Vivek T
17397

DS284 Assignment-0

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(1.a) Given:

$$\mathbb{V} = \left\{ \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} \middle| \; w - x - y + z = 0; w, x, y, z \in \mathbb{R} \right\} \tag{1}$$

Consider three vectors u_1, u_2 and u_3 such that $u_1, u_2, u_3 \in \mathbb{V}$.
Let

$$u_1 = \begin{pmatrix} w_1 \\ x_1 \\ y_1 \\ z_1 \end{pmatrix} \text{, } u_2 = \begin{pmatrix} w_2 \\ x_2 \\ y_2 \\ z_2 \end{pmatrix} \texttt{ and } u_3 = \begin{pmatrix} w_3 \\ x_3 \\ y_3 \\ z_3 \end{pmatrix}$$

Checking for the conditions necessary for satisfying a real vector space,

1. Closure under addition property:

$$u_1 + u_2 = \begin{pmatrix} w_1 + w_2 \\ x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{pmatrix}$$

$$(w_1 + w_2) - (x_1 + x_2) - (y_1 + y_2) + (z_1 + z_2)$$

$$= (w_1 - x_1 - y_1 + z_1) + (w_2 - x_2 - y_2 + z_2)$$

Since $u_1,\ u_2 \in \mathbb{V}$, $(w_1 - x_1 - y_1 + z_1) = 0$ `and` $(w_2 - x_2 - y_2 + z_2) = 0$.

$$\implies (w_1 - x_1 - y_1 + z_1) + (w_2 - x_2 - y_2 + z_2) = 0$$

$$\implies u_1 + u_2 \in \mathbb{V}$$

Thus the closure under addition property is satisfied.

2. Commutative property

$$u_1 + u_2 = \begin{pmatrix} w_1 + w_2 \\ x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{pmatrix}$$

$$= \begin{pmatrix} w_2 + w_1 \\ x_2 + x_1 \\ y_2 + y_1 \\ z_2 + z_1 \end{pmatrix} \quad \because \ w, x, y, z \in \mathbb{R} \ \texttt{obeys commutativity}$$

$$= \begin{pmatrix} w_2 \\ x_2 \\ y_2 \\ z_2 \end{pmatrix} + \begin{pmatrix} w_1 \\ x_1 \\ y_1 \\ z_1 \end{pmatrix}$$

$$= u_2 + u_1$$

Hence commutativity is satisfied.

3. Associative property

$$u_1 + (u_2 + u_3) = \begin{pmatrix} w_1 \\ x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} w_2 + w_3 \\ x_2 + x_3 \\ y_2 + y_3 \\ z_2 + z_3 \end{pmatrix}$$

$$= \begin{pmatrix} w_1 + w_2 + w_3 \\ x_1 + x_2 + x_3 \\ y_1 + y_2 + y_3 \\ z_1 + z_2 + z_3 \end{pmatrix}$$

$$= \begin{pmatrix} w_1 + w_2 \\ x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{pmatrix} + \begin{pmatrix} w_3 \\ x_3 \\ y_3 \\ z_3 \end{pmatrix}$$

$$= (u_1 + u_2) + u_3$$

Hence associativity is satisfied.

4. Zero vector

$$u_1 + 0 = \begin{pmatrix} w_1 \\ x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} w_1 + 0 \\ x_1 + 0 \\ y_1 + 0 \\ z + 1 + 0 \end{pmatrix}$$

$$= \begin{pmatrix} w_1 \\ x_1 \\ y_1 \\ z_1 \end{pmatrix}, \because w, x, y, z \in \mathbb{R} \texttt{ obeys additive property}$$

$$= u_1$$

Hence additivity property is satisfied for any $u_1 \in \mathbb{V}$

5. Additive inverse $u_1 \in \mathbb{V}$.
Then $-1 \cdot u_1 \in \mathbb{V}$,
since $w_1 - x_1 - y_1 + z_1 = 0 \implies -1 \cdot w_1 + 1 \cdot x_1 + 1 \cdot y_1 - 1 \cdot z_1 = 0$
and $-w_1, -x_1, -y_1, -z_1 \in \mathbb{R}$.
$\implies \forall u_1 \in \mathbb{V}, \exists - u_1 \in \mathbb{V}$ such that $u_1 + (-u_1) = 0$, where $-u_1$ is the additive inverse.

6. Closure under scalar Multiplication Let $c \in \mathbb{R}$.
Then,

$$c \cdot u_1 = c \cdot \begin{pmatrix} w_1 \\ x_1 \\ y_1 \\ z_1 \end{pmatrix}$$

$$= \begin{pmatrix} c \cdot w_1 \\ c \cdot x_1 \\ c \cdot y_1 \\ c \cdot z_1 \end{pmatrix}$$

$\because u_1 \in \mathbb{V}$, $w_1 - x_1 - y_1 + z_1 = 0$.
Multiplying both LHS and RHS by scalar constant c,
$c \cdot (w_1 - x_1 - y_1 + z_1) = 0$.
$\implies c \cdot w_1 - c \cdot x_1 - c \cdot y_1 + c \cdot z_1 = 0$
$\implies c \cdot u_1 \in \mathbb{V}$.

7. Distributive property for scalar addition
   Let $c, d \in \mathbb{R}$.
   Then,

$$
\begin{aligned}
(c+d) \cdot u_1 &= (c+d) \cdot \begin{pmatrix} w_1 \\ x_1 \\ y_1 \\ z_1 \end{pmatrix} \\
&= \begin{pmatrix} (c+d) \cdot w_1 \\ (c+d) \cdot x_1 \\ (c+d) \cdot y_1 \\ (c+d) \cdot z_1 \end{pmatrix} \\
&= \begin{pmatrix} c \cdot w_1 + d \cdot w_1 \\ c \cdot x_1 + d \cdot x_1 \\ c \cdot y_1 + d \cdot y_1 \\ c \cdot z_1 + d \cdot z_1 \end{pmatrix} \\
&= \begin{pmatrix} c \cdot w_1 \\ c \cdot x_1 \\ c \cdot y_1 \\ c \cdot z_1 \end{pmatrix} + \begin{pmatrix} d \cdot w_1 \\ d \cdot x_1 \\ d \cdot y_1 \\ d \cdot z_1 \end{pmatrix} \\
&= c \cdot \begin{pmatrix} w_1 \\ x_1 \\ y_1 \\ z_1 \end{pmatrix} + d \cdot \begin{pmatrix} w_1 \\ x_1 \\ y_1 \\ z_1 \end{pmatrix} \\
&= c \cdot u_1 + d \cdot u_1
\end{aligned}
$$

8. Distribute property for vector addition Let $c \in \mathbb{R}$.
   Then,

$$c(u_1 + u_2) = c \cdot \left(\begin{pmatrix} w_1 \\ x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} w_2 \\ x_2 \\ y_2 \\ z_2 \end{pmatrix} \right)$$

$$= c \cdot \begin{pmatrix} w_1 + w_2 \\ x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{pmatrix}$$

$$= \begin{pmatrix} c \cdot (w_1 + w_2) \\ c \cdot (x_1 + x_2) \\ c \cdot (y_1 + y_2) \\ c \cdot (z_1 + z_2) \end{pmatrix}$$

$$= \begin{pmatrix} c \cdot w_1 + c \cdot w_2 \\ c \cdot x_1 + c \cdot x_2 \\ c \cdot y_1 + c \cdot y_2 \\ c \cdot z_1 + c \cdot z_2 \end{pmatrix}$$

$$= \begin{pmatrix} c \cdot w_1 \\ c \cdot x_1 \\ c \cdot y_1 \\ c \cdot z_1 \end{pmatrix} + \begin{pmatrix} c \cdot w_2 \\ c \cdot x_2 \\ c \cdot y_2 \\ c \cdot z_2 \end{pmatrix}$$

$$= c \cdot \begin{pmatrix} w_1 \\ x_1 \\ y_1 \\ z_1 \end{pmatrix} + c \cdot \begin{pmatrix} w_2 \\ x_2 \\ y_2 \\ z_2 \end{pmatrix}$$

$$= c \cdot u_1 + c \cdot u_2$$

9. Identity operation

$$
\begin{aligned}
1 \cdot u_1 &= 1 \cdot \begin{pmatrix} w_1 \\ x_1 \\ y_1 \\ z_1 \end{pmatrix} \\
&= \begin{pmatrix} 1 \cdot w_1 \\ 1 \cdot x_1 \\ 1 \cdot y_1 \\ 1 \cdot z_1 \end{pmatrix} \\
&= \begin{pmatrix} w_1 \\ x_1 \\ y_1 \\ z_1 \end{pmatrix} \\
&= u_1
\end{aligned}
$$

Hence, $\mathbb{V}$ forms a vector space as it satisfies all the conditions required for a vector space.

(1.b) Given:

$$\mathbb{M}^{2\times 2} = \left\{ \begin{pmatrix} a & 1 \\ b & c \end{pmatrix} \;\middle|\; a, b, c \in \mathbb{R} \right\} \tag{2}$$

Let $u_1 \in \mathbb{M}^{2\times 2}$, such that

$$u_1 = \begin{pmatrix} a_1 & 1 \\ b_1 & c_1 \end{pmatrix}$$

Checking for the conditions for vector space,

1. Additive inverse

   $$u_1 = \begin{pmatrix} a_1 & 1 \\ b_1 & c_1 \end{pmatrix}$$

   In order to have an additive inverse, say $v \in \mathbb{M}^{2\times 2}$, the condition $u_1 + v = 0$ must be satisfied.
   Then $v = -u_1$.

   $$-u_1 = \begin{pmatrix} -a_1 & -1 \\ -b_1 & -c_1 \end{pmatrix}$$

   But $-u_1 \notin \mathbb{M}^{2\times 2}$ as it is not of the form:

   $$\begin{pmatrix} a & 1 \\ b & c \end{pmatrix}$$

Hence, $\mathbb{M}^{2\times 2}$ does not form a vector space.

(1.c) Given:

$$\mathbb{N} = \left\{ f : \mathbb{R} \to \mathbb{R} \;\middle|\; \frac{df}{dx} + 2f = 1 \right\}$$

Checking for closure under scalar multiplication:

Let $c \in \mathbb{R}$ and $f \in \mathbb{N}$. Then,

$\frac{df}{dx} + 2f = 1$

So for closure under scalar multiplication, $cf \in \mathbb{N}$.

Then,
$d(cf)/dx + 2(cf) = 1$
$\implies cd(f)/dx + 2c(f) = 1$
$\implies c(df/dx + 2f) = 1$

But we already know that $df/dx + 2f = 1$
$\implies c(1) = 1$
or $c = 1$

So the given set of functions will not satisfy closure under scalar multiplication for any real scalar values other than 1. Hence $\mathbb{N}$ doesn't form a vector space.

(2.a) Given:

$$\mathbb{V} = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \;\middle|\; x, y, z \geq 0 \right\}$$

No, $\mathbb{V}$ doesn't form a subspace of $\mathbb{R}^3$.
Reason:
Take the following counter example. Let,

$$v = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$\because 1, 2, 3 \in \mathbb{R}^3$ and $1, 2, 3 \geq 0$, $v \in \mathbb{V}$.
But the 2nd condition states that :
$a \in \mathbb{W}, \alpha \in \mathbb{F} \implies \alpha \cdot a \in \mathbb{W}$.
Here $\mathbb{F} = \mathbb{R}$. Take $\alpha = -1$ as $-1 \in \mathbb{R}$.
Then,

$$\alpha \cdot a = b = \begin{pmatrix} -1 \\ -2 \\ -3 \end{pmatrix}$$

But $b \notin \mathbb{V}$ which violates the 2nd condition. Hence $\mathbb{V}$ cannot form a subspace of $\mathbb{R}^3$.

(2.b) Given:

$$\mathbb{V} = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \;\middle|\; x, y, z \in \mathbb{R},\; x^2 = z^2 \right\}$$

No, $\mathbb{V}$ cannot form a subspace of $\mathbb{R}^3$.
Reason:
It violates 1st condition. Take a counter example.
Let $u_1, u_2 \in \mathbb{V}$ be the following:

$$u_1 = \begin{pmatrix} 3 \\ 1 \\ -3 \end{pmatrix},\; u_2 = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}$$

Then 1st condition states that:
$a \in \mathbb{W}, b \in \mathbb{W} \implies a + b \in \mathbb{W}$.

$$u_1 + u_2 = \begin{pmatrix} 5 \\ 4 \\ -1 \end{pmatrix}$$

Here, $x^2 = 5^2 = 25$ and $z^2 = (-1)^2 = 1$. Clearly $x^2 \neq z^2$.
Hence $\mathbb{V}$ cannot form a subspace of $\mathbb{R}^3$.
(2.c) Given:

$$\mathbb{V} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, \texttt{det} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 0 \right\}$$

No, $\mathbb{V}$ cannot form a subspace of $\mathbb{R}^{2\times 2}$.
Reason:
Take the following counter example.
Let $A, B \in \mathbb{R}^{2\times 2}$ be given by:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \; B = \begin{pmatrix} 9 & 3 \\ 3 & 1 \end{pmatrix}$$

Both A and B satisfies $\texttt{det(A)} = 0$ and $\texttt{det(B)} = 0$ respectively.
But 1st condition says $A + B \in \mathbb{V}$ if $\mathbb{V}$ is a subspace of $\mathbb{R}^{2\times 2}$.

$$A + B = \begin{pmatrix} 10 & 3 \\ 3 & 1 \end{pmatrix}$$

$\texttt{det}(A + B) = (10 \cdot 1 - 3 \cdot 3) = 1 \neq 0$. Hence 1st condition is false for this example. $\because$ there is atleast a case where one of the subspace condition fails,
$\mathbb{V}$ cannot be a subspace of $\mathbb{R}^{2\times 2}$.

(2.d) To prove: Intersection of two subspaces of a vector space $\mathbb{V}$ over a field $\mathbb{F}$ is a subspace of $\mathbb{V}$.
In order to have $U_1 \cap U_2$ (where U_1, U_2 are subspaces of $\mathbb{V}$) to be a subspace of $\mathbb{V}$ over a field $\mathbb{F}$, the following conditions must be met:

- $\mathbf{0}$, the zero vector is in $U_1 \cap U_2$.
- $\forall u, v \in U_1 \cap U_2, u + v \in U_1 \cap U_2$.
- $\forall u \in U_1 \cap U_2$ and $\alpha \in \mathbb{F}$, $\alpha \cdot u \in U_1 \cap U_2$.

Checking for these conditions,

- Since $U_1, U_2 \in \mathbb{V}$ and $\mathbf{0} \in U_1$ and $\mathbf{0} \in U_2$, by default $\mathbf{0} \in U_1 \cap U_2$.
- Let $u, v \in U_1 \cap U_2$.
  This means $u \in U_1$ and $u \in U_2$. Similarly $v \in U_1$ and $v \in U_2$.
  Since it is given that U_1 is a subspace and $u, v \in U_1$, this implies $u + v \in U_1$.
  Similarly for U_2, $u + v \in U_2$.
  $\therefore u + v \in U_1 \cap U_2$.

- Let $u \in U_1 \cap U_2$ and $\alpha \in \mathbb{F}$.
  As $u \in U_1 \cap U_2$, the vector u lies in U_1 as well as U_2.
  Scalar multiplication is closed in U_1 and U_2 as both U_1 and U_2 are subspaces of a common vector space.
  Thus $\alpha \cdot u \in U_1$ and $\alpha \cdot u \in \mathbb{U_2}$.
  $\therefore\ \alpha \cdot u \in U_1 \cap U_2$.

  Since all the three conditions are satisfied, it is proved that the intersection of two subspaces of a vector space $\mathbb{V}$ over a field $\mathbb{F}$ is a subspace of $\mathbb{V}$.

(3.a) Let:

$$a = \begin{pmatrix} 5 \\ 3 \\ 7 \end{pmatrix},\ b = \begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix},\ c = \begin{pmatrix} 0 \\ -26 \\ -9 \end{pmatrix},\ d = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}$$

a and b basis vectors will span a $\mathbb{R}^2$ space in $\mathbb{R}^3$, if they are not parallel. If c and d lies on this specific $\mathbb{R}^2$ plane, then it implies that linear combinations of a and b can be used to describe c and d.
Checking if a and b are parallel:

$$\begin{aligned} cos(\theta) &= \frac{a \cdot b}{|a||b|} \\ &= \frac{5 \cdot 2 + 3 \cdot -4 + 7 \cdot 1}{\sqrt{25+9+49}\sqrt{4+16+1}} \\ &= \frac{5}{\sqrt{83}\sqrt{21}} \\ &\implies \theta \neq 0 \end{aligned}$$

$\therefore$ the vectors aren't parallel and hence spans an $\mathbb{R}^2$ space.
Inorder to check if the vectors c and d lies on the plane, first determine the normal to the plane and then find the angle between the normal and each of c and d respectively. If the angles turns out to be 90 (or 270) deg, then the normal vector is perpendicular to c and d and hence both the vectors lie on the plane.
Normal to the $\mathbb{R}^2$ plane:

$$\begin{aligned} n &= \frac{a \times b}{|a| \cdot |b|} \\ &= \begin{pmatrix} 0.7425 \\ 0.2156 \\ -0.6228 \end{pmatrix} \end{aligned}$$

Checking if c lies in this $\mathbb{R}^2$ plane:

$$\begin{aligned} \theta_1 &= cos^{-1}\left(\frac{n \cdot c}{|n||c|}\right) \\ &= cos^{-1}(0) \\ &= 90\ deg \end{aligned}$$

Since the angle between the normal to the plane and c is 90 deg, c lies in the plane spanned by a and b. Hence c can be expressed as a linear combination of a and b.

Checking if d lies in this $\mathbb{R}^2$ plane:

$$
\begin{aligned}
\theta_2 &= cos^{-1}\left(\frac{n \cdot d}{|n||d|}\right) \\
&= cos^{-1}(-0.2936) \\
&= 107.07\ deg
\end{aligned}
$$

Since the angle between the normal to the plane and d is 107.07 deg, d doesn't lie in the plane spanned by a and b. Hence d cannot be expressed as a linear combination of a and b.

(3.b) Given,

$$U = \left\{ \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} \in \mathbb{R}^4 \;\middle|\; 3w + x - 7z = 0 \right\}$$

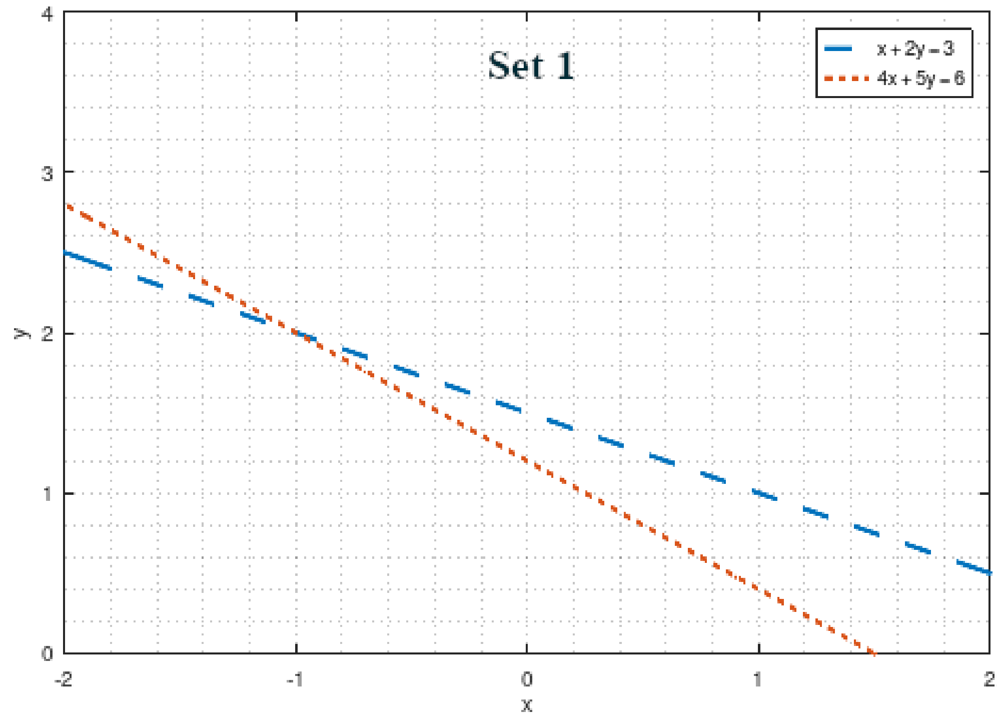
$3w + x - 7z = 0$
$\implies x = 7z - 3w$

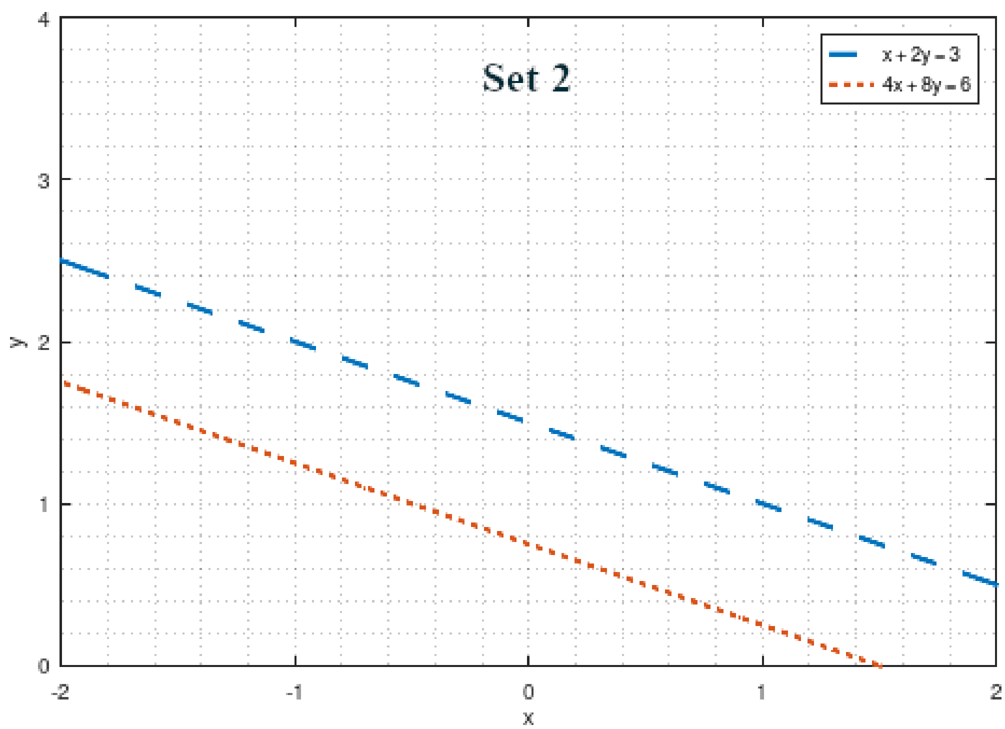
$$
\begin{aligned}
\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} &= \begin{pmatrix} w \\ 7z - 3w \\ y \\ z \end{pmatrix} \\
&= \begin{pmatrix} w \\ -3w \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 7z \\ 0 \\ z \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ y \\ 0 \end{pmatrix} \\
&= w \cdot \begin{pmatrix} 1 \\ -3 \\ 0 \\ 0 \end{pmatrix} + z \cdot \begin{pmatrix} 0 \\ 7 \\ 0 \\ 1 \end{pmatrix} + y \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}
\end{aligned}
$$

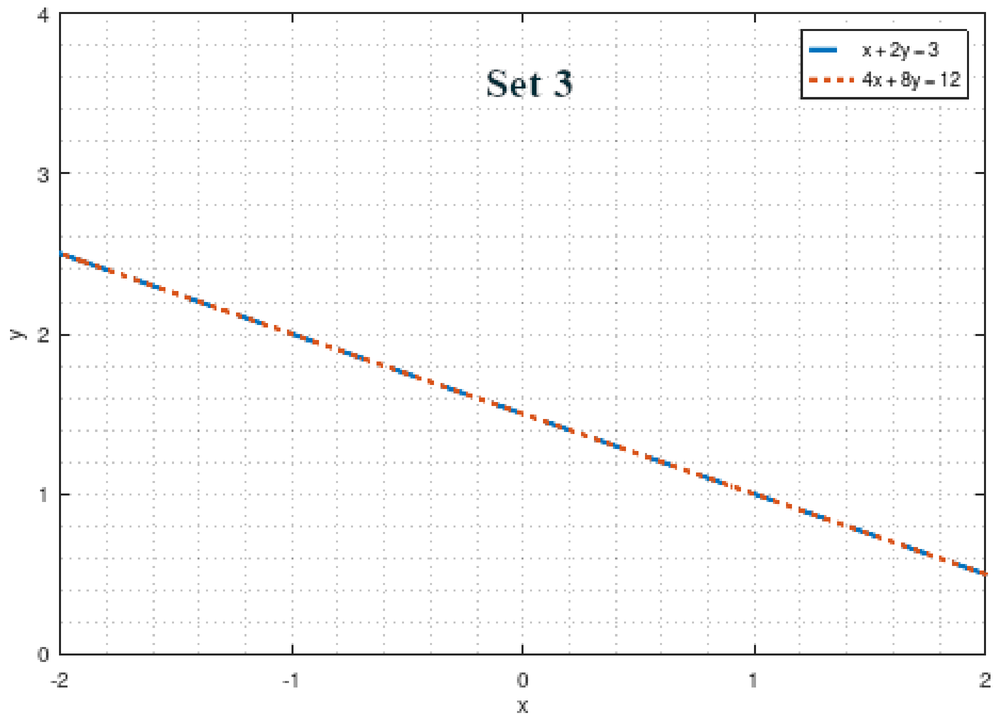
Hence a basis for the given subspace can be the following set:

$$\left\{ \begin{pmatrix} 1 \\ -3 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 7 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\} \tag{3}$$

(4.a)







Given system of linear equations for each set can be represented as straight lines on a 2D plane. Their intersection points corresponds to the possible solutions.
Set 1: Unique solution at $(x, y) = (-1, 2)$
Set 2: no solution (since both lines are parallel and not coincident)
Set 3: infinite solutions (since both lines are parallel and coincident)

(4.b) Each set of linear system of equations can be algebraically solved via elimination method or by using their matrix representation.

Set 1 (i) Elimination method:

$$x + 2y = 3$$

$$\implies x = 3 - 2y$$

$$4x + 5y = 6$$

$$\texttt{So,}\ 4(3 - 2y) + 5y = 6$$

$$12 - 8y + 5y = 6$$

$$3y = 6$$

$$y = 2$$

$$\implies x + 2 \cdot 2 = 3$$

$$x = -1$$

(ii) Matrix method:
The system of equations can be written in matrix representation form as:

$$\begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$$

which is of the form $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$.

$$\texttt{det}(\mathbf{A}) = (5 \cdot 1 - 2 \cdot 4)$$

$$= -3$$

$\det(\mathbf{A})$ is non zero. Hence the square matrix $\mathbf{A}$ is non-singular and $\mathbf{A}^{-1}$ exists.
Then $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$.

$$\mathbf{A}^{-1} = \frac{1}{-3} \times \begin{pmatrix} 5 & -2 \\ -4 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -5/3 & 2/3 \\ 4/3 & -1/3 \end{pmatrix}$$

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

$$= \begin{pmatrix} -5/3 & 2/3 \\ 4/3 & -1/3 \end{pmatrix} \times \begin{pmatrix} 3 \\ 6 \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

$\therefore$ $x = -1$ and $y = 2$ is the solution.

Set 2 (i) Elimination method:

$$x + 2y = 3$$

$$\implies x = 3 - 2y$$

$$4x + 8y = 6$$

$$\texttt{So,}\ 4(3 - 2y) + 8y = 6$$

$$12 - 8y + 8y = 6$$

$$12 = 6$$

which is not true. Hence the system of equations has no solution.

(ii) Matrix method:

The system of equations can be written in matrix representation form as:

$$\begin{pmatrix} 1 & 2 \\ 4 & 8 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$$

which is of the form $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$.

$$\texttt{det}(\mathbf{A}) = (8 \cdot 1 - 2 \cdot 4)$$

$$= 0$$

det($\mathbf{A}$) is zero. Hence the square matrix $\mathbf{A}$ is singular and $\mathbf{A}^{-1}$ does not exist.
Simplifying the system of equations,

$$x + 2y = 3$$

$$x + 2y = 3/2$$

Subtracting first equation from the second equation, we get $0 = -3/2$. Since this is not possible for any combination of x and y, this system of equations has no solutions.
Set 3 (i) Elimination method:

$$x + 2y = 3$$

$$\implies x = 3 - 2y$$

$$4x + 8y = 12$$

$$\texttt{So,}\ 4(3 - 2y) + 8y = 12$$

$$12 - 8y + 8y = 12$$

$$12 = 12$$

which is true for infinitely many values of x and y which satisfies any one of the equations. Hence the system of equations has infinitely many solutions.

(ii) Matrix method:
The system of equations can be written in matrix representation form as:

$$\begin{pmatrix} 1 & 2 \\ 4 & 8 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 12 \end{pmatrix}$$

which is of the form $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$.

$$\texttt{det}(\mathbf{A}) = (8 \cdot 1 - 2 \cdot 4)$$

$$= 0$$

det($\mathbf{A}$) is zero. Hence the square matrix $\mathbf{A}$ is singular and $\mathbf{A}^{-1}$ does not exist.
Simplifying the system of equations,

$$x + 2y = 3$$

$$x + 2y = 3$$

Since both the equations in the system simplifies to the same equation, both lines have infinitely many coincident points and thus the system has infinitely many solutions.

(4.c) For each set of linear system of equations, of the form:

$$a_1x + b_1y = c_1$$

$$a_2x + b_2y = c_2$$

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = x \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + y \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

where,

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \text{ , } \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \texttt{ and } \mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

This can be thought of as checking whether the vector space spanned by the basis vectors **a** and **b** contains the vector **c**. If it does contain **c**, then that implies that atleast a solution exists for x and y.
For the given sets, a and b may either span an $\mathbb{R}$ line or an $\mathbb{R}^2$ plane depending upon whether a and b are parallel or not.

Set 1

$$\begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$$

Here,

$$a = \begin{pmatrix} 1 \\ 4 \end{pmatrix}, \ b = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

Checking if a and b are parallel:

$$\theta = cos^{-1}\left(\frac{a \cdot b}{|a||b|}\right)$$

$$= cos^{-1}\left(\frac{2 + 20}{\sqrt{17} \cdot \sqrt{29}}\right)$$

$$\neq 0$$

Since they are non-parallel, the vectors spans the entire $\mathbb{R}^2$ plane. So any vector $r \in \mathbb{R}^2$ like $c = (3\ 6)^T$ also lies on the R^2 plane and a unique solution exists.

Set 2

$$\begin{pmatrix} 1 & 2 \\ 4 & 8 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$$

Here,

$$a = \begin{pmatrix} 1 \\ 4 \end{pmatrix},\ b = \begin{pmatrix} 2 \\ 8 \end{pmatrix}$$

But $b = 2 \cdot a$, which means b is a scaled version of the vector a. So a and b are parallel. $c = (3\ 6)^T$ doesn't lie on the $\mathbb{R}$ line. Hence a solution $(x\ y)^T$ cannot be obtained in this case. Or in other words, no solutions exist.

Set 3

$$\begin{pmatrix} 1 & 2 \\ 4 & 8 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 12 \end{pmatrix}$$

This is similar to **Set 2**, but the difference is that $c = (3\ 12)^T = 3 \cdot (1\ 4)^T$ lies on the specific $\mathbb{R}$ line. Which means that an infinte no of combinations of x, y in $(x\ y)^T$ is possible to arrive at c. Hence the given set of linear equations has infinite number of solutions.

5.

$$\mathbf{a} = \begin{pmatrix} -209/362 \\ -209/362 \\ 209/362 \end{pmatrix},\ \mathbf{b} = \begin{pmatrix} 0 \\ -408/577 \\ -408/577 \end{pmatrix},\ \mathbf{c} = \begin{pmatrix} 396/485 \\ -198/485 \\ 198/485 \end{pmatrix}$$

Rewriting,

$$\mathbf{a} = 209/362 \times \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix},\ \mathbf{b} = 408/577 \times \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix},\ \mathbf{c} = 198/485 \times \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

(5.a) Dot product:

$$\mathbf{a.b} = \sum_{i=1}^{n} a_i b_i,\ \mathbf{a,b} \in \mathbb{R}^n$$

$$= (209/362) \times (408/577) \times ((-1 \cdot 0) + (-1 \cdot -1) + (1 \cdot -1))$$

$$= 0$$

$$\mathbf{b.c} = \sum_{i=1}^{n} b_i c_i,\ \mathbf{b,c} \in \mathbb{R}^n$$

$$= (408/577) \times (198/485) \times ((0 \cdot 2) + (-1 \cdot -1) + (-1 \cdot 1))$$

$$= 0$$

$$\mathbf{c.a} = \sum_{i=1}^{n} c_i a_i,\ \mathbf{c,a} \in \mathbb{R}^n$$

$$= (198/485) \times (209/362) \times ((2 \cdot -1) + (-1 \cdot -1) + (1 \cdot 1))$$

$$= 0$$

(5.b) Geometric length:

$$\sqrt{\mathbf{a.a}} = \sqrt{\sum_{i=1}^{n} a_i a_i}, \;\; \mathbf{a} \in \mathbb{R}^n$$

$$= \sqrt{(209/362) \times (209/362) \times ((-1 \cdot -1) + (-1 \cdot -1) + (1 \cdot 1))}$$

$$= 1$$

$$\sqrt{\mathbf{b.b}} = \sqrt{\sum_{i=1}^{n} b_i b_i}, \;\; \mathbf{b} \in \mathbb{R}^n$$

$$= \sqrt{(408/577) \times (408/577) \times ((0 \cdot 0) + (-1 \cdot -1) + (-1 \cdot -1))}$$

$$= 1$$

$$\sqrt{\mathbf{c.c}} = \sqrt{\sum_{i=1}^{n} c_i c_i}, \;\; \mathbf{c} \in \mathbb{R}^n$$

$$= \sqrt{(198/485) \times (198/485) \times ((2 \cdot 2) + (-1 \cdot -1) + (1 \cdot 1))}$$

$$= 1$$

$$\mathbf{x} = \begin{pmatrix} 2 \\ -40/57 \\ 8/77 \end{pmatrix}$$

$$\sqrt{\mathbf{x.x}} = \sqrt{\sum_{i=1}^{n} x_i x_i}, \;\; \mathbf{x} \in \mathbb{R}^n$$

$$= \sqrt{(2 \cdot 2) + (-40/57 \cdot -40/57) + (8/77 \cdot 8/77)}$$

$$\approx \sqrt{4 + 1600/3249 + 64/5929}$$

$$\approx 2.12$$

(5.c)

$$\mathbf{A} = \begin{pmatrix} -209/362 & 0 & 396/485 \\ -209/362 & -408/577 & -198/485 \\ 209/362 & -408/577 & 198/485 \end{pmatrix}$$

$$\mathbf{Ax} = \begin{pmatrix} -209/362 & 0 & 396/485 \\ -209/362 & -408/577 & -198/485 \\ 209/362 & -408/577 & 198/485 \end{pmatrix} \times \begin{pmatrix} 2 \\ -40/57 \\ 8/77 \end{pmatrix}$$

$$= \begin{pmatrix} -1.070 \\ -0.701 \\ 1.693 \end{pmatrix}$$

$$\sqrt{\mathbf{Ax} \cdot \mathbf{Ax}} = \sqrt{(-1.070 \cdot -1.070) + (-0.701 \cdot -0.701) + (1.693 \cdot 1.693)}$$

$$\approx \sqrt{1.145 + 0.491 + 2.866}$$

$$\approx 2.12$$

It is observed that both $\sqrt{\mathbf{Ax.Ax}}$ and $\sqrt{\mathbf{x.x}}$ are equal.

(5.d)

$$\mathbf{A}^T\mathbf{A} = \begin{pmatrix} -209/362 & -209/362 & 209/362 \\ 0 & -408/577 & -408/577 \\ 396/485 & -198/485 & 198/485 \end{pmatrix} \times \begin{pmatrix} -209/362 & 0 & 396/485 \\ -209/362 & -408/577 & -198/485 \\ 209/362 & -408/577 & 198/485 \end{pmatrix}$$

$$\approx \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{A}\mathbf{A}^T = \begin{pmatrix} -209/362 & 0 & 396/485 \\ -209/362 & -408/577 & -198/485 \\ 209/362 & -408/577 & 198/485 \end{pmatrix} \times \begin{pmatrix} -209/362 & -209/362 & 209/362 \\ 0 & -408/577 & -408/577 \\ 396/485 & -198/485 & 198/485 \end{pmatrix}$$

$$\approx \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(5.e) To prove: If $\mathbf{A}^T\mathbf{A} = \mathbf{A}\mathbf{A}^T = \mathbf{I}$, then $\sqrt{\mathbf{Ax} \cdot \mathbf{Ax}} = \sqrt{\mathbf{x} \cdot \mathbf{x}}$.

$$\begin{aligned} \mathbf{Ax} \cdot \mathbf{Ax} &= (\mathbf{Ax})^T (\mathbf{Ax}) \\ &= \left(\mathbf{x}^T\mathbf{A}^T\right)(\mathbf{Ax}) \\ &= \mathbf{x}^T \left(\mathbf{A}^T\mathbf{A}\right)\mathbf{x} \\ &= \mathbf{x}^T\mathbf{I}\mathbf{x} \\ &= \mathbf{x}^T\mathbf{x} \\ &= \mathbf{x} \cdot \mathbf{x} \end{aligned}$$

which means

$$\sqrt{\mathbf{Ax} \cdot \mathbf{Ax}} = \sqrt{\mathbf{x} \cdot \mathbf{x}}$$

Vivek T
17397
DS284 Assignment-0
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Why do you want to take this course and what are your expectations?

This course is relevant to my field of research. I expect to get a better grasp of how numerical linear algebra translates to practical applications.