To prove:

(a) $\frac{1}{2}\nabla(u\cdot u) = (\nabla u)^T u$

$$\left(\frac{1}{2}\nabla(u\cdot u)\right)_j = \frac{1}{2}\frac{\partial}{\partial x_j}(u_i u_i)$$

$$= \frac{1}{2}\left[u_i\frac{\partial u_i}{\partial x_j} + u_i\frac{\partial u_i}{\partial x_j}\right]$$

$$= u_i\frac{\partial u_i}{\partial x_j}$$

$$= (\nabla u)^T_{ji} u_i$$

$$= \left((\nabla u^T) u\right)_j$$

(b) To prove:

$$\nabla(\phi T v) = \phi\nabla(Tv) + (Tv)\otimes\nabla\phi.$$

$$\left(\nabla(\phi Tv)\right)_{ij} = \frac{\partial}{\partial x_j}(\phi T_{ik} V_k)$$

$$= \phi\frac{\partial}{\partial x_j}(T_{ik}V_k) + (T_{ik}V_k)\frac{\partial\phi}{\partial x_j}$$

$$= (\phi\nabla(Tv))_{ij} + (Tv)_i(\nabla\phi)_j$$

$$= \left(\phi\nabla(Tv) + Tv\otimes\nabla\phi\right)_{ij}$$

(c) $\nabla(\phi T) = \phi \nabla T + T \otimes \nabla \phi$

$$(\nabla(\phi T))_{ijk} = \frac{\partial}{\partial x_j}(\phi T_{ik})$$

$$= \phi \frac{\partial T_{ik}}{\partial x_j} + T_{ik} \frac{\partial \phi}{\partial x_j}$$

$$= (\phi \nabla T)_{ijk} + T_{ik} \cdot (\nabla \phi)_j$$

$$= (\phi \nabla T + T \otimes \nabla \phi)_{ijk}$$

(d) $\nabla \cdot (\nabla \times U) = 0$

$$(\nabla \cdot (\nabla \times U))_{ijk} = \frac{\partial}{\partial x_j} \cdot \left(\frac{\partial}{\partial x_k} \times U_i\right)$$

$$= \frac{\partial}{\partial x_j} \cdot \left(\varepsilon_{kij} \frac{\partial U_i}{\partial x_k}\right)$$

$$= \left(\frac{\partial^2 U_i}{\partial x_j \partial x_k}\right) \cdot \varepsilon_{kij}$$

$$= \left(\frac{\partial^2 U_i}{\partial x_k \partial x_j}\right) \varepsilon_{jik}$$

$$= -\left(\frac{\partial^2 U_i}{\partial x_k \partial x_j}\right) \varepsilon_{kij}$$

Since $\frac{\partial^2 U_i}{\partial x_j \partial x_k} \varepsilon_{kij} = -\frac{\partial^2 U_i}{\partial x_j \partial x_k} \varepsilon_{kij}$

$$\Rightarrow \frac{\partial^2 U_i}{\partial x_j \partial x_k} \varepsilon_{kij} = 0.$$

$$\Rightarrow (\nabla \cdot (\nabla \times U))_{ijk} = 0$$

(c) $U \times (V \times \omega)$

Let $Z = V \times \omega$

Then $(U \times (V \times \omega))_k$

$= (U \times Z)_k$

$= \varepsilon_{ijk} U_i Z_j \quad \ldots ①$

$Z_j = (V \times \omega)_j = \varepsilon_{lmj} V_l \omega_m \quad \ldots ②$

Sub ② in ①

$\Rightarrow (U \times (V \times \omega))_k = \varepsilon_{ijk} U_i \varepsilon_{lmj} V_l \omega_m$

$= (\varepsilon_{ijk} \varepsilon_{lmj}) U_i V_l \omega_m$

$\because \varepsilon_{ijk} = \varepsilon_{jki} = \varepsilon_{kij}$ and

$\varepsilon_{lmj} = \varepsilon_{mjl} = \varepsilon_{jml}$

$\varepsilon_{ijk} \varepsilon_{lmj} = \varepsilon_{kij} \varepsilon_{jml}$

Using the identity $\varepsilon_{ijk} \varepsilon_{klm}$

$= -\delta_{il}\delta_{jm} + \delta_{im}\delta_{jl}$,

we have

$\varepsilon_{kij} \varepsilon_{jml} = -\delta_{km}\delta_{il} + \delta_{kl}\delta_{im}$

$\rightarrow (U \times (V \times \omega))_k = (-\delta_{km}\delta_{il} + \delta_{kl}\delta_{im}) U_i V_l \omega_m$

Using the δ_{ij} as substitution operator,:

$(U \times (V \times \omega))_k = -U_i V_i \omega_k + U_i V_k \omega_i$

$(-(U \cdot V)\omega + (\omega \cdot U) \cdot V)_k$

$\Rightarrow U \times (V \times \omega) = (\omega \cdot U) V - (V \cdot U) \omega$

2) Given ~~square~~ skew-symmetric matrix $a \otimes b - b \otimes a$.

Let $c = a \otimes b - b \otimes a$.

Let the axial vector required be d.

Then for an arbitrary u,

$cu = d \times u$.

$(a \otimes b - b \otimes a)u = d \times u$

$(a \otimes b - b \otimes a)u$

$= (a \otimes b)u - (b \otimes a)u$.

$= (b \cdot u)a - (a \cdot u)b$.

But this is same as $u \times (a \times b)$. (Since, from the identity, $x \times (y \times z) = (y \cdot x)z + (z \cdot x)y$)

or here

$u \times (a \times b)$

$= (b \cdot u)a - (a \cdot u)b$.

$\therefore$ $u \times (a \times b) = d \times u$.

$p \times q = -q \times p$, p, q vectors.

$\Rightarrow -(a \times b) \times u = d \times u$.

or $\underline{d = -(a \times b)}$

or $\underline{\underline{d = b \times a}}$

3). ~~I prove~~ Given: $W_1, W_2 \rightarrow$ skew symmetric tensors.

$w_1, w_2 \rightarrow$ axial vectors.

To prove, $W_1 W_2 = w_2 \otimes w_1 - (w_1 \cdot w_2) I$

Take an arbitrary vector v.

Then $W_1 W_2 v$

$= W_1 (W_2 v)$

$= W_1 (w_2 \times v)$

If $w_2 \times v = u$,

then $W_1 (w_2 \times v)$

$= W_1 u$

$= w_1 \times u$

$= w_1 \times (w_2 \times v)$

$\Rightarrow$

Using the triple cross product identity,

$w_1 \times (w_2 \times v) = (v \cdot w_1) w_2 - (w_2 \cdot w_1) v$.

But $(v \cdot w_1) w_2$ is same as $(w_2 \otimes w_1) v$.

$\Rightarrow$.

$\Rightarrow W_1 W_2 v = (w_2 \otimes w_1) v - (w_2 \cdot w_1) v$

or $(W_1 W_2) v = (w_2 \otimes w_1 - w_2 \cdot w_1) v$

$\Rightarrow W_1 W_2 = w_2 \otimes w_1 - (w_2 \cdot w_1) I$.

4) To prove: $\log(\det(T)) = Tr(\log(T))$

Diagonalizing T gives

$$T = X \Lambda X^{-1}$$, Λ being the diagonal matrix.

Then $\det(T) = \det(X) \det(\Lambda) \cdot \det(X^{-1})$

$= \det(\Lambda) = \lambda_1 \lambda_2 \lambda_3$.

$\because \Lambda$ has eigenvalues $\lambda_1, \lambda_2, \lambda_3$ as its elements.

Also $Tr(\log(T)) = \log(X \Lambda X^{-1})$

$= \cancel{\log(X) + \log(\Lambda) + \log(X^{-1})}$

$= \cancel{\log(\Lambda)}$

$$= \sum_{i=1}^{3} \log(\lambda_i)$$

$= \log(\lambda_1 \lambda_2 \lambda_3)$, λ_i being the eigenvalues of T.

Thus $\log(\det(T)) = \log(\lambda_1 \lambda_2 \lambda_3)$

and $Tr(\log(T)) = \log(\lambda_1 \lambda_2 \lambda_3)$

$\Rightarrow \log(\det(T)) = Tr(\log(T))$

5) To show : $\int_V \nabla \phi \, dV = \int_S \phi n \, dS$

$$\left(\int_V \nabla \phi \, dV\right)_i = \int_V \frac{\partial(\phi)}{\partial x_i} dV$$

$$= \int_V \frac{\partial \phi}{\partial x_j} \delta_{ij} \, dV$$, using δ_{ij} as a substitution operator.

$$= \int_S \phi \, \delta_{ij} n_j \, dS$$

$$= \int_S \phi \, n_i \, dS.$$

$$\Rightarrow \int_V \nabla \phi \, dV = \int_S \phi \underline{\underline{n \, dS}}.$$

6) $\frac{d}{dx}\left(\log\left(\text{trace}(A(x))\right)\right) = ?$

$$\left[\frac{d}{dx}\left(\log\left(\text{trace}\left(A(x)\right)\right)\right)\right]_j = \frac{d}{dx_j}\left(\log\left(A_{ii}(x)\right)\right)$$

$$\left\{\because \operatorname{tr}(A)_{N\times N} = \sum_{i=1}^{N} A_{ii}\right\}$$

Differentiating,

$$= \frac{1}{A_{ii}(x)} \frac{dA_{ii}(x)}{dx_j}$$

$$\therefore \frac{d}{dx}\left(\log\left(\text{trace}(A(x))\right)\right) = \frac{1}{\text{trace}(A(x))} \cdot \nabla\left(\text{trace}(A(x))\right)$$