$$

$$= \frac{x-1}{-7} = \frac{y+1}{5} = \frac{z-2}{-2}$$

5) Find the local maxima & minima.

① $f(x, y) = 3x^2 + y^2 - 3xy + 6x - 4y$

$f_x = 6x + 0 - 3y + 6 - 0$
$= 6x - 3y + 6$

$f_y = 0 + 2y - 3x + 0 - 4$
$= \cancel{6x - 3y + 6}\ 2y - 3x - 4$

$f_x = 0$

$6x - 3y + 6 = 0$
$3(2x - y + 2) = 0$
$2x - y + 2 = 0$
$2x - y = -2$ —— ①

$f_y = 0$

$2y - 3x - 4 = 0.$
$2y - 3x = 4$ —— ②

Multiply eq^n 1 with 2

$4x - 2y = -4$

$2y - 2x = 4$

$x = 0$

Substitute value of x in eq^n (1)

$2(0) - y = -2$

$-y = -2 \quad \therefore y = 2$

$\therefore$ Critical points are $(0, 2)$.

$r = f_{xx} = 6$

$t = f_{yy} = 2$

$s = f_{xy} = -3$

Here $r > 0$.

$= rt - s^2$

$= 6(2) - (-3)^2$

$= 12 - 9$

$= 3 > 0$

$\therefore f$ has maximum at $(0, 2)$

$3x^2 + y^2 - 3xy + 6x - 4y$ at $(0, 2)$.

$3(0)^2 + (2)^2 - 3(0)(2) + 6(0) - 4(2)$.

$0 + 4 - 0 + 0 - 8$

$= -4$

(ii) $f(x, y) = 2x^4 + 3x^2y - y^2$.

$f_x = 8x^3 + 6xy$

$f_y = 3x^2 - 2y$

$f_x = 0$

$\therefore 8x^3 + 6xy = 0$

$2x(4x^2 + 3y) = 0$

$4x^2 + 3y = 0 \quad \text{—— (1)}$

$f_y = 0. \quad 3x^2 - 2y = 0 \quad \text{—— (2)}$

Multiply eq (1) with 3 & (2) with 4

$\cancel{12x^2} + 9y = 0$

$-\cancel{12x^2} - 8y = 0$

$\therefore \boxed{y = 0}$

substitute value of y in eq^n (1)

$4x^2 + 3(0) = 0$

$4x^2 = 0$

$\boxed{x = 0}$

critical point is $(0,0)$

$r = f_{xx} = 24x^2 + 6x$

$t = f_{yy} = 0 - 2 = -2$

$s = f_{xy} = 6x - 0 = 6x = 6(0) = 0$

r at $(0,0)$

$= 24(0) + 6(0) = 0$

$\therefore r = 0$

$rt - s^2 = 0(-2) - (s)^2$

$= 0 - 0 = 0$

$r = 0$ & $rt - s^2 = 0$

(nothing to say)

(ii) $f(x,y) = x^2 - y^2 + 2x + 8y - 70$

$f_x = 2x + 2$

$f_y = -2y + 8$

$f_x = 0$ $\therefore 2x + 2 = 0$.

$x = -2/2 = -1$ $\therefore \boxed{x = -1}$

$f_y = 0$ $-2y + 8 = 0$

$y = \frac{-8}{-2} = 4$.

$\boxed{\therefore y = 4}$

∴ Critical point is $(-1, 4)$.

$r = f_{xx} = 2$

$t = f_{yy} = -2$

$s = f_{xy} = 0$.

$r > 0$

$rt - s^2 = 2(-2) - (0)^2$

$= -4 - 0$

$= -4 < 0$

$f(x, y)$ at $(-1, 4)$

$(-1)^2 - (4)^2 + 2(-1) + 8(4) - 70$

$= 1 + 16 - 2 + 32 - 70$

$= 17 + 30 - 70$

$= 37 - 70 = \underline{33}$

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