When Simple Pendulums Get Complicated

Why We Need Computers for Physics

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"The pendulum is perhaps the most studied object in all of physics. And yet, after three centuries, it still has something to teach us about the limits of the human mind—and the power of the machine."

1   A mass, a string, and gravity

Consider the humblest of physical systems: a mass m suspended from a rigid, massless rod of length L, free to swing in a plane under gravity. Let θ denote the angle the rod makes with the downward vertical. Applying Newton's second law along the tangential direction gives us the **equation of motion**:

$$\ddot{\theta} + \frac{g}{L}\sin\theta = 0 \tag{1}$$

This is a second-order, nonlinear ordinary differential equation. It looks deceptively simple—just five symbols—but that $\sin\theta$ hides a world of trouble.

2   The textbook trick: small angles

2.1   The approximation

Every introductory physics course makes the same move: assume the angle θ is small enough that

$$\sin\theta \approx \theta \qquad \text{(valid for } |\theta| \ll 1 \text{ rad).} \tag{2}$$

Substituting (2) into (1) yields the **linearised equation**:

$$\ddot{\theta} + \frac{g}{L}\theta = 0. \tag{3}$$

2.2   The exact solution

Equation (3) is the simple harmonic oscillator—one of the few differential equations we can solve completely by hand. Define the natural frequency

$$\omega_0 = \sqrt{\frac{g}{L}}\,. \tag{4}$$

The characteristic equation $r^2 + \omega_0^2 = 0$ has roots $r = \pm i\omega_0$, so the general solution is

$$\theta(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t). \tag{5}$$

With the natural initial conditions $\theta(0) = \theta_0$ and $\dot{\theta}(0) = 0$ (released from rest), we obtain

$$\theta(t) = \theta_0 \cos\left(\sqrt{\frac{g}{L}}\, t\right) \tag{6}$$

and the famous period formula

$$T = 2\pi \sqrt{\frac{L}{g}} \, . \tag{7}$$

2.3   Why this is beautiful

Equation (6) is remarkable. It tells us that:

- The period T depends *only* on the length L and gravity g—not on the mass, not on the amplitude. This is Galileo's isochronism.

- The motion is a pure cosine: perfectly periodic, perfectly symmetric.

- We have a *closed-form expression*. Given any t, we can compute $\theta(t)$ with a pocket calculator.

This is physics at its most elegant. One equation, one exact answer.

3   When elegance breaks down

3.1   The honest equation

Now suppose we push the pendulum to $\theta_0 = 90°$, or $120°$, or release it from nearly inverted. The small-angle approximation collapses: at $\theta = \pi/2$, the error in $\sin\theta \approx \theta$ is already 36%.

We are forced back to the full nonlinear equation (1):

$$\ddot{\theta} + \frac{g}{L}\sin\theta = 0.$$

Can we solve this exactly? *Sort of.* The exact period turns out to be

$$T_{\text{exact}} = 4\sqrt{\frac{L}{g}} \, K\left(\sin\frac{\theta_0}{2}\right), \tag{8}$$

where $K(k)$ is the **complete elliptic integral of the first kind**:

$$K(k) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - k^2\sin^2\phi}} \, . \tag{9}$$

This integral has no closed-form expression in terms of elementary functions. There is no neat formula like Eq. (7)—just an infinite series, a table of values, or a numerical routine.

3.2   The deeper problem

And it gets worse. Even T_{exact} only gives the *period*. If we want the full trajectory $\theta(t)$ at every instant, we need the **Jacobi elliptic functions**—objects that most physicists have heard of but few have used by hand.

The honest truth is this: *even the simplest mechanical system resists exact, elementary solution once we remove a single approximation.*

This is not a failure of physics. It is a fact about nonlinear differential equations.

4 Enter the computer

4.1 Reformulating as a system

What we *can* do is convert Eq. (1) into a first-order system by introducing the angular velocity $\omega = \dot{\theta}$:

$$\begin{cases} \dfrac{d\theta}{dt} = \omega \\[2ex] \dfrac{d\omega}{dt} = -\dfrac{g}{L}\sin\theta \end{cases} \tag{10}$$

This is the form that numerical integrators love. Given an initial state (θ_0, ω_0), an algorithm like the classical **Runge–Kutta method** (RK4) can march forward in time, step by step, to any desired accuracy.

4.2 The code

The following Python script solves system (10) on a grid of initial conditions and renders the **phase portrait**—a map of all possible trajectories in the (θ, ω) plane.

```python
import numpy as np
import matplotlib.pyplot as plt

# Parameters
g = 9.81    # gravitational acceleration (m/s^2)
L = 1.0     # pendulum length (m)

# Phase-space grid
theta = np.linspace(-2 * np.pi, 2 * np.pi, 400)
omega = np.linspace(-8, 8, 400)
THETA, OMEGA = np.meshgrid(theta, omega)

# Vector field (right-hand side of the ODE system)
dtheta_dt = OMEGA
domega_dt = -(g / L) * np.sin(THETA)
speed = np.sqrt(dtheta_dt**2 + domega_dt**2)

# Plot
fig, ax = plt.subplots(figsize=(12, 8))
strm = ax.streamplot(
    THETA, OMEGA, dtheta_dt, domega_dt,
    color=speed, cmap="coolwarm", density=2.5,
    linewidth=0.7, arrowsize=0.7,
)
fig.colorbar(strm.lines, ax=ax, label="Phase-space speed")

# Separatrix: energy E = gL (unstable equilibrium)
theta_sep = np.linspace(-2*np.pi, 2*np.pi, 1000)
omega_sep = np.sqrt(2*g/L * np.maximum(1+np.cos(theta_sep), 0))
ax.plot(theta_sep,  omega_sep, "k--", linewidth=1.8)
ax.plot(theta_sep, -omega_sep, "k--", linewidth=1.8)

ax.set_xlabel("theta (rad)")
ax.set_ylabel("omega (rad/s)")
ax.set_title("Phase Portrait: Nonlinear Pendulum")
plt.savefig("pendulum_phase_portrait.png", dpi=150)
```

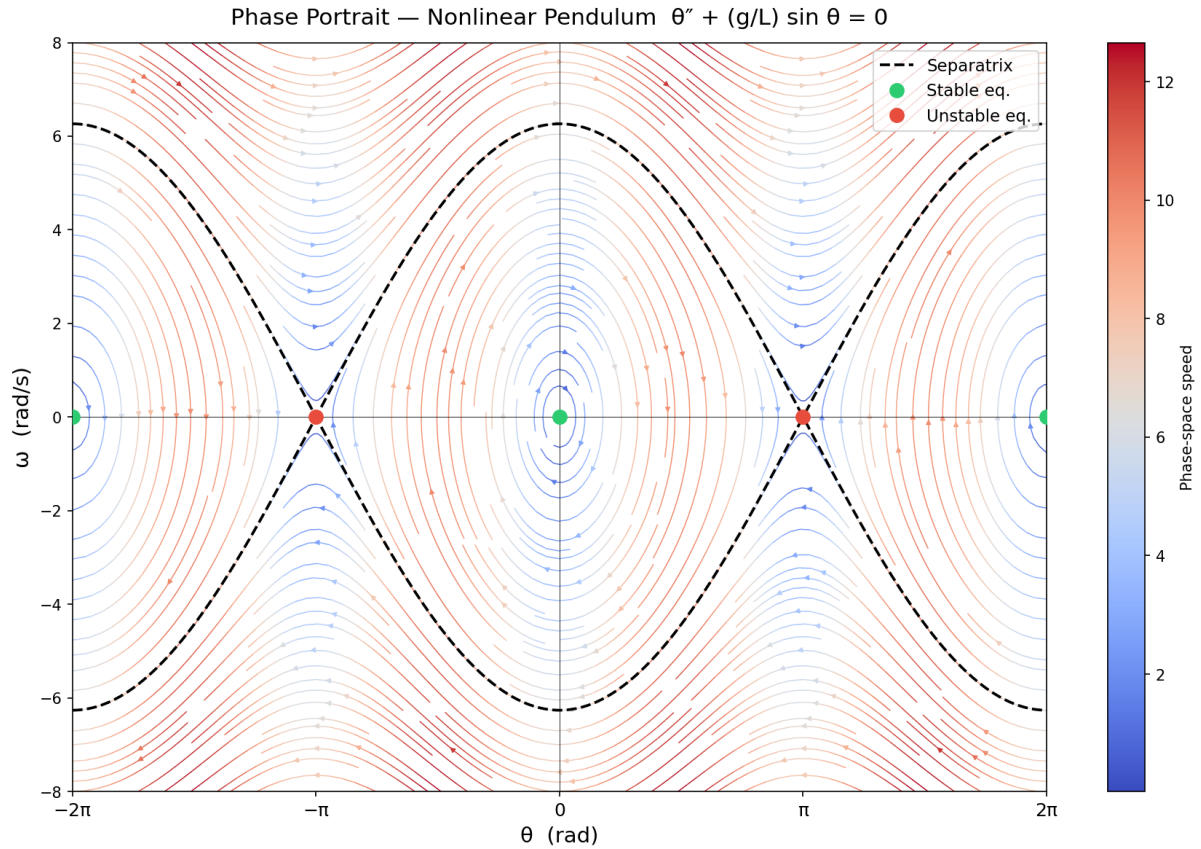Listing 1: Phase portrait of the nonlinear pendulum.

5 The phase portrait

Figure 1: Phase portrait of the nonlinear pendulum $\ddot{\theta} + (g/L)\sin\theta = 0$. Streamlines are coloured by speed in phase space. The dashed black curves are the **separatrices**. Green dots mark stable equilibria (pendulum hanging down); red dots mark unstable equilibria (pendulum inverted).

5.1 Reading the portrait

Figure 1 encodes the complete dynamics of the pendulum in a single image. Three qualitatively distinct regimes are visible:

1. **Libration** (closed orbits inside the separatrix). The pendulum swings back and forth without going over the top. Smaller loops correspond to smaller amplitudes—approaching simple harmonic motion near the origin.

2. **Rotation** (wavy curves outside the separatrix). The pendulum has enough energy to spin continuously. Curves above the separatrix represent counter-clockwise rotation; curves below, clockwise.

3. **The separatrix** (dashed curves). This is the critical boundary—the trajectory of a pendulum released with *exactly* enough energy to reach the inverted position. It would take infinite time to arrive there, asymptotically approaching the unstable equilibrium.

Notice that the closed orbits are *not* perfect ellipses. In the small-angle limit, they would be—but the nonlinearity of $\sin\theta$ distorts them, stretching the orbits as the amplitude grows. This distortion *is* the large-angle physics that no linearisation can capture.

6 The moral

The pendulum is a first-year physics problem, and yet its exact behaviour lies beyond elementary mathematics. The small-angle solution (6) is a beautiful special case, but it is just that: a special case.

The general solution requires either elliptic functions (an 18th-century invention that most scientists never encounter) or numerical computation. The phase portrait in Figure 1 would have taken Euler weeks of hand calculation. A modern laptop produces it in under a second.

This is not a story about pendulums. It is a story about the vast majority of differential equations in physics, engineering, and biology. Nonlinearity is the rule; exact solutions are the exception. And that is why we need computers for physics.