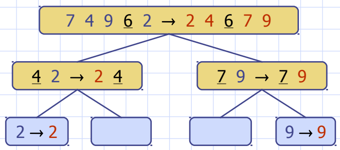
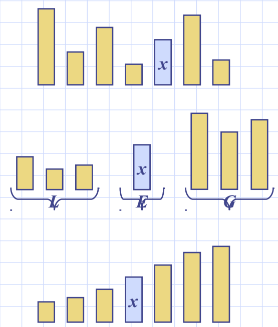
Quick-Sort



Quick-Sort

- Quick-sort is a randomized sorting algorithm based on the divide-and-conquer paradigm:
  - Divide: pick a random element x (called pivot) and partition S into
    - L elements less than x
    - E elements equal x
    - G elements greater than x
  - Recur: sort L and G
  - Conquer: join L, E and G



Partition

- We partition an input sequence as follows:
  - We remove, in turn, each element y from S and
  - We insert y into L, E or G, depending on the result of the comparison with the pivot x
- Each insertion and removal is at the beginning or at the end of a sequence, and hence takes $O(1)$ time
- Thus, the partition step of quick-sort takes $O(n)$ time

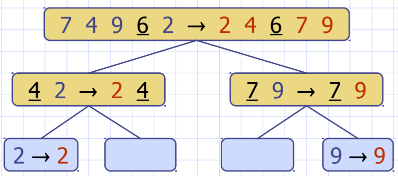
Algorithm *partition*(S, p)
  Input sequence S, position p of pivot
  Output subsequences L, E, G of the elements of S less than, equal to, or greater than the pivot, resp.
  L, E, $G \leftarrow$ empty sequences
  $x \leftarrow S.remove(p)$
  while $\neg S.isEmpty()$
    $y \leftarrow S.remove(S.first())$
    if $y < x$
      $L.addLast(y)$
    else if $y = x$
      $E.addLast(y)$
    else { $y > x$ }
      $G.addLast(y)$
  return L, E, G

Python Implementation

```
def quick_sort(S):
  """Sort the elements of queue S using the quick-sort algorithm."""
  n = len(S)
  if n < 2:
    return                                # list is already sorted
  # divide
  p = S.first( )                          # using first as arbitrary pivot
  L = LinkedQueue( )
  E = LinkedQueue( )
  G = LinkedQueue( )
  while not S.is_empty( ):                # divide S into L, E, and G
    if S.first( ) < p:
      L.enqueue(S.dequeue( ))
    elif p < S.first( ):
      G.enqueue(S.dequeue( ))
    else:                                 # S.first() must equal pivot
      E.enqueue(S.dequeue( ))
  # conquer (with recursion)
  quick_sort(L)                           # sort elements less than p
  quick_sort(G)                           # sort elements greater than p
  # concatenate results
  while not L.is_empty( ):
    S.enqueue(L.dequeue( ))
  while not E.is_empty( ):
    S.enqueue(E.dequeue( ))
  while not G.is_empty( ):
    S.enqueue(G.dequeue( ))
```

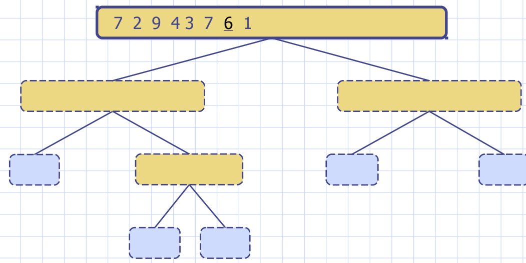
Quick-Sort Tree

- An execution of quick-sort is depicted by a binary tree
  - Each node represents a recursive call of quick-sort and stores
    - Unsorted sequence before the execution and its pivot
    - Sorted sequence at the end of the execution
  - The root is the initial call
  - The leaves are calls on subsequences of size 0 or 1



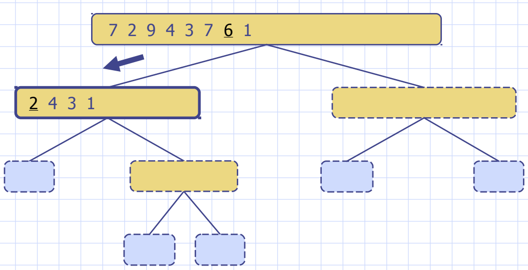
Execution Example

- Pivot selection



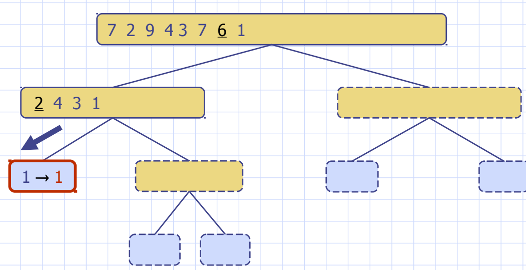
Execution Example (cont.)

- Partition, recursive call, pivot selection



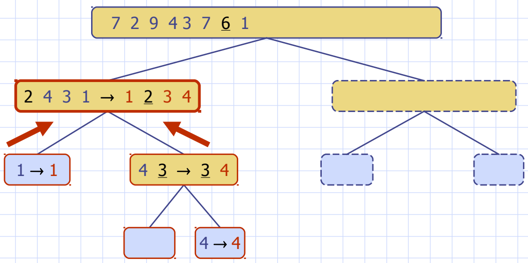
Execution Example (cont.)

- Partition, recursive call, base case



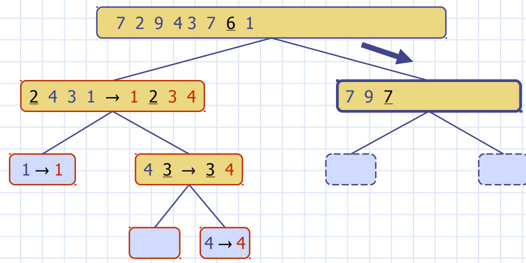
Execution Example (cont.)

◆ Recursive call, …, base case, join



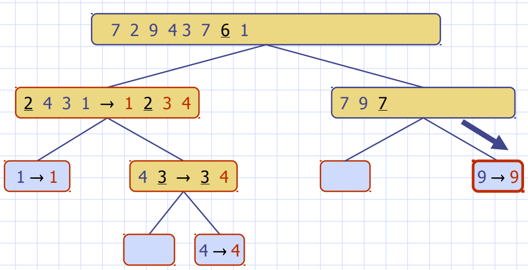
Execution Example (cont.)

◆ Recursive call, pivot selection



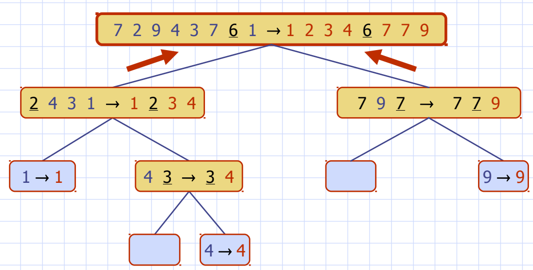
Execution Example (cont.)

◆ Partition, …, recursive call, base case



Execution Example (cont.)

◆ Join, join

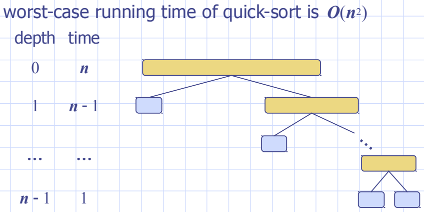


Worst-case Running Time

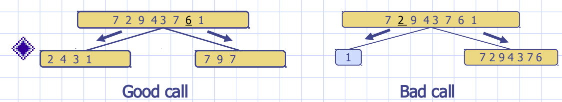
- The worst case for quick-sort occurs when the pivot is the unique minimum or maximum element
- One of L and G has size $n - 1$ and the other has size 0
- The running time is proportional to the sum

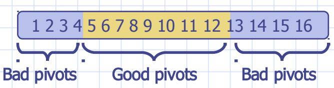
$$n + (n - 1) + \ldots + 2 + 1$$

- Thus, the worst-case running time of quick-sort is $O(n^2)$



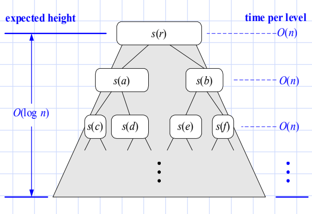
Expected Running Time



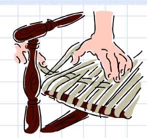


Expected Running Time, Part 2

- Probabilistic Fact: The expected number of coin tosses required in order to get k heads is $2k$
- For a node of depth i, we expect
  - $i/2$ ancestors are good calls
  - The size of the input sequence for the current call is at most $(3/4)^{i/2}n$
- Therefore, we have
  - For a node of depth $2\log_{4/3}n$, the expected input size is one
  - The expected height of the quick-sort tree is $O(\log n)$
- The amount or work done at the nodes of the same depth is $O(n)$
- Thus, the expected running time of quick-sort is $O(n \log n)$



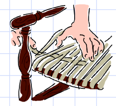
In-Place Quick-Sort



- Quick-sort can be implemented to run in-place
- In the partition step, we use replace operations to rearrange the elements of the input sequence such that
  - the elements less than the pivot have rank less than h
  - the elements equal to the pivot have rank between h and k
  - the elements greater than the pivot have rank greater than k
- The recursive calls consider
  - elements with rank less than h
  - elements with rank greater than k

Algorithm *inPlaceQuickSort*(S, l, r)
  Input sequence S, ranks l and r
  Output sequence S with the elements of rank between l and r rearranged in increasing order
  if $l \geq r$
    return
  $i \leftarrow$ a random integer between l and r
  $x \leftarrow S.elemAtRank(i)$
  $(h, k) \leftarrow inPlacePartition(x)$
  $inPlaceQuickSort(S, l, h - 1)$
  $inPlaceQuickSort(S, k + 1, r)$

In-Place Partitioning



- Perform the partition using two indices to split S into L and E U G (a similar method can split E U G into E and G).

j k

3 2 5 1 0 7 3 5 9 2 7 9 8 9 7 **6** 9 (pivot = 6)

- Repeat until j and k cross:
  - Scan j to the right until finding an element ≥ x.
  - Scan k to the left until finding an element < x.
  - Swap elements at indices j and k

j k

3 2 5 1 0 7 3 5 9 2 7 9 8 9 7 **6** 9

Python Implementation

```
def inplace_quick_sort(S, a, b):
  """Sort the list from S[a] to S[b] inclusive using the quick-sort algorithm."""
  if a >= b: return                                    # range is trivially sorted
  pivot = S[b]                                         # last element of range is pivot
  left = a                                             # will scan rightward
  right = b−1                                          # will scan leftward
  while left <= right:
    # scan until reaching value equal or larger than pivot (or right marker)
    while left <= right and S[left] < pivot:
      left += 1
    # scan until reaching value equal or smaller than pivot (or left marker)
    while left <= right and pivot < S[right]:
      right −= 1
    if left <= right:                                  # scans did not strictly cross
      S[left], S[right] = S[right], S[left]                 # swap values
      left, right = left + 1, right − 1                     # shrink range

  # put pivot into its final place (currently marked by left index)
  S[left], S[b] = S[b], S[left]
  # make recursive calls
  inplace_quick_sort(S, a, left − 1)
  inplace_quick_sort(S, left + 1, b)
```


Summary of Sorting Algorithms

Algorithm	Time	Notes
selection-sort	$O(n^2)$	▪ in-place ▪ slow (good for small inputs)
insertion-sort	$O(n^2)$	▪ in-place ▪ slow (good for small inputs)
quick-sort	$O(n \log n)$ expected	▪ in-place, randomized ▪ fastest (good for large inputs)
heap-sort	$O(n \log n)$	▪ in-place ▪ fast (good for large inputs)
merge-sort	$O(n \log n)$	▪ sequential data access ▪ fast (good for huge inputs)