Homework 1

Theoretical problems (no calculator/computer is needed):

T1. Consider the following boundary value problem:

$$-(p(x)u'(x))' + q(x)u(x) = f(x), \quad a < x < b, \quad u(a) = \alpha,\ u(b) = \beta,$$

where $p \in C^4([a,b])$ and $p(x) \geq \tilde{c} > 0$, $q(x) \in C([a,b])$, $q(x) \geq 0$, and $f(x) \in C([a,b])$. Divide the interval $[a,b]$ into $N+1$ equal cells with cell size $h = (b-a)/(N+1)$. Define the grid points $x_i = a + ih$, $i = 0, \dots, N+1$, with $x_0 = a$, $x_{N+1} = b$. Define also the half grid points as $x_{i+1/2} = x_i + h/2$. A natural finite difference scheme to approximate this problem is given as follows:

$$-\frac{1}{h^2}\left[p(x_{i+1/2})U_{i+1} - (p(x_{i+1/2}) + p(x_{i-1/2}))U_i + p(x_{i-1/2})U_{i-1}\right] + q(x_i)U_i = f(x_i),$$

for $i = 1, \dots, N$, with $U_0 = \alpha$ and $U_{N+1} = \beta$.

(a) Show that the consistency error T_i of the scheme is $O(h^2)$ as $h \to 0$, provided $u \in C^4([a,b])$.

(b) Formulate the finite difference scheme as $AU = F$, and show that this linear system has a unique solution.

T2. (a) For the same problem as in [T1.], if we expand the left hand side as

$$-p(x)u''(x) - p'(x)u'(x) + q(x)u(x) = f(x),$$

and then approximate by the finite difference scheme

$$-p(x_i)\frac{U_{i+1} - 2U_i + U_{i-1}}{h^2} - p'(x_i)\frac{U_{i+1} - U_{i-1}}{2h} + q(x_i)U_i = f(x_i).$$

Show that this yields a tridiagonal but not symmetric matrix.

(b) For the equation

$$-u''(x) + a(x)u'(x) + b(x)u(x) = f(x), \quad a < x < b, \quad u(a) = \alpha,\ u(b) = \beta,$$

where a, b, and f all belong to $C[a,b]$, and $b(x) > 0$. Formulate a similar finite difference scheme as in part (a) of this problem, and find a sufficient condition for the resulting linear system to have a unique solution.

T3. Consider the two-point boundary value problem

$$-u'' = f(x), \quad x \in (0,1), \quad u'(0) = \alpha, \quad u'(1) = \beta.$$

(a) When does this equation have a solution? Is the solution unique?

(b) Formulate a second order finite difference scheme for this problem as $AU = F$. Is the matrix A you obtain singular? If yes, suggest a way to make it nonsingular.

T4. Consider the elliptic boundary value problem on the unit square $\Omega = (0,1)^2$ in $\mathbb{R}^2$:

$$-\Delta u = f(x,y) \text{ in } \Omega, \quad u = g(x,y) \text{ on } \partial\Omega.$$

A natural five-point finite difference scheme for this problem on the mesh $x_i = ih$, $y_j = jh$, $i,j = 0,\dots,N+1$ with spacing $h = 1/(N+1)$ is given by

$$L_h U_{i,j} = f(x_i,y_j),\ (x_i,y_j) \in \Omega_h, \quad U_{i,j} = g(x_i,y_j),\ (x_i,y_j) \in \partial\Omega_h,$$

where

$$L_h U_{i,j} := -\frac{U_{i-1,j} - 2U_{i,j} + U_{i+1,j}}{h^2} - \frac{U_{i,j-1} - 2U_{i,j} + U_{i,j+1}}{h^2}.$$

Define the global error as $e_{i,j} = u(x_i,y_j) - U_{i,j}$, where u is the exact solution, then $e_{i,j} = 0$ on the boundary $\partial\Omega_h$. We thus have $L_h e_{i,j} = \Delta u(x_i,y_j) + L_h u(x_i,y_j) := T_{i,j}$, where $T_{i,j}$ is the local truncation error. Using the Taylor expansion, it can be shown that

$$|T_{i,j}| \le \frac{1}{12}h^2 \left(\max_{\bar{\Omega}} |\partial_x^4 u| + \max_{\bar{\Omega}} |\partial_y^4 u| \right) := T.$$

To obtain a bound for $e_{i,j}$, we proceed as follows: define a comparison function $\Phi_{i,j} = \frac{1}{4}((x_i - \frac{1}{2})^2 + (y_j - \frac{1}{2})^2)$, and consider an auxiliary function $\phi_{i,j} := e_{i,j} + T\Phi_{i,j}$.

(a) Show that $L_h \Phi_{i,j} = -1$ and $L_h \phi_{i,j} \le 0$ for all $(i,j) \in \Omega_h$. Then use the discrete maximum principle to show that $e_{i,j} \le \frac{1}{8}T$ for all $(i,j) \in \Omega_h$.

(b) Repeat the same thing by considering the auxiliary function $\bar{\phi}_{i,j} := e_{i,j} - T\Phi_{i,j}$ and use the discrete minimum principle to show that $e_{i,j} \ge -\frac{1}{8}T$ for all $(i,j) \in \Omega_h$.

(Note that combining (a) and (b), we finally obtain $|e_{i,j}| \le \frac{1}{8}T$ for all $(i,j) \in \Omega_h$.)

Coding problems (attach the code you used to generate the results):

C1. Consider following elliptic boundary value problem:

$$-\Delta u = 2\pi^2 \sin(\pi x)\cos(\pi y), \quad (x,y) \in (0,1)^2,$$

subject to the boundary condition

$$\begin{aligned} &u(x,0) = \sin(\pi x),\ u(x,1) = -\sin(\pi x),\ x \in [0,1], \\ &u(0,y) = u(1,y) = 0,\ y \in [0,1]. \end{aligned}$$

Implement a second order finite difference scheme for this problem on the $(N+1) \times (N+1)$ uniform grid (same as described in [T4.]). Compare your numerical solution with the exact solution $u(x,y) = \sin(\pi x)\cos(\pi y)$. Try a few different meshes $N = 20, 40, 80, 160$ and compute the corresponding errors in discrete L^1, L^∞, L^2 and H^1 norms. Plot all these errors in the same figure vs $h = 1/(N+1)$ using loglog scale and verify the second order accuracy. (You may use a direct method or iterative method to solve the resulting linear system.)