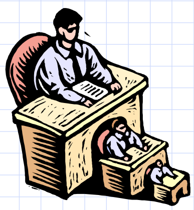
Recursion



The Recursion Pattern

- **Recursion:** when a method calls itself
- Classic example--the factorial function:
  - n! = 1· 2· 3· ··· · (n-1)· n
- Recursive definition:

$$f(n) = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot f(n-1) & \textit{else} \end{cases}$$

- As a Python method:

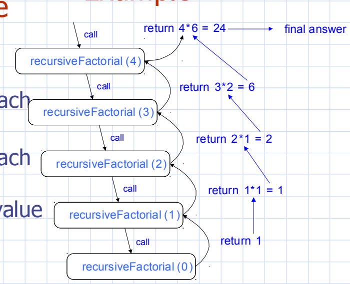
```
def factorial(n):
    if n == 0:
        return 1
    else:
        return n * factorial(n−1)
```


Content of a Recursive Method

- Base case(s)
  - Values of the input variables for which we perform no recursive calls are called base cases (there should be at least one base case).
  - Every possible chain of recursive calls must eventually reach a base case.
- Recursive calls
  - Calls to the current method.
  - Each recursive call should be defined so that it makes progress towards a base case.

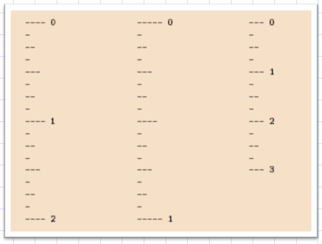
Visualizing Recursion

- Recursion trace
  - A box for each recursive call
  - An arrow from each caller to callee
  - An arrow from each callee to caller showing return value
- Example



Example: English Ruler

- Print the ticks and numbers like an English ruler:



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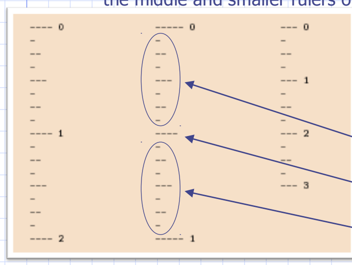
Using Recursion

drawTicks(length)
Input: length of a 'tick'
Output: ruler with tick of the given length in the middle and smaller rulers on either side



drawTicks(length)

if(length > 0) then

drawTicks(length - 1)

draw tick of the given length

drawTicks(length - 1)

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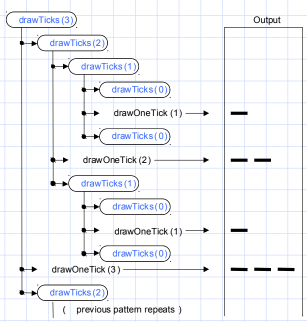
Recursive Drawing Method

- The drawing method is based on the following recursive definition
- An interval with a central tick length L $\geq$1 consists of:
  - An interval with a central tick length L-1
  - An single tick of length L
  - An interval with a central tick length L-1



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A Recursive Method for Drawing Ticks on an English Ruler

```
def draw_line(tick_length, tick_label=''):
  """Draw one line with given tick length (followed by optional label)."""
  line = '-' * tick_length
  if tick_label:
    line += ' ' + tick_label
  print(line)

def draw_interval(center_length):
  """Draw tick interval based upon a central tick length."""
  if center_length > 0:                    # stop when length drops to 0
    draw_interval(center_length - 1)       # recursively draw top ticks
    draw_line(center_length)               # draw center tick
    draw_interval(center_length - 1)       # recursively draw bottom ticks

def draw_ruler(num_inches, major_length):
  """Draw English ruler with given number of inches, major tick length."""
  draw_line(major_length, '0')             # draw inch 0 line
  for j in range(1, 1 + num_inches):
    draw_interval(major_length - 1)        # draw interior ticks for inch
    draw_line(major_length, str(j))        # draw inch j line and label
```

Note the two recursive calls

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Binary Search

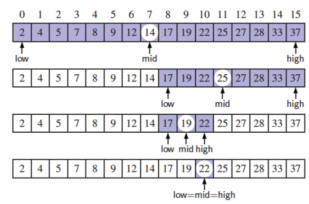
- Search for an integer, target, in an ordered list.

```
def binary_search(data, target, low, high):
  """Return True if target is found in indicated portion of a Python list.

  The search only considers the portion from data[low] to data[high] inclusive.
  """
  if low > high:
    return False                          # interval is empty; no match
  else:
    mid = (low + high) // 2
    if target == data[mid]:               # found a match
      return True
    elif target < data[mid]:
      # recur on the portion left of the middle
      return binary_search(data, target, low, mid − 1)
    else:
      # recur on the portion right of the middle
      return binary_search(data, target, mid + 1, high)
```


Visualizing Binary Search

- We consider three cases:
  - If the target equals data[mid], then we have found the target.
  - If target < data[mid], then we recur on the first half of the sequence.
  - If target > data[mid], then we recur on the second half of the sequence.



Analyzing Binary Search

- Runs in O(log n) time.
  - The remaining portion of the list is of size high – low + 1.
  - After one comparison, this becomes one of the following:

$$(\text{mid}-1)-\text{low}+1=\left\lfloor\frac{\text{low}+\text{high}}{2}\right\rfloor-\text{low}\leq\frac{\text{high}-\text{low}+1}{2}$$

$$\text{high}-(\text{mid}+1)+1=\text{high}-\left\lfloor\frac{\text{low}+\text{high}}{2}\right\rfloor\leq\frac{\text{high}-\text{low}+1}{2}.$$

  - Thus, each recursive call divides the search region in half; hence, there can be at most log n levels.

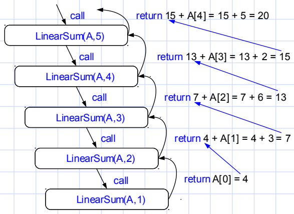
Linear Recursion

- Test for base cases
  - Begin by testing for a set of base cases (there should be at least one).
  - Every possible chain of recursive calls must eventually reach a base case, and the handling of each base case should not use recursion.
- Recur once
  - Perform a single recursive call
  - This step may have a test that decides which of several possible recursive calls to make, but it should ultimately make just one of these calls
  - Define each possible recursive call so that it makes progress towards a base case.

Example of Linear Recursion

Algorithm LinearSum(A, n):
Input:
A integer array A and an integer $n = 1$, such that A has at least n elements
Output:
The sum of the first n integers in A
if $n = 1$ **then**
  return $A[0]$
else
  return LinearSum(A, n - 1) + $A[n$ - 1$]$

Example recursion trace:



Reversing an Array

Algorithm ReverseArray(A, i, j):
Input: An array A and nonnegative integer indices i and j
Output: The reversal of the elements in A starting at index i and ending at j
if $i < j$ **then**
  Swap $A[i]$ and $A[j]$
  ReverseArray(A, $i + 1$, $j - 1$)
return

Defining Arguments for Recursion

- In creating recursive methods, it is important to define the methods in ways that facilitate recursion.
- This sometimes requires we define additional paramaters that are passed to the method.
- For example, we defined the array reversal method as ReverseArray(A, i, j), not ReverseArray(A).
- Python version:

```
def reverse(S, start, stop):
  """Reverse elements in implicit slice S[start:stop]."""
  if start < stop − 1:                         # if at least 2 elements:
    S[start], S[stop−1] = S[stop−1], S[start]  # swap first and last
    reverse(S, start+1, stop−1)                # recur on rest
```


Computing Powers

- The power function, p(x,n)=x^n, can be defined recursively:

$$p(x,n) = \begin{cases} 1 & \text{if } n = 0 \\ x \cdot p(x, n-1) & \text{else} \end{cases}$$

- This leads to an power function that runs in O(n) time (for we make n recursive calls).
- We can do better than this, however.

Recursive Squaring

- We can derive a more efficient linearly recursive algorithm by using repeated squaring:

$$p(x,n)=\begin{cases} 1 & \text{if } x = 0 \\ x \cdot p(x,(n-1)/2)^2 & \text{if } x>0 \text{ is odd} \\ p(x,n/2)^2 & \text{if } x>0 \text{ is even} \end{cases}$$

- For example,

$2^4 = 2^{(4/2)2} = (2^{4/2})^2 = (2^2)^2 = 4^2 = 16$
$2^5 = 2^{1+(4/2)2} = 2(2^{4/2})^2 = 2(2^2)^2 = 2(4^2) = 32$
$2^6 = 2^{(6/2)2} = (2^{6/2})^2 = (2^3)^2 = 8^2 = 64$
$2^7 = 2^{1+(6/2)2} = 2(2^{6/2})^2 = 2(2^3)^2 = 2(8^2) = 128.$

Recursive Squaring Method

Algorithm Power(x, n):
    Input: A number x and integer $n = 0$
    Output: The value x^n
  if $n = 0$ **then**
    return 1
  if n is odd **then**
    y = Power(x, $(n - 1)/2$)
    return $x \cdot y \cdot y$
  else
    y = Power(x, $n/2$)
    return $y \cdot y$

Analysis

Algorithm Power(x, n):
    Input: A number x and integer $n = 0$
    Output: The value x^n
  if $n = 0$ **then**
    return 1
  if n is odd **then**
    y = Power(x, $(n - 1)/2$)
    return $x \cdot y \cdot y$
  else
    y = Power(x, $n/2$)
    return $y \cdot y$

Each time we make a recursive call we halve the value of n; hence, we make log n recursive calls. That is, this method runs in O(log n) time.

It is important that we use a variable twice here rather than calling the method twice.

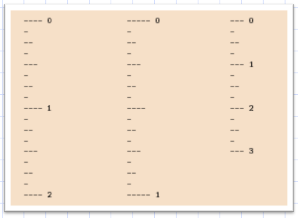
Tail Recursion

- Tail recursion occurs when a linearly recursive method makes its recursive call as its last step.
- The array reversal method is an example.
- Such methods can be easily converted to non-recursive methods (which saves on some resources).
- Example:

Algorithm IterativeReverseArray(A, i, j):
    Input: An array A and nonnegative integer indices i and j
    Output: The reversal of the elements in A starting at index i and ending at j
  while $i < j$ **do**
    Swap $A[i]$ and $A[j]$
    $i = i + 1$
    $j = j - 1$
  return

Binary Recursion

- Binary recursion occurs whenever there are **two** recursive calls for each non-base case.
- Example from before: the DrawTicks method for drawing ticks on an English ruler.

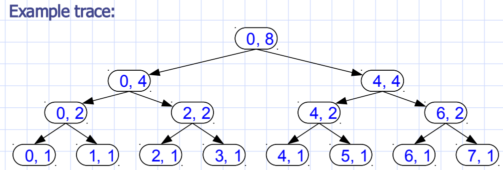


Another Binary Recusive Method

- Problem: add all the numbers in an integer array A:

Algorithm BinarySum(*A, i, n*):
Input: An array *A* and integers *i* and *n*
Output: The sum of the *n* integers in *A* starting at index *i*
if $n = 1$ **then**
return $A[i]$
return BinarySum($A, i, n/2$) + BinarySum($A, i + n/2, n/2$)

- Example trace:



Computing Fibonacci Numbers

- Fibonacci numbers are defined recursively:

$F_0 = 0$
$F_1 = 1$
$F_i = F_{i-1} + F_{i-2}$ for $i > 1$.

- Recursive algorithm (first attempt):

Algorithm BinaryFib(k):
Input: Nonnegative integer k
Output: The kth Fibonacci number F_k
if $k = 1$ **then**
return k
else
return BinaryFib(k - 1) + BinaryFib(k - 2)

Analysis

- Let n_k be the number of recursive calls by BinaryFib(k)
  - $n_0 = 1$
  - $n_1 = 1$
  - $n_2 = n_1 + n_0 + 1 = 1 + 1 + 1 = 3$
  - $n_3 = n_2 + n_1 + 1 = 3 + 1 + 1 = 5$
  - $n_4 = n_3 + n_2 + 1 = 5 + 3 + 1 = 9$
  - $n_5 = n_4 + n_3 + 1 = 9 + 5 + 1 = 15$
  - $n_6 = n_5 + n_4 + 1 = 15 + 9 + 1 = 25$
  - $n_7 = n_6 + n_5 + 1 = 25 + 15 + 1 = 41$
  - $n_8 = n_7 + n_6 + 1 = 41 + 25 + 1 = 67$.
- Note that n_k at least doubles every other time
- That is, $n_k > 2^{k/2}$. It is exponential!

A Better Fibonacci Algorithm

- Use linear recursion instead

```
Algorithm LinearFibonacci(k):
   Input: A nonnegative integer k
   Output: Pair of Fibonacci numbers (Fk , Fk-1)
   if k = 1 then
      return (k, 0)
   else
      (i, j) = LinearFibonacci(k - 1)
      return (i +j, i)
```

- LinearFibonacci makes k-1 recursive calls

Multiple Recursion

- Motivating example:
  - summation puzzles
    - *pot + pan = bib*
    - *dog + cat = pig*
    - *boy + girl = baby*
- Multiple recursion:
  - makes potentially many recursive calls
  - not just one or two

Algorithm for Multiple Recursion

```
Algorithm PuzzleSolve(k,S,U):
 Input: Integer k, sequence S, and set U (universe of elements to
    test)
 Output: Enumeration of all k-length extensions to S using elements
    in U without repetitions
 for all e in U do
    Remove e from U      {e is now being used}
    Add e to the end of S
    if k = 1 then
    Test whether S is a configuration that solves the puzzle
    if S solves the puzzle then
    return "Solution found: " S
    else
    PuzzleSolve(k - 1, S,U)
    Add e back to U      {e is now unused}
    Remove e from the end of S
```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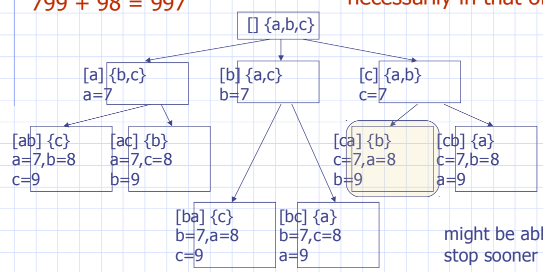
Example

cbb + ba = abc
799 + 98 = 997

a,b,c stand for 7,8,9; not necessarily in that order



might be able to stop sooner

Visualizing PuzzleSolve