F Business and Economic Applications

- **Understand basic business terms and formulas, determine marginal revenues, costs, and profits, find demand functions, and solve business and economic optimization problems.**

Business and Economic Applications

Previously, you learned that one of the most common ways to measure change is with respect to time. In this section, you will study some important rates of change in economics that are not measured with respect to time. For example, economists refer to **marginal profit, marginal revenue,** and **marginal cost** as the rates of change of the profit, revenue, and cost with respect to the number of units produced or sold.

SUMMARY OF BUSINESS TERMS AND FORMULAS

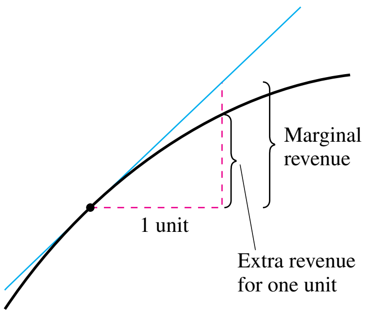
Basic Terms	**Basic Formulas**
x is the number of units produced (or sold).	
p is the price per unit.	
R is the total revenue from selling x units.	$R = xp$
C is the total cost of producing x units.	
$\overline{C}$ is the average cost per unit.	$\overline{C} = \dfrac{C}{x}$
P is the total profit from selling x units.	$P = R - C$
The **break-even point** is the number of units for which $R = C$.	

Marginals

$\dfrac{dR}{dx}$ = marginal revenue ≈ *extra* revenue from selling one additional unit

$\dfrac{dC}{dx}$ = marginal cost ≈ *extra* cost of producing one additional unit

$\dfrac{dP}{dx}$ = marginal profit ≈ *extra* profit from selling one additional unit



A revenue function
Figure F.1

In this summary, note that marginals can be used to approximate the *extra* revenue, cost, or profit associated with selling or producing one additional unit. This is illustrated graphically for marginal revenue in Figure F.1.

EXAMPLE 1 Finding the Marginal Profit

A manufacturer determines that the profit P (in dollars) derived from selling x units of an item is given by

$$P = 0.0002x^3 + 10x.$$

a. Find the marginal profit for a production level of 50 units.

b. Compare the marginal profit with the actual gain in profit obtained by increasing production from 50 to 51 units.

Solution

a. Because the profit is $P = 0.0002x^3 + 10x$, the marginal profit is given by the derivative

$$\frac{dP}{dx} = 0.0006x^2 + 10.$$

When $x = 50$, the marginal profit is

$$\frac{dP}{dx} = (0.0006)(50)^2 + 10 \qquad \text{Substitute 50 for } x.$$

$$= \$11.50 \text{ per unit.} \qquad \text{Marginal profit for } x = 50$$

b. For $x = 50$, the actual profit is

$$P = (0.0002)(50)^3 + 10(50) \qquad \text{Substitute 50 for } x.$$

$$= 25 + 500$$

$$= \$525.00 \qquad \text{Actual profit for } x = 50$$

and for $x = 51$, the actual profit is

$$P = (0.0002)(51)^3 + 10(51) \qquad \text{Substitute 51 for } x.$$

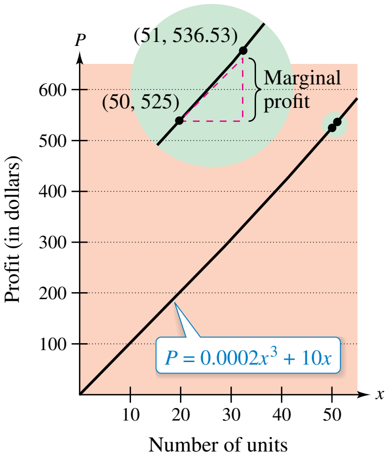
$$= 26.53 + 510$$

$$= \$536.53. \qquad \text{Actual profit for } x = 51$$

So, the additional profit obtained by increasing the production level from 50 to 51 units is

$$536.53 - 525.00 = \$11.53. \qquad \text{Extra profit for one unit}$$

Note that the actual profit increase of \$11.53 (when x increases from 50 to 51 units) can be approximated by the marginal profit of \$11.50 per unit (when $x = 50$), as shown in Figure F.2.



Marginal profit is the extra profit from selling one additional unit.
Figure F.2

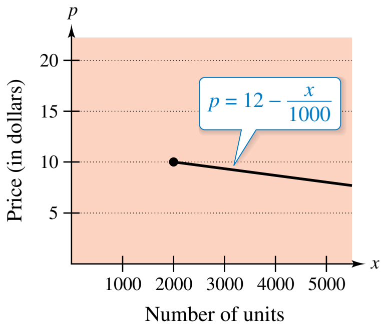
The profit function in Example 1 is unusual in that the profit continues to increase as long as the number of units sold increases. In practice, it is more common to encounter situations in which sales can be increased only by lowering the price per item. Such reductions in price ultimately cause the profit to decline.

The number of units x that consumers are willing to purchase at a given price per unit p is given by the **demand function**

$$p = f(x). \qquad \text{Demand function}$$

EXAMPLE 2 Finding a Demand Function

A business sells 2000 items per month at a price of \$10 each. It is estimated that monthly sales will increase by 250 items for each \$0.25 reduction in price. Find the demand function corresponding to this estimate.



A demand function p

Figure F.3

Solution From the estimate, x increases 250 units each time p drops \$0.25 from the original cost of \$10. This is described by the equation

$$\begin{aligned} x &= 2000 + 250\left(\frac{10 - p}{0.25}\right) \\ &= 12{,}000 - 1000p \end{aligned}$$

or

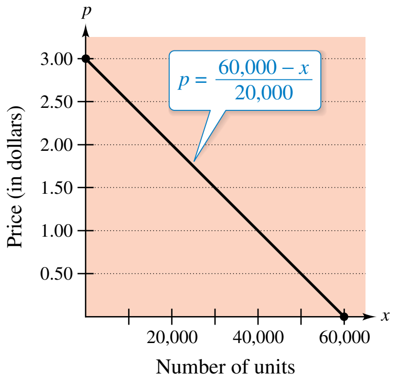
$$p = 12 - \frac{x}{1000}, \quad x \geq 2000. \qquad \text{Demand function}$$

The graph of the demand function is shown in Figure F.3.

EXAMPLE 3 Finding the Marginal Revenue

A fast-food restaurant has determined that the monthly demand for its hamburgers is

$$p = \frac{60{,}000 - x}{20{,}000}$$



As the price decreases, more hamburgers are sold.

Figure F.4

where p is the price per hamburger (in dollars) and x is the number of hamburgers. Find the increase in revenue per hamburger (marginal revenue) for monthly sales of 20,000 hamburgers. (See Figure F.4.)

Solution Because the total revenue is given by $R = xp$, you have

$$\begin{aligned} R &= xp && \text{Formula for revenue} \\ &= x\left(\frac{60{,}000 - x}{20{,}000}\right) && \text{Substitute for } p. \\ &= \frac{1}{20{,}000}(60{,}000x - x^2). && \text{Revenue function} \end{aligned}$$

By differentiating, you can find the marginal revenue to be

$$\frac{dR}{dx} = \frac{1}{20{,}000}(60{,}000 - 2x).$$

When $x = 20{,}000$, the marginal revenue is

$$\begin{aligned} \frac{dR}{dx} &= \frac{1}{20{,}000}[60{,}000 - 2(20{,}000)] && \text{Substitute 20,000 for } x. \\ &= \frac{20{,}000}{20{,}000} \\ &= \$1 \text{ per unit.} && \text{Marginal revenue for } x = 20{,}000 \end{aligned}$$

The demand function in Example 3 is typical in that a high demand corresponds to a low price, as shown in Figure F.4.

EXAMPLE 4 Finding the Marginal Profit

For the fast-food restaurant in Example 3, the cost C (in dollars) of producing x hamburgers is

$$C = 5000 + 0.56x, \quad 0 \le x \le 50{,}000.$$

Find the total profit and the marginal profit for 20,000, 24,400, and 30,000 hamburgers.

Solution Because $P = R - C$, you can use the revenue function in Example 3 to obtain

$$P = \frac{1}{20{,}000}(60{,}000x - x^2) - 5000 - 0.56x$$

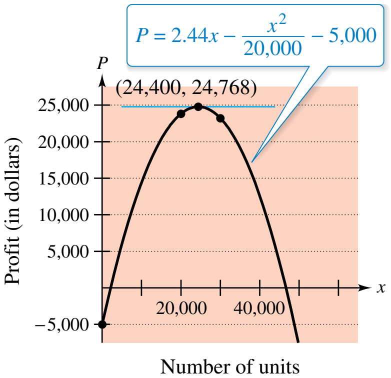
$$= 2.44x - \frac{x^2}{20{,}000} - 5000.$$

So, the marginal profit is

$$\frac{dP}{dx} = 2.44 - \frac{x}{10{,}000}.$$

The table shows the total profit and the marginal profit for each of the three indicated demands. Figure F.5 shows the graph of the profit function.

Demand	20,000	24,400	30,000
Profit	\$23,800	\$24,768	\$23,200
Marginal profit (per hamburger)	\$0.44	\$0.00	−\$0.56



The maximum profit corresponds to the point where the marginal profit is 0. When more than 24,400 hamburgers are sold, the marginal profit is negative—increasing production beyond this point will *reduce* rather than increase profit.

Figure F.5

EXAMPLE 5 Finding the Maximum Profit

The marketing department of a business has determined that the demand for a product is

$$p = \frac{50}{\sqrt{x}} \qquad \text{Demand function}$$

where p is the price per unit (in dollars) and x is the number of units. The cost C (in dollars) of producing x units is given by $C = 0.5x + 500$. Find the price per unit that yields a maximum profit.

Solution From the cost function, you obtain

$$P = R - C = xp - (0.5x + 500).$$

Substituting for p (from the demand function) produces

$$P = x\left(\frac{50}{\sqrt{x}}\right) - (0.5x + 500) = 50\sqrt{x} - 0.5x - 500.$$

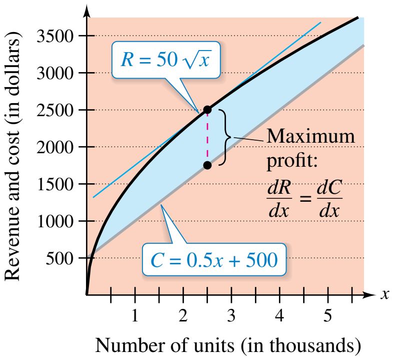
Setting the marginal profit equal to 0

$$\frac{dP}{dx} = \frac{25}{\sqrt{x}} - 0.5 = 0$$

yields $x = 2500$. From this, you can conclude that the maximum profit occurs when the price is

$$p = \frac{50}{\sqrt{2500}} = \frac{50}{50} = \$1. \qquad \text{Price per unit}$$

See Figure F.6.



Maximum profit occurs when $\frac{dR}{dx} = \frac{dC}{dx}$.

Figure F.6

To find the maximum profit in Example 5, the profit function, $P = R - C$, was differentiated and set equal to 0. From the equation

$$\frac{dP}{dx} = \frac{dR}{dx} - \frac{dC}{dx} = 0$$

it follows that the maximum profit occurs when the marginal revenue is equal to the marginal cost, as shown in Figure F.6.

EXAMPLE 6 Minimizing the Average Cost

A company estimates that the cost C (in dollars) of producing x units of a product is given by $C = 800 + 0.04x + 0.0002x^2$. Find the production level that minimizes the average cost per unit.

Solution Substituting from the equation for C produces

$$\overline{C} = \frac{C}{x} = \frac{800 + 0.04x + 0.0002x^2}{x} = \frac{800}{x} + 0.04 + 0.0002x.$$

Next, find $d\overline{C}/dx$.

$$\frac{d\overline{C}}{dx} = -\frac{800}{x^2} + 0.0002$$

Then set $d\overline{C}/dx$ equal to 0 and solve for x.

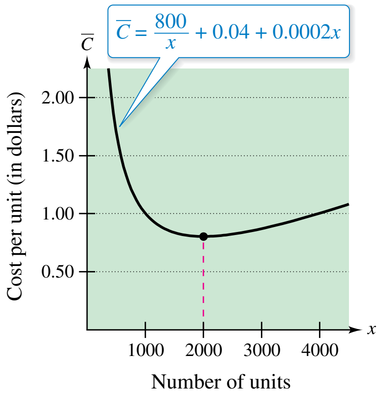
$$-\frac{800}{x^2} + 0.0002 = 0$$

$$0.0002 = \frac{800}{x^2}$$

$$x^2 = 4{,}000{,}000$$

$$x = 2000 \text{ units}$$

A production level of 2000 units minimizes the average cost per unit. See Figure F.7.



Minimum average cost occurs when $\frac{d\overline{C}}{dx} = 0$.

Figure F.7

F Exercises

1. **Think About It** The figure shows the cost C of producing x units of a product.
   (a) What is $C(0)$ called?
   (b) Sketch a graph of the marginal cost function.
   (c) Does the marginal cost function have an extremum? If so, describe what it means in economic terms.

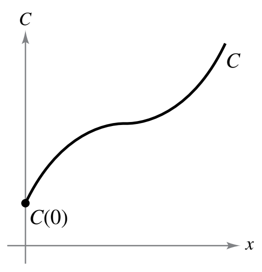


Figure for 1

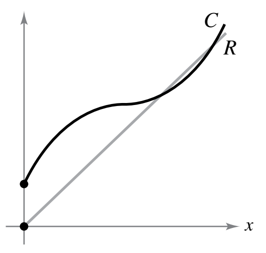


Figure for 2

2. **Think About It** The figure shows the cost C and revenue R for producing and selling x units of a product.
   (a) Sketch a graph of the marginal revenue function.
   (b) Sketch a graph of the profit function. Approximate the value of x for which profit is maximum.

Maximum Revenue **In Exercises 3–6, find the number of units x that produces a maximum revenue R.**

3. $R = 900x - 0.1x^2$
4. $R = 600x^2 - 0.02x^3$
5. $R = \dfrac{1{,}000{,}000x}{0.02x^2 + 1800}$
6. $R = 30x^{2/3} - 2x$

Average Cost **In Exercises 7–10, find the number of units x that produces the minimum average cost per unit $\overline{C}$.**

7. $C = 0.125x^2 + 20x + 5000$
8. $C = 0.001x^3 - 5x + 250$
9. $C = 3000x - x^2\sqrt{300 - x}$
10. $C = \dfrac{2x^3 - x^2 + 5000x}{x^2 + 2500}$

Maximum Profit **In Exercises 11–14, find the price per unit p (in dollars) that produces the maximum profit P.**

	Cost Function	Demand Function
11.	$C = 100 + 30x$	$p = 90 - x$
12.	$C = 2400x + 5200$	$p = 6000 - 0.4x$
13.	$C = 4000 - 40x + 0.02x^2$	$p = 50 - \dfrac{x}{100}$
14.	$C = 35x + 2\sqrt{x - 1}$	$p = 40 - \sqrt{x - 1}$

Average Cost **In Exercises 15 and 16, use the cost function to find the value of x at which the average cost is a minimum. For that value of x, show that the marginal cost and average cost are equal.**

15. $C = 2x^2 + 5x + 18$
16. $C = x^3 - 6x^2 + 13x$

17. **Proof** Prove that the average cost is a minimum at the value of x where the average cost equals the marginal cost.

18. **Maximum Profit** The profit P for a company is

$$P = 230 + 20s - \tfrac{1}{2}s^2$$

where s is the amount (in hundreds of dollars) spent on advertising. What amount of advertising produces a maximum profit?

19. **Numerical, Graphical, and Analytic Analysis** The cost per unit for the production of a product is \$60. The manufacturer charges \$90 per unit for orders of 100 or less. To encourage large orders, the manufacturer reduces the charge by \$0.15 per unit for each unit ordered in excess of 100 (for example, there would be a charge of \$87 per unit for an order size of 120).

(a) Analytically complete six rows of a table such as the one below. (The first two rows are shown.)

x	Price	Profit
102	$90 - 2(0.15)$	$102[90 - 2(0.15)] - 102(60) = 3029.40$
104	$90 - 4(0.15)$	$104[90 - 4(0.15)] - 104(60) = 3057.60$

(b) Use a graphing utility to generate additional rows of the table. Use the table to estimate the maximum profit. (*Hint:* Use the *table* feature of the graphing utility.)

(c) Write the profit P as a function of x.

(d) Use calculus to find the order size that produces a maximum profit.

(e) Use a graphing utility to graph the function in part (c) and verify the maximum profit from the graph.

20. **Maximum Profit** A real estate office handles 50 apartment units. When the rent is \$720 per month, all units are occupied. However, on average, for each \$40 increase in rent, one unit becomes vacant. Each occupied unit requires an average of \$48 per month for service and repairs. What rent should be charged to obtain a maximum profit?

21. **Minimum Cost** A power station is on one side of a river that is $\frac{1}{2}$-mile wide, and a factory is 6 miles downstream on the other side. It costs \$18 per foot to run power lines over land and \$25 per foot to run them underwater. Find the most economical path for the transmission line from the power station to the factory.

22. **Maximum Revenue** When a wholesaler sold a product at \$25 per unit, sales were 800 units per week. After a price increase of \$5, the average number of units sold dropped to 775 per week. Assume that the demand function is linear, and find the price that will maximize the total revenue.

23. **Minimum Cost** The ordering and transportation cost C (in thousands of dollars) of the components used in manufacturing a product is

$$C = 100\left(\frac{200}{x^2} + \frac{x}{x + 30}\right), \quad x \geq 1$$

where x is the order size (in hundreds). Find the order size that minimizes the cost. (*Hint:* Use Newton's Method or the *zero* feature of a graphing utility.)

24. **Average Cost** A company estimates that the cost C (in dollars) of producing x units of a product is

$$C = 800 + 0.4x + 0.02x^2 + 0.0001x^3.$$

Find the production level that minimizes the average cost per unit. (*Hint:* Use Newton's Method or the *zero* feature of a graphing utility.)

25. **Revenue** The revenue R for a company selling x units is

$$R = 900x - 0.1x^2.$$

Use differentials to approximate the change in revenue when sales increase from $x = 3000$ to $x = 3100$ units.

26. **Analytic and Graphical Analysis** A manufacturer of fertilizer finds that the national sales of fertilizer roughly follow the seasonal pattern

$$F = 100{,}000\left\{1 + \sin\left[\frac{2\pi(t - 60)}{365}\right]\right\}$$

where F is measured in pounds and time t is measured in days, with $t = 1$ corresponding to January 1.

(a) Use calculus to determine the day of the year when the maximum amount of fertilizer is sold.

(b) Use a graphing utility to graph the function and approximate the day of the year when sales are minimum.

27. Modeling Data The table shows the monthly sales G (in thousands of gallons) of gasoline at a gas station in 2016. The time in months is represented by t, with $t = 1$ corresponding to January 2016.

t	1	2	3	4	5	6
G	8.91	9.18	9.79	9.83	10.37	10.16

t	7	8	9	10	11	12
G	10.37	10.81	10.03	9.97	9.85	9.51

A model for these data is

$$G = 9.90 - 0.64\cos\left(\frac{\pi t}{6} - 0.62\right).$$

(a) Use a graphing utility to plot the data and graph the model.

(b) Use the model to approximate the month when gasoline sales were greatest.

(c) What factor in the model causes the seasonal variation in sales of gasoline? What part of the model gives the average monthly sales of gasoline?

(d) The gas station adds the term $0.02t$ to the model. What does the inclusion of this term mean? Use this model to estimate the maximum monthly sales in 2020.

28. Airline Revenues The annual revenue R (in millions of dollars) for an airline for the years 2007–2016 can be modeled by

$$R = 4.6t^4 - 193.5t^3 + 2941.7t^2 - 19{,}294.7t + 48{,}500$$

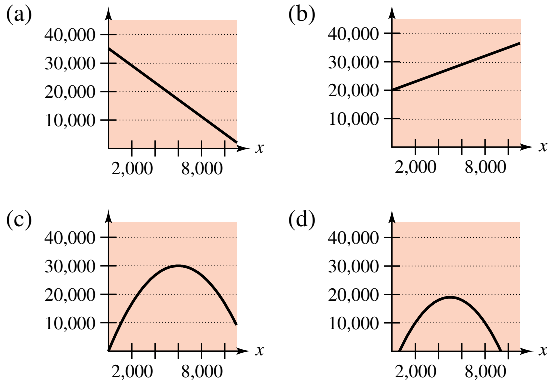
where $t = 7$ corresponds to 2007.

(a) During which year (between 2007 and 2016) was the revenue of the airline a minimum? What was the minimum revenue?

(b) During which year (between 2007 and 2016) was the revenue a maximum? What was the maximum revenue?

(c) Use a graphing utility to confirm the results in parts (a) and (b).

29. Think About It Match each graph with the function it best represents—a demand function, a revenue function, a cost function, or a profit function. Explain your reasoning. [The graphs are labeled (a), (b), (c), and (d).]



Elasticity **The relative responsiveness of consumers to a change in the price of an item is called the *price elasticity of demand*. If $p = f(x)$ is a differentiable demand function, then the price elasticity of demand is**

$$\eta = \frac{p/x}{dp/dx}$$

where η is the lowercase Greek letter eta. For a given price, if $|\eta| < 1$, then the demand is *inelastic*. If $|\eta| > 1$, then the demand is *elastic*. In Exercises 30–33, find η for the demand function at the indicated x-value. Is the demand elastic, inelastic, or neither at the indicated x-value?

30. $p = 400 - 3x$

$x = 20$

31. $p = 5 - 0.03x$

$x = 100$

32. $p = 400 - 0.5x^2$

$x = 20$

33. $p = \dfrac{500}{x + 2}$

$x = 23$