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MTech, AE

ME261 Assignment-1

1\. (a) Let S be the given set of matrices.

To prove: S forms a group under matrix multiplication.

Let $A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \in S$ where $a = \pm 1$, $b = \pm 1$.

$B = \begin{bmatrix} 0 & c \\ d & 0 \end{bmatrix} \in S$ where $c = \pm 1$, $d = \pm 1$

- Then $AB = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} 0 & c \\ d & 0 \end{bmatrix} = \begin{bmatrix} 0 & ac \\ bd & 0 \end{bmatrix}$

Then for given a, b, c, d, $ac = \pm 1$, $bd = \pm 1$.

$\Rightarrow AB \in S \quad \forall A, B \in S.$

$\Rightarrow$ closure property satisfied.

- Let $C = \begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix} \in S$ where $p = \pm 1$, $q = \pm 1$

Then $(AB)C = \begin{bmatrix} 0 & ac \\ bd & 0 \end{bmatrix}\begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix} = \begin{bmatrix} 0 & acq \\ bdq & 0 \end{bmatrix} \in S.$

$A(BC) = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} 0 & cq \\ dp & 0 \end{bmatrix} = \begin{bmatrix} 0 & acq \\ bdp & 0 \end{bmatrix} \in S.$

So $(AB)C = A(BC) \in S$

The same holds if $C = \begin{bmatrix} 0 & p \\ q & 0 \end{bmatrix}$.

$\Rightarrow$ associativity satisfied.

- $A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \in S \Rightarrow |A| = ab$

$A^{-1} = \frac{1}{|A|} \operatorname{adj}(A) = \frac{1}{ab}\begin{bmatrix} b & 0 \\ 0 & a \end{bmatrix} = \begin{bmatrix} 1/a & 0 \\ 0 & 1/b \end{bmatrix}$

$\because a, b = \pm 1, \; 1/a, 1/b = \pm 1.$

$\therefore A^{-1} \in S \quad \forall A \in S.$

llly for $B = \begin{bmatrix} 0 & c \\ d & 0 \end{bmatrix}, \; B^{-1} = \begin{bmatrix} 0 & 1/d \\ 1/c & 0 \end{bmatrix},$

$\because c, d = \pm 1, \; 1/c, 1/d = \pm 1$

and $B^{-1} \in S \quad \forall B \in S.$

$\Rightarrow$ multiplicative inverse exists.

- $AI = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = A \quad \forall A \in S.$

$BI = \begin{bmatrix} 0 & c \\ d & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & c \\ d & 0 \end{bmatrix} = B \quad \forall B \in S.$

$\Rightarrow$ Multiplicative identity element $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ exists.

Since all 4 conditions - closure, associativity, existence of identity, existence of multiplicative inverse, ~~exists~~ is satisfied under matrix multiplication,

$\boxed{\text{S forms a group under matrix multiplication}}$

- For S to be an Abelian Group, commutativity under matrix multiplication must be satisfied.

$A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}, \quad B = \begin{bmatrix} 0 & c \\ d & 0 \end{bmatrix} \in S \quad \left.\right\} \begin{matrix} a = \pm 1 \\ b = \pm 1 \\ c = \pm 1 \\ d = \pm 1 \end{matrix}$

$AB = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} 0 & c \\ d & 0 \end{bmatrix} = \begin{bmatrix} 0 & ac \\ bd & 0 \end{bmatrix} \in S$

$BA = \begin{bmatrix} 0 & c \\ d & 0 \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} 0 & cb \\ da & 0 \end{bmatrix} \in S$

$\Rightarrow AB \neq BA \quad \forall A, B \in S$

$\boxed{\text{Thus Group S isn't an Abelian Group}}$

(b) Let, $\mathbb{C}$ = set of complex numbers.

To prove : $\mathbb{C}$ forms a group under multiplication

Let $\left.\begin{array}{l} x_1 = a+ib \\ x_2 = c+id \\ x_3 = p+iq \end{array}\right\}$ $\begin{array}{l} x_1, x_2, x_3 \in \mathbb{C} \\ a, b, c, d, p, q \in \mathbb{R}. \end{array}$

• Then $x_1 \cdot x_2 = (a+ib)(c+id)$
$= (ac-bd) + \cancel{ab}(ad+bc)\,i \in \mathbb{C}$
$\because (ac-bd), (ad+bc) \in \mathbb{R}$.

$\therefore x_1 \cdot x_2 \in \mathbb{C} \quad \forall\, x_1, x_2 \in \mathbb{C}$.

$\Rightarrow$ multiplication satisfies closure property for $\mathbb{C}$.

• $x_1 \cdot (x_2 \cdot x_3) = (a+ib)\,((c+id)(p+iq))$
$= (a+ib)\cdot(c+id)\cdot(p+iq)$
$= [(a+ib)\cdot(c+id)]\cdot(p+iq)$
$= (x_1 \cdot x_2)\cdot x_3 \in \mathbb{C}$

$\Rightarrow$ associativity satisfied

• $x_1 \cdot 1 = (a+ib)\cdot 1 = (a+ib) = x_1$ & $1 \in \mathbb{C}$.

$\Rightarrow$ multiplicative identity exists $\forall\, x \in \mathbb{C}$.

• $x_1 \cdot \dfrac{1}{x_1} = 1$ where $\dfrac{1}{x_1} = (x_1)^{-1} \Rightarrow$ multiplicative inverse

$(x_1)^{-1} = \dfrac{1}{a+ib} = \dfrac{a-ib}{a^2+b^2} \in \mathbb{C}$.

as $\left(\dfrac{a}{a^2+b^2}\right), \left(\dfrac{-b}{a^2+b^2}\right) \in \mathbb{R}$.

and a, b both not simultaneously 0 i.e. $x_1 \neq 0+0i$.

$\Rightarrow$ Multiplicative inverse exists

$\forall\, x_1 \in \mathbb{C} - \{0\}$.

Since all conditions are satisfied for $\mathbb{C} - \{0\}$,

$\boxed{\mathbb{C} - \{0\} \text{ forms a group under multiplication}}$.

• Checking for commutativity,

$x_1 \cdot x_2 = (a+ib)(c+id) = (ac-bd)+(ad+bc)i \in \mathbb{C}$.

$x_2 \cdot x_1 = (c+id)(a+ib) = \cancel{ac}\ (ca-db)+(da+cb)i \in \mathbb{C}$.

$\Rightarrow x_1 \cdot x_2 = x_2 \cdot x_1 \quad \forall\, x_1, x_2 \in \mathbb{C}$.

$\Rightarrow$ multiplication satisfies commutativity $\forall\, x_1, x_2 \in \mathbb{C}$.

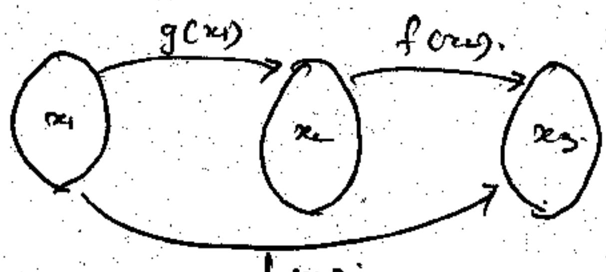
$\therefore$ $\boxed{\mathbb{C} - \{0\} \text{ forms an Abelian group under multiplication.}}$

(c) ~~Let sym~~

$f : A \to A$ is bijective if f is both one to one and onto function.

Given binary operation: $f(g(x_1))$.

• $f(g(x_1))$, $\qquad g(x_1), f(x_2), h(x_1) \in \text{sym}(X)$



$g(x_1)$ maps to a unique element $x_2 \in X$.

& $f(g(x_1)) = f(x_2) = x_3 \in X$.

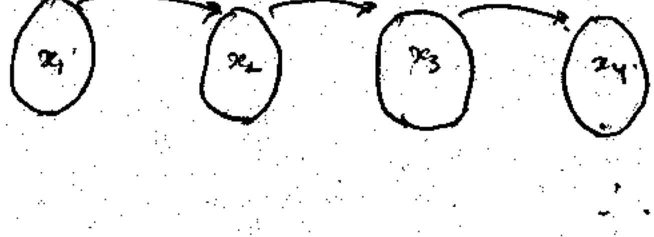
This belongs to the set of bijective functions as another bijective function can exist in sym(X) which maps x_1 directly to x_3 i.e. $h(x_1)$.

$\Rightarrow$ closure property satisfied.

• To prove: $f[g(h)](x) = [f(g)]h(x)$

~~$f(g(h(x_1))) = \ldots$~~

~~LHS~~



Shifting of brackets doesn't affect the ~~function~~ overall function representation.

$\therefore f[g(h)](x) = [f(g)]h(x)$

ex

- Let $\exists$ a unique function $I(x_1) = x_1$, $x_1 \in X$ and $I(x_1) \in sym(X)$
  ~~that is~~ It is a bijection and it maps every element of X to itself.

- If $f \in sym(X)$
  i.e. f is bijective function $f : X \longrightarrow X$.
  Then f^{-1} exists: $f^{-1} : X \longrightarrow X$.

  $$\text{eg. if } f(x_1) = x_2 \Rightarrow f^{-1}(x_2) = x_1.$$

  $$\therefore f^{-1}(f(x_1)) = I(x_1) = \underline{\underline{x_1}}$$

Thus $\boxed{\text{sym}(X) \text{ forms a group}}$

- Commutativity:

  $f(g(x_1))$: $x_1 \xrightarrow{g} x_2 \xrightarrow{f} x_3$

  $g(f(x_1))$: $x_1 \xrightarrow{f} x_2 \xrightarrow{g} x_3$

~~$f(x_1) \neq g(x_1)$~~

$f(x_1)$ need not be the same as $g(x_1)$ as each function maps x_1 in unique way to some element in X.

$\Rightarrow$ $\boxed{\text{Not an Abelian group}}$

2 (a) $\boxed{\text{False.}}$

Reason: matrix multiplication is not commutative.
if $x_1, x_2 \in R$, to be a commutative ring, it must satisfy $x_1 \cdot x_2 = x_2 \cdot x_1$.

Take for example $x_1 = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$, $x_2 = \begin{bmatrix} 1 & 3 \\ 2 & 9 \end{bmatrix}$.

Then $x_1 x_2 = \begin{bmatrix} 8 & 33 \\ 9 & 39 \end{bmatrix}$ and $x_2 x_1 = \begin{bmatrix} 5 & 15 \\ 13 & 42 \end{bmatrix}$.

Thus $x_1 x_2 \neq x_2 x_1$

(b) $\boxed{\text{False}}$

Reason: Closure property under addition is violated. So it cannot form a group and hence not a ring.
For example, Let ~~odd~~ 1 and 3 be the odd integers. Then $1+3 = 4$ is not an odd integer $\Rightarrow$ closure property violated.

(c) $\boxed{\text{True}}$

Reason:
— Under addition:
· Closure :- $f, g \in R = C(f, R)$,
Thus $(f+g)(x) = f(x) + g(x) = h(x)$ also h is a function from R to R.
Thus $h(x) \in R$.
Closure satisfied.

· Commutativity ~~Associativity~~ $(f+g)(x) = f(x) + g(x)$
$= g(x) + f(x)$
$= (g+f)(x) \rightarrow$ Commutativity ~~Associativity~~ satisfied.

· Identity element existence :- Take $k(x) = 0 \in R$.
Thus $f(x) + k(x) = f(x)$
Thus $k(x)$ is an identity element $= 0$.

- Inverse existence: For $f(x) \in R$, we can also have $f'(x) = -f(x) \in R$ such that $f(x) + f'(x) = f(x) - f(x) = 0 = k(x)$.

  Thus inverse exists.

- ~~Commutativity~~ Associativity: ~~$(f+g)(x) =$~~

$$\begin{aligned}(f+(g+h))(x) &= f(x) + (g(x) + h(x)) \\ &= f(x) + g(x) + h(x) \\ &= (f(x) + g(x)) + h(x) \\ &= ((f+g)+h)(x) \Rightarrow \text{Associativity satisfied.}\end{aligned}$$

Thus $(R, +)$ forms an Abelian group.

III^{rly} under multiplication,

- $(f \cdot g)(x) = f(x) \cdot g(x) \in R \Rightarrow$ closure
- $$\begin{aligned}(f \cdot (g \cdot h))(x) &= f(x)(g(x) \cdot h(x)) \\ &= (f(x) \cdot g(x)) \cdot h(x) \\ &= ((f \cdot g) h)(x) \Rightarrow \text{associative.}\end{aligned}$$
- $(f \cdot k')(x) = f(x) \cdot k'(x) = f(x)$ for $k'(x) = 1$.

  $\Rightarrow$ identity exists.

- ~~$(f \cdot f^{-1})(x) = \ldots \Rightarrow$ inverse exists~~

  Thus $(G, \cdot)$ satisfies closure, associativity and existence of identity.

- Distribution laws:

(a) $(f \cdot (g+h))(x) =$ ~~$(f \cdot g + f \cdot h)(x)$~~
$$\begin{aligned}&= f(x) \cdot (g+h)(x) \\ &= f(x) \cdot g(x) + f(x) \cdot h(x) \\ &= (f \cdot g)(x) + (f \cdot h)(x) \\ &= (f \cdot g + f \cdot h)(x).\end{aligned}$$

(b) $((f+g) \cdot h)(x) =$
$$\begin{aligned}&(f+g)(x) \cdot h(x) \\ &= [f(x) + g(x)]\, h(x) \\ &= f(x) h(x) + g(x) h(x) \\ &= (f \cdot h)(x) + (g \cdot h)(x) = (f \cdot h + g \cdot h)(x).\end{aligned}$$

$\Rightarrow (R, +, \cdot)$ forms a ring//.

3) To prove: a set of complex numbers ($\mathbb{C}$) forms a field.

Proof: From the definition of a field:

field is an integral domain such that inverse exists for every $a \in (G-0)$.

To show that ($\mathbb{C}$) forms an integral domain, one must show that $\mathbb{C}$ forms a commutative ring which satisfies $a \cdot c = b \cdot c \Rightarrow a = b$ for, $a, b, c \in \mathbb{C}$.

Checking if $\mathbb{C}$ forms an Abelian group under addition:

- Closure:

  Let $a = x_1 + iy_1$, $b = x_2 + iy_2$ & $a, b \in \mathbb{C}$, $x_1, y_1, x_2, y_2 \in \mathbb{R}$ and $i = \sqrt{-1}$.

  From the definition of complex addition,

  $$a+b = (x_1 + x_2) + i(y_1 + y_2)$$

  $\because$ real numbers under addition forms a group, real addition is closed.

  $\therefore (x_1 + x_2), (y_1 + y_2) \in \mathbb{R}$ and

  $$a+b \in \mathbb{C}.$$

- Associativity:

  Let $a = (x_1, y_1)$, $b = (x_2, y_2)$, $c = (x_3, y_3)$ & $a, b, c \in \mathbb{C}$ & $x_1, x_2, x_3, y_1, y_2, y_3 \in \mathbb{R}$.

  Thus $a + (b + c)$

  $$= (x_1 + iy_1) + (x_2 + iy_2 + x_3 + iy_3)$$

  $$= (x_1 + iy_1) + ((x_2 + x_3) + i(y_2 + y_3))$$

  $$= (x_1 + x_2 + x_3) + i(y_1 + y_2 + y_3)$$

  $$= (x_1 + x_2) + x_3 + i(y_1 + y_2) + iy_3$$

  $$= (\underline{x_1 + iy_1} + \underline{x_2 + iy_2}) + (\underline{x_3 + iy_3})$$

  $$= (a+b) + c$$

- Existence of identity element:

For the complex numbers $0+0i$, $x+iy$,

$$(x+iy)+(0+0i) = (x+0)+i(y+0)$$
$$= \underline{x+iy}$$

Similarly $(0+0i) = (0+x)+i(0+y)$
$$= x+iy$$

Hence $0 = 0+0i$ is the identity element.

- Existence of inverse element:

Inverse of an arbitrary complex number $x+iy \in (\mathbb{C},+)$ is $-x-iy \in \mathbb{C}$.

as $(x+iy)+(-x-iy)$
$$= (x-x)+i(y-y)$$
$$= \underline{0+0i}$$

Similarly $(-x-iy)+(x+iy)$
$$= (-x+x)+i(-y+y)$$
$$= \underline{0+0i}$$

- Commutativity:

Let $a, b \in \mathbb{C}$, $a = x_1+iy_1$, $b = x_2+iy_2$.

Then $a+b = (x_1+iy_1)+(x_2+iy_2)$
$$= (x_1+x_2)+i(y_1+y_2)$$
$$= (x_2+x_1)+i(y_2+y_1)$$
$$= (x_2+iy_2)+(x_1+iy_1)$$
$$= \underline{b+a}$$

Hence commutative under addition

Hence $(\mathbb{C},+)$ is an Abelian group.

- Checking if $(\mathbb{C}, \cdot)$ ~~forms~~ satisfies closure, associativity and existence of identity elements: (53)

- Closure:

Let $a, b \in \mathbb{C}$, $a = x_1 + iy_1$ $b = x_2 + iy_2$
$= (x_1, y_1)$ $= (x_2, y_2)$

Then $a \cdot b = (x_1, y_1) \cdot (x_2, y_2)$

$= x_1 x_2 - y_1 y_2 + i x_1 y_2 + i x_2 y_1$

$= (x_1 x_2 - y_1 y_2) + i (x_1 y_2 + x_2 y_1) \in \mathbb{C}$.

- Associativity.

Let $a, b, c \in \mathbb{C}$: $a = (x_1, y_1)$, $b = (x_2, y_2)$, $c = (x_3, y_3)$.

Then $a \cdot (b \cdot c) = (x_1, y_1) \cdot \left((x_2, y_2) \cdot (x_3, y_3) \right)$

$= (x_1, y_1) \cdot (x_2 x_3 - y_2 y_3 , x_2 y_3 + \overset{y_2 x_3}{\cancel{x_3 y_2}})$

$= (x_1 x_2 x_3 - x_1 y_2 y_3 - y_1 x_2 y_3 - y_1 y_2 x_3 +$

$\cancel{i [- y_1 x_2 y_3 - y_1 y_2 x_3}$

$+ i (y_1 x_2 x_3 - y_1 y_2 y_3 + x_1 x_2 y_3 + x_1 y_2 x_3)$

$= [(x_1 x_2 - y_1 y_2) x_3 - (x_1 y_2 + y_1 x_2) y_3] +$

$i [(x_1 x_2 - y_1 y_2) y_3 + (x_1 y_2 + y_1 x_2) x_3]$

$= (x_1 x_2 - y_1 y_2 , x_1 y_2 + y_1 x_2) (x_3, y_3)$

$= ((x_1, y_1) (x_2, y_2)) \cdot (x_3, y_3)$

$= (a \cdot b) \cdot c$

$\Rightarrow$

- Existence of identity element:

Let $a = x+iy \in \mathbb{C}$. Also $(1+0i) \in \mathbb{C}$.

Then $(x+iy)\cdot(1+0i)$

$= (x\cdot 1 - y\cdot 0) + i\,(y\cdot 1 + 0\cdot x)$

$= (x+iy)$

Similarly, $(1+0i)\cdot(x+iy)$

$= (1\cdot x - 0\cdot y) + i\,(0\cdot x + 1\cdot y)$

$= (x+iy)$

Hence $1+0i$ is an identity element for $\mathbb{C}$ under multiplication.

- Distribution laws:

Let $a = x_1+iy_1$, $b = x_2+iy_2$ & $c = x_3+iy_3 \in \mathbb{C}$.

Then $a\cdot(b+c) = (x_1+iy_1)\cdot(x_2+iy_2+x_3+iy_3)$

$= (x_1+iy_1)\cdot(x_2+x_3+i(y_2+y_3))$

$= [x_1\cdot(x_2+x_3) - y_1(y_2+y_3)] + i[y_1\cdot(x_2+x_3) + x_1\cdot(y_2+y_3)]$

$= [x_1x_2 + x_1x_3 - y_1y_2 - y_1y_3] + i[x_2y_1 + x_3y_1 + x_1y_2 + x_1y_3]$

$= [x_1x_2 - y_1y_2 + x_1x_3 - y_1y_3] + i[x_2y_1 + x_1y_2 + x_3y_1 + x_1y_3]$

$= [x_1x_2 - y_1y_2 + i(x_2y_1 + x_1y_2)] + [x_1x_3 - y_1y_3 + i(x_3y_1 + x_1y_3)]$

$= (x_1+iy_1)\cdot(x_2+iy_2) + (x_1+iy_1)\cdot(x_3+iy_3)$

$= a\cdot b + a\cdot c$

$(a+b)\cdot c = (x_1+iy_1 + x_2+iy_2)\cdot(x_3+iy_3)$

$= (x_1+x_2,\ y_1+y_2)\cdot(x_3, y_3)$

$= (x_1x_3 + x_2x_3 - y_1y_3 - y_2y_3,\ x_1y_3 + x_2y_3 + x_3y_1 + x_3y_2)$

$= (\underline{x_1x_3 - y_1y_3} + \underline{x_2x_3 - y_2y_3},\ \underline{x_1y_3 + x_3y_1} + \underline{x_2y_3 + x_3y_2})$

$= (x_1x_3 - y_1y_3,\ x_1y_3 + x_3y_1) + (x_2x_3 - y_2y_3,\ x_2y_3 + x_3y_2)$

$= (x_1, y_1)\cdot(x_3, y_3) + (x_2, y_2)\cdot(x_3, y_3)$

$= a\cdot c + b\cdot c$

$\because$ $(\mathbb{C}, +)$ forms an Abelian group, $(\mathbb{C}, \cdot)$ satisfies closure, associativity and existence of identity element and $(+, \cdot)$ satisfies distribution laws,

$(\mathbb{C}, +, \cdot)$ is a ring.

Checking for a commutative ring:

Let $a, b \in \mathbb{C}$ & [illegible]

Then $ab = (x_1+iy_1)(x_2+iy_2)$

$= (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$

$= (x_2x_1 - y_2y_1) + i(x_2y_1 + x_1y_2)$

$= (x_2x_1 + ix_2y_1) - y_2y_1 + ix_1y_2$

$= x_2(x_1+iy_1) + i(y_2(x_1+iy_1))$

$= (x_2+iy_2)(x_1+iy_1)$

$= b\cdot a$

Thus $a\cdot b = b\cdot a$

Checking if $a \cdot c = b \cdot c \Rightarrow a = b$

$$a \cdot c = (x_1 + iy_1) \cdot (x_3 + iy_3)$$
$$= (x_1 x_3 - y_1 y_3) + i(y_1 x_3 + y_3 x_1)$$
$$b \cdot c = (x_2 + iy_2) \cdot (x_3 + iy_3)$$
$$= (x_2 x_3 - y_2 y_3) + i(y_2 x_3 + y_3 x_2)$$

$$a \cdot c = b \cdot c$$

$\Rightarrow$ $x_1 x_3 - y_1 y_3 = x_2 x_3 - y_2 y_3 \cdots ①$ and

$y_1 x_3 + y_3 x_1 = y_2 x_3 + y_3 x_2 \cdots ②$

From ①,

$$(x_1 - x_2) x_3 = (y_1 - y_2) y_3$$

From ②,

$$(y_1 - y_2) x_3 = -(x_1 - x_2) y_3.$$

$$\Rightarrow \frac{(x_1 - x_2)}{(y_1 - y_2)} = \frac{(y_1 - y_2)}{-(x_1 - x_2)}$$

$$\Rightarrow (x_1 - x_2)^2 + (y_1 - y_2)^2 = 0.$$

But this is only possible when $x_1 = x_2 = y_1 = y_2 = 0$ or

if $(x_1 - x_2) = 0$ and $(y_1 - y_2) = 0$.

In both scenarios, $x_1 = x_2$ and $y_1 = y_2$.

$\Rightarrow$ $\underline{a = b}$ if $a \cdot c = b \cdot c$.

Thus $(\mathbb{C}, +, \cdot)$ forms an integral domain.

For $\mathbb{C}$ to be a field, an inverse must exist for $\forall a \in (G-0)$.

For $a = x+iy \in \mathbb{C}$ it was already proven that additive inverse $a' = -x-iy \in \mathbb{C}$ exists, such that $a+a' = a'+a = 0+0i$.

For multiplicative inverse,

$$\text{let } a \cdot a'' = 1+0i = a'' \cdot a.$$

$$(x_1+iy_1)a'' = 1+0i = a''(x_1+iy_1)$$

$$\therefore \cancel{a'' = \frac{1+0i}{x+iy}}$$

$$a'' = \frac{(x-iy)(1+0i)}{x^2+y^2}$$

$$= \frac{(x+0y) - i(y+0x)}{x^2+y^2}$$

$$= \frac{x-iy}{x^2+y^2}$$

$$\Rightarrow a'' = \left(\frac{x}{x^2+y^2}\right) - i\left(\frac{y}{x^2+y^2}\right)$$

$$(x_1+iy_1)a'' = (x_1+iy_1)(x_2+iy_2)$$

$$= (x_1x_2 - y_1y_2) + i(y_1x_2 + y_2x_1) = 1+0i$$

$x_1x_2 - y_1y_2 = 1$ and

$$y_1x_2 + y_2x_1 = 0 \Rightarrow x_2 = \frac{-y_2x_1}{y_1}$$

$$x_1\left(\frac{-y_2x_1}{y_1}\right) - y_1y_2 = 1$$

$$(x_1^2+y_1^2)\,y_2 = -y_1$$

$$\Rightarrow \boxed{y_2 = \frac{-y_1}{(x_1^2+y_1^2)}}$$

For ~~all~~ non zero ~~complex~~ numbers, there exist ~~a'' such that~~ $a a'' = a'' a = 1+0i$, where $1+0i$ is the multiplicative identity.

$$y_2 = \frac{-y_1}{(x_1^2+y_1^2)}$$

$$\Rightarrow x_2 = -\frac{y_2}{y_1} \cdot x_1$$

$$= +\frac{y_1 x_1}{(x_1^2+y_1^2) \cdot y_1}$$

$$= \frac{x_1}{(x_1^2+y_1^2)} \Rightarrow a'' = \frac{x_1}{x_1^2+y_1^2} - \frac{i y_1}{x_1^2+y_1^2}$$

III^{ly} $a^{\prime}a = 1+0i$

$\Rightarrow (x_2+iy_2)(x_1+iy_1) = 1+0i$

$(x_2x_1 - y_2y_1) + i(y_2x_1 + y_1x_2) = 1+0i$

$x_2x_1 - y_2y_1 = 1$

$y_2x_1 + y_1x_2 = 0 \Rightarrow y_2 = \frac{-y_1}{x_1}x_2$

$x_2x_1 - \left(\frac{-y_1}{x_1}x_2\right)y_1 = 1$

$\Rightarrow x_2 = \frac{x_1}{x_1^2+y_1^2}$

$y_2 = \frac{-y_1}{x_1}\cdot x_2 = \frac{-y_1}{x_1}\cdot\frac{x_1}{x_1^2+y_1^2}$

$= \frac{-y_1}{x_1^2+y_1^2}$

Thus $a^{\prime} = x_2+iy_2 = \frac{x_1}{x_1^2+y_1^2} - \frac{iy_1}{x_1^2+y_1^2}$

$\Rightarrow$ Inverse exists and for $a \in (C-0)$ $\boxed{a^{-1} = a^{\prime} = \frac{x_1}{x_1^2+y_1^2} - \frac{iy_1}{x_1^2+y_1^2}}$

Thus $\boxed{\text{set of complex numbers forms a field.}}$

(4.1)

4) Let $M^{n\times n}$ be the set of all $N\times N$ symmetric matrices.

Let $A, B \in M^{n\times n}$.

- $\Rightarrow a_{ij} = a_{ji},\ b_{ij} = b_{ji}$, $1 \le i,j \le N$. unique

Also $A^T = A$, $B^T = B$ ....①.

Let $C = A+B$.

$(A+B)^T = A^T + B^T$.

But from ①,

$= A+B$.

Thus $(A+B)^T = (A+B)$

$\Rightarrow$ $C = A+B = (A+B)^T$, we have

$C \in M^{n\times n}$ and $c_{ij} = c_{ji}$ for $1 \le i,j \le N$.

$\Rightarrow$ closure under addition

- Let $\lambda \in \mathbb{R}$.

Then $D = \lambda A \Rightarrow \lambda a_{ij} = \lambda a_{ji}$

$\Rightarrow d_{ij} = d_{ji}$ $1 \le i,j \le N$, where d_{ij} is $(i,j)^{th}$ element of $N\times N$ matrix D.

$\therefore D = \lambda A \in M^{n\times n}$.

$\Rightarrow$ closure under scalar multiplication.

- Let $\underset{\sim}{O} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & & & \vdots \\ 0 & \cdots & & \cdots & 0 \end{bmatrix}_{N\times N}$ be the $N\times N$ zero matrix.

Then $\underset{\sim}{O}_{ij} = \underset{\sim}{O}_{ji} = 0$, $1 \le i,j \le N$.

$\therefore \underset{\sim}{O}$ is a $N\times N$ symmetric matrix.

or $\underset{\sim}{O} \in M^{N\times N}$.

~~Since $N\times N$ matrices follow commutativity, associativity.~~

$\Rightarrow$ $N\times N$ symmetric matrices form a vector space.

* Let $M'^{n\times n}$ be the set of all anti-symmetric matrices of size $N\times N$.

Let $A, B \in M'^{n\times n}$.

Then $A^T = -A$, $B^T = -B$ and

$a_{ij} = -a_{ji}$, $b_{ij} = -b_{ji}$, $1 \le i, j \le N$

- Let $C = A + B$.

Then $C^T = (A+B)^T$
$= A^T + B^T$
$= -A - B$
$= -(A+B)$
$= \underline{\underline{-C}}$

$\therefore$ $C^T = -C$ and $c_{ij} = -c_{ji}$, $1 \le i, j \le N$,

$\underline{\underline{C \in M'^{n\times n}}}$.

- Let $\lambda \in \mathbb{R}$.

Then $D = \lambda C$ still implies

$\lambda c_{ij} = -\lambda c_{ji}$ or

$d_{ij} = -d_{ji}$

$\Rightarrow D \in M'^{n\times n} \Rightarrow \lambda C \in M'^{n\times n}$.

- The $N\times N$ zero matrix $\underset{\sim}{O}$ also satisfies $O^T = -O$.

$\therefore \underset{\sim}{O} \in M'^{n\times n}$.

$\therefore$ set of all anti-symmetric matrices of size $N\times N$ forms a vector space.

5. Given:

Generating set: $\{v_1, v_2, v_3, \dots v_k\}$

To prove: $\text{Span}\{v_1, v_2, \dots v_k\}$ is a subspace of V if

$\{v_1, v_2, \dots v_k\} \in V$.

$$u = \sum_{1}^{k} \alpha_i v_i$$

- Closure under addition:

$$u_1 + u_2 = \sum_{i=1}^{k} \alpha_i v_i + \sum \beta_i v_i$$
$$= \sum_{i=1}^{k} (\alpha_i + \beta_i) v_i$$

One can ~~see~~ observe that $u_1 + u_2$ is still a linear combination of v_i, hence this satisfies closure property.

- Closure under scalar multiplication:

$$\lambda u = \lambda \sum_{i=1}^{k} \alpha_i v_i \Rightarrow \sum_{i=1}^{k} (\lambda \alpha_i) v_i \in V.$$

- For $\alpha = 0$, $u = 0 \in V$.
- For $\beta = -\alpha$, $u = -\sum_{i=1}^{k} \alpha_i v_i \in V$.

$\Rightarrow$ $\text{span}\{v_1, v_2 \dots v_k\}$ is a subspace of V.

6) $V = U \oplus W$ if $V = U + W$
and $U \cap W = \{0\}$.

Given $U = \text{span}\{(0,1,1)\}$.
$W = \text{span}\{(1,0,1), (1,1,1)\}$.
$V = \mathbb{R}^3$.

Let $u \in U$.
Then $u = \lambda(0,1,1)$, where $\lambda \in \mathbb{R}$.
Taking $\lambda = 0$,
we have $u = (0,0,0)$
$\Rightarrow (0,0,0) \in U$.

Let $w \in W$,
Then $w = \lambda_1(1,0,1) + \lambda_2(1,1,1)$, where $\lambda_1, \lambda_2 \in \mathbb{R}$.
Take $\lambda_1 = \lambda_2 = 0$
Then $w = (0,0,0)$
$\Rightarrow$ ~~or~~ $(0,0,0) \in W$.

Representing each element of U & W as column vectors,
$U = \text{span}\left\{\begin{bmatrix}0\\1\\1\end{bmatrix}\right\}$ $\quad W = \text{span}\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}\right\}$

Angle b/w $\begin{bmatrix}1\\0\\1\end{bmatrix}$ and $\begin{bmatrix}1\\1\\1\end{bmatrix}$

$\theta = \cos^{-1}\left(\dfrac{1\cdot1 + 0\cdot1 + 1\cdot1}{\sqrt{2}\cdot\sqrt{3}}\right) = \cos^{-1}\left(\dfrac{\sqrt{2}}{\sqrt{3}}\right)$. as $\cos($

$\neq 0$.

Hence $\begin{bmatrix}1\\0\\1\end{bmatrix}$ and $\begin{bmatrix}1\\1\\1\end{bmatrix}$ are basis vectors for W, whose elements ~~which~~ geometrically form a 2D-plane in $\mathbb{R}^3$.

Similarly $\begin{bmatrix}0\\1\\1\end{bmatrix}$ being the only basis vector for U, ~~forms~~ whose elements geometrically forms a line in $\mathbb{R}^3$.

To check whether this line lies on the 2D plane, one must determine the angle between the plane and the line.

If w_a, w_b are the basis vectors for W,
the $w_a \times w_b$ gives the direction of normal to the plane spanned by W.

Suppose U_a is the basis vector for U.
Thus if $U_a \cdot (w_a \times w_b) = 0$, this implies U is ~~normal~~ coplanar with W.

$w_a \times w_b$ = ~~$|w_a||w_b|\sin\theta$~~, ~~θ being the angle between w_a and w_b~~

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{vmatrix} = -\hat{i} + 0\hat{j} + 1\hat{k}$$

$$= (-1, 0, 1)$$

$$U_a \cdot (w_a \times w_b) = (0,1,1) \cdot (-1,0,1)$$

$$= 1 \neq 0.$$

$\therefore$ U line is at an angle to the W plane.

A line can intersect a plane at only a point.

$\because$ $(0,0,0) \in U$ and $(0,0,0) \in W$.

$\Rightarrow$ $(0,0,0)$ is the only intersection point.

$\Rightarrow$ $U \cap W = (0,0,0)$. ...①

Also, since U isn't coplanar with W,

$U + W$ spans the entire $\mathbb{R}^3$.

or in other words $\mathbb{R}^3 = \text{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ ... ②

$\Rightarrow$ $\boxed{V = U \oplus W}$ from ① and ②.

7) * Given:-

Schwarz inequality $|\vec{U}\cdot\vec{V}| \leq \|\vec{U}\| \, \|\vec{V}\| \quad \ldots (1)$

To prove;

If $\vec{U}, \vec{V} \in V^n$, where V^n is an n-dim vector spaces

$$\|\vec{U}+\vec{V}\| \leq \|\vec{U}\| + \|\vec{V}\|.$$

Proof:-

$\|\vec{U}+\vec{V}\|^2 = \langle \vec{U}+\vec{V}, \vec{U}+\vec{V} \rangle$, from The definition of norm: $\|\vec{x}\| = \sqrt{\langle \vec{x}, \vec{x} \rangle}$

$= \langle \vec{U}, \vec{U} \rangle + \langle \vec{V}, \vec{V} \rangle + \langle \vec{U}, \vec{V} \rangle + \langle \vec{V}, \vec{U} \rangle$, which follows from the additivity property of inner products

$(\langle \vec{x}+\vec{y}, \vec{z} \rangle = \langle \vec{x}, \vec{z} \rangle + \langle \vec{y}, \vec{z} \rangle, \ \vec{x}, \vec{y}, \vec{z} \in V^n)$

$= \langle \vec{U}, \vec{U} \rangle + \langle \vec{V}, \vec{V} \rangle + 2Re\langle \vec{U}, \vec{V} \rangle$

$\leq \|\vec{U}\|^2 + \|\vec{V}\|^2 + 2|\langle \vec{U}, \vec{V} \rangle|$

$\leq \|\vec{U}\|^2 + \|\vec{V}\|^2 + 2\|\vec{U}\| \, \|\vec{V}\|$, from (1).

$= (\|\vec{U}\| + \|\vec{V}\|)^2 \quad \ldots\ldots (2)$

$\Rightarrow \|\vec{U}+\vec{V}\|^2 \leq (\|\vec{U}\| + \|\vec{V}\|)^2$

or $\boxed{\|\vec{U}+\vec{V}\| \leq \|\vec{U}\| + \|\vec{V}\|.} \quad \ldots (A)$

* Given: $\|\vec{U}+\vec{V}\| \leq \|\vec{U}\| + \|\vec{V}\|$, $\vec{U}, \vec{V} \in V^n$

Prove: $\|\vec{z}\| \leq \|\vec{x}\| + \|\vec{y}\|$, where $\vec{x}, \vec{y}, \vec{z} \in V^n$ forms the sides of a right triangle with $\vec{z}$ being the hypotenuse.

From Pythagorus' theorem,

$$\|\vec{z}\|^2 = \|\vec{x}\|^2 + \|\vec{y}\|^2 \quad \cdots ③ ,$$

$\because \langle \vec{x}, \vec{y} \rangle_E = \langle \vec{x}, \vec{y} \rangle = 0$ as $\vec{x}$ & $\vec{y}$ are at right angles.

$$\|\vec{x} + \vec{y}\|^2 = \|\vec{x}\|^2 + \|\vec{y}\|^2 + 2\|\vec{x}\|\,\|\vec{y}\| \quad \text{from } ②.$$

$$\Rightarrow \|\vec{x}+\vec{y}\|^2 > \|\vec{x}\|^2 + \|\vec{y}\|^2 \quad \cdots ④ \quad (\because 2\|x\|\,\|y\| > 0)$$

$\downarrow$ as $\vec{x}$ and $\vec{y}$ are sides of the right triangle $\|\vec{x}\|, \|\vec{y}\|, \|\vec{z}\|$ cannot be 0.

From ③ and ④,

$$\|\vec{x}+\vec{y}\|^2 > \|\vec{z}\|^2$$

Taking square roots,

$$\|\vec{x}+\vec{y}\| > \|\vec{z}\| \quad \cdots ⑤$$

~~Hence~~
From Ⓐ,

$$\|\vec{x}+\vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|.$$

Using it in ⑤,

$$\boxed{\|\vec{z}\| < \|\vec{x}\| + \|\vec{y}\|}$$

Hence proved!

8) Graham–Schmidt algorithm says,

given a linearly independent set $\{x_1, x_2, \cdots x_p\}$ in W, then there exists an orthogonal list $\{v_1, v_2, \cdots v_p\}$ of vectors in W

s. $Span\{x_1, x_2, \cdots x_j\} = Span\{v_1, v_2, \cdots v_j\}$ for $j = 1, 2, \cdots p$.

More specifically,

$$v_1 = x_1$$

$$v_2 = x_2 - \frac{\langle x_2, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1$$

$$v_3 = x_3 - \frac{\langle x_3, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 - \frac{\langle x_3, v_2 \rangle}{\langle v_2, v_2 \rangle} v_2$$

$$\vdots$$

$$v_p = x_p - \frac{\langle x_p, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 - \frac{\langle x_p, v_2 \rangle}{\langle v_2, v_2 \rangle} v_2 \cdots - \frac{\langle x_p, v_{p-1} \rangle}{\langle v_{p-1}, v_{p-1} \rangle} v_{p-1}.$$

Normalize each of $v_1, v_2, \cdots v_p$ to get the orthonormal set.

Using this definition for the given problems,

(i) $\{(4,0), (2,1)\}$

Here $x_1 = (4,0)$, $x_2 = (2,1)$.

Then $v_1 = x_1 = (4,0)$

$$v_2 = x_2 - \frac{\langle x_2, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1$$

$$= \overset{(2,1)}{\cancel{(4,0)}} - \frac{\langle (2,1), (4,0) \rangle}{\langle (4,0), (4,0) \rangle} (4,0)$$

$$\langle (2,1), (4,0) \rangle = 2\cdot4 + 1\cdot0 = \underline{8}$$

$$\langle (4,0), (4,0) \rangle = 4\cdot4 + 0\cdot0 = \underline{16}$$

{Taking ~~inner product~~ inner product to be the dot product}

$$\therefore v_2 = (2,1) - \frac{8}{16}(4,0)$$

$$= (2,1) - (2,0)$$

$$= \underline{(0,1)}$$

Normalizing both V_1 and V_2.

$$\frac{V_1}{\|V_1\|} = \frac{(4,0)}{\sqrt{16}} = (1,0)$$

$$\frac{V_2}{\|V_2\|} = \frac{(0,1)}{\sqrt{1}} = (0,1)$$

Thus the required orthonormal set is $\boxed{\{(1,0),(0,1)\}}$

(ii) Here

$x_1 = (6,-1,1,2,1)$

$x_2 = (2,3,-1,1,4)$

$v_1 = x_1 = (6,-1,1,2,1)$

$$v_2 = x_2 - \frac{\langle x_2, v_1\rangle}{\langle v_1, v_1\rangle} v_1$$

$\langle x_2, v_1\rangle = \langle (2,3,-1,1,4), (6,-1,1,2,1)\rangle$ { Taking inner product as the dot product }

$= 12 - 3 - 1 + 2 + 4$

$= \underline{14}$

$\langle v_1, v_1\rangle = \langle (6,-1,1,2,1), (6,-1,1,2,1)\rangle$

$= 36 + 1 + 1 + 4 + 1$

$= \underline{43}$

$\therefore\ V_2 = \cancel{6,-1,1}\ (2,3,-1,1,4) - \frac{14}{43}(6,-1,1,2,1)$

$= (2,3,-1,1,4) - \left(\frac{84}{43}, \frac{-14}{43}, \frac{14}{43}, \frac{28}{43}, \frac{14}{43}\right)$

$= \left(\frac{2}{43}, \frac{143}{43}, \frac{-57}{43}, \frac{15}{43}, \frac{158}{43}\right)$

Normalizing,

$$\hat{V}_1 = \frac{V_1}{\|V_1\|} = \frac{(6,-1,1,2,1)}{\sqrt{43}} = (0.915, -0.1525, 0.1525, 0.3050, 0.1525)$$

$$\hat{v}_2 = \frac{v_2}{\|v_2\|} = [9.045\times10^{-3},\ 0.6467,\ -0.2578,\ 0.06784,\ 0.7146]$$

Thus ~~$\hat{v}_1$ and $\hat{v}_2$ are~~ $\{\hat{v}_1, \hat{v}_2\}$ is ~~are~~ the required orthonormal set.

9) Given,

starting basis : $\{1, x, x^2\}$

Inner product definition: $(f(x), g(x)) = \int_{-1}^{1} \frac{f(x)\,g(x)}{\sqrt{1-x^2}}\,dx$

To find: set of orthogonal polynomials ~~start~~ from the given basis. $\{v_1, v_2, v_3\}$.

$\cancel{(1,1) = \|1\|^2 = \int_{-1}^{1} \frac{1\cdot 1}{\sqrt{1-x^2}}\,dx = \sin^{-1}(1) - \sin^{-1}(-1)}$

Take $x = \cos\phi$. $\{\because -1 \le x \le 1\}$. Then starting basis becomes $\{1, \cos\phi, \cos^2\phi\}$

$$\text{Then } \langle p, q\rangle = \int_{-1}^{1} \frac{p(x)\cdot q(x)}{\sqrt{1-x^2}}\,dx$$

$$= -\int_{0}^{\pi} \frac{p(\cos\phi)\, q(\cos\phi)}{\sqrt{1-\cos^2\phi}} \cos'\phi \, d\phi$$

$$= \int_{0}^{\pi} p(\cos\phi)\, q(\cos\phi)\, d\phi .$$

Take $v_1 = x_1 = 1$.

$$v_2 = x_2 - \frac{\langle x_2, v_1\rangle}{\langle v_1, v_1\rangle} v_1 .$$

$$\langle x_2, v_1 \rangle = \int_0^{\pi} \cos\phi \cdot 1 \, d\phi$$

$$= \sin(\pi) - \sin(0) = 0$$

$$\langle v_1, v_1 \rangle = \int_0^{\pi} 1 \cdot 1 \cdot d\phi$$

$$= \pi$$

$$\therefore v_2 = x_2 - \frac{0}{\pi} v_1$$

$$= x_2$$

$$= x \quad (\text{or } \cos\phi).$$

$$v_3 = x_3 - \frac{\langle x_3, v_2 \rangle}{\langle v_2, v_2 \rangle} v_2 - \frac{\langle x_3, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1.$$

$$\langle x_3, v_2 \rangle = \int_0^{\pi} \cos^2\phi \cdot \cos\phi \, d\phi$$

$$= \sin(\pi) - \sin(0) - \left[\frac{\sin^3(\pi) - \sin^3(0)}{3}\right].$$

$$= 0$$

$$\langle x_3, v_1 \rangle = \int_0^{\pi} \cos^2\phi \cdot 1 \, d\phi$$

$$= \frac{1}{2}(\pi - 0) + 0$$

$$= \frac{\pi}{2}.$$

$$\langle v_1, v_1 \rangle = \pi.$$

$$\langle v_2, v_2 \rangle = \int_0^{\pi} \cos^4\phi \, d\phi = \frac{3}{8}\pi + \frac{1}{4}(0) + \frac{1}{32}(0)$$

$$= \frac{3}{8}\pi.$$

$$\therefore v_3 = x^2 - \frac{0}{(3\pi/8)} x - \frac{(\pi/2)}{\pi}$$

$$= x^2 - \frac{1}{2}$$

$\Rightarrow \{1, x, x^2 - \frac{1}{2}\}$ forms a set of orthogonal polynomials.

If one were to scale 3rd polynomial by 2, we get the first three Chebyshev polynomials:

$T_0 = 1, \; T_1 = x, \; T_2 = 2x^2 - 1$