Chapter 2
RELATIONS

2.0 INTRODUCTION

Every day we deal with relationships such as those between a business and its telephone number, an employee and his or her work, a person and other person, and so on. Relationships such as that between a program and a variable it uses and that between a computer language and a valid statement in this language often arise in computer science. The relationship between the elements of the sets is represented by a structure, called relation, which is just a subset of the Cartesian product of the sets.

Relations are used to solve many problems such as determining which pairs of cities are linked by airline flights in a network or producing a useful way to store information in computer databases.

In this chapter, we will study equivalence relation, equivalence class, composition of relations, matrix of relations, and closure of relations.

2.1 RELATION

A relation is a set of ordered pairs. Let A and B be two sets. Then a relation from A to B is a subset of $A \times B$.

Symbolically, R is a relation from A to B iff $R \subseteq A \times B$.

If $(x, y) \in R$, then we can express it by writing xRy and say that "x is related to y with relation R".

Thus, $\quad (x, y) \in R \Leftrightarrow xRy$

2.2 RELATION ON A SET

A relation R on a set A is the subset of $A \times A$, i.e., $R \subseteq A \times A$. Here both the sets A and B are same.

Example 1 Let $A = \{2, 3, 4, 5\}$ and $B = \{2, 4, 6, 10, 12\}$. Then find a relation R from A to B defined as

$$R = \{x, y) : x \text{ divides } y, x \in A, y \in B\}$$

Solution $R = \{(2, 2), (2, 4), (2, 6), (2, 10), (2, 12), (3, 6), (3, 12), (4, 4), (4, 12), (5, 10)\}$

2.3 DOMAIN AND RANGE OF RELATION

Let R be a relation from A to B. Then the set of all first coordinates of the ordered pairs in R is called the domain of R and the set of all second coordinates of the ordered pairs in R is called the range of R.

Thus,

$$\text{Domain of } R = \{x : (x, y) \in R\}$$
$$\text{Range of } R = \{y : (x, y) \in R\}$$

Example 2 If $A = \{1, 2, 3, 4\}$ and $B = \{2, 4, 6, 8, 10\}$, then find a relation from A to B, defined as

$$xRy \Leftrightarrow x + y \text{ is an even number.}$$

Also find the domain and range of R.

Solution According to the given relation, we have

$$R = \{(2, 2), (2, 4), (2, 6), (2, 8), (2, 10), (4, 2), (4, 4), (4, 6), (4, 8), (4, 10)\}$$

Domain of $R = \{2, 4\}$ and

Range of $R = \{2, 4, 6, 8, 10\} = B$

2.4 EMPTY AND UNIVERSAL RELATION

Let A and B be two sets. Then we know that a relation from A to B is a subset of $A \times B$. Since ϕ is a subset of every set, therefore, $\phi \subseteq A \times B$. Thus, ϕ is a relation from A, to B, and it is called empty relation.

Again we know that every set is a subset of itself, therefore

$$A \times B \subseteq A \times B$$

Thus, $A \times B$ is a relation from A to B and it is called universal relation.

2.5 IDENTITY RELATION

An identity relation on a set A is denoted by I_A and is defined as

$$I_A = \{(x, x) : x \in A\}$$

2.6 INVERSE OF A RELATION

Let R be a relation from A to B then the inverse of R is denoted by R^{-1} and it is also a relation from B to A, i.e.

$$R^{-1} = \{(y, x) : (x, y) \in R\}$$

Example 3 If $R = \{(1, 2), (2, 3), (3, 3), (4, 2), (4, 3)$ is a relation on set $A = \{1, 2, 3, 4\}$, then $R^{-1} = \{(2, 1), (3, 2), (3, 3), (2, 4), (3, 4)\}$.

2.7 NUMBER OF RELATIONS

If R is a relation from a set A to set B, where $|A| = m$ and $|B| = n$. Then the total number of relations from A to B is 2^{mn} as $|A \times B| = m.n$.

If R is a relation on set A with $|A| = n$, then the total number of relations on A is 2^{n^2}.

Example 4 How many relations are possible from a set $A = \{1, 2, 3\}$ to the set $B = \{a, b, c, d, e\}$?

Solution It is given that $|A| = 3$ and $|B| = 5$, therefore, $|A \times B| = 3.5 = 15$.

Thus, 2^{15} relations are possible from A to B.

2.8 REPRESENTATION OF A RELATION

There are so many methods to represent a relation R from set A to set B.

2.8.1 Roster Method

In this method, all the ordered pairs of the relation are enclosed within curly brackets.

For example, if $A = \{1, 2\}$ and $B = \{x, y, z\}$ then the relation $R = \{(1, x), (1, y), (2, z)\}$ is in roster form.

2.8.2 Matrix Method

Let A and B be two non-empty sets with $|A| = m$ and $|B| = n$. Let R be a relation from A to B. Then the relation R can be represented by a $m \times n$ matrix denoted as M_R and this matrix is called adjacency matrix or Boolean matrix, i.e.

$$M_R = [m_{ij}]_{m \times n}$$

where

$$m_{ij} = \begin{cases} 1, \text{if } (a_i, b_j) \in R \\ 0, \text{if } (a_i, b_j) \notin R \end{cases}$$

Example 5 If $R = \{(1, x), (1, y), (2, y), (3, z), (4, x), (4, y), (4, z)$ is relation from set $A = \{1, 2, 3, 4\}$ to set $B = \{x, y, z)$, then find the matrix of R.

Solution Since $|A| = 4$ and $|B| = 3$, therefore, there will be 4×3 matrix of the relation R, i.e.

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccc} x & y & x \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \end{array}$$

Example 6 Find the relation R on set $A = \{1, 2, 3, 4\}$, whose matrix is given below:

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

Solution We know that any element $(a_i, b_j) \in R$ iff $(i, j)^{\text{th}}$ element of M_R, i.e., $m_{ij} = 1$

By writing the given matrix as

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

We get $R = \{(1, 1), (1, 3), (1, 4), (2, 2), (3, 2), (3, 3), (4, 1), (4, 2)\}$

2.8.3 Digraph of a Relation on Sets

When a relation is defined on a set A then we can represent the relation by a digraph. First the elements of A are written down. Then arrows are drawn from each element x to each element y whenever $(x, y) \in R$.

Example 7 If $R = \{(1, 1), (1, 2), (1, 3), (2, 2), (2, 4), (3, 1), (3, 3), (3, 4), (4, 2), (4, 3)$ is a relation on set $A = \{1, 2, 3, 4\}$. Then represent the relation R by its digraph.

Solution First of all, we represent all the elements of A by small circles. Then we will show the relations.

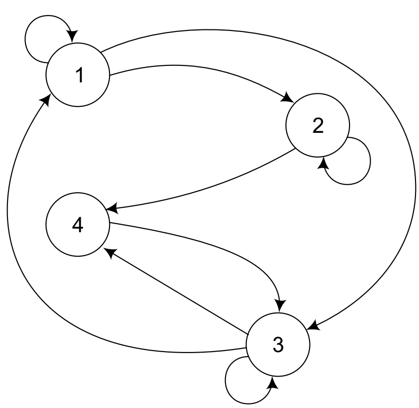

Fig. 2.1

2.8.4 Set Builder Form of a Relation

In this method, the rule that associates the first and second coordinates of each ordered pair is given.

Example 8 Let $A = \{2, 3, 4\}$ and $B = \{2, 6, 8, 11, 12\}$.

If $R = \{2, 2\}, \{2, 6\}, \{2, 8\}, \{2, 12\}, (3, 6\}, (3, 12), (4, 8), (4, 12)\}$ is a relation from A to B.

Then find the set builder form of A.

Solution The set builder form of the given relation R is given as below:

$$R = \{x, y) : x \text{ divides } y\}.$$

2.9 UNION AND INTERSECTION OF TWO RELATIONS

If R and S are two relations from set A to set B, then

$$R \cup S = \{(a, b) : (a, b) \in R \text{ or } (a, b) \in S$$

and

$$R \cap S = \{(a, b) : (a, b) \in R \text{ and } (a, b) \in S\}$$

where $a \in A$ and $b \in B$

Example 9 If $R = \{(1, 1), (1, 2), (2, 2), (2, 3), (3, 4), (4, 1), (4, 4)\}$ and $S = \{(1, 1), (1, 3), (2, 3), (3, 4), (4, 2), (4, 3)\}$ are two relations on set $A = \{1, 2, 3, 4\}$, then find $R \cup S$ and $R \cap S$.

Solution $R \cup S = \{(1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 4), (4, 1), (4, 2), (4, 3), (4, 4)\}$

and $R \cap S = \{(1, 1), (2, 3), (3, 4)\}$

2.10 COMPOSITION OF TWO RELATIONS

Let A, B, and C be three non-empty sets. Let R and S be the relations from A to B and B to C respectively, i.e., $R \subseteq A \times B$ and $S \subseteq B \times C$.

Then we define a relation from A to C denoted by RoS (or SoR as used by certain authors) given by

$$RoS = \{(a, \text{c}) : (a, b) \in R \text{ and } (b, c) \in S \text{ for some } b \in B\}.$$

This relation is called a composition of R and S or a composite relation of R and S.

Example 10 Let $A = \{1, 2, 3, 4\}$, $B = \{p, q, r, s\}$ and $C = \{x, y, z\}$ and also $R = \{(1, p), (1, r), (1, s), (2, r), (2, s), (3, p), (4, s)\}$ and $S = \{(p, x), (p, z), (q, y), (r, y), (r, z), (s, x), (s, z)\}$. Find composition of R and S.

Solution According to the definition of RoS, we have

$$RoS = \{(1, x), (1, z), (1, y), (2, y), (2, z), (2, x), (3, x), (3, z), (4, x)\}$$

2.11 USE OF BOOLEAN MATRIX TO FIND UNION, INTERSECTION, COMPOSITION AND INVERSE OF RELATIONS

Let A, B, and C be three non-empty sets and let R and S be the relations from A to B and B to C respectively. Then we have to find $R \cup S$, $R \cap S$, RoS and R^{-1} with the help of Boolean matrix.

Let M_R and M_S be the matrices of the relations R and S respectively. Then

(i) The matrix of $R \cup S$ is given by

$M_{R \cup S} = M_R \vee M_S$, i.e., the join of M_R and M_S.

(ii) The matrix of $R \cap S$ is given by

$M_{R \cap S} = M_R \wedge M_S$, i.e., the meet of M_R and M_S.

(iii) The matrix of RoS is given by

$M_{RoS} = M_R \,.\, M_S$, i.e., the multiplication of M_R and M_S

(iv) The matrix of R^{-1} is given by

$M_{R\text{-}1} = (M_R)^T$, i.e., the transpose of matrix M_R

Example 11 Let $A = \{1, 2, 3, 4\}$ and let $R = \{(1, 1), (1, 2), (1, 3), (2, 4), (3, 2)\}$ and $S = \{(1, 3), (1, 4), (2, 3), (3, 1), (4, 1)\}$ be two relations on set A. Thus, use Boolean matrix to find $R \cup S$, $R \cap S$, RoS and R^{-1}.

Solution The matrices of the relations R and S are given as below:

$$M_R = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \text{and} \quad M_S = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

(i) The matrix of $R \cup S$ is given by

$$M_{R \cup S} = M_R \vee M_S = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \vee \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

i.e., $R \cup S = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 1),, (3, 2), (4, 1)\}$

(ii) The matrix of $R \cap S$ is given by

$$M_{R \cap S} = M_R \wedge M_S = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \wedge \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

i.e., $R \cap S = \{(1, 3)\}$.

(iii) The matrix of RoS is given by

$$M_{RoS} = M_R \,.\, M_S = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

i.e., $RoS = \{(1, 1), (1, 3), (1, 4), (2, 1), (3, 3)\}$

(iv) The matrix of R^{-1} is given by

$$M_{R^{-1}} = (M_R)^T = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

i.e., $R^{-1} = \{(1, 1), (2, 1), (2, 3), (3, 1), (4, 2)\}$

Theorem 1: Let A, B, C and D be four non-empty sets. Let R, S and T be three relations from A to B, B to C and C to D respectively. Then

$$(RoS)oT = Ro(SoT)$$

Proof: The relation $(RoS)oT$ and $Ro(SoT)$ are determined by Boolean matrices $M_{(RoS)\,oT}$ and $M_{Ro(SoT)}$.

Now, $$M_{(RoS)\,oT} = M_{RoS} \,.\, M_T = (M_R \,.\, M_S)\, M_T \qquad \text{...(i)}$$

and $$M_{Ro(SoT)} = M_R \,.\, M_{SoT} = M_R \,.\, (M_S \,.\, M_T) \qquad \text{...(ii)}$$

Since the Boolean matrix multiplication is associative, therefore

$$M_{(RoS)\,oT} = M_{Ro(SoT)}$$

$\Rightarrow$ $(RoS)oT = Ro(SoT)$. **Hence proved.**

Theorem 2: If R^{-1} and S^{-1} are the inverses of the relations R and S respectively, then

$$(RoS)^{-1} = S^{-1}oR^{-1} \text{ or } (SoR)^{-1} = R^{-1}oS^{-1}$$

Proof: Let A, B, and C be three non-empty sets and let R and S be the relation from A to B and B to C respectively. Then $R \subseteq A \times B$ and $S \subseteq B \times C$

$\Rightarrow$ $RoS \subseteq A \times C$

$\Rightarrow$ $(RoS)^{-1} \subseteq C \times A$

Now, $(c, a) \in (RoS)^{-1}$

$\Rightarrow$ $(a, c) \in RoS$

$\Rightarrow \quad \exists\, b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$

$\Rightarrow \qquad (b, a) \in R^{-1}$ and $(c, b) \in S^{-1}$

$\Rightarrow \qquad (c, b) \in S^{-1}$ and $(b, a) \in R^{-1}$

$\Rightarrow \qquad (c, a) \in S^{-1}oR^{-1}$

i.e., $\qquad (c, a) \in (RoS)^{-1} \Rightarrow (c, a) \in S^{-1}oR^{-1}$

Therefore, $\qquad (RoS)^{-1} \subseteq S^{-1}oR^{-1}$ ...(i)

Similarly, we can prove that

$$S^{-1}oR^{-1} \subseteq (RoS)^{-1} \qquad \text{...(ii)}$$

From (i) and (ii), we have

$$(RoS)^{-1} = S^{-1}oR^{-1}$$

2.12 PROPERTIES OF RELATIONS ON A SET

The following are the main properties of the relation.

(i) Reflexive Relation: A relation R on a set A is said to be reflexive if each $x \in A$, $(x, x) \in R$.

For example, if $A = \{1, 2, 3\}$, then $R = \{(1, 1), (2, 2), (3, 1), (3, 2), (3, 3)\}$ is a reflexive relation.

A reflexive relation R on set A contains the identity relation I_A, i.e., $I_A \subseteq R$.

(ii) Irreflexive Relation: A relation R on a set A is said to be irreflexive if for each $x \in A$, $(x, x) \notin R$.

For example, consider, $A = \{1, 2, 3\}$ and $R = \{(1, 2), (1, 3), (2, 1), (3, 2)\}$, then R is irreflexive since $(x, x) \notin r$ for each $x \in A$.

(iii) Non-reflexive Relation: A relation R is said to be non-reflexive if it is neither reflexive nor irreflexive.

For example, the relation $R = \{(1, 1), (1, 2), (2, 2), (3, 1), (3, 2)\}$ is a non-reflexive relation as it is neither reflexive nor irreflexive.

(iv) Symmetric Relation: A relation R on a set A is said to be symmetric if $xRy \Rightarrow yRx$,i.e., whenever $(x, y) \in R$ then $(y, x) \in R$ for all $x, y \in A$.

Example 12 Let L be the set of all straight lines in a plane. Then the relation R is defined as $R = \{(x, y) : x$ is perpendicular to y for all, $x, y \in L\}$ is a symmetric relation because $x \perp y \Rightarrow y \perp x$ for all $x, y \in L$

Example 13 Then relation $R = \{(1, 1), (1, 2), (2, 1), (2, 3), (3, 2), (3, 3)\}$ on set $A = \{1, 2, 3\}$ is a symmetric relation.

(v) Asymmetric Relation: A relation R on a set A is said to be asymmetric if $(x, y) \in R \Rightarrow (y, x) \notin R$ for all $x, y \in A$.

Example 14 The relation $R = \{(1, 2), (2, 3), (3, 4), (4, 2)\}$ is an asymmetric relation on the set $A = \{1, 2, 3, 4\}$.

Example 15 If R is a relation on N (the set of all natural numbers), defined as $R = \{(x, y) : x < y$, where $x, y \in N\}$ is an asymmetric relation because if $x < y$, then y is not less than x.

(vi) Antisymmetric Relation: A relation R on a set A is said to be antisymmetric if

$$xRy \text{ and } yRx \Rightarrow x = y \text{ for all } x, y \in A$$

Example 16 Let N be the set of natural numbers and let R be a relation on N, defined by "x divides y" $\forall\ x, y \in N$. Then R is an antisymmetric relation as x divides y and y divides $x \Rightarrow x = y$.

Example 17 The relation '≤' is an antisymmetric relation on the set of all natural numbers, N because $x \leq y$ and $y \leq x \Rightarrow x = y$ for all $x, y \in N$.

(vii) Transitive Relation: A relation R on a set A is said to be transitive relation if xRy and $yRz \Rightarrow xRz$ for all $x, y, z \in A$.

Example 18 The relation R on set N, defined by "x divides y" for all $x, y \in N$ is a transitive relation because if x divides y and y divides z, then x divides z.

Example 19 The relation R of parallelism in the set L of straight lines in a plane is a transitive relation because $x \parallel y$ and $y \parallel z \Rightarrow x \parallel z$ for all $x, y, z \in L$.

2.13 EQUIVALENCE RELATION

A relation R on a set A is said to be an equivalence relation if it is reflexive, symmetric and transitive.

Example 20 The relation R of parallelism on the set L, of all straight lines in the plane is an equivalence relation as R is reflexive, symmetric and transitive.

2.14 PARTIAL ORDER RELATION

A relation R on the set A is said to be a partial order relation if R is reflexive, anti-symmetric and transitive.

Example 21 The relation R on the set of all natural numbers (N) defined as "x divides y" for all $x, y \in N$, is a partial order relation as R is reflexive, antisymmetric and transitive.

Theorem 3: The inverse of an equivalence relation is also an equivalence relation.

Proof: Let R be an equivalence relation on a set A, i.e., R is reflexive, symmetric and transitive on A. To prove that R^{-1} is an equivalence relation on A.

R^{-1} is reflexive: Let $x \in A$. Then $(x, x) \in R$ as R is reflexive

$\Rightarrow \quad (x, x) \in R^{-1}$

i.e., for all $x \in A$, $(x, x) \in R^{-1}$. Thus, R^{-1} is reflexive.

R^{-1} is symmetric: Let $x, y \in A$

Let $(x, y) \in R^{-1}$

$\Rightarrow$ $(y, x) \in R$

$\Rightarrow$ $(x, y) \in R$ as R is symmetric

$\Rightarrow$ $(y, x) \in R^{-1}$

Thus, R^{-1} is symmetric.

R^{-1} is transitive: Let $x, y, z \in A$. Let (x, y) and $(y, z) \in R^{-1}$

$\Rightarrow$ (y, x) and $(z, y) \in R$

$\Rightarrow$ (z, y) and $(y, x) \in R$

$\Rightarrow$ $(z, x) \in R$

$\Rightarrow$ $(x, z) \in R^{-1}$

Thus, R^{-1} is transitive.

Hence, R^{-1} is an equivalence relation on set A.

Theorem 4: The intersection of two equivalence relations is also an equivalence relation.

Proof: Let R and S be two equivalence relations on a set A. To prove that $R \cap S$ is an equivalence relation on set A.

$R \cap S$ is reflexive: Let $x \in A$. Then $(x, x) \in R$ and $(x, x) \in S$ as R and S are reflexive on A.

This implies $(x, x) \in R \cap S$.

Thus, $R \cap S$ is reflexive on A.

$R \cap S$ is symmetric: Let $x, y \in A$. Let $(x, y) \in R \cap S$.

$\Rightarrow$ $(x, y) \in R$ and $(x, y) \in S$

$\Rightarrow$ $(y, x) \in R$ and $(y, x) \in S$ as R and S are symmetric

$\Rightarrow$ $(y, x) \in R \cap S$

Thus, $R \cap S$ is symmetric

$R \cap S$ is transitive: Let $x, y, z \in A$. Let (x, y) and $(y, z) \in R \cap S$. Then (x, y) and $(y, z) \in R$ and (z, y) and $(y, z) \in S$

$\Rightarrow$ $(x, y) \in R$ and $(x, z) \in S$

$\Rightarrow$ $(x, z) \in R \cap S$.

Thus, $R \cap S$ is a transitive relation.

Hence, $R \cap S$ is an equivalence relation on set A.

Theorem 5: A relation R on a set is symmetric iff $R = R^{-1}$.

Proof: First suppose that R is symmetric, then we have to prove that $R = R^{-1}$.

Let $(x, y) \in R$. Then

$$\begin{aligned}(x, y) \in R &\Rightarrow (y, x) \in R\\ &\Rightarrow (x, y) \in R^{-1}\end{aligned}$$

$\therefore \quad R \subseteq R^{-1}$ ...(i)

Let $(x, y) \in R^{-1} \Rightarrow (y, x) \in R$

$\Rightarrow$ $(x, y) \in R$ as R is symmetric.

$\therefore \quad R^{-1} \subseteq R$ ...(ii)

From (i) and (ii), we have $R = R^{-1}$.

Conversely, suppose that $R = R^{-1}$. Then we have to prove that R is symmetric.

Let $(x, y) \in R$. Then

$$(x, y) \in R \Rightarrow (y, x) \in R^{-1}$$

$$\Rightarrow \quad (y, x) \in R \text{ as } R = R^{-1}.$$

Thus, R is symmetric.

Theorem 6: A relation R on a set A is antisymmetric iff $R \cap R^{-1} \subseteq I_A$.

Proof: Suppose that the relation R on set A is antisymmetric. To prove that $R \cap R^{-1} \subseteq I_A$

Let $(x, y) \in R \cap R^{-1} \Rightarrow (x, y) \in R$ and $(x, y) \in R^{-1}$.

$$\Rightarrow \quad (x, y) \in R \text{ and } (y, x) \in R$$

$$\Rightarrow \quad x = y \qquad [\because R \text{ is antisymmetric}]$$

$$\Rightarrow \quad (x, y) \in I_A$$

$$\Rightarrow \quad R \cap R^{-1} \subseteq I_A$$

Conversely, suppose that $R \cap R^{-1} \subseteq I_A$. To prove that R is antisymmetric.

Let (x, y) and $(y, x) \in R$

$$\Rightarrow \quad (x, y) \in R \text{ and } (x, y) \in R^{-1}$$

$$\Rightarrow \quad (x, y) \in R \cap R^{-1}$$

$$\Rightarrow \quad (x, y) \in I_A$$

$\Rightarrow$ $x = y$, i.e., R is antisymmetric.

Theorem 7: A relation R on set A is asymmetric if and only if $R \cap R^{-1} = \phi$

Proof: Let R be an asymmetric relation on set A. To prove that $R \cap R^{-1} = \phi$.

Now, let $(x, y) \in R$, then $(y, x) \in R^{-1}$

But if $(x, y) \in R$ then $(y, x) \notin R$ as R is asymmetric, therefore, $R \cap R^{-1} = \phi$.

Conversely, suppose that $R \cap R^{-1} = \phi$. To prove that R is asymmetric.

Let $(x, y) \in R$ then $(y, x) \in R^{-1}$.

Now because $R \cap R^{-1} = \phi$, therefore, $(y, x) \notin R$

Thus, R is asymmetric.

Example 22 Give an example of a relation which is

(i) Both reflexive and symmetric

(ii) Neither reflexive nor irreflexive

(iii) Both symmetric and antisymmetric

(iv) Each reflexive, symmetric and transitive.

(v) Both symmetric and transitive but not reflexive.

Solution

(i) The relation $R = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3)\}$ on a set $A = \{1, 2, 3\}$ is both reflexive and symmetric.

(ii) The relation $R = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 2)$ on set $A = \{1, 2, 3\}$ is neither reflexive nor irreflexive.

(iii) Then identity relation $I_A = \{(1, 1), (2, 2), (3, 3)\}$ on set $A = \{1, 2, 3\}$ is both symmetric and antisymmetric.

(iv) The relation $R = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (3, 4), (4, 3), (4, 4)\}$ is reflexive, symmetric and transitive.

(v) The empty relation ϕ on any finite set. A symmetric and transitive but not reflexive.

Example 23 Which of the following defines a relation on the set N of natural numbers.

$R_1 : x > y$, $R_2 : x + y = 10$, $R_3 = x + 4y = 10$ for all $x, y \in N$. Determine which of the relations are

(i) reflexive

(ii) symmetric

(iii) antisymmetric

(iv) transitive

Solution

(i) Reflexive: None is reflexive. For example, $(2, 2) \notin R, S$ and T

(ii) Symmetric: R is not symmetric because if $x < y$ then $y \not< x$.

The relation S is symmetric because if $(x, y) \in S$ then $x + y = 10$

$$\Rightarrow \quad y + x = 10$$

$$\Rightarrow \quad (y, x) \in S'$$

The relation T is not symmetric because if $(x, y) \in T$ then $x + 4y = 10$, but $(y, x) \notin T$ as $y + 4x \neq 10$.

(iii) Antisymmetric: The relation R is antisymmetric as if $x < y$ and $y < x \Rightarrow x = y$.

The relation S is not antisymmetric

The relation T is antisymmetric as if (x, y) and $(y, x) \in T$

$$\Rightarrow \quad x + 4y = 10 \quad \text{and} \quad y + 4x = 10$$

$$\Rightarrow \quad x + 4y = y + 4x$$

$$\Rightarrow \quad 3x = 3y$$

$$\Rightarrow \quad x = y \ \forall \ x, y \in N.$$

(iv) Transitive: The relation R is transitive because if $x < y$ and $y < z$

$\Rightarrow$ $$x < y \ \forall x, y, z \in N.$$

The relation S is not transitive because if (x, y) and $(y, z) \in S$

$\Rightarrow$ $$x + y = 10 \quad \text{and} \quad y + z = 10$$

$\nRightarrow$ $$x + z = 10.$$

Similarly, we can prove that T is not transitive

Example 24 Prove with an example that the union of two equivalence relations is not necessarily an equivalence relation.

Solution It can be proved by giving a counter-example.

Let $A = \{1, 2, 3, 4\}$. Let $R = \{(1, 1), (2, 2), (3, 3), (4, 4), (2, 3), (2, 4), (3, 2), (4, 2)\}$ and $S = \{(1, 1), (2, 2), (3, 3), (4, 4), (3, 4), (4, 3), (1, 2), (2, 1)\}$ be two relations on A.

It can be easily verified that R and S are equivalence relations.

$R \cup S = \{(1, 1), (1, 2), (2, 1), (2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4), (4, 2), (4, 3), (4, 4)\}$

Now, $(1, 2) \in R \cup S$ and $(2, 3) \in R \cup S$, but $(1, 3) \notin R \cup S$

$\therefore$ $R \cup S$ is not transitive.

Hence, we conclude that $R \cup S$ is not necessarily an equivalence relation.

Example 25 Let L be the set of all straight lines in the plane. Let R be a relation on L, defined as

$$R = \{(x, y) : x \text{ is parallel to } y,\ x, y \in L\}$$

Then prove that R is an equivalence relation.

Solution Given that for all $x, y \in L$, xRy iff x is parallel to y.

***R* is reflexive:** Let $x \in L$. Then we can say that the line x is parallel to itself, i.e., $(x, x) \in R$. Thus, R is reflexive.

***R* is symmetric:** Let $x, y \in L$. Let $(x, y) \in R$. Then

$$\begin{aligned} (x, y) \in R &\Rightarrow x \text{ is parallel to } y \\ &\Rightarrow y \text{ is parallel to } x \\ &\Rightarrow (y, x) \in R \end{aligned}$$

i.e., R is symmetric.

***R* is transitive:** Let $x, y, z \in L$

Let (x, y) and $(y, z) \in R$

$\Rightarrow$ x is parallel to y and y is parallel to z

$\Rightarrow$ x is parallel to z

$\Rightarrow$ $(x, z) \in R$

i.e. R is transitive

Hence, R is an equivalence relation on L.

Example 26 Let I be the set of all integers. Let R be a relation on a set I, defined as "xRy iff $x - y$ is divisible by 4".

Then prove that R is an equivalence relation on I.

Solution Given that for all $x, y \in I$,

$$R = \{(x, y) : x - y \text{ is divisible by } 4\}$$

***R* is reflexive:** Let $x \in I$. Then $x - x = 0$, which is divisible by 4. Thus, $(x, x) \in R$, i.e., R is reflexive.

***R* is symmetric:** Let $(x, y) \in R$, where $x, y \in I$

$\Rightarrow$ $x - y$ is divisible by 4

$\Rightarrow$ $y - x$ is divisible by 4

$\Rightarrow$ $(y, x) \in R$

i.e., R is symmetric.

***R* is transitive:** Let (x, y) and $(y, z) \in R$, where $x, y, z \in I$

$\Rightarrow$ $x - y$ and $y - z$ are divisible by 4

$\Rightarrow$ $x - y + y - z$ is divisible by 4

$\Rightarrow$ $x - z$ is divisible by 4

$\Rightarrow$ $(x, z) \in R$

i.e., R is transitive.

Hence, R is an equivalence relation on I.

Example 27 Let A relation R on the set of real numbers be defined as

$$aRb \Leftrightarrow 1 + ab > 0$$

Prove that R is reflexive, symmetric but not transitive.

Solution **(i) Reflexive**: Let R' be the set of real numbers.

Let $a \in R' \Rightarrow 1 + a.a = 1 + a^2 > 0$

$\Rightarrow$ $(a, a) \in R$, i.e., R is reflexive.

(ii) Symmetric: Let $a, b \in R'$. Let $(a, b) \in R$

$\Rightarrow$ $1 + a.b > 0$

$\Rightarrow$ $1 + b.a > 0$

$\Rightarrow$ $(b, a) \in R$

i.e., R is symmetric.

(iii) Transitive: The relation R is not transitive because we find that $\left(2, \frac{1}{2}\right) \in R$ and $\left(\frac{1}{2}, -2\right) \in R$ but $(2, -2) \notin R$ since $1 + 2(-2) = -3$ (which is not positive).

Hence, R is reflexive, symmetric but not transitive.

Example 28 Prove that the relation R on the set $N \times N$, where N is the set of natural numbers, defined as follows:

$$(a, b)\ R\ (c, d) \Leftrightarrow a + d = b + c$$

is an equivalence relation for $a, b, c, d \in N$

Solution **(i) Reflexive:** Let $(a, b) \in N \times N$, then $a + b = b + a$

$\Rightarrow$ $(a, b)\ R(a, b)$

$\therefore$ R is reflexive.

(ii) Symmetric: Let $(a, b), (c, d) \in N \times N$, where $a, b, c, d \in N$.

Let $(a, b)\ R(c, d)$

$\Rightarrow$ $a + d = b + c$

$\Rightarrow$ $b + c = a + d$

$\Rightarrow$ $c + b = d + a$

$\Rightarrow$ $(c, d)\ R(a, b)$

i.e., R is symmetric.

(iii) Transitive: Let $(a, b), (c, d)$ and $(e, f) \in N \times N$, where $a, b, c, d, e, f \in N$.

Let $(a, b)\ R(c, d)$ and $(c, d)\ R(e, f)$

$\Rightarrow$ $a + d = b + c$

and $c + f = d + e$

$\Rightarrow$ $(a + d) + (c + f) = (b + c) + (d + e)$

$\Rightarrow$ $a + f = b + e$

$\Rightarrow$ $(a, b)\ R(e, f)$

$\therefore$ R is transitive.

Hence, R is an equivalence relation on $N \times N$.

Example 29 Prove that the relation R on set $N \times N$, defined as

$$(a, b)\ R(c, d) \Leftrightarrow ad = bc\ \forall\ (a, b), (c, d) \in N \times N.$$

+ is an equivalence relation for $a, b, c, d \in N$

Solution Similarly, as in the previous example.

Example 30 Suppose that A is a non-empty set and f is a function that has A as its domain. Let R be the relation on A consisting of all ordered pairs (x, y), where $f(x) = f(y)$. Show that R is an equivalence relation on A.

Solution It is given that $R = \{(x, y) : f(x) = f(y)\}$ for all $x, y \in A$.

To prove that R is an equivalence relation.

(i) Reflexive: Let $x \in A$. Then $f(x) = f(x)$

$\Rightarrow$ $(x, x) \in R$

i.e., R is reflexive.

(ii) Symmetric: Let $x, y \in A$. Let $(x, y) \in R$

$\Rightarrow \quad f(x) = f(y)$

$\Rightarrow \quad f(y) = f(x)$

$\Rightarrow \quad (y, x) \in R$.

i.e., R is symmetric.

(iii) Transitive: Let $x, y, z \in A$. Let (x, y) and $(y, z) \in R$

$\Rightarrow \quad f(x) = f(y)$

and $\quad f(y) = f(z)$

$\Rightarrow \quad f(x) = f(y) = f(z)$

$\Rightarrow \quad f(x) = f(z)$

$\Rightarrow \quad (x, z) \in R$

i.e., R is transitive.

Hence, R is an equivalence relation on set A.

Example 31 Let $A = \{1, 2, 3, 4, 5\}$ and $P_1 = \{1, 2\}$, $P_2 = \{3\}$, $P_3 = \{4, 5\}$ be three partitions of A. Obtain an equivalence relation R *pertaining* to the partition.

Solution Let R be an equivalence relation on A.

Now all elements of P_1 are in relation R

$\Rightarrow \quad \{(1, 1), (2, 2), (1, 2), (2, 1)\} \subseteq R$

All elements of P_2 and P_3 are to be in R, so, we must have $\{(3, 3)\} \subseteq R$ and $\{(4, 4), (5, 5), (4, 5), (5, 4) \subseteq R$.

Thus, $R = \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (2, 1), (4, 5), (5, 4), (5, 5)\}$

Example 32 Consider the set Z of integers and an integer $m > 1$. We say that x is congruent to y modulo m, written as $x \equiv y \pmod{m}$ if $x - y$ is divisible by m or $x - y = km$, where k is any integer. Show that $\equiv$ is an equivalence relation on Z.

Solution **(i) Reflexive**: Let $x \in I$. Then $x \equiv x \pmod{m}$, since $x - x = 0$ is divisible by m. Hence $\equiv$ is reflexive.

(ii) Symmetric: Let $x, y \in I$. Let $x \equiv y \pmod{m} \Rightarrow x - y$ is divisible by $m \Rightarrow y - x$ is divisible by $m \Rightarrow y \equiv x \pmod{m}$

Hence $\equiv$ is symmetric.

(iii) Transitive: Let $x, y, z \in I$.

Let $x \equiv y \pmod{m}$ and $y \equiv z \pmod{m}$

$\Rightarrow \quad x - y$ and $y - z$ are divisible by m

$\Rightarrow \quad x - y + y - z$ is also divisible by m

$\Rightarrow \quad x - z$ is divisible by m

$\Rightarrow \quad x \equiv z \pmod{m}$

$\therefore \quad \equiv$ is transitive.

Hence $\equiv$ is an equivalence relation.

2.15 EQUIVALENCE CLASS

Let R be an equivalence relation on the non-empty set A and let $a \in A$. Then the equivalence class of a is denoted by $[a]$ is the set of all elements of A to which a is related, i.e.,

$$[a] = \{x : aRx\}$$

The set of all equivalence classes is denoted by A/R and read as "A modulo R" or "A mod R". This is called the quotient of A by R, i.e.,

$$A/R = \{[a] : a \in A\}$$

Example 33 The relation $R = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (3, 4), (4, 3), (4, 4), (5, 5)\}$ is an equivalence relation on set $A = \{1, 2, 3, 4, 5\}$. Find the equivalence class of each element of A. Also find the quotient set A/R.

Solution The equivalence class of the elements of A are as follows:

$$[1] = \{x : 1Rx\} = \{1, 2\}$$

$$[2] = \{x : 2Rx\} = \{1, 2\}$$

$$[3] = \{x : 3Rx\} = \{3, 4\}$$

$$[4] = \{x : 4Rx\} = \{3, 4\}$$

$$[5] = \{5\}$$

Now, the quotient set

$$\begin{aligned} A/R &= \{[a] : a \in A\} \\ &= \{[1], [2], [3], [4], [5]\} \\ &= \{\{1, 2\}, \{3, 4\}, \{5\}\} \end{aligned}$$

Remark: The set A/R is also called the partition of A induced by R. In the above example, the set $\{\{1, 2\}, \{3, 4\}, \{5\}\}$ is the partition of A induced by R.

2.16 PROPERTIES OF EQUIVALENCE CLASSES

Theorem 8: Let A be any non-empty set and let R be an equivalence relation on set A. Let $a, b \in A$. Then

(i) $a \in [a]$

(ii) If $b \in [a]$, then $[a] = [b]$

(iii) $[a] = [b]$ if and only if $(a, b) \in R$

(iv) Either $[a] = [b]$ or $[a] \cap [b] = \phi$, i.e., two equivalence classes are either disjoint or identical.

Proof: (i) Since R is reflexive, therefore for each $x \in A$, $(x, x) \in R$, i.e., xRx

But $\quad [a] = \{x \in A : aRx\}$

Thus, $\quad aRa \Rightarrow a \in [a]$

(ii) Given that $b \in [a]$

$\Rightarrow$ bRa or $(b, a) \in R \Rightarrow aRb$

Let $x \in [a] \Rightarrow xRa$

Now xRa and $aRb \Rightarrow xRb$ as R is transitive

$\Rightarrow$ $x \in [b]$

i.e., $[a] \subseteq [b]$ ...(i)

Now, let $y \in [b] \Rightarrow yRb$

Now yRb and $bRa \Rightarrow yRa$

$\Rightarrow$ $y \in [a]$

i.e., $[b] \subseteq [a]$ ...(ii)

From (i) and (ii), we have

$$[a] = [b].$$

(iii) Let $[a] = [b]$ to prove that aRb

Let $x \in [a] \Rightarrow xRa \Rightarrow aRx$

Now, $x \in [b] \Rightarrow xRb$

Now, aRx and $xRb \Rightarrow aRb$.

Conversely, suppose that aRb. To prove that $[a] = [b]$.

Let $x \in [a] \Rightarrow xRa$.

But it is given that aRb

Thus, xRb

$\Rightarrow$ $x \in [b]$

i.e., $[a] \subseteq [b]$ ...(i)

Again, let $y \in [b]$

$\Rightarrow$ yRB

But it is given that $aRb \Rightarrow bRa$

Now yRb and $bRa \Rightarrow yRa$

$\Rightarrow$ $y \in [a]$

i.e., $[b] \subseteq [a]$ ...(ii)

From (i) and (ii), we have

$$[a] = [b]$$

(iv) Suppose that $[a] \cap [b] \neq \phi$, then we have to show that $[a] = [b]$.

Now, $[a] \cap [b] \neq \phi$

$\Rightarrow$ There exists at least one element $x \in [a] \cap [b]$ such that $x \in [a]$ and $x \in [b]$

$\Rightarrow$ xRa and xRb

$\Rightarrow$ aRx and xRb [$\because$ R is transitive]

$\Rightarrow$ $aRb \Rightarrow [a] = [b]$. [By part (iii)]

Hence, two equivalence classes are either disjoint or identical.

Theorem 9: An equivalence relation on a set A decomposing the set into disjoint classes.

Proof: Let R be an equivalence relation on a set A. Let $a \in A$ and B be a subset of A consisting of all those elements which are equivalent to a i.e.,

$$B = \{x \in A : aRx\}$$

Then $a \in B$, for aRa (R is reflexive). Any two elements of B are equivalent to each other, for if $x, y \in B$, then xRa and yRa

Now, $\quad xRa, yRa \Rightarrow xRa, aRy \quad$ [R is symmetric]

$\Rightarrow \quad xRy$

Thus, B is an equivalence class.

Suppose that B_1 is another equivalence class

i.e., $\quad B_1 = \{x \in A : xRb\}$

where b is not equivalent to a. Then the classes B and B_1 must be disjoint. For, if c is a common element of B and B_1, cRa and cRb, so that bRa which is contrary to our hypothesis.

Now, the set A can be decomposed into equivalence classes $B, B_1, B_2, \ldots$ such that every element of A belongs to one of these classes and we obtain the required partition of A as these classes are mutually disjoint.

Example 34 Let $A = \{1, 2, 3, 4, 5, 6\}$ and $R = \{(1, 1), (1, 5), (2, 2), (2, 3), (2, 6), (3, 2), (3, 3), (3, 6), (4, 4), (5, 1), (5, 5), (6, 2), (6, 3), (6, 6)\}$ on A. Find the partition of A induced by R, i.e., the quotient set A/R.

Solution Given that

$$A = \{1, 2, 3, 4, 5, 6\}$$

Now,

$$[1] = \{x \in A : 1Rx\} = \{1, 5\}$$
$$[2] = \{x \in A : 2Rx\} = \{2, 3, 6\}$$
$$[3] = \{x \in A : 3Rx\} = \{2, 3, 6\}$$
$$[4] = \{x \in A : 4Rx\} = \{4\}$$
$$[5] = \{x \in A : 5Rx\} = \{1, 5\}$$
$$[6] = \{x \in A : 6Rx\} = \{2, 3, 6\}$$

The partition of A induced by R is given by

$$A/R = \{[1], [2], [3], [4], [5], [6]\}$$
$$= \{\{1, 5\}, \{2, 3, 6\}, \{4\}\}$$

2.17 PATH IN A RELATION

Let R be a relation on set A. Then a path of length n in R from a to b is a finite sequence a, $x_1, x_2, x_3, \ldots, x_{n-1}, b$ which begins with a and ends with b such that

$$aRx_1, x_1Rx_2, x_2Rx_3, \ldots., x_{n-1}Rb$$

The path of length n must have $n + 1$ elements of A. The elements may be distinct or same.

2.18 CYCLE IN A RELATION

A path that begins and ends at the same vertex is called a cycle.

Remark: The length of a path is the number of edges in the path.

Example 35 Consider the relation whose digraph is given below.

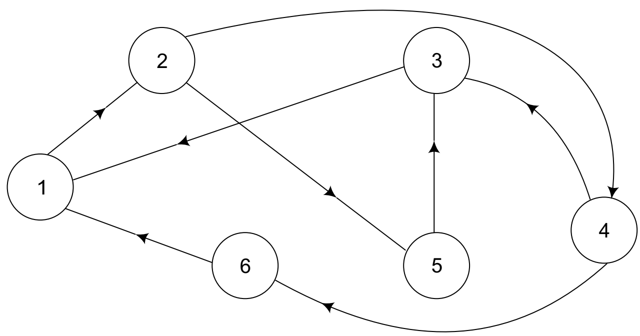

Fig. 2.2

The graph has a path $P_1 = 1, 2, 4, 6$ of length 3.

The graph has a path $P_2 = 1, 2, 5, 3, 4, 6$ of length 5.

The graph has a path $P_3 = 1, 2, 4, 6, 1$ of length 4.

The graph has a cycle $P_4 = 1, 2, 5, 3, 4, 6, 1$ of length 6.

Paths in a relation R can be used to define following 1 relation on the set A.

$\boldsymbol{R^n}$**:** The relation R^n on set A is defined as a path of length n from a to b in R.

$\boldsymbol{R^\infty}$**:** The relation R^∞ on set A is defined as some path from a to b in R and the length of path depends on a and b. This is called the connectivity relation for R.

$\boldsymbol{R^n(x)}$**:** It is defined as a set, having all the vertices that can be reached from x by means of a path in R of length n.

$\boldsymbol{R^\infty(x)}$**:** It is defined as a set, having all the vertices that can be reached from x by some path in R.

$\boldsymbol{R^*}$**:** Let R be a relation on set A, where $|A| = n$. Then the relation R^* of R is defined as a relation such that aR^*b if $a = b$ or $aR^\infty b$. This is also called the reachability relation. In other words, we can say that b is reachable from a if either $b = a$ or there is some path from a to b.

Example 36 Let R be a relation on set $A = \{1, 2, 3, 4, 5\}$ whose digraph is shown in Figure 2.3. Determine the digraph of the relations R^2 and R^3 on A.

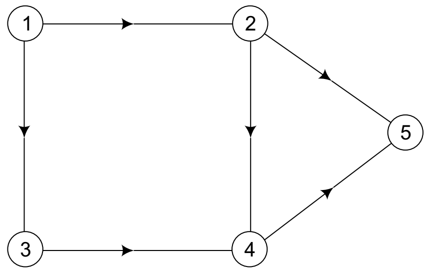

Fig. 2.3

Solution To obtain the digraph of the relation R^2, we find all the edges that connect two vertices in R^2, when there is a path of length two connecting those vertices in R. The digraph of R^2 is shown in Fig. 2.4.

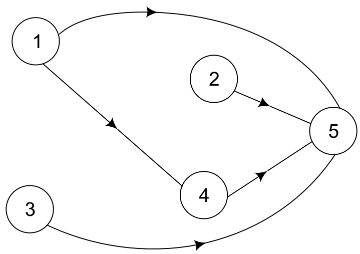

Fig. 2.4 Digraph of R^2

To obtain the digraph of the relation R^3, we find all paths of length 3 in R from any vertex to any other vertex and then join those vertices in R^3 by a direct line. The digraph of R^3 as shown in Fig. 2.5.

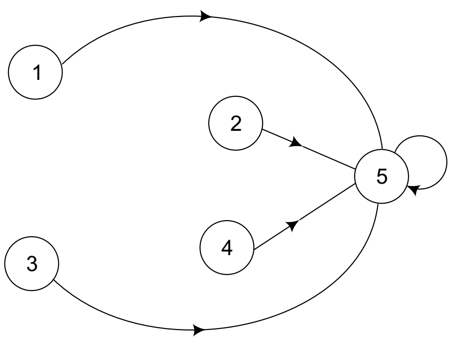

Fig. 2.5 Digraph of R^3

Example 37 Let R be a relation on set $A = \{1, 2, 3, 4, 5, 6, 7\}$, whose digraph is shown in Fig. 2.6.

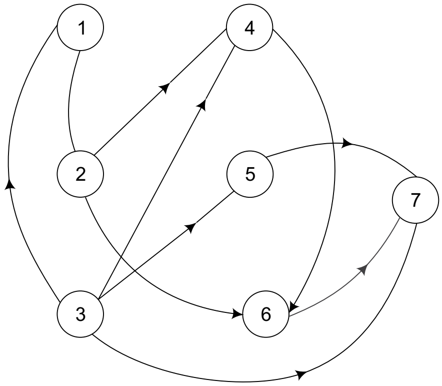

Fig. 2.6

Determine (a) $R^2(4)$ (b) $R^2(7)$ (c) $R^\infty(5)$ (d) $R^\infty(2)$.

Solution To find $R^2(4)$ and $R^2(7)$, we find all the vertices which can be reached from 4 and 7 by means of a path of length 2 respectively. Thus,

(i) $R^2(4) = \{7\}$

(ii) $R^2(7) = \{3\}$.

Now, to find $R^\infty(5)$ and $R^\infty(2)$, we find all the vertices which can be reached from 5 and 2 by means of a path of some length respectively. Thus,

(iii) $R^\infty(5) = \{3, 4, 6, 7\}$

(iv) $R^\infty(2) = \{4, 6, 7, 5, 3\}$

2.19 COMPUTATION OF THE MATRIX OF R^n, R^∞ AND R^*

(i) Let R be a relation on a finite set A. Then the matrix of R^n is given by

$M_R n = M_R \,.\, M_R \,.\, M_R \,....\, M_R$ (n times), $n \geq 2$

(ii) Let R be a relation on finite set A. Then the matrix of R^∞ is given by

$M_{R^\infty} = M_R \vee M_{R^2} \vee M_{R^3} \vee \ldots$

(iii) Let R be a relation on a finite set A, then the matrix of R^* is given by

$$M_R* = I_n \vee M_{R^\infty}$$

where I_n is the identity matrix of $n \times n$ and n is the number of elements in A.

i.e., $$M_{R^*} = I_n \vee M_R \vee M_{R^2} \vee M_{R^3} \vee \ldots..$$

Example 38 Let $A = \{1, 2, 3, 4\}$ and let $R = \{(1, 1), (1, 2), (2, 1), (2, 3), (3, 1), (3, 3), (4, 3)\}$ be a relation on R. Then find the matrix of R^n, R^∞ and R^*.

Solution The matrix of the given relation R is given by

$$M_R = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

(i) The matrix of R^n is given by

$$M_R{}^n = M_R \,.\, M_R \,.\, M_R \,.\, M_R.$$

$$= \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

(ii) The matrix of R^∞ is given by

$$M_{R^\infty} = M_R \vee M_{R^2} \vee M_{R^3} \vee M_{R^4}$$

Now, $$M_{R^2} = M_R \,.\, M_R = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

$$M_{R^3} = M_{R^2} \,.\, M_R = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}, \quad M_{R^4} = M_{R^3} M_R = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

Now, $$M_R{}^\infty = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

(iii) The matrix of R^* is given by

$$M_R^* = I_n \vee M_{R^\infty}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

2.20 CLOSURE OF RELATIONS

In the present study we want to consider the situation of constructing a new relation R^c from the given previous known relation R such that $R \subseteq R^c$. The relation R^c is called the closure of R.

Let R be a relation on a set A. R may or may not have some property P(say) such as reflexivity, symmetry or transitivity. If there is a relation R^c with property P and $R \subseteq R^c$ such that R^c is a subset of each relation with property P containing R, then R^c is called closure of R with property P. Now we will give the methods of obtaining some closures of the given relations.

2.20.1 Reflexive Closure

Let R be a relation on a set A, and suppose that R is not reflexive. Now we produce a reflexive relation containing R that is as small as possible. This resulting relation is called reflexive closure of R and it is denoted as R_r. The reflexive closure of R can be obtained as follows:

$$R_r = R \cup I_A, \text{ where } I_A \text{ is identity relation on set } A.$$

Example 39 If $R = \{(1, 1), (1, 2), (2, 3), (3, 1), (3, 2)\}$ is a relation on the set $A = \{1, 2, 3\}$, then find the reflexive closure of R.

Solution First we find the identity relation on set A

i.e., $$I_A = \{(1, 1), (2, 2), (3, 3)\}$$

The reflexive closure of the given relation R is given by

$$\begin{aligned} R_r &= R \cup I_A \\ &= \{(1, 1), (1, 2), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\} \end{aligned}$$

Example 40 Find the reflexive closure of the relation $R = \{(x, y) : x < y\}$ on the set of natural numbers.

Solution Here $$I_A = \{(x, x) : x \in N\}$$

The reflexive closure of R is given as

$$R_r = R \cup I_A = \{(x, y) : x \le y\}$$

2.20.2 Symmetric Closure

The symmetric closure of a relation R on a set A is denoted by R_s and it is defined as follows:

$R_s = R \cup R^{-1}$, where R^{-1} is the inverse of the relation R.

Example 41 Find the symmetric closure of the relation $R = \{(a, b), (a, c), (b, a), (b, d), (c, d), (d, c), (d, d)$ on the set $A = \{a, b, c, d\}$.

Solution First we find R^{-1} i.e.

$$R^{-1} = \{(b, a), (c, a), (a, b), (d, b), (d, c), (c, d), (d, d)\}.$$

Now, the symmetric closure of R is given as

$$R_s = R \cup R^{-1} = \{(a, b), (a, c), (b, a), (b, d), (c, a), (c, d), (d, b), (d, c), (d, d)\}$$

Example 42 Find the symmetric closure of the relation $R = \{(x, y) : x < y\}$ on the set of natural numbers.

Solution The symmetric closure of R is given as

$$\begin{aligned} R_s = R \cup R^{-1} &= \{(x, y) : x < y\} \cup \{(y, x) : x < y\} \\ &= \{(x, y) : x \neq y\} \end{aligned}$$

2.20.3 Transitive Closure

Let R be a relation on a set A. Then the transitive closure of R is denoted by R_t and it is the smallest relation which contains R as subset and which is transitive.

If the cardinality of the set A is n, i.e., $|A| = n$, then the transitive closure of R, i.e., R_t is given by

$$R_t = R \cup R^2 \cup R^3 \cup .... \cup R^n, \text{ where}$$
$$R^2 = RoR, R^3 = R^2oR, \text{ etc.}$$

The transitive closure of the above given relation can be calculated by an alternative method, i.e., matrix method.

The matrix of the transitive closure of R, i.e., R_t is given as below.

$$M_{R_t} = M_R \vee M_{R2} \vee M_{R3} \vee ...... \vee M_{Rn},$$

where addition is done using Boolean arithmetic.

Example 43 Let $R = \{(1, 2), (2, 1), (2, 2), (3, 1), (4, 3)\}$ be a relation on set $A = \{1, 2, 3, 4\}$. Find the transitive closure of R.

Solution Let M_R, M_{R2}, M_{R3} and M_{R4} be the matrices representing the relations R, R^2, R^3 and R^4 respectively and M_{Rt} be the matrix of the transitive closure of R. Then

$$M_{Rt} = M_R \vee M_{R2} \vee M_{R3} \vee M_{R4}$$

Now, $$M_R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$M_{R^2} = M_R \,.\, M_R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} . \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

i.e., $\{(1, 1), (1, 2), (2, 1), (2, 2), (3, 2), (4, 1)\} = R^2$

$$M_{R^3} = M_{R^2} \,.\, M_R = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

i.e., $\{(1, 1), (1, 2), (2, 1), (2, 2), (3, 1), (3, 2), (4, 2)\} = R^3$

$$M_{R^4} = M_{R^3} \,.\, M_R = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

i.e., $\{(1, 1), (1, 2), (2, 1), (2, 2), (3, 1), (3, 2), (4, 1), (4, 2)\}$

Now, $M_{R_t} = M_R \vee M_{R^2} \vee M_{R^3} \vee M_{R^4}$

$$= \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} \quad \text{(by using Boolean Algebra)}$$

i.e., $R_t = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\}$ which is the required transitive closure of R.

2.21 WARSHAL'S ALGORITHM

If the number of elements in the set A is large, then the above method of finding the transitive closure of the relation R on set A is very difficult. Fortunately a more efficient algorithm for computing transitive closure is available. It is known as Warshal's algorithm. We describe this algorithm as follows:

Let R be a relation on a set $A = \{a_1, a_2, ..., a_n\}$. Now for $1 \le k \le n$, we define a Boolean matrix w_k as follows. w_k has a 1 in position i, j, if and only if there is a path from a_i to a_j in R whose interior vertices, if any, come from the set $\{a_1, a_2, ..., a_k\}$.

Since any vertex come from the set A, therefore, the matrix w_n has a 1 in position i, j if and only if some path in R connects a_i with a_j. If we define w_0 to M_R, then we will have a sequence $w_0, w_1, w_2, \ldots, w_n$ whose first term is M_R and whose last term is M_{R^∞}, where

$$M_{R^\infty} = M_R \vee M_{R^2} \vee M_{R^3} \vee \ldots$$

Now, we will show how to compute each matrix w_k from the previous matrix w_{k-1}. Then we can start with the matrix M_R and proceed one step at a time until, in n steps, we reach the matrix M_{R^∞}. This procedure is called Warshal's algorithm. We have the following procedure for computing w_k from w_{k-1}.

Step 1: First copy to w_k all 1's from w_{k-1}

Step 2: List the locations $c_1, c_2, c_3, \ldots$, in column k of w_{k-1}, where the entry is 1, and the locations $r_1, r_2, r_3, \ldots$, in row k of w_{k-1}, where the entry is 1.

Step 3: Put 1's in all the positions c_i, r_j of w_k (if they are not already there).

Example 44 If $R = \{(a, b), (b, c), (c, d), (b, a)\}$ is a relation on set $A = \{a, b, c, d\}$, then find the transitive closure of R by using Warshal's algorithm.

Solution The matrix w_0 is given as

$$w_0 = M_R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

First we compute w_1 so that $k = 1.w_0$ has 1's in location 2 of column 1 and location 2 of row 1. Thus, w_1 is just same as w_0 with a new 1 in position (2, 2).

$$w_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Now, we compute w_2, i.e., $k = 2$. The matrix w_1 has 1's in column locations 1 and 2 of column 2 and locations 1, 2 and 3 of row 2.

Thus, to obtain w_2, we must put 1's in positions (1, 1), (1, 2), (1, 3), (2, 1), (2, 2) and (2, 3) of matrix w_1 (if 1's are not already there), i.e.

$$w_2 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Now, we compute w_3 i.e., $k = 3$. We see that column 3 of w_2 has 1's in locations 1 and 2 and row 3 of w_2 has a 1 in location 4. Thus to obtain w_3, we must put 1's in positions (1, 4) and (2, 4) of w_2, i.e.

$$w_3 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Finally, w_3 has 1's in locations 1, 2, 3 of column 4 and no 1's in row 4, so no new 1's are added and $M_{R^\infty} = M_{R_t} = w_4 = w_3$.

Thus, the transitive closure of the given relation is given as

$$R_t = \{(a, a), (a, b), (a, c), (a, d), (2, a), (2, b), (2, c), (2, d), (3, d)\}.$$

Theorem 10: If R and S are two equivalence relations on a set A, then the smallest equivalence relation containing both R and S is $(R \cup S)^\infty$.

Proof: Recall that I_A is the identity relation on set A and that a relation is reflexive iff it contains I_A.

Since both R and S is also symmetric. Because of this, all paths in $R \cup S$ are reflexive, therefore, $I_A \subseteq R$ and $I_A \subseteq S$

$\Rightarrow \quad I_A \subseteq R \cup S \subseteq (R \cup S)^\infty$, and $(R \cup S)^\infty$ is also reflexive.

Since R and S are symmetric, therefore, $R = R^{-1}$ and $S = S^{-1}$, so $(R \cup S')^{-1} = R^{-1} \cup S^{-1} = R \cup S$, and $R \cup S$ is also symmetric. Because of this, all path in $R \cup S$ are "two-way streets" and it follows from the definitions that $(R \cup S)^\infty$ must also be symmetric. As we know that $(R \cup S)^\infty$ is transitive, therefore, $(R \cup S)^\infty$ is an equivalence relation containing $R \cup S$.

Example 45 Let $A = \{1, 2, 3, 4, 5\}$ and $R = \{(1, 1), (1, 2), (2,. 1), (2, 2), (3, 3), (3, 4), (4, 3), (4, 4), (5, 5)\}$ and $S = \{(1, 1), (2, 2), (3, 3), (4, 4), (4, 5), (5, 4), (5, 5)\}$ be two equivalence relations on set A. The partition A/R of A corresponding to R is $\{\{1, 2\}, \{3, 4\}, \{5\}\}$ and the partition A/S of A corresponding to R is $\{\{1\}, \{2\}, \{3\}, \{4, 5\}$. Find the smallest equivalence relation containing R and S, and compute the partition of A that it produces.

Solution We have

$$M_R = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad M_S = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$M_{R \cup S} = M_R \vee M_S = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Now, we compute $M_{(R\cup S)^\infty}$ by Warshal's algorithm.

$$\text{First} \qquad w_0 = M_{R\cup S} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Now, we compute w_1, i.e., $k = 1$. Since w_0 has 1's in locations 1 and 2 of column 1 and in locations 1 and 2 of row 1. We can see that there is no change, i.e.

$$w_1 = w_0$$

Similarly, we can see that

$$w_2 = w_1 \quad \text{and} \quad w_3 = w_2$$

Now, we compute w_4, i.e., $k = 4$. Since w_3 has 1's in locations 3, 4, and 5 of column 4 and in locations 3, 4 and 5 of row 4. We must add new 1's to w_3 in partition (3, 5) and (5, 3). Thus.

$$w_4 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

Now, we can verify that $w_5 = w_4$ and thus

$$(R \cup S)^\infty = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (3, 4), (3, 5), (4, 3), (4, 4), (4, 5), (5, 3), (5, 4), (5, 5)\}$$

Now find the corresponding partition of A.

$$[1] = \{1, 2\}$$
$$[2] = \{1, 2\}$$
$$[3] = \{3, 4, 5\}$$
$$[4] = \{3, 4, 5\}$$
$$[5] = \{3, 4, 5\}$$

Thus,

$$A/(R \cup S)^\infty = \{\{1, 2\}, \{3, 4, 5\}\}$$

EXERCISE

1. A relation R on the set $A = \{2, 3, 5, 8, 10, 15\}$ is given by "x divides y", for all $x, y \in A$. Then find the domain, range, and digraph of R.

2. If $A = \{1, 2, 3\}$ and $B = \{a, b, c, d\}$, then find $A \times B$, $B \times A$, and $A \times A$.
3. If $A = \{a, b, c, d\}$ and $B = \{1, 2, 3\}$. If R is the relation from A to B, then find the matrix of R^{-1} and R^2.
4. Which of the following relations are reflexive, symmetric and transitive on set $A = \{1, 2, 3, 4\}$
   (i) $R_1 = \{(1, 1), (1, 2), (2, 3), (2, 4), (3, 4)\}$
   (ii) $R_2 = \phi$, the empty relation.
   (iii) $R_3 = A \times A$
   (iv) R_4 = Relation $\leq$ (less than or equal)
   (v) R_5 = Relation | (divides)
5. Let R be a relation on set N, defined by the equation $x + 3y = 12$, i.e. $R = \{(x, y) : x + 3y = 12\}$. Write.
   (i) R as the set of ordered pairs
   (ii) Find R^{-1}
   (iii) Find the composition of RoR.
6. Let $A = \{1, 2, 3\}$ and let $P(A)$ be the power set of A. Define the relation R on $P(A)$ as follows. If B and C are subsets of A, then BRC if $A \subset B$. Draw a digraph for this relation. Determine its reflexivity, transitivity and antisymmetry.
7. Which of the following relations are equivalence relations on the set of integers.
   (i) aRb iff $a > b$
   (ii) aRb iff $a + b = 6$
   (iii) aRb iff $a + b < 6$
8. Let $R = \{(a, a), (a, b), (b, b), (b, c), (c, a), (c, c), (d, a), (d, b)\}$ be a relation on set $A = \{a, b, c, d\}$. Then find.
   (i) reflexive, symmetric and transitive closure of R.
   (ii) R^∞, R^n and R^*.
9. Let R be a relation defined by xRy if and only if $|x - y|$ is even. Show that R is an equivalence relation on set N.
10. Partition the set $A = \{1, 2, 3, 4, 5, 6\}$ by $\{\{1, 2\}, \{3\}, \{4, 5\}, \{6\}\}$. List the symmetric relation determined by this relation.
11. If $A = \{1, 2, 3, 4, 5, 6, 7\}$ and $R = \{(x, y) : x - y$ is divisible by $3\}$, show that R is an equivalence relation. Also find the equivalence class of each element of A.
12. Consider a set $A = \{1, 3, 5\}$. Let R be a relation defined by $xRy \Leftrightarrow y = x + 2$ and S be a relation defined by $xSy \Leftrightarrow x \leq y$. Then find
   (i) RoS
   (ii) SoR
   (iii) Is $RoS = SoR$.

13. In Q.13 find $R \cup S$, $R \cap S$, RoS and R^{-1} by using Boolean matrices.
14. Consider the set A = {1, 5, 6, 7} and the relation R on set A is given by R = {(4, 5), (5, 6), (5, 7), (6, 6), (6, 7), (7, 6), (7, 7). Determine (i) R^3 (ii) R^∞.
15. Consider the digraph of the relation R on set A = {1, 2, 3, 4, 5} as shown in Fig. 2.7.

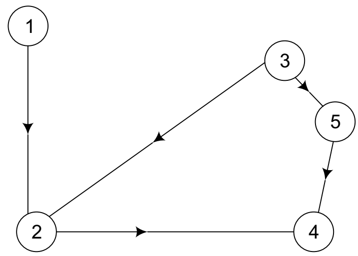

Fig. 2.7

Determine (i) R^* (ii) R^∞.

16. Using Warshal's algorithm, find the transitive closure of R defined on A = {1, 2, 3, 4} and R = {(1, 1), (2, 1), (2, 2), (2, 3), (3, 4), (4, 4)}
17. Let A = {a, b, c, d, e} and let R and S be two relations on set A described by

$$M_R = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad M_S = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

Use Warshal's algorithm to compute the transitive closure of $R \cup S$.

18. Let A = {1, 2, 3, ..., 15}. Let R be the equivalence relation on A, defined by (a, b) R(c, d) if $ad = bc$. Find the equivalence class of (3, 2).
19. Show that the relation R, consisting of all pairs (x, y) where x and y are bits strings of length 3 or more that agree in their first three bits, is an equivalence relation on the set of all bit strings of length three or more.
20. Obtain the distinct equivalence classes of the relation R "congruence modulo 5" on the set I.