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ME 261 : Assignment - 2
(Linear Algebra)

Q.1).

Let V be the set of all $N \times N$ real matrices & $A, B \in V$.

$$V = \{ M \mid M \in \mathbb{R}^{N \times N} \}$$

Let F denote the field, which in here is the set of all real numbers.

$$F = \{ x \mid x \in \mathbb{R} \}.$$

* Given definition of inner product ~~space~~ for A & $B \in V \Rightarrow$

$$(A,B) = \sum_{i=1}^{N} \sum_{j=1}^{N} A_{ij} B_{ij} \text{ — scalar function say } \mathcal{L}.$$

~~(A,B)~~ $\mathcal{L} \in F \quad \forall A, B \in V \quad \because A_{ij}$ & B_{ij} are real.

$\Rightarrow$ scalar function $\mathcal{L}$ satisfies closure property — ①

* $$(\alpha A, B) = \sum_{i=1}^{N} \sum_{j=1}^{N} (\alpha A_{ij}, B_{ij}) = \alpha \sum_{i=1}^{N} \sum_{j=1}^{N} (A_{ij} B_{ij})$$
$$= \alpha (A,B), \quad \text{here } \alpha \in \mathbb{R}.$$

$\Rightarrow \mathcal{L}$ satisfies associativity. — ②

* Let $C \in V$.

Then
$$(A+B, C) = \sum_{i=1}^{N} \sum_{j=1}^{N} (A_{ij} + B_{ij}) C_{ij}$$
$$= \sum_{i=1}^{N} \sum_{j=1}^{N} (A_{ij} C_{ij} + B_{ij} C_{ij})$$
$$= \sum_{i=1}^{N} \sum_{j=1}^{N} A_{ij} C_{ij} + \sum_{i=1}^{N} \sum_{j=1}^{N} B_{ij} C_{ij}$$
$$= (A, C) + (B, C)$$

Similarly $(A, B+C) = \cancel{(A+B)} (A,B) + (A,C)$

$\Rightarrow \mathcal{L}$ satisfies distributivity. — ③

* $(A,B) = \sum_{i=1}^{N} \sum_{j=1}^{N} A_{ij} B_{ij}$

$$= \sum_{i=1}^{N} \sum_{j=1}^{N} B_{ij} A_{ij} = (B,A)$$

$$\Rightarrow (A,B) = (B,A)$$

$\Rightarrow h$ satisfies commutativity. — ④

* $(A,A) = \sum_{i=1}^{N} \sum_{j=1}^{N} (A_{ij})(A_{ij})$

$$= \sum_{i=1}^{N} \sum_{j=1}^{N} (A_{ij})^2 \geq 0 \text{ always.}$$

$(A,A) = 0$ only when $A_{ij} = 0 \ \forall \ 1 \leq i,j \leq N; \ i,j \in \mathbb{N}$.

$\Rightarrow (A,A) = 0 \rightarrow A = 0$.

$\Rightarrow h$ satisfies positivity condition. — ⑤

From ①, ②, ③, ④, ⑤, it can be concluded that set of all $N \times N$ real matrices forms an inner product space under given definition of scalar function h.

Q.2) Let $A, B, C \in L(V; U)$

~~Then the following is true.~~

Let $\cancel{\forall} v, \bar{v}_1, \bar{v}_2 \in V$, $\cancel{\forall} u, \bar{u}_1, \bar{u}_2 \in U$,

such That

~~Let~~ $A v_1 = u_1$

$B v_2 = u_2$

$A v = u$.

* $(A+B) v = Av + Bv$.

$\downarrow$ addition of linear transformations

$\downarrow$ Addition of vectors in V (vector space).

$\Rightarrow$ Closure satisfied. — ①

* $(\lambda A) v = \lambda (Av) = A(\lambda v)$, where $\lambda \in F$.

Scaling of linear transformations

scaling of vector in V.

$\Rightarrow$ Scaling satisfied. — ②

* $(A+0) v = Av + 0v = Av$.

where 0 is a linear transformation.

where $A, 0 \in L(V;U)$ and $v \in V$, $Av \in U$.

$\Rightarrow$ Identity satisfied. — ③

*
$$[(A+B)+C] v = (A+B) v + Cv$$
$$= Av + Bv + Cv.$$
$$= Av + (B+C) v$$
$$= [A + (B+C)] v.$$

$\Rightarrow$ associativity satisfied. — ④

* Additive inverse of a linear transformation A exists in $L(V;U)$ such That: $(A + (-A)) v = Av - Av = 0$ zero vector $\in U$.

$\forall (v \in V)$

$\Rightarrow$ inverse condition satisfied. — ⑤

* Commutativity: $(A+B)v = Av + Bv$

$= Bv + Av$ ($\because$ vector addition is commutative)

$= (B+A)v$

$\Rightarrow$ commutativity satisfied —— ⑥

* For $\lambda, \mu \in F$, field,

$(\lambda+\mu)Av$

$= (\lambda+\mu)v$

$= \lambda v + \mu v$ ~~in vectors~~

$= \lambda Av + \mu Av$ —— ⑦

* $\lambda(Av + Bv)$

$= \lambda(v_A + v_B)$ where $Av = v_A$, $Bv = v_B$, $v_A, v_B \in U$

$= \lambda v_A + \lambda v_B$

$= \lambda Av + \lambda Bv$ —— ⑧

* $1 \cdot Av = Av$ —— ⑨
* $0 \cdot Av = 0$ —— ⑩

Using ① — ⑩, one can conclude that set of all ~~the~~ linear transformations $\mathcal{L}(V; U)$ forms a vector space

Q.3) Given: $A : V \longrightarrow V$, $V = \mathbb{R}^3$.

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 4 & 6 & 3 \\ 1 & 0 & 0 \end{bmatrix}$$

- $K(A) = ?$

$K(A) = \{ v \mid Av = 0,\ v \in V \}$

Take $v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $x, y, z \in \mathbb{R}$, $v \in \cancel{\mathbb{R}}V$.

Then $Av = 0$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 4 & 6 & 3 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Obtaining the row reduced form:-

$R_2 \longrightarrow R_2 - 3R_1$: $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$R_3 \longrightarrow R_3 - R_2$: $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

We get the linear system of equations:

$\therefore\ x + 2y + z = 0$

$x = 0.$

$\Rightarrow 2y + z = 0$ or $z = -2y$.

$$\therefore \cancel{\begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x + 2y + z \\ x \\ 0 \end{bmatrix}}$$

$$= \cancel{x} \cancel{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}} + y \cancel{\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}} + z \cancel{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}$$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ y \\ -2y \end{bmatrix} = y \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$$, here $y \in \mathbb{R}$ can be any value in $\mathbb{R}$.

$$\therefore \boxed{K(A) = \operatorname{Span}\left\{ \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} \right\}}$$

Also, $\dim K(A) = 1$.

- $R(A) = ?$

$$R(A) = \{ u \mid Av = u, \; u \in V, v \in V \}$$

$$\therefore \begin{bmatrix} 1 & 2 & 1 \\ 4 & 6 & 3 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = u, \quad \text{where } v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \; v \in V, \; x, y, z \in \mathbb{R}.$$

$$\Rightarrow u = \begin{bmatrix} x + 2y + z \\ 4x + 6y + 3z \\ x \end{bmatrix}$$

$$= x \underbrace{\begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}}_{u_1} + y \underbrace{\begin{bmatrix} 2 \\ 6 \\ 0 \end{bmatrix}}_{u_2} + z \underbrace{\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}}_{u_3}, \quad u_1, u_2, u_3 \in \mathbb{R}^3.$$

But $u_2 = 2u_3 = \begin{bmatrix} 2 \\ 6 \\ 0 \end{bmatrix}$

$\Rightarrow$ u_2 and u_3 are not linearly independent.

Removing the redundant vector,

$$u = x \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} + (2y + z) \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \quad x, (2y+z) \in \mathbb{R}^3.$$

$\therefore$ $R(A)$ can be spanned by $\begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$

or in other words, $\boxed{R(A) = \text{span} \left\{ \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} \right\}}$

$\dim (R(A)) = 2$

- $R(A^T) = ?$

$$A^T = \begin{bmatrix} 1 & 4 & 1 \\ 2 & 6 & 0 \\ 1 & 3 & 0 \end{bmatrix}$$

$$R(A^T) = \{ w \mid w = A^T v, \; w \in V, v \in V \}$$

$$w = \begin{bmatrix} 1 & 4 & 1 \\ 2 & 6 & 0 \\ 1 & 3 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad v \in V, \; v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \; x, y, z \in \mathbb{R}.$$

$$= \begin{bmatrix} x + 4y + z \\ 2x + 6y \\ x + 3y \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}}_{w_1} x + \underbrace{\begin{bmatrix} 4 \\ 6 \\ 3 \end{bmatrix}}_{w_2} y + \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{w_3} z$$

~~But~~ But $\omega_2 = 3\omega_1 + \omega_3$

$$\begin{bmatrix}4\\6\\3\end{bmatrix} = 3\times\begin{bmatrix}1\\2\\1\end{bmatrix} + \begin{bmatrix}1\\0\\0\end{bmatrix}$$

So $\omega_1, \omega_2, \omega_3$ are not linearly independent.

Remove one of the redundant dependent vector,

the The set $\left\{ \begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix} \right\}$ is linearly independent set of vectors.

$\Rightarrow$ $R(A^T)$ can be spanned by the basis vectors $\begin{bmatrix}1\\2\\1\end{bmatrix}$ and $\begin{bmatrix}1\\0\\0\end{bmatrix}$.

Verifying,

$$\begin{bmatrix}1\\2\\1\end{bmatrix}x + \begin{bmatrix}4\\6\\3\end{bmatrix}y + \begin{bmatrix}1\\0\\0\end{bmatrix}z = \begin{bmatrix}1\\2\\1\end{bmatrix}x + \left\{3\times\begin{bmatrix}1\\2\\1\end{bmatrix} + 1\times\begin{bmatrix}1\\0\\0\end{bmatrix}\right\}\times y + \begin{bmatrix}1\\0\\0\end{bmatrix}z$$

$$= (x+3y)\begin{bmatrix}1\\2\\1\end{bmatrix} + (y+z)\begin{bmatrix}1\\0\\0\end{bmatrix}.$$

$x+3y,\ y+z \in \mathbb{R}$ as $x, y, z \in \mathbb{R}$.

Thus $\boxed{R(A^T) = \text{span}\left\{ \begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix} \right\}}$ Also, $\dim(R(A^T)) = 2$.

~~Similarly~~

- $V = K(A) \oplus R(A^T)$ since

  $K(A) \cap R(A^T) = \{0\}$

  $K(A) + R(A^T)$ spans $V = \mathbb{R}^3$.

This is evident because the set of basis ~~vectors~~ from $K(A)$ & $R(A^T)$,

$\left\{ \begin{bmatrix}0\\1\\-2\end{bmatrix}, \begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix} \right\}$ in ~~it~~ itself can serve as the basis vectors for spanning $V = \mathbb{R}^3$.

So $K(A) \oplus R(A^T) = V$

Now to show $K(A)^{\perp} = R(A^T)$.

let $v \in K(A)$, $w \in R(A^T)$,

Then $\bar{v} \cdot \bar{w} = 0$. will be true if $R(A^T) = K(A^{\perp})$.

~~Ans~~ $\bar{v} = \alpha \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$ $\left\{ \alpha, \beta, \gamma \text{, are being scalars.} \right\}$

$$\bar{w} = \beta \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + \gamma \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{v} \cdot \bar{w} = \alpha\beta\,(0\cdot 1 + 1\cdot 2 + (2\cdot -1)) + \alpha\gamma\,(0\cdot 1 + 1\cdot 0 - 2\cdot 0)$$

$$= \underline{\underline{0}}$$

$\Rightarrow$ $\underline{\underline{R(A^T) = K(A^{\perp})}}$.

(Q. 4)

Let $L \in \mathcal{L}(V;V)$

and $L = S + A$, $S \in \mathbb{S}(V;V)$ & $A \in \mathbb{A}(V;V)$

$\uparrow$ Symmetric linear subspace $\qquad$ $\uparrow$ Antisymmetric linear subspace.

$$L = \frac{L + L^T + L - L^T}{2} = \frac{L + L^T}{2} + \frac{L - L^T}{2}$$

- If $\frac{L + L^T}{2} = S$, then $S^* = S$.

Here $S^* = \left(\frac{L + L^T}{2}\right)^T = \frac{L^T + (L^T)^T}{2}$

$$= \frac{L^T + L}{2} = \frac{L + L^T}{2} = \underline{S}$$

$\Rightarrow \underline{S^* = S}$

or $\frac{L + L^T}{2}$ is a symmetric linear subspace

- If $\frac{L - L^T}{2} = A$.

Then checking $A^* = -A$,

$$A^* = \left(\frac{L - L^T}{2}\right)^T = \frac{L^T - (L^T)^T}{2}$$

$$= \frac{L^T - L}{2} = -\left(\frac{L - L^T}{2}\right)$$

$$= \underline{-A}$$

$\Rightarrow \frac{L - L^T}{2}$ is an antisymmetric linear subspace.

This shows, $\therefore$ $L = S \oplus A$ i.e. a set of linear transformations $\mathcal{L}(V; V)$ is a direct sum of symmetric and ~~two~~ antisymmetric linear transformation subspaces as $S \cap A = \{0\}$ as zero transformation is the only one which is both symmetric and antisymmetric.

Q. 5) Finished reading Lemma 1 and Lemma 2 from the given article.

Q. 6) Given A is a real matrix.

Let a_{ij} be the $(i,j)^{th}$ entry of A.

Also $(i,j)^{th}$ entry of A is the $(j,i)^{th}$ entry of A^T.

$\therefore$ $(i,j)^{th}$ entry of $(A^T)^T$ will be ~~$(i,j)^{th}$ entry~~ $(j,i)^{th}$ entry of A^T. But we know already that $(j,i)^{th}$ entry of A^T corresponds to $(i,j)^{th}$ entry of A.

$\Rightarrow$ if a'_{ij} be the $(i,j)^{th}$ entry of $(A^T)^T$, we have

$$a'_{ij} = a_{ij} \;\forall \text{ elements of } (A^T)^T \text{ and } (A^T).$$

Since each elements in A matches with their corresponding element in $(A^T)^T$, we have $A = (A^T)^T$.

Let A be an $m \times n$ matrix

$$\therefore A_{m\times n} = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & A_{23} \cdots & A_{2n} \\ \vdots & & \ddots & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{bmatrix}$$

$$\Rightarrow A^T_{n\times m} = \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{m1} \\ A_{12} & A_{22} & & \vdots \\ \vdots & \vdots & & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{mn} \end{bmatrix}$$

$$(A^T)^T_{m\times n} = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{bmatrix}$$

$$\therefore \boxed{(A^T)^T = A}$$

• $(A^{-1})^T = (A^T)^{-1}$.

we know that $A \cdot A^{-1} = I = A^{-1}A$, if A is invertible.

Taking transpose,

$$(A \cdot A^{-1})^T = I^T$$

$$(A^{-1})^T \cdot A^T = I^T$$

Multiply both LHS and RHS by $(A^T)^{-1}$ on the right side,

$$\Rightarrow (A^{-1})^T A^T \cdot (A^T)^{-1} = I^T \cdot (A^T)^{-1}$$

$\because$ $I^T = I$, and $IA = AI = A$,

we have $(A^{-1})^T \underbrace{(A^T)(A^T)^{-1}}_{} = (A^T)^{-1}$

this is of the form

$P \cdot P^{-1}$, P being a invertible matrix.

$= I$.

$$\Rightarrow (A^{-1})^T \cdot I = (A^T)^{-1}$$

or $\boxed{(A^{-1})^T = (A^T)^{-1}}$

(Q.7) $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 6 & 3 \\ 0 & 2 & 5 \end{bmatrix}$.

Using row elementary operations,

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 2 & 6 & 3 & 0 & 1 & 0 \\ 0 & 2 & 5 & 0 & 0 & 1 \end{array}\right]$$

$R_2 \to R_2 - 2R_1$: $\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & -2 & 1 & 0 \\ 0 & 2 & 5 & 0 & 0 & 1 \end{array}\right]$

$R_1 \to R_1 - R_2$: $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -1 & 0 \\ 0 & 2 & 1 & -2 & 1 & 0 \\ 0 & 2 & 5 & 0 & 0 & 1 \end{array}\right]$

$R_3 \to R_3 - R_2$: $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -1 & 0 \\ 0 & 2 & 1 & -2 & 1 & 0 \\ 0 & 0 & 4 & 2 & -1 & 1 \end{array}\right]$

$R_3 \to R_3/4$: $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -1 & 0 \\ 0 & 2 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{4} & \frac{1}{4} \end{array}\right]$

$R_2 \to R_2 - R_3$: $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -1 & 0 \\ 0 & 2 & 0 & -5/2 & 5/4 & -1/4 \\ 0 & 0 & 1 & 1/2 & -1/4 & 1/4 \end{array}\right]$

$R_2 \to R_2/2$: $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -1 & 0 \\ 0 & 1 & 0 & -5/4 & 5/8 & -1/8 \\ 0 & 0 & 1 & 1/2 & -1/4 & 1/4 \end{array}\right]$

Thus $\boxed{A^{-1} = \begin{bmatrix} 3 & -1 & 0 \\ -5/4 & 5/8 & -1/8 \\ 1/2 & -1/4 & 1/4 \end{bmatrix}}$

8) Given oblique projection $P = \begin{bmatrix} 1 & 0 \\ -\alpha & 0 \end{bmatrix}$.

$K(A) = \{ \breve{u} \mid A\breve{u} = 0 \}$ for $A : U \longrightarrow V$, $u \in U$; $A_{ij} \in F$, (field)

To determine basis for $K(P)$, one ~~can~~ has to first determine $K(P)$:

$K(P) = ?$

$P\breve{u} = 0$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ -\alpha & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad u = \begin{bmatrix} x \\ y \end{bmatrix}, \quad u \in U.$$

$$\Rightarrow \left. \begin{array}{l} x = 0. \\ -\alpha x = 0 \end{array} \right\} \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ y \end{bmatrix} = y \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad y \in F_{\text{(field)}}. \text{ is any scalar in } F.$$

$$\therefore K(P) = \operatorname{span}\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

or in other words, $\boxed{\begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ is a basis for the null space}}$

To find basis for range, $R(P)$,

$R(P) = \{ v \mid Au = v,\ A : U \longrightarrow V,\ u \in U,\ v \in V \}$.

Let $u \in U$ be $u = \begin{bmatrix} x \\ y \end{bmatrix}$, $x, y \in$ field F.

Then $\begin{bmatrix} 1 & 0 \\ -\alpha & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = v$.

$$v = \begin{bmatrix} x \\ -\alpha x \end{bmatrix} = x \begin{bmatrix} 1 \\ -\alpha \end{bmatrix}$$

$$\therefore \text{range } R(P) = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ -\alpha \end{bmatrix} \right\}$$

or in other words, $\boxed{\begin{bmatrix} 1 \\ -\alpha \end{bmatrix} \text{ is a basis for } R(P)}$

9) Given $H, S, A, T, M \in \mathcal{L}(V;V)$ are invertible linear transformations

Then $\left((HS)^T (AT)^T M^{-1}\right)^{-1}$

$$= \left((S^T H^T)(T^T A^T) M^{-1}\right)^{-1}$$

$$= (S^T H^T T^T A^T M^{-1})^{-1}$$

$$= (M^{-1})^{-1} (A^T)^{-1} (T^T)^{-1} (H^T)^{-1} (S^T)^{-1} \qquad \left(\because (U^{-1})^T = (U^T)^{-1}\right)$$

$$= M (A^{-1})^T (T^{-1})^T (H^{-1})^T (S^{-1})^T.$$

$$\therefore \boxed{\left((HS)^T (AT)^T M^{-1}\right)^{-1} = M (A^{-1})^T (T^{-1})^T (H^{-1})^T (S^{-1})^T}$$