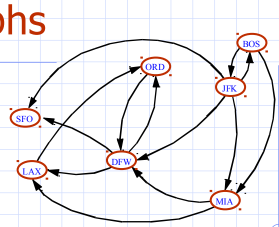
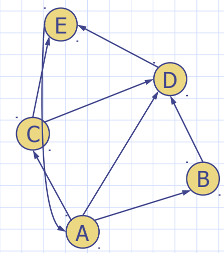
Directed Graphs

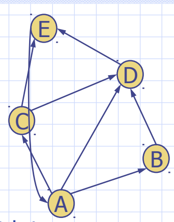


Digraphs

- A digraph is a graph whose edges are all directed
  - Short for "directed graph"
- Applications
  - one-way streets
  - flights
  - task scheduling



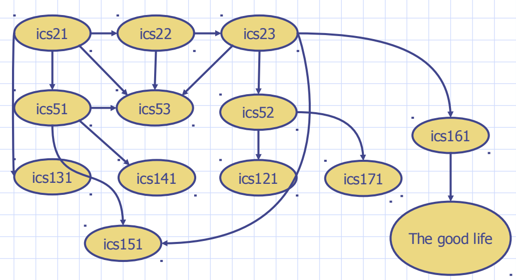
Digraph Properties



- A graph G=(V,E) such that
  - Each edge goes in one direction:
  - Edge (a,b) goes from a to b, but not b to a
- If G is simple, $m \leq n \cdot (n - 1)$
- If we keep in-edges and out-edges in separate adjacency lists, we can perform listing of incoming edges and outgoing edges in time proportional to their size

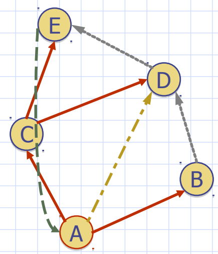
Digraph Application

- Scheduling: edge (a,b) means task a must be completed before b can be started

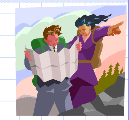


Directed DFS

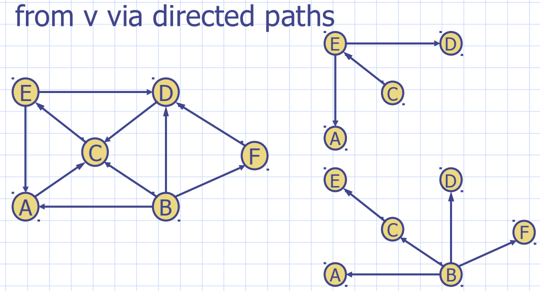
- We can specialize the traversal algorithms (DFS and BFS) to digraphs by traversing edges only along their direction
- In the directed DFS algorithm, we have four types of edges
  - discovery edges
  - back edges
  - forward edges
  - cross edges
- A directed DFS starting at a vertex s determines the vertices reachable from s



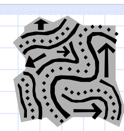
Reachability



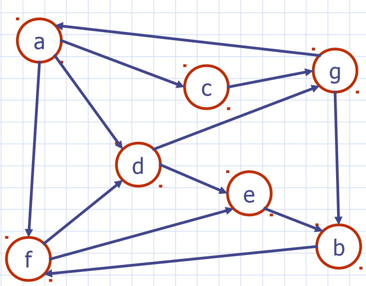
- DFS tree rooted at v: vertices reachable from v via directed paths



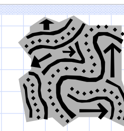
Strong Connectivity



- Each vertex can reach all other vertices

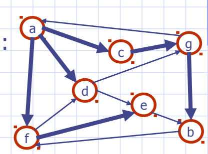


Strong Connectivity Algorithm

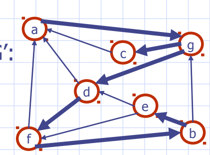


- Pick a vertex v in G
- Perform a DFS from v in G
  - If there’s a w not visited, print “no”
- Let G’ be G with edges reversed
- Perform a DFS from v in G’
  - If there’s a w not visited, print “no”
  - Else, print “yes”
- Running time: O(n+m)

G:

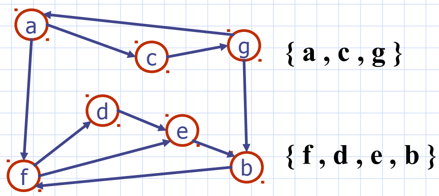


G’:



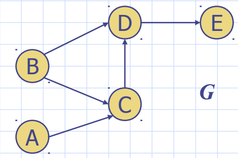
Strongly Connected Components

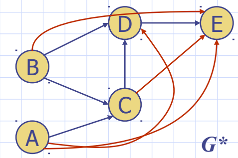
- Maximal subgraphs such that each vertex can reach all other vertices in the subgraph
- Can also be done in O(n+m) time using DFS, but is more complicated (similar to biconnectivity).



Transitive Closure

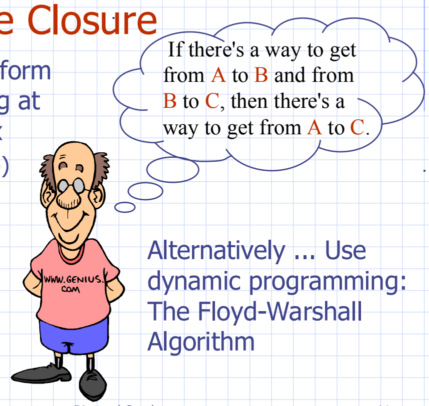
- Given a digraph G, the transitive closure of G is the digraph G^* such that
  - G^* has the same vertices as G
  - if G has a directed path from u to v ($u \neq v$), G^* has a directed edge from u to v
- The transitive closure provides reachability information about a digraph





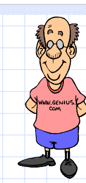
Computing the Transitive Closure

- We can perform DFS starting at each vertex
  - O(n(n+m))

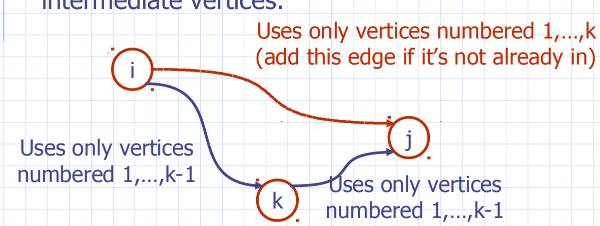


Alternatively ... Use dynamic programming: The Floyd-Warshall Algorithm

Floyd-Warshall Transitive Closure



- Idea #1: Number the vertices 1, 2, …, n.
- Idea #2: Consider paths that use only vertices numbered 1, 2, …, k, as intermediate vertices:



Floyd-Warshall's Algorithm

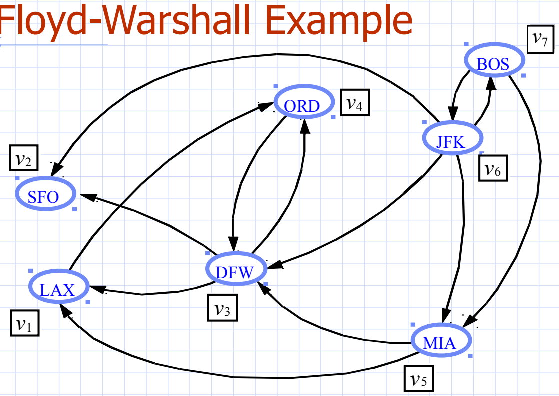
- Number vertices $v_1, \ldots, v_n$
- Compute digraphs $G_0, \ldots, G_n$
  - $G_0 = G$
  - G_k has directed edge (v_i, v_j) if G has a directed path from v_i to v_j with intermediate vertices in $\{v_1, \ldots, v_k\}$
- We have that $G_n = G^*$
- In phase k, digraph G_k is computed from G_{k-1}
- Running time: $O(n^3)$, assuming areAdjacent is $O(1)$ (e.g., adjacency matrix)

Algorithm *FloydWarshall*(G)
Input digraph G
Output transitive closure G^* of G
$i \leftarrow 1$
for all $v \in G.vertices()$
  denote v as v_i
  $i \leftarrow i + 1$
$G_0 \leftarrow G$
for $k \leftarrow 1$ **to** n **do**
  $G_k \leftarrow G_{k-1}$
  for $i \leftarrow 1$ **to** n $(i \neq k)$ **do**
    for $j \leftarrow 1$ **to** n $(j \neq i, k)$ **do**
      if $G_{k-1}.areAdjacent(v_i, v_k) \wedge G_{k-1}.areAdjacent(v_k, v_j)$
        if $\neg G_k.areAdjacent(v_i, v_j)$
          $G_k.insertDirectedEdge(v_i, v_j, k)$
return G_n

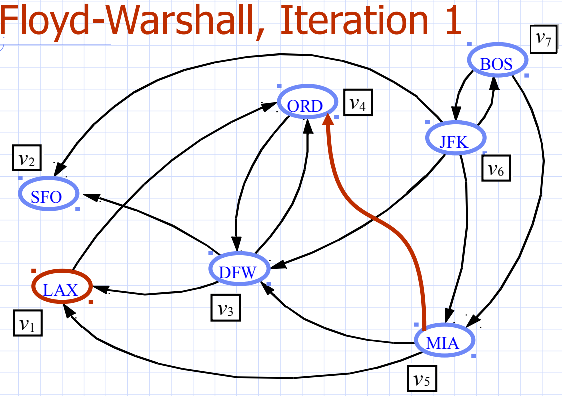
Python Implementation

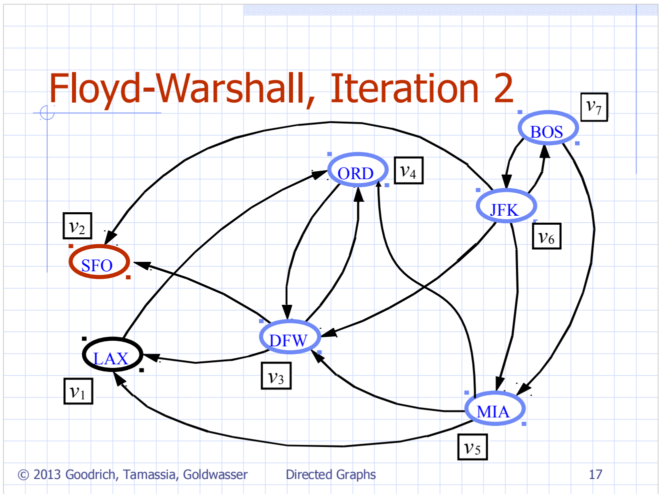
```
def floyd_warshall(g):
  """Return a new graph that is the transitive closure of g."""
  closure = deepcopy(g)                       # imported from copy module
  verts = list(closure.vertices())            # make indexable list
  n = len(verts)
  for k in range(n):
    for i in range(n):
      # verify that edge (i,k) exists in the partial closure
      if i != k and closure.get_edge(verts[i],verts[k]) is not None:
        for j in range(n):
          # verify that edge (k,j) exists in the partial closure
          if i != j != k and closure.get_edge(verts[k],verts[j]) is not None:
            # if (i,j) not yet included, add it to the closure
            if closure.get_edge(verts[i],verts[j]) is None:
              closure.insert_edge(verts[i],verts[j])
  return closure
```


Floyd-Warshall Example



Floyd-Warshall, Iteration 1




Floyd-Warshall, Iteration 2
BOS
v_7
ORD
v_4
JFK
v_6
v_2
SFO
DFW
LAX
v_3
v_1
MIA
v_5
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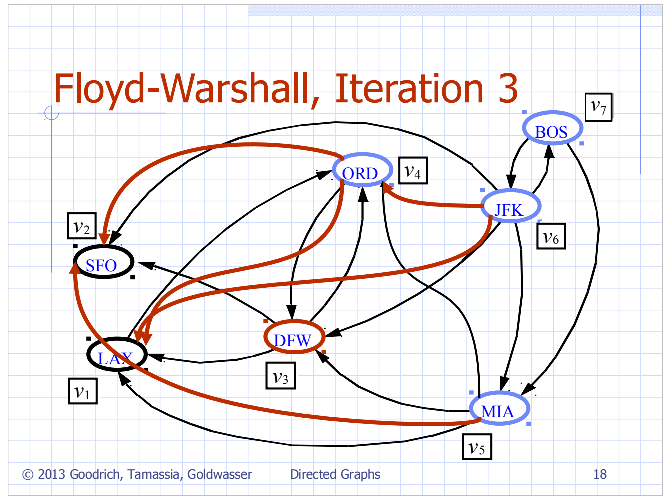
Floyd-Warshall, Iteration 3
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v_4
JFK
v_6
v_2
SFO
DFW
LAX
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v_1
MIA
v_5
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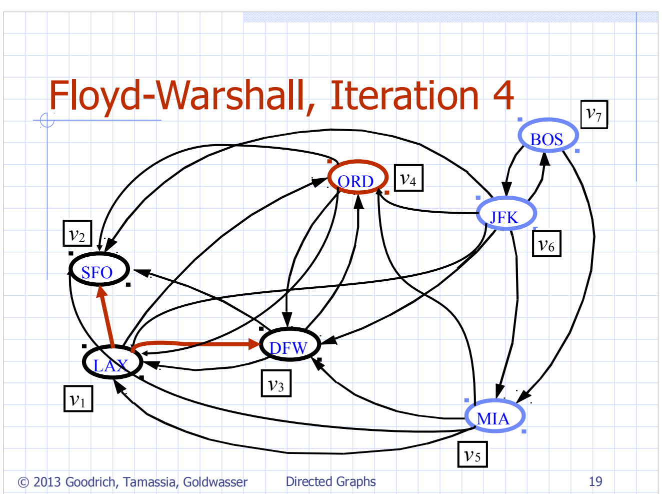
Floyd-Warshall, Iteration 4
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JFK
v_6
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DFW
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MIA
v_5
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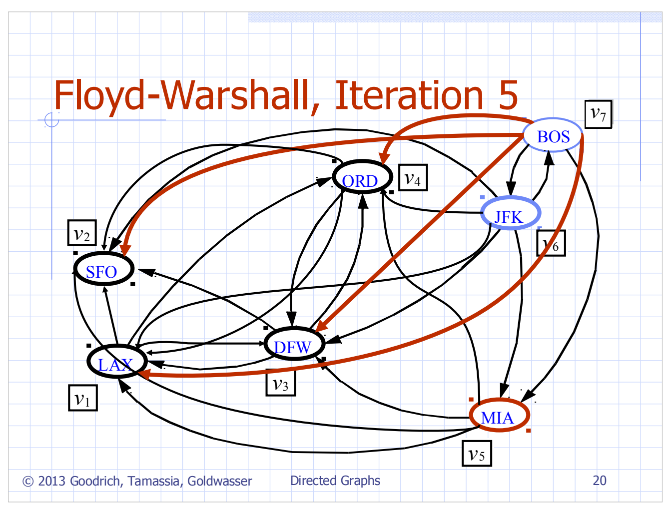
Floyd-Warshall, Iteration 5
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v_2
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v_5
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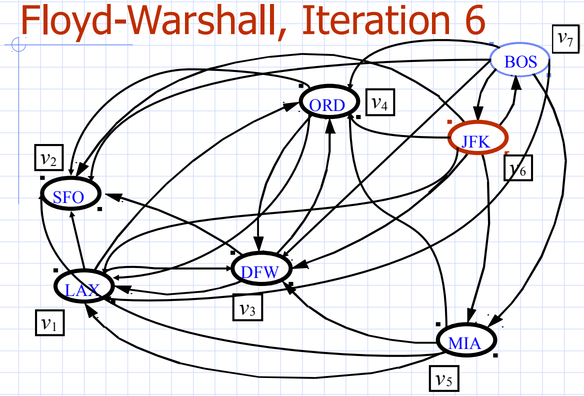
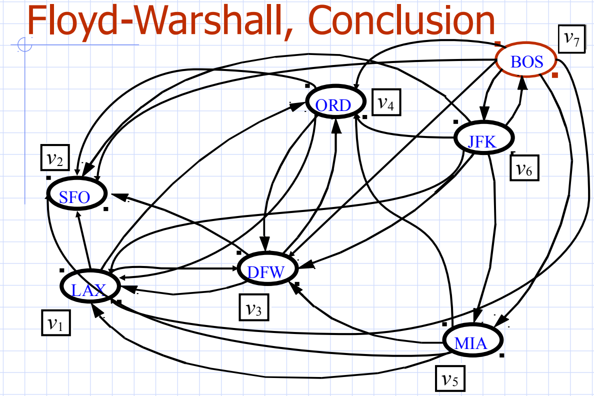
Floyd-Warshall, Iteration 6



Floyd-Warshall, Conclusion



DAGs and Topological Ordering

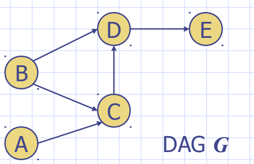
- A directed acyclic graph (DAG) is a digraph that has no directed cycles
- A topological ordering of a digraph is a numbering

  $v_1, \ldots, v_n$

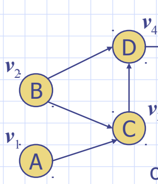
  of the vertices such that for every edge (v_i, v_j), we have $i < j$
- Example: in a task scheduling digraph, a topological ordering a task sequence that satisfies the precedence constraints

Theorem

A digraph admits a topological ordering if and only if it is a DAG



DAG G

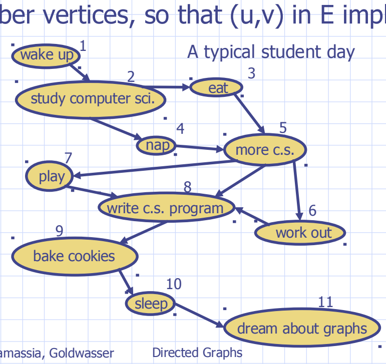


Topological ordering of G

Topological Sorting

- Number vertices, so that (u,v) in E implies u < v

A typical student day



Algorithm for Topological Sorting

- Note: This algorithm is different than the one in the book

```
Algorithm TopologicalSort(G)
    H ← G          // Temporary copy of G
    n ← G.numVertices()
    while H is not empty do
        Let v be a vertex with no outgoing edges
        Label v ← n
        n ← n - 1
        Remove v from H
```

- Running time: O(n + m)

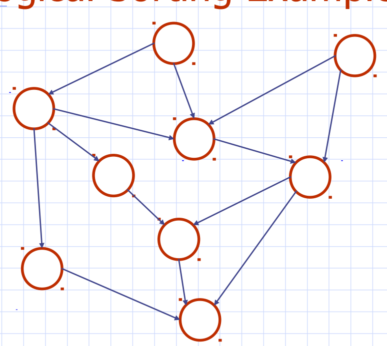
Implementation with DFS

- Simulate the algorithm by using depth-first search
- O(n+m) time.

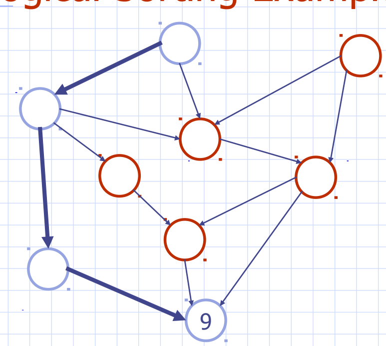
```
Algorithm topologicalDFS(G)
    Input dag G
    Output topological ordering of G
  n ← G.numVertices()
  for all u ∈ G.vertices()
    setLabel(u, UNEXPLORED)
  for all v ∈ G.vertices()
    if getLabel(v)
 = UNEXPLORED
      topologicalDFS(G, v)
```

```
Algorithm topologicalDFS(G, v)
  Input graph G and a start vertex v of G
  Output labeling of the vertices of G
    in the connected component of v
setLabel(v, VISITED)
for all e ∈ G.outEdges(v)
  { outgoing edges }
  w ← opposite(v,e)
  if getLabel(w) = UNEXPLORED
    { e is a discovery edge }
    topologicalDFS(G, w)
  else
    { e is a forward or cross edge }
Label v with topological number n
 n ← n - 1
```

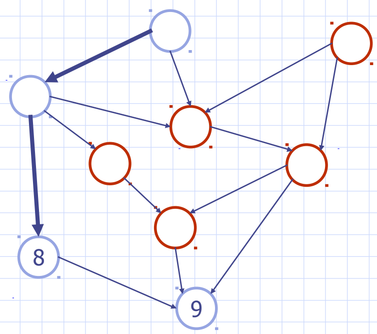

Topological Sorting Example



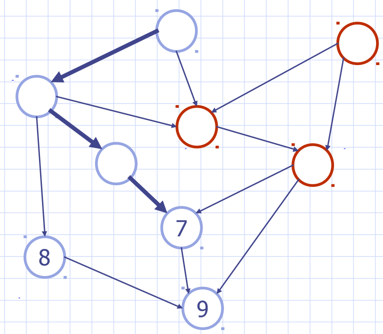
Topological Sorting Example



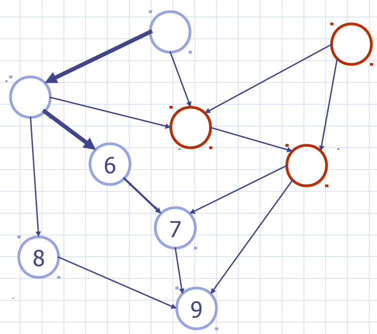
Topological Sorting Example



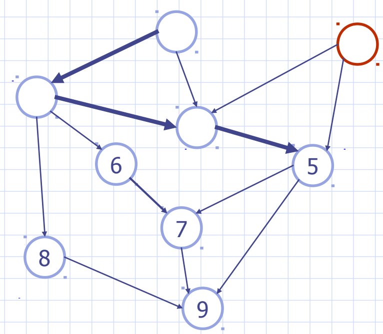
Topological Sorting Example

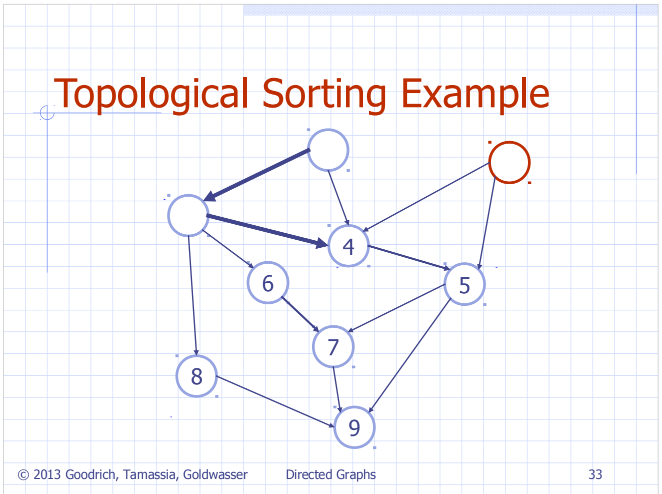


Topological Sorting Example



Topological Sorting Example




Topological Sorting Example
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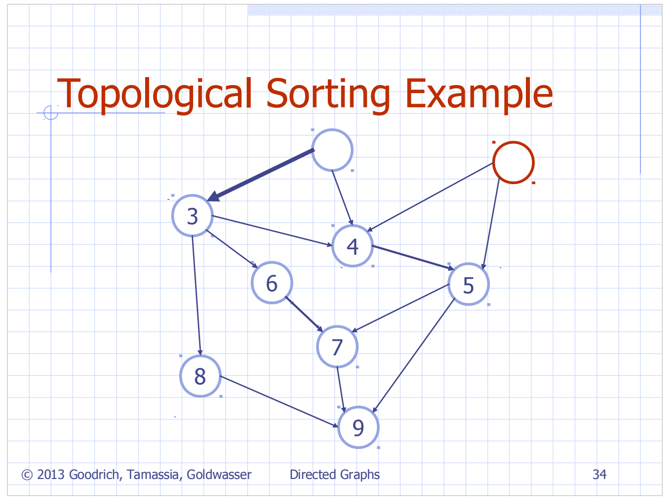
Topological Sorting Example
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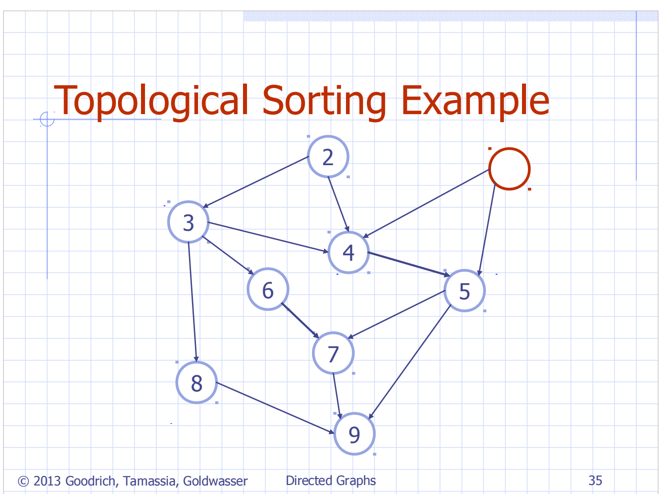
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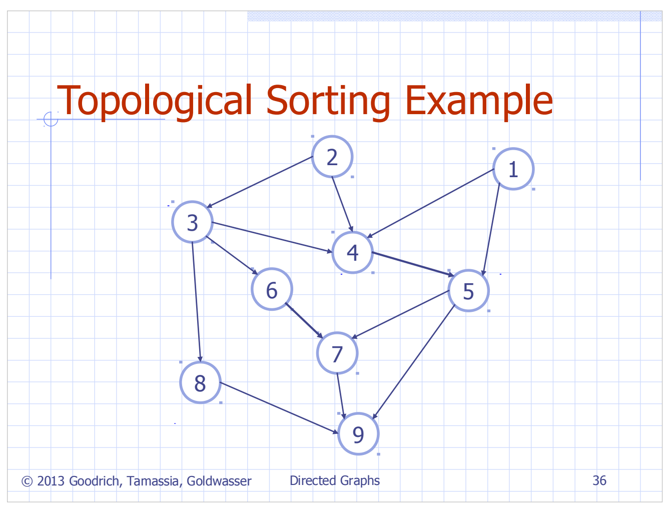
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