Chapter 2 Manifolds

We are concerned mainly with smooth manifolds.

5. Smooth Functions on a Euclidean Space

5.1. Topological Manifolds

Definition 5.1 A topological space is **second countable** if it has a countable base.

Definition 5.2 A **neighborhood** of a point p in a topological space M is any open set $U \subset M$ such that $p \in U$.

Definition 5.3 An **open cover** of a topological space M is a collection $\{U_\alpha\}_{\alpha \in A}$ of open sets such that $M = \bigcup_{\alpha \in A} U_\alpha$.

Definition 5.4 A topological space M is **locally Euclidean of dimension n** if every point $p \in M$ has a neighborhood U such that there is a homeomorphism ϕ from U to an open subset of R^n.

The pair $(U, \phi : U \to \mathbb{R}^n)$ is called a **chart** of M at p, U is called the **coordinate neighborhood** or **coordinate open set** of p, and ϕ is called the **coordinate map** or **coordinate system** on U. A chart (U, ϕ) is **centered** at $p \in U$ if $\phi(p) = 0$.

Definition 5.5 A **topological manifold** is a Hausdorff, second countable, locally Euclidean space. It is of **dimension n** if it is locally Euclidean of dimension n.

Corollary 5.6 (invarience of dimension) An open subset of $\mathbb{R}^n$ is not homeomorphic to an open subset of $\mathbb{R}^m$ if $n \neq m$.

If a topological space has several connected components, it is possible for each component to have a different dimension.

Example 5.7 The Euclidean space $\mathbb{R}^n$ is covered by a single chart $(\mathbb{R}^n, \mathbb{1}_{\mathbb{R}^n})$, where $\mathbb{1}_{\mathbb{R}^n} : \mathbb{R}^n \to \mathbb{R}^n$ is the identity map. Every open subset of $\mathbb{R}^n$ is also a topological manifold, with chart $(U, \mathbb{1}_U)$.

Proposition 5.8 The Hausdorff condition and second countability are "hereditary" properties, i.e.,
(i) A subspace of a Hausdorff space is Hausdorff,
(ii) A subspace of a second-countable space is second countable.

Example 5.9 (A cusp) The graph of $y = x^{\frac{2}{3}}$ in $\mathbb{R}^2$ is a topological manifold. By virtue of being a subspace of $\mathbb{R}^2$, it is Hausdorff and second countable. It is locally Euclidean of dimension 1, since it is homeomorphic to $\mathbb{R}$ via $\left(x, x^{\frac{2}{3}}\right) \mapsto x$

Example 5.10 (A cross) The cross in $\mathbb{R}^2$ with subspace topology is not locally Euclidean at the intersection p, and so cannot be a topological manifold.

Solution. Suppoese the cross is locally Euclidean of dimension n at p. Then there is a neighborhood U of p homeomorphic to an open ball $B \coloneqq B(0, \varepsilon) \subset \mathbb{R}^n$ with p mapping to 0. Then $U - \{p\}$ is homeomorphic to $B - \{0\}$. Since $B - \{0\}$ is connected if $n \geq 2$ or has two connected components if $n = 1$, but $U - \{p\}$ has 4 connected components, $U - \{p\}$ cannot be homeomorphic to $B - \{0\}$, contradicting the assumption that U is homeomorphic to B.

5.2. Compatible Charts

Suppose $(U, \phi : U \to \mathbb{R}^n)$ and $(V, \psi : V \to \mathbb{R}^n)$ are two charts of a topological manifold. Since $U \cap V$ is open in U, the image $\phi(U \cap V)$ is open in $\mathbb{R}^n$. Similarly, $\psi(U \cap V)$ is open in $\mathbb{R}^n$.

Definition 5.11 Two charts $(U, \phi : U \to \mathbb{R}^n), (V, \psi : V \to \mathbb{R}^n)$ of a topological manifold are ***C*$^\infty$-compatible** if the two maps

$$\phi \circ \psi^{-1} : \psi(U \cap V) \to \phi(U \cap V), \quad \psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V) \tag{1}$$

are C^∞. These two maps are called the **transition functions** between the charts (U, ϕ) and (V, ψ). If $U \cap V$ is empty, then the two charts are automatically C^∞-compatible. To simplify the notation, we will write $U_{\alpha\beta}$ for $U_\alpha \cap U_\beta$ and $U_{\alpha\beta\gamma}$ for $U_\alpha \cap U_\beta \cap U_\gamma$. Since we are interested only in C^∞-compatible charts, we often omit mention of "C^∞" and speak simply of "compatible charts".

Definition 5.12 A $\boldsymbol{C^\infty}$ **atlas** or simply an **atlas** on a locally Euclidean space M is a collection $\mathfrak{U} = \{(U_\alpha, \phi_\alpha)\}$ of pairwise C^∞-compatible charts that cover M, i.e., $M = \bigcup_\alpha U_\alpha$.

Example 5.13 The Unit circle S^1 in the complex plane $\mathbb{C}$ may by described as the set of points $\{e^{it} \in \mathbb{C} \mid 0 \le t \le 2\pi\}$. Let U_1 and U_2 be the open sets of S^1 defined by

$$\begin{aligned} U_1 &= \{e^{it} \in \mathbb{C} \mid -\pi < t < \pi\}, \\ U_2 &= \{e^{it} \in \mathbb{C} \mid 0 < t < 2\pi\}, \end{aligned} \tag{2}$$

and $\phi_\alpha : U_\alpha \to \mathbb{R}$ for $\alpha = 1, 2$ by

$$\begin{aligned} \phi_1(e^{it}) &= t, \quad -\pi < t < \pi, \\ \phi_2(e^{it}) &= t, \quad 0 < t < 2\pi, \end{aligned} \tag{3}$$

where ϕ_1 and ϕ_2 are both homeomorphisms from U_1 and U_2 to open subsets of $\mathbb{R}$. Thus, (U_1, ϕ_1) and (U_2, ϕ_2) are charts on S^1. The intersection $U_1 \cap U_2$ consists of two connected components,

$$\begin{aligned} A &= \{e^{it} \mid -\pi < t < 0\}, \\ B &= \{e^{it} \mid 0 < t < \pi\}, \end{aligned} \tag{4}$$

with

$$\begin{aligned} \phi_1(U_1 \cap U_2) &= \phi_1(A \sqcup B) = \phi_1(A) \sqcup \phi_1(B) = (-\pi, 0) \sqcup (0, \pi), \\ \phi_2(U_1 \cap U_2) &= \phi_2(A \sqcup B) = \phi_2(A) \sqcup \phi_2(B) = (\pi, 2\pi) \sqcup (0, \pi), \end{aligned} \tag{5}$$

where $\sqcup$ denotes the disjoint union. The transition functions are given by

$$\begin{aligned} (\phi_2 \circ \phi_1^{-1})(t) &= \begin{cases} t + 2\pi & \text{for } t \in (-\pi, 0) \\ t & \text{for } t \in (0, \pi) \end{cases}, \\ (\phi_1 \circ \phi_2^{-1})(t) &= \begin{cases} t - 2\pi & \text{for } t \in (\pi, 2\pi) \\ t & \text{for } t \in (0, \pi) \end{cases}, \end{aligned} \tag{6}$$

which means that the charts (U_1, ϕ_1) and (U_2, ϕ_2) are C^∞-compatible and then form a C^∞ atlas.

Remark 5.14 Although the C^∞ compatibility of charts is clearly reflexive and symmetric, it is not transitive. For example, Suppose (U_1, ϕ_1) is C^∞-compatible with (U_2, ϕ_2) and (U_2, ϕ_2) is C^∞-compatible with (U_3, ϕ_3). Then the composite

$$\phi_3 \circ \phi_1^{-1} = (\phi_3 \circ \phi_2^{-1}) \circ (\phi_2 \circ \phi_1^{-1}) \tag{7}$$

is C^∞, but only on $\phi_1(U_{123})$, not necessarily on $\phi_1(U_{13})$.

Definition 5.15 A chart (V, ψ) is **compatible** with an atlas $\{(U_\alpha, \phi_\alpha)\}$ if it is compatible with all the charts (U_α, ϕ_α) of the atlas.

Lemma 5.16 Let $\{(U_\alpha, \phi_\alpha)\}$ be an atlas on a locally Euclidean space M. If two charts (V, ψ) and (W, σ) are both compatible with the atlas $\{(U_\alpha, \phi_\alpha)\}$, then they are compatible with each other.

Proof. Let $p \in V \cap W$. Since $\{(U_\alpha, \phi_\alpha)\}$ is an atlas for M, $p \in U_\alpha$ for some α, then $p \in V \cap W \cap U_\alpha$. The composite

$$\sigma \circ \psi^{-1} = \left(\sigma \circ \phi_\alpha^{-1}\right) \circ \left(\phi_\alpha \circ \psi^{-1}\right) \tag{8}$$

is C^∞ on $\psi(V \cap W \cap U_\alpha)$, hence at $\psi(p)$. Since p is arbitrary, the composite $\sigma \circ \psi^{-1}$ is C^∞ on $\psi(V \cap W)$. Similarly, $\psi \circ \sigma^{-1}$ is C^∞ on $\psi(V \cap W)$. Thus, the charts (V, ψ) and (W, σ) are compatible. □

5.3. Smooth Manifolds

Definition 5.17 An atlas $\mathfrak{M}$ on a locally Euclidean space M is **maximal** if it is not contained in a larger atlas, i.e., if $\mathfrak{M} \subseteq \mathfrak{U}$, then $\mathfrak{M} = \mathfrak{U}$.

Definition 5.18 A **smooth manifold** or $\boldsymbol{C^\infty}$ **manifold** is a topological manifold M together with a maximal atlas $\mathfrak{M}$. The maximal atlas $\mathfrak{M}$ is called the **differential structure** on M. A manifold is said to have dimension n if all of its connected components have dimension n. A 1-dimensional manifold is called a **curve**, a 2-dimensional manifold is called a **surface**, and an n-dimensional manifold is called an $\boldsymbol{n}$**-manifold**.

In parctice, to check that a topological manifold M is a smooth manifold, it is not necessary to exhibit a maximal atlas. The existence of *any* atlas on M will do.

Proposition 5.19 Any atlas $\mathfrak{U} = \{(U_\alpha, \phi_\alpha)\}$ on a locally Euclidean space M is contained in a unique maximal atlas.

Proof. Adjoin to the atlas $\mathfrak{U}$ all charts (V_i, ψ_i) that are compatible with $\mathfrak{U}$. By Lemma 5.16 the charts (V_i, ψ_i) are compatible with each other. So the enlarged collection $\mathfrak{U} \cup \{(V_i, \psi_i)\}$ is an atlas on M. Any chart compatible with the new atlas is compatible with $\mathfrak{U}$, and so is contained in the new atlas. Thus, the new atlas is maximal.
Let $\mathfrak{M}_1, \mathfrak{M}_2$ be two maximal atlases containing $\mathfrak{U}$. Then all the charts of $\mathfrak{M}_1$ are compatible with $\mathfrak{U}$, then must belong to $\mathfrak{M}_2$, i.e., $\mathfrak{M}_1 \subseteq \mathfrak{M}_2$. Similarly, $\mathfrak{M}_2 \subseteq \mathfrak{M}_1$. Thus, $\mathfrak{M}_1 = \mathfrak{M}_2$, i.e., the maximal atlas containing $\mathfrak{U}$ is unique. □

In summary, to show that a topological manifold M is a C^∞ manifold, it is sufficient to check that
(i) M is Hausdorff and second countable,
(ii) M has a C^∞ atlas (not necessarily maximal).

From now on, a "manifold" will mean a C^∞ manifold, with the terms "smooth" and "C^∞" used interchangeably.

Definition 5.20 In the context of manifolds, we denote the standard coordinates on $\mathbb{R}^n$ by $r^1, \cdots, r^n$. If $(U, \phi : U \to \mathbb{R}^n)$ is a chart of a manifold, we let $x^i = r^i \circ \phi$ be the ith component of ϕ and write $\phi = \left(x^1, \cdots, x^n\right)$ and $(U, \phi) = \left(U, x^1, \cdots, x^n\right)$. For $p \in U$, $x^1(p), \cdots, x^n(p)$ is a point in $\mathbb{R}^n$. The functions $x^1, \cdots, x^n$ are called **coordinates** or **local coordinates** on U. By abuse of notation, we sometimes omit the p. So the notation

$(x^1, \cdots, x^n)$ stands alternately for local coordinates on U or for a point in $\mathbb{R}^n$. A chart (U, ϕ) **about** $p \in M$ is a chart in the differential structure of M such that $p \in U$.

5.4. Examples of Smooth Manifolds

Example 5.21 (Euclidean space) The Euclidean space $\mathbb{R}^n$ is a smooth manifold with a single chart $(\mathbb{R}^n, r^1, \cdots, r^n)$, where $r^1, \cdots, r^n$ are the standard coordinates on $\mathbb{R}^n$.

Example 5.22 (Open subsets of a manifold) Any open subset $V \subset M$ of a manifold M is also a manifold. If $\{(U_\alpha, \phi_\alpha)\}$ is an atlas on M, then $\left\{\left(U_\alpha \cap V, \phi_\alpha|_{U_\alpha \cap V}\right)\right\}$ is an atlas on V, where $\phi_\alpha|_{U_\alpha \cap V} : U_\alpha \cap V \to \mathbb{R}^n$ denotes the restriction of ϕ_α to $U_\alpha \cap V$.

Example 5.23 (Manifolds of dimension zero) In a manifold of dimension zero, every singleton subset is homeomorphic to $\mathbb{R}^0$ and so is open. Thus, a zero-dimensional manifold is a discrete set. By second countability, this discrete set is countable.

Example 5.24 (Graph of a smooth function) For a subset $A \in \mathbb{R}^n$ and a function $f : A \to \mathbb{R}^m$, the **graph** of f is defined to be the subset

$$\Gamma(f) = \{(x, f(x)) \in A \times \mathbb{R}^m\}. \tag{9}$$

If U is an open subset of $\mathbb{R}^n$ and $f : U \to \mathbb{R}^m$ is C^∞, then the two maps

$$\begin{aligned} \phi : \Gamma(f) &\to U \\ (x, f(x)) &\mapsto x, \end{aligned} \tag{10}$$

and

$$\begin{aligned} (1, f) : U &\to \Gamma(f) \\ x &\mapsto (x, f(x)), \end{aligned} \tag{11}$$

are continuous and inverse to each other, and so are homeomorphisms. The graph $\Gamma(f)$ of $f : U \to \mathbb{R}^m$ has an atlas with a single chart $(\Gamma(f), \phi)$, and is therefore a C^∞ manifold.

Example 5.25 (General linear group) For any two positive integers m, n, let $\mathbb{R}^{m\times n}$ be the vector space of all $m \times n$ matrices. The **general linear group** $\mathrm{GL}(n, \mathbb{R})$ is defined by

$$\mathrm{GL}(n, \mathbb{R}) = \{A \in \mathbb{R}^{n\times n} \mid \det A \neq 0\} = \det^{-1}(\mathbb{R} - \{0\}). \tag{12}$$

Since the determinant function is continuous, $\mathrm{GL}(n, \mathbb{R})$ is an open subset of $\mathbb{R}^{n\times n} \simeq \mathbb{R}^{n^2}$ and is therefore a manifold of dimension n^2.
The **complex general linear group** $\mathrm{GL}(n, \mathbb{C})$ is defined to be the group of nonsingular $n \times n$ complex matrices. Since an $n \times n$ matrix A is nonsingular if and only if $\det A \neq 0$, $\mathrm{GL}(n, \mathbb{C})$ is an open subset of $\mathbb{C}^{n\times n} \simeq \mathbb{R}^{2n^2}$ and is therefore a manifold of dimension $2n^2$.

Example 5.26 (Unit circle in the (x, y)-plane) We now view S^1 as the unit circle in the real plane $\mathbb{R}^2$ with defining equation

$$x^2 + y^2 = 1. \tag{13}$$

We can cover S^1 with four open sets

$$\begin{aligned} U_1 &= \Big\{\Big(x, \sqrt{1-x^2}\Big) \mid -1 < x < 1\Big\}, \\ U_2 &= \Big\{\Big(x, -\sqrt{1-x^2}\Big) \mid -1 < x < 1\Big\}, \\ U_3 &= \Big\{\Big(\sqrt{1-y^2}, y\Big) \mid -1 < y < 1\Big\}, \\ U_4 &= \Big\{\Big(-\sqrt{1-y^2}, y\Big) \mid -1 < y < 1\Big\}, \end{aligned} \tag{14}$$

with maps

$$\begin{aligned} \phi_1(x,y) &= \phi_2(x,y) = x, \\ \phi_3(x,y) &= \phi_4(x,y) = y. \end{aligned} \tag{15}$$

The transition functions are given by

$$\begin{aligned} (\phi_3 \circ \phi_1^{-1})(x) &= \phi_3\Big(x, \sqrt{1-x^2}\Big) = \sqrt{1-x^2}, \\ (\phi_4 \circ \phi_1^{-1})(x) &= \phi_4\Big(x, \sqrt{1-x^2}\Big) = \sqrt{1-x^2}, \\ (\phi_3 \circ \phi_2^{-1})(x) &= \phi_3\Big(x, -\sqrt{1-x^2}\Big) = -\sqrt{1-x^2}, \\ (\phi_4 \circ \phi_2^{-1})(x) &= \phi_4\Big(x, -\sqrt{1-x^2}\Big) = -\sqrt{1-x^2}, \end{aligned} \tag{16}$$

etc. which are all C^∞. Thus $\{(U_i, \phi_i)\}_{i=1}^4$ is a C^∞ atlas on S^1.

Proposition 5.27 (An atlas for a product manifold) If $\{(U_\alpha, \phi_\alpha)\}$ and $\{(V_i, \psi_i)\}$ are C^∞ atlases for the manifolds M and N of dimensions m and n, respectively, then the collection

$$\{(U_\alpha \times V_i, \phi_\alpha \times \psi_i : U_\alpha \times V_i \to \mathbb{R}^m \times \mathbb{R}^n)\} \tag{17}$$

of charts is a C^∞ atlas on $M \times N$. Therefore $M \times N$ is a C^∞ manifold of dimension $m+n$.

Proof. Since $\{(U_\alpha, \phi_\alpha)\}$ and $\{(V_i, \psi_i)\}$ are C^∞ atlases for the manifolds M and N, respectively, the charts (U_α, ϕ_α) and (V_i, ψ_i) are C^∞-compatible and cover M and N, respectively, i.e., $M = \bigcup_\alpha U_\alpha$ and $N = \bigcup_i V_i$.
For any $p \times q \in M \times N$, there are $p \in U_\alpha$ and $q \in V_i$, then $p \times q \in U_\alpha \times V_i$, i.e., $M \times N = \bigcup_{\alpha,i}(U_\alpha \times V_i)$.
For $\big(U_\alpha, \phi_\alpha : U_\alpha \to \tilde{U}_\alpha \subset \mathbb{R}^m\big)$ and $\big(V_i, \psi_i : V_i \to \tilde{V}_i \subset \mathbb{R}^n\big)$, the product map $\phi_\alpha \times \psi_i : U_\alpha \times V_i \to \tilde{U}_\alpha \times \tilde{V}_i \subset \mathbb{R}^m \times \mathbb{R}^n \simeq \mathbb{R}^{m+n}$ is a homeomorphism as the product of homeomorphisms.
For $U_\alpha \times V_i, U_\beta \times V_j \subset M \times N$, and suppose $(U_\alpha \times V_i) \cap \big(U_\beta \times V_j\big) \neq \varnothing$. The transition functions

$$\begin{aligned} \big(\phi_\beta \times \psi_j\big) \circ (\phi_\alpha \times \psi_i)^{-1} &= \big(\phi_\beta \circ \phi_\alpha^{-1}\big) \times \big(\psi_j \circ \psi_i^{-1}\big), \\ (\phi_\alpha \times \psi_i) \circ \big(\phi_\beta \times \psi_j\big)^{-1} &= \Big(\phi_\alpha \circ \phi_\beta^{-1}\Big) \times \big(\psi_i \circ \psi_j^{-1}\big), \end{aligned} \tag{18}$$

are C^∞ because the compositions are C^∞ and the products of C^∞ functions are C^∞. Thus, the collection $\{(U_\alpha \times V_i, \phi_\alpha \times \psi_i)\}$ is a C^∞ atlas on $M \times N$. The dimension of $M \times N$ is $m + n$. $\square$

Example 5.28 The infinite cylinder $S^1 \times \mathbb{R}$ and the torus $S^1 \times S^1$ are smooth manifolds of dimensions 2.

Since $M \times N \times P = (M \times N) \times P$ is the successive product of spaces, if M, N, P are manifolds, then so is $M \times N \times P$. Thus, the n- dimensional torus $S^1 \times \cdots \times S^1$ is a manifold of dimension n.

Remark 5.29 Let S^n be the unit sphere

$$\left(x^1\right)^2 + \left(x^2\right)^2 + \cdots + \left(x^{n+1}\right)^2 = 1 \tag{19}$$

in $\mathbb{R}^{n+1}$. Using Example 5.26, it is easy to write down a C^∞ atlas for S^n, showing that S^n has a differential structure. The manifold S^n with this differential structure is called the **standard n-sphere**.

6. Smooth Maps on a Manifold

By the C^∞ compatibility of charts in an atlas, the smoothness of a map between two manifolds is independent of the choice of charts and is therefore well defined.

6.1. Smooth Functions on a Manifold

Definition 6.1 Let M be a smooth manifold of dimension n. A function $f : M \to \mathbb{R}$ is said to be **smooth** or **C^∞** at a point $p \in M$ if there is a chart (U, ϕ) about p such that the composite

$$f \circ \phi^{-1} : \phi(U) \to \mathbb{R} \tag{20}$$

is C^∞ at $\phi(p)$. The function f is said to be C^∞ on M if it is C^∞ at every point of M.

Remark 6.2 The definition of smoothness of a function f at a point is independent of the chart (U, ϕ), for if $f \circ \phi^{-1}$ is C^∞ at $\phi(p)$, and (V, ψ) any other chart about p, then on $\psi(U \cap V)$, the composite

$$f \circ \psi^{-1} = \left(f \circ \phi^{-1}\right) \circ \left(\phi \circ \psi^{-1}\right) \tag{21}$$

is C^∞ at $\psi(p)$.

In Definition 6.1, $f : M \to \mathbb{R}$ is not assumed to be continuous. However, if f is C^∞ at $p \in M$, then $f \circ \phi^{-1} : \phi(U) \to \mathbb{R}$, being a C^∞ function at $\phi(p)$ in an open subset of $\mathbb{R}^n$, is continuous at $\phi(p)$. As a composite of continuous functions, $f = \left(f \circ \phi^{-1}\right) \circ \phi$ is continuous at p. Since we are interested only in functions that are C^∞ on an open set, there is no loss of generality in assuming at the outset that f is continuous.

Proposition 6.3 (Smoothness of a real-valued function) Let M be a smooth manifold of dimension n and $f : M \to \mathbb{R}$ a real-valued function on M. The following are equivalent:

(i) The function $f : M \to \mathbb{R}$ is C^∞.

(ii) The manifold M has an atlas such that for every chart (U, ϕ) in the atlas, the composite

$$f \circ \phi^{-1} : \mathbb{R}^n \supset \phi(U) \to \mathbb{R} \tag{22}$$

is C^∞.

(iii) For every chart (V, ψ) on M, the composite

$$f \circ \psi^{-1} : \mathbb{R}^n \supset \psi(U) \to \mathbb{R} \tag{23}$$

is C^∞.

Proof. We can prove the proposition as a cycle chain of implications.
(ii) $\Rightarrow$ (i): This follows directly from the definition of a C^∞ function, since by (ii) every point $p \in M$ has a chart (U, ϕ) about it such that $f \circ \phi^{-1}$ is C^∞ at $\phi(p)$.
(i) $\Rightarrow$ (iii): Let $p \in M$ and (V, ψ) be a chart about p. By Remark 6.2, $f \circ \psi^{-1}$ is C^∞ at $\psi(p)$. Since p is arbitrary, $f \circ \psi^{-1}$ is C^∞ on $\psi(V)$.
(iii) $\Rightarrow$ (ii): Obvious. $\square$

Definition 6.4 Let $F : N \to M$ be a map and h a function on M. The **pullback** of h by F,denoted by F^*h, is the composite function

$$F^*h = h \circ F : N \to \mathbb{R}. \tag{24}$$

In this terminology, a function f on M is C^∞ on a chart (U, ϕ) if and only if its pullback $(\phi^{-1})^* f$ is C^∞ on the subset $\phi(U) \subset \mathbb{R}^n$.

6.2. Smooth Maps Between Manifolds

Definition 6.5 Let N and M be manifolds of dimensions n and m, respectively. A continuous map $F : N \to M$ is C^∞ at a point $p \in N$ if there are charts (V, ψ) about $F(p) \in M$ and (U, ϕ) about $p \in N$ such that the composite

$$\psi \circ F \circ \phi^{-1} : \mathbb{R}^n \supset \phi\big(F^{-1}(V) \cap U\big) \to \mathbb{R}^m \tag{25}$$

is C^∞ at $\phi(p)$. The continuous map $F : N \to M$ is said to be C^∞ on N if it is C^∞ at every point of N.

In Definition 6.5, we assume $F : N \to M$ is continuous to ensure that $F^{-1}(V)$ is open in N. Thus, C^∞ maps are by definition continuous.

Remark 6.6 (Smooth maps into $\mathbb{R}^m$) In case $M = \mathbb{R}^m$, we can take $(\mathbb{R}^m, \mathbb{1}_{\mathbb{R}^m})$ as a chart about $F(p)$ in M. According to Definition 6.5, $F : N \to \mathbb{R}^m$ is C^∞ at $p \in N$ if and only if there is a chart (U, ϕ) about $p \in N$ such that the composite

$$F \circ \phi^{-1} : \mathbb{R}^n \supset \phi(U) \to \mathbb{R}^m \tag{26}$$

is C^∞ at $\phi(p)$. Letting $m = 1$, we recover the definition of a function being C^∞ at a point in Definition 6.1.

Proposition 6.7 Suppose $F : N \to M$ is C^∞ at $p \in N$. If (U, ϕ) is any chart about $p \in N$ and (V, ψ) is any chart about $F(p) \in M$, then $\psi \circ F \circ \phi^{-1}$ is C^∞ at $\phi(p)$.

Proof. Since F is C^∞ at p, there are charts (U_α, ϕ_α) about $p \in N$ and (V_i, ψ_i) about $F(p) \in M$ such that $V_i \circ F \circ \phi_\alpha^{-1}$ is C^∞ at $\phi_\alpha(p)$. By the C^∞ compatibility of charts in a differential structure, both $\phi_\alpha \circ \phi^{-1}$ and $\psi \circ \psi_i^{-1}$ are C^∞. Hence, the composite

$$\psi \circ F \circ \phi^{-1} = \left(\psi \circ \psi_i^{-1}\right) \circ \left(V_i \circ F \circ \phi_\alpha^{-1}\right) \circ \left(\phi_\alpha \circ \phi^{-1}\right) \tag{27}$$

is C^∞ at $\phi(p)$. □

Proposition 6.8 (Smoothness of a map in terms of charts) Let N and M be manifolds of dimensions n and m, respectively and $F : N \to M$ a continuous map. The following are equivalent:

(i) The map $F : N \to M$ is C^∞.

(ii) There are atlases $\mathfrak{U}$ for N and $\mathfrak{V}$ for M such that for every chart $(U, \phi) \in \mathfrak{U}$ and $(V, \psi) \in \mathfrak{V}$, the composite

$$\psi \circ F \circ \phi^{-1} : \mathbb{R}^n \supset \phi\left(U \cap F^{-1}(V)\right) \to \mathbb{R}^m \tag{28}$$

is C^∞.

(iii) For every chart (U, ϕ) on N and (V, ψ) on M, the composite

$$\psi \circ F \circ \phi^{-1} : \mathbb{R}^n \supset \phi\left(U \cap F^{-1}(V)\right) \to \mathbb{R}^m \tag{29}$$

is C^∞.

Proof. We can prove the proposition as a cycle chain of implications.
(ii) ⇒ (i): Let $p \in N$ and $(U, \phi) \in \mathfrak{U}$ be a chart about p and $(V, \psi) \in \mathfrak{V}$ a chart about $F(p)$, then $\psi \circ F \circ \phi^{-1}$ is C^∞ at $\phi(p)$. By Definition 6.5, $F : N \to M$ is C^∞ at p. Since p is arbitrary, $F : N \to M$ is C^∞ on N.
(i) ⇒ (iii): Let (U, ϕ) be a chart on N and (V, ψ) a chart on M such that $U \cap F^{-1}(V) \neq \emptyset$. Let $p \in U \cap F^{-1}(V)$, then (U, ϕ) is a chart about p and (V, ψ) is a chart about $F(p)$. By Proposition 6.7, $\psi \circ F \circ \phi^{-1}$ is C^∞ at $\phi(p)$. Since p is arbitrary, $\phi(p)$ is arbitrary, $\psi \circ F \circ \phi^{-1}$ is C^∞ on $\phi(U \cap F^{-1}(V))$.
(iii) ⇒ (ii): Obvious. □

Proposition 6.9 (Composition of C^∞ maps) If $F : N \to M$ and $G : M \to P$ are C^∞ maps between manifolds, then the composite $G \circ F : N \to P$ is C^∞.

Proof. Let $(U, \phi), (V, \psi), (W, \sigma)$ be charts on N, M, P, respectively. Then

$$\sigma \circ (G \circ F) \circ \phi^{-1} = \left(\sigma \circ G \circ \psi^{-1}\right) \circ \left(\psi \circ F \circ \phi^{-1}\right). \tag{30}$$

Since F and G are C^∞, by Proposition 6.8(i) ⇒ (iii), $\sigma \circ G \circ \psi^{-1}$ and $\psi \circ F \circ \phi^{-1}$ are C^∞. As a composite of C^∞ maps of open subsets of Euclidean spaces, $\sigma \circ (G \circ F) \circ \phi^{-1}$ is C^∞. By Proposition 6.8(iii) ⇒ (i), $G \circ F : N \to P$ is C^∞. □

6.3. Diffeomorphisms

Definition 6.10 A **diffeomorphism** of manifolds is a bijective C^∞ map $F : N \to M$ whose inverse F^{-1} is also C^∞.

According to the next two propositions, coordinate maps are diffeomorphisms, and conversely, every diffeomorphism of an open subset of a manifold with an open subset of Euclidean space can serve as a coordinate map.

Proposition 6.11 If (U, ϕ) is a chart on a manifold M of dimension n, then the coordinate map $\phi : U \to \phi(U) \subset \mathbb{R}^n$ is a diffeomorphism.

Proof. By definition, ϕ is a homeomorphism, so it suffices to check that both ϕ and ϕ^{-1} are C^∞.
We use the atlas $\{(U, \phi)\}$ with a single chart on U and atlas $\left\{\left(\phi(U), \mathbb{1}_{\phi(U)}\right)\right\}$ with a single chart on $\phi(U)$. Since

$$\mathbb{1}_{\phi(U)} = \mathbb{1}_{\phi(U)} \circ \phi \circ \phi^{-1} : \phi(U) \to \phi(U) \tag{31}$$

is the identity map, it is C^∞. By Proposition 6.8(ii) $\Rightarrow$ (i), ϕ is C^∞.
Similarly, ϕ^{-1} is C^∞ because

$$\mathbb{1}_U = \phi \circ \phi^{-1} \circ \mathbb{1}_{\phi(U)} : U \to U \tag{32}$$

is C^∞. □

Proposition 6.12 Let U be an open subset of a manifold M of dimension n. If $F : U \to F(U) \subset \mathbb{R}^n$ is a diffeomorphism onto an open subset of $\mathbb{R}^n$, then (U, F) is a chart in the differential structure of M.

Proof. For any chart (U_α, ϕ_α) in the maximal atlas of M, both ϕ_α and ϕ_α^{-1} are C^∞ by Proposition 6.11. As composites of C^∞ maps, $F \circ \phi_\alpha^{-1}$ and $\phi_\alpha \circ F^{-1}$ are C^∞. Hence, (U, F) is compatible with the maximal atlas, i.e., (U, F) is a chart in the differential structure of M. □

6.4. Smoothness in Terms of Components

Proposition 6.13 (Smoothness of a vector-valued function) Let N be a manifold and $F : N \to \mathbb{R}^m$ a continuous map. The following are equivalent:

(i) The map $F : N \to \mathbb{R}^m$ is C^∞.

(ii) The manifold N has an atlas such that for every chart (U, ϕ) in the atlas, the composite

$$F \circ \phi^{-1} : \phi(U) \to \mathbb{R}^m \tag{33}$$

is C^∞.

(iii) For every chart (U, ϕ) on N, the composite

$$F \circ \phi^{-1} : \phi(U) \to \mathbb{R}^m \tag{34}$$

is C^∞.

Proof. We can prove the proposition as a cycle chain of implications.
(ii) $\Rightarrow$ (i): Let $\{(\mathbb{R}^m, \mathbb{1}_{\mathbb{R}^m})\}$ be the atlas on $\mathbb{R}^m$ with a single chart, then

$$F \circ \phi^{-1} = \mathbb{1}_{\mathbb{R}^m} \circ F \circ \phi^{-1} : \phi(U) \to \mathbb{R}^m \tag{35}$$

is C^∞. By Proposition 6.8(ii) $\Rightarrow$ (i), $F : N \to \mathbb{R}^m$ is C^∞.
(i) $\Rightarrow$ (iii): Let $\{(\mathbb{R}^m, \mathbb{1}_{\mathbb{R}^m})\}$ be the atlas on $\mathbb{R}^m$ with a single chart, then by Proposition 6.8(i) $\Rightarrow$ (iii),

$$\mathbb{1}_{\mathbb{R}^m} \circ F \circ \phi^{-1} = F \circ \phi^{-1} : \phi(U) \to \mathbb{R}^m \tag{36}$$

is C^∞.
(iii) $\Rightarrow$ (ii): Obvious. $\square$

Proposition 6.14 (Smoothness in terms of components) Let N be a manifold. A vector-valued function $F : N \to \mathbb{R}^m$ is C^∞ if and only if its components functions $F^1, \cdots, F^m : N \to \mathbb{R}$ are all C^∞.

Proof. The map $F : N \to \mathbb{R}^m$ is C^∞.
$\Longleftrightarrow$ For every chart (U, ϕ) on N, $F \circ \phi^{-1} : \phi(U) \to \mathbb{R}^m$ is C^∞.
$\Longleftrightarrow$ For every chart (U, ϕ) on N, the functions $F^i \circ \phi^{-1} : \phi(U) \to \mathbb{R}$ are C^∞.
$\Longleftrightarrow$ The functions $F^i : N \to \mathbb{R}$ are C^∞. $\square$

Example 6.15 (Smoothness of a map to a circle).
Prove the map $F : \mathbb{R} \to S^1, F(t) = (\cos t, \sin t)$ is C^∞.

Solution. Let $\{(U_i, \phi_i) | i = 1, \cdots, 4\}$ be the atlas on S^1. On $F^{-1}(U_1)$,

$$(\phi_1 \circ F)(t) = \cos t \tag{37}$$

is C^∞. Similar computations show that $\phi_i \circ F$ is C^∞.

Proposition 6.16 (Smoothness of a map in terms of vector-valued functions) Let $F : N \to M$ be a continuous map between manifolds of dimensions n and m, respectively. The following are equivalent:

(i) The map $F : N \to M$ is C^∞.

(ii) The manifold M has an atlas such that for every chart $(V, \psi) = \left(V, y^1, \cdots, y^m\right)$ in the atlas, the vector-valued function

$$\psi \circ F : F^{-1}(V) \to \mathbb{R}^m \tag{38}$$

is C^∞.

(iii) For every chart $(V, \psi) = \left(V, y^1, \cdots, y^m\right)$ on M, the vector-valued function

$$\psi \circ F : F^{-1}(V) \to \mathbb{R}^m \tag{39}$$

is C^∞.

Proof. We can prove the proposition as a cycle chain of implications.
(ii) $\Rightarrow$ (i): Let $\mathfrak{V}$ be the atlas for M and $\mathfrak{U} = \{(U, \phi)\}$ any arbitrary atlas for N. Then for each chart $(V, \psi) \in \mathfrak{V}$, the collection

$$\left\{\left(U \cap F^{-1}(V), \phi|_{U \cap F^{-1}(V)}\right)\right\} \tag{40}$$

is an atlas for $F^{-1}(V)$. Since $\psi \circ F$ is C^∞, by Proposition 6.13(i) $\Rightarrow$ (iii),

$$\psi \circ F \circ \phi^{-1} : \phi(U \cap F^{-1}(V)) \to \mathbb{R}^m \tag{41}$$

is C^∞. It then follows from Proposition 6.8(ii) $\Rightarrow$ (i) that $F : N \to M$ is C^∞.
(i) $\Rightarrow$ (iii): Being a coordinate map, ψ is C^∞. As a composite of C^∞ maps, $\psi \circ F$ is C^∞.
(iii) $\Rightarrow$ (ii): Obvious. $\square$

Combining Proposition 6.16 and Proposition 6.14, the smoothness of a map $F : N \to M$ can be expressed in terms of the smoothness of its components.

Proposition 6.17 (Smoothness of a map in terms of components) Let $F : N \to M$ be a continuous map between manifolds of dimensions n and m, respectively. The following are equivalent:

(i) The map $F : N \to M$ is C^∞.

(ii) The manifold M has an atlas such that for every chart $(V, \psi) = (V, y^1, \cdots, y^m)$ in the atlas, the components

$$y^i \circ F : F^{-1}(V) \to \mathbb{R} \tag{42}$$

of F relative to the chart are all C^∞.

(iii) For every chart $(V, \psi) = (V, y^1, \cdots, y^m)$ on M, the components

$$y^i \circ F : F^{-1}(V) \to \mathbb{R} \tag{43}$$

of F relative to the chart are all C^∞.

6.5. Examples of Smooth Maps

Example 6.18 (Smoothness of a projection map) Let M and N be manifolds and

$$\begin{aligned} \pi : M \times N &\to M \\ (p, q) &\mapsto p \end{aligned} \tag{44}$$

the projection to the first factor. Prove that π is a C^∞ map.

Solution. Let $(p, q) \in M \times N$. Suppose $(U, \phi) = (U, x^1, \cdots, x^m)$ and $(V, \psi) = (V, y^1, \cdots, y^n)$ are charts about $p \in M$ and $q \in N$, respectively. Then

$$(U \times V, \phi \times \psi) = (U \times V, x^1, \cdots, x^m, y^1, \cdots, y^n) \tag{45}$$

is a chart about $(p, q) \in M \times N$. Then

$$(\phi \circ \pi \circ (\phi \times \psi)^{-1})(a^1, \cdots, a^m, b^1, \cdots, b^n) = (a^1, \cdots, a^m) \tag{46}$$

is a C^∞ map from $\phi(U \times V) \in \mathbb{R}^{m+n}$ to $\phi(U) \in \mathbb{R}^m$. So π is C^∞ at (p, q). Since (p, q) is arbitrary, $\pi : M \times N \to M$ is C^∞.

Example 6.19 (Smoothness of a map to a Cartesian Product) Let M_1, M_2 and N be manifolds of dimensions m_1, m_2 and n, respectively. Prove that the map

$$(f_1, f_2) : N \to M_1 \times M_2 \tag{47}$$

is C^∞ if and only if $f_i : N \to M_i, i = 1, 2$, are both C^∞.

Solution. Let $p \in N$ and (U, ϕ) be a chart about p. Let (V_1, ψ_1) and (V_2, ψ_2) be charts about $f_1(p) \in M_1$ and $f_2(p) \in M_2$, respectively.

(i) Assuming both f_1 and f_2 are C^∞, then they are both continuous. Then

$$\begin{aligned}(\psi_1 \times \psi_2) \circ (f_1, f_2) \circ \phi^{-1} &= \left(\psi_1 \circ f_1 \circ \phi^{-1}, \psi_2 \circ f_2 \circ \phi^{-1}\right) \\ &: \phi(U \cap f_1^{-1}(V_1) \cap f_2^{-1}(V_2)) \to \mathbb{R}^{m_1+m_2}\end{aligned} \tag{48}$$

is C^∞. Thus (f_1, f_2) is C^∞ at p. Since p is arbitrary, (f_1, f_2) is C^∞ on N.

(ii) Conversely, if (f_1, f_2) is C^∞, then

$$(\psi_1 \times \psi_2) \circ (f_1, f_2) \circ \phi^{-1} = \left(\psi_1 \circ f_1 \circ \phi^{-1}, \psi_2 \circ f_2 \circ \phi^{-1}\right) \tag{49}$$

is C^∞. Thus, $\psi_1 \circ f_1 \circ \phi^{-1}$ and $\psi_2 \circ f_2 \circ \phi^{-1}$ are both C^∞, then f_1 and f_2 are both C^∞ at p. Since p is arbitrary, f_1 and f_2 are both C^∞ on N.

Example 6.20 Prove that a C^∞ function $f(x, y)$ on $\mathbb{R}^2$ restricts to a C^∞ function S^1.

Solution. We denote a point on S^1 as $p = (a, b)$ and x, y as the standard coordinate functions on $\mathbb{R}^2$, i.e., $x(a, b) = a$ and $y(a, b) = b$. Suppose we can show that x and y restrict to C^∞ functions on S^1, then the inclusion map

$$\begin{aligned} i : S^1 &\to \mathbb{R}^2 \\ p &\mapsto (x(p), y(p)) \end{aligned} \tag{50}$$

is C^∞ on S^1. Therefore the restriction of f to S^1, $f|_{S^1} = f \circ i$, is C^∞ on S^1.

Consider the first function X, we use the atlas

$$\begin{aligned}
(U_1, \phi_1) &= \left(\left\{\left(x, \sqrt{1-x^2}\right) \mid -1 < x < 1\right\}, x\right) \\
(U_2, \phi_2) &= \left(\left\{\left(x, -\sqrt{1-x^2}\right) \mid -1 < x < 1\right\}, x\right) \\
(U_3, \phi_3) &= \left(\left\{\left(\sqrt{1-y^2}, y\right) \mid -1 < y < 1\right\}, y\right) \\
(U_4, \phi_4) &= \left(\left\{\left(-\sqrt{1-y^2}, y\right) \mid -1 < y < 1\right\}, y\right).
\end{aligned} \tag{51}$$

Since x is a coordinate function on U_1 and U_2, x is C^∞ on $U_1 \cup U_2$. The composite

$$(x \circ \phi_3^{-1})(b) = x\left(\sqrt{1-b^2}, b\right) = \sqrt{1-b^2} \tag{52}$$

is C^∞ on U_3, thus x is C^∞ on U_3.
Similarly, the composite

$$(x \circ \phi_4^{-1})(b) = x\left(-\sqrt{1-b^2}, b\right) = -\sqrt{1-b^2} \tag{53}$$

is C^∞ on U_4, thus x is C^∞ on U_4.
Since x is C^∞ on U_1, U_2, U_3, U_4, and $S^1 = U_1 \cup U_2 \cup U_3 \cup U_4$, x is C^∞ on S^1.
The same argument shows that y is C^∞ on S^1.

Armed with the definition of a smooth map between manifolds, we can define a Lie group.

Definition 6.21 A **Lie group** is a C^∞ manifold G having a group structure such that the multiplication map

$$\mu : G \times G \to G \tag{54}$$

and the inverse map

$$\iota : G \to G, \quad \iota(x) = x^{-1}, \tag{55}$$

are both C^∞.

Similarly, A **topological group** is a topological space having a group structure such that the multiplication map and the inverse map are both continuous. Noting that a topological group is required to be a topological space, but not a topological manifold.

Example 6.22
(i) The Euclidean space $\mathbb{R}^n$ is a Lie group under addition.
(ii) The set $C^\times$ of nonzero complex numbers is a Lie group under multiplication.
(iii) The unit circle S^1 in $C^\times$ is a Lie group under multiplication.
(iv) The Cartesian product $G_1 \times G_2$ of two Lie groups (G_1, μ_1) and (G_2, μ_2) is a Lie group under coordinatewise multiplication $\mu_1 \times \mu_2$.

Example 6.23 (General linear group) The general linear group

$$\mathrm{GL}(n, \mathbb{R}) = \left\{A \in \left[a_{ij}\right] \in \mathbb{R}^{n\times n} \mid \det(A) \neq 0\right\} \tag{56}$$

is a manifold, as an open subset of $\mathbb{R}^{n\times n}$. Since the (i, j)-entry of the product of two matrices $A, B \in \mathrm{GL}(n, \mathbb{R})$,

$$(A \times B)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}, \tag{57}$$

is a polynomial in the coordinates of A and B, matrix multiplication

$$\mu : \mathrm{GL}(n, \mathbb{R}) \times \mathrm{GL}(n, \mathbb{R}) \to \mathrm{GL}(n, \mathbb{R}) \tag{58}$$

is a C^∞ map.
By Cramer's rule from linear algebra, the (i, j)-entry of A^{-1} is

$$\left(A^{-1}\right)_{ij} = \frac{1}{\det A} \cdot (-1)^{i+j}((j, i) - \text{minor of } A), \tag{59}$$

which is a C^∞ function provided $\det A \neq 0$. Thus, the inverse map

$$\iota : \mathrm{GL}(n, \mathbb{R}) \to \mathrm{GL}(n, \mathbb{R}) \tag{60}$$

is also C^∞. Therefore, $\mathrm{GL}(n, \mathbb{R})$ is a Lie group.

6.6. Partial Derivatives

Definition 6.24 On a manifold M of dimension n, let $(U,\phi) = (U, x^1, \cdots, x^n) = (U, r^1 \circ \phi, \cdots, r^n \circ \phi)$ be a chart and $f : M \to \mathbb{R}^n$ a C^∞ function, where $r^1, \cdots, r^n$ are the standard coordinates on $\mathbb{R}^n$. For $p \in U$, the **partial derivative** $\frac{\partial f}{\partial x^i}$ of f with respect to x^i at p is defined as

$$\begin{aligned}\left.\frac{\partial}{\partial x^i}\right|_p f &:= \frac{\partial f}{\partial x^i}(p) \\ &:= \frac{\partial (f \circ \phi^{-1})}{\partial r^i}(\phi(p)) \\ &:= \left.\frac{\partial}{\partial r^i}\right|_{\phi(p)} (f \circ \phi^{-1}).\end{aligned} \tag{61}$$

Since $p = \phi^{-1}(\phi(p))$, this equation can rewritten as

$$\frac{\partial f}{\partial x^i}(\phi^{-1}(\phi(p))) = \frac{\partial (f \circ \phi^{-1})}{\partial r^i}(\phi(p)). \tag{62}$$

Thus, as functions on $\phi(U)$,

$$\frac{\partial f}{\partial x^i} \circ \phi^{-1} = \frac{\partial (f \circ \phi^{-1})}{\partial r^i}. \tag{63}$$

The partial derivative $\frac{\partial f}{\partial x^i}$ is C^∞ on U because its pullback by ϕ^{-1}, $\frac{\partial f}{\partial x^i} \circ \phi^{-1}$ is C^∞ on $\phi(U)$.

Proposition 6.25 Suppose $(U, x^1, \cdots, x^n)$ is a chart on a manifold. Then $\frac{\partial x^i}{\partial x^j} = \delta^i_j$.

Proof. At a point $p \in U$, by the definition of $\frac{\partial}{\partial x^j}\big|_p$,

$$\begin{aligned}\frac{\partial x^i}{\partial x^j}(p) &= \frac{\partial (x^i \circ \phi^{-1})}{\partial r^j}(\phi(p)) \\ &= \frac{\partial (r^i \circ \phi \circ \phi^{-1})}{\partial r^j}(\phi(p)) \\ &= \frac{\partial r^i}{\partial r^j}(\phi(p)) \\ &= \delta^i_j.\end{aligned} \tag{64}$$

□

Definition 6.26 Let $F : N \to M$ be a C^∞ map, and let $(U,\phi) = (U, x^1, \cdots, x^n)$ and $(V,\psi) = (V, y^1, \cdots, y^m)$ be charts on N and M respectively such that $F(U) \subset V$. Denote by

$$\begin{aligned}F^i &:= y^i \circ F \\ &= r^i \circ \psi \circ F : U \to \mathbb{R}\end{aligned} \tag{65}$$

the ith component of F in the chart (V, ψ). Then the matrix $\left[\frac{\partial F^i}{\partial x^j}\right]$ is called the **Jacobian matrix** of F relative to the charts (U, ϕ) and (V, ψ). In case N and M have the same dimension, the determinant $\det\left[\frac{\partial F^i}{\partial x^j}\right]$ is called the **Jacobian determinant** of F relative to the two charts. The Jacobian determinant also written as

$$\frac{\partial(F^1, \cdots, \partial F^n)}{\partial(x^1, \cdots, \partial x^n)} \tag{66}$$

When N and M are open subsets of Euclidean spaces and the charts are $(U, r^1, \cdots, r^n)$ and $(V, r^1, \cdots, r^m)$, the Jacobian matrix $\left[\frac{\partial F^i}{\partial r^j}\right]$, where $F^i = r^i \circ F$, is the usual Jacobian matrix from calculus.

Example 6.27 (Jacobian matrix of a transition map) Let $(U, \phi) = (U, x^1, \cdots, x^n)$ and $(V, \psi) = (V, y^1, \cdots, y^n)$ be overlapping charts on a manifold M. The transition map

$$\psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V) \tag{67}$$

is a diffeomorphism of open subsets of $\mathbb{R}^n$. Show that the Jacobian matrix $J(\psi \circ \phi^{-1})$ at $\phi(p)$ is the matrix $\left[\frac{\partial y^i}{\partial x^j}\right]$ of partial derivatives at p.

Solution. By definition, $J(\psi \circ \phi^{-1}) = \left[\frac{\partial(\psi \circ \phi^{-1})^i}{\partial r^j}\right]$, where

$$\begin{aligned} \frac{\partial(\psi \circ \phi^{-1})^i}{\partial r^j}(\phi(p)) &= \frac{\partial(r^i \circ \psi \circ \phi^{-1})}{\partial r^j}(\phi(p)) \\ &= \frac{\partial(y^i \circ \phi^{-1})}{\partial r^j}(\phi(p)) \\ &= \frac{\partial y^i}{\partial x^j}(p) \end{aligned} \tag{68}$$

6.7. The Inverse Function Theorem

Definition 6.28 A C^∞ map $F : N \to M$ is **locally invertible** or a **local diffeomorphism** at $p \in N$ if p has a neighborhood U on which $F|_U : U \to F(U)$ is a diffeomorphism.

Theorem 6.29 (Inverse function theorem for $\mathbb{R}^n$) Let $F : W \to \mathbb{R}^n$ be a C^∞ map defined on an open subset $W \subset \mathbb{R}^n$. For any point $p \in W$, the map F is locally invertible at p if and only if the Jacobian determinant $\det\left[\frac{\partial F^i}{\partial r^j}(p)\right] \neq 0$.

Because the inverse function theorem for $\mathbb{R}^n$ is a local result, it easily translates to manifolds.

Theorem 6.30 (Inverse function theorem for manifolds) Let $F : N \to M$ be a C^∞ map between two manifolds of the same dimension, say n, and $p \in N$. Suppose for some chart $(U, \phi) = (U, x^1, \cdots, x^n)$ about $p \in N$ and $(V, \psi) = (V, y^1, \cdots, y^n)$ about $F(p) \in M$, $F(U) \subset V$. Set $F^i = y^i \circ F$. Then F is locally invertible at p if and only if the Jacobian determinant $\det\left[\frac{\partial F^i}{\partial x^j}(p)\right] \neq 0$.

Proof.

$$\begin{aligned}\left[\frac{\partial F^i}{\partial x^j}(p)\right] &= \left[\frac{\partial(y^i\circ F)}{\partial x^j}(p)\right] \\ &= \left[\frac{\partial(r^i\circ\psi\circ F)}{\partial x^j}(p)\right] \\ &= \left[\frac{\partial(r^i\circ\psi\circ F\circ\phi^{-1})}{\partial r^j}(\phi(p))\right] \\ &= \left[\frac{\partial(\psi\circ F\circ\phi^{-1})^i}{\partial r^j}(\phi(p))\right], \end{aligned} \tag{69}$$

which is the Jacobian matrix at $\phi(p)$ of the map

$$\psi\circ F\circ\phi^{-1}:\mathbb{R}^n\supset\phi(U)\to\psi(V)\subset\mathbb{R}^n \tag{70}$$

between two open subsets of $\mathbb{R}^n$. By the inverse function theorem for $\mathbb{R}^n$,

$$\det\left[\frac{\partial F^i}{\partial x^j}(p)\right] = \det\left[\frac{\partial(\psi\circ F\circ\phi^{-1})^i}{\partial r^j}(\phi(p))\right]\neq 0 \tag{71}$$

if and only if $\psi\circ F\circ\phi^{-1}$ is locally invertible at $\phi(p)$. Since ψ and ϕ are diffeomorphisms, this is equivalent to F being locally invertible at p. □

We usually apply the inverse function theorem in the following form.

Corollary 6.31 Let N be a manifold of dimension n. A set of n smooth functions $F^1,\cdots,F^n$ defined on a coordinate neighborhood $(U,x^1,\cdots,x^n)$ of a point $p\in N$ forms a coordinate system about p if and only if the Jacobian determinant $\det\left[\frac{\partial F^i}{\partial x^j}(p)\right]\neq 0$.

Proof. Let $F=(F^1,\cdots,F^n):U\to\mathbb{R}^n$. Then
$\det\left[\frac{\partial F^i}{\partial x^j}(p)\right]\neq 0$.
$\iff F:U\to\mathbb{R}^n$ is locally invertible at p. (By Theorem 6.30)
$\iff$ There is a neighborhood W of $p\in N$ such that $F:W\to F(W)$ is a diffeomorphism. (By Definition 6.28)
$\iff (U,F^1,\cdots,F^n)$ is a coordinate chart about p in the differential structure of N. (By Proposition 6.12)

□

Example 6.32 Find all points in $\mathbb{R}^2$ of which the functions x^2+y^2-1, y can serve as a coordinate system in a neighborhood.

Solution. Define $F:\mathbb{R}^2\to\mathbb{R}^2$ by

$$F(x,y)=(x^2+y^2-1,y). \tag{72}$$

The map F can serve as a coordinate map in a neighborhood of p if and only if it is a local diffeomorphism at p. The Jacobian determinant of F is

$$\begin{aligned}\frac{\partial(F^1, F^2)}{\partial(x, y)} &= \det\begin{bmatrix} 2x & 2y \\ 0 & 1 \end{bmatrix} \\ &= 2x. \end{aligned} \tag{73}$$

By the inverse function theorem, F is a local diffeomorphism at $p = (x, y)$ if and only if $x \neq 0$, i.e., F can serve as a coordinate system at any point p not on the y-axis.

7. Quotients

7.1. The Quotient Topology

Definition 7.1 For an equivalence relation $\sim$ on a set S, the **equivalence class** of $x \in S$, denoted by $[x]$, is the set of all elements in S equivalent to x. An equivalence relation on S partitions S into disjoint equivalence classes. The **quotient** of S by the equivalence relation $\sim$, denoted by $S/\sim$, is the set of equivalence classes. There is a natural **projection map** $\pi : S \to S/\sim$ defined by

$$\pi(x) = [x], \quad x \in S. \tag{74}$$

Definition 7.2 Assume now that S is a topological space. The **quotient topology** on $S/\sim$ is defined as follows: A subset $U \subset S/\sim$ is open if and only if $\pi^{-1}(U)$ is open in S. With this topology, $S/\sim$ is called the **quotient space** of S by the equivalence relation $\sim$, and the projection map $\pi : S \to S/\sim$ is automatically continuous.

Proof.

(i) $\emptyset = \pi^{-1}(\emptyset)$ is open in S, so $\emptyset$ is open in $S/\sim$; $S = \pi^{-1}(S/\sim)$ is open in S, so $S/\sim$ is open in $S/\sim$.

(ii) Let U_α be open in $S/\sim$, $\alpha = 1, 2, \cdots$. Then

$$\pi^{-1}\left(\bigcup_\alpha U_\alpha\right) = \bigcup_\alpha \pi^{-1}(U_\alpha) \tag{75}$$

is open in S, so $\bigcup\limits_\alpha U_\alpha$ is open in $S/\sim$.

(iii) Let U_i be open in $S/\sim$, $i = 1, \cdots, n$. Then

$$\pi^{-1}\left(\bigcap_{i=1}^n U_i\right) = \bigcap_{i=1}^n \pi^{-1}(U_i) \tag{76}$$

is open in S, so $\bigcap\limits_{i=1}^n U_i$ is open in $S/\sim$. □

7.2. Continuity of a Map on a Quotient

Let $\sim$ be an equivalence relation on a topological space S and $S/\sim$ the quotient space, with the quotient topology. Suppose a function $f : S \to Y$ from S to another topological space Y is constant on each equivalence class, i.e., $f(x) = f(y)$ whenever $x \sim y$. Then f induces a function $\bar{f} : S/\sim \to Y$ defined by

$$\bar{f}([p]) = f(p), \quad p \in S. \tag{77}$$

Proposition 7.3 The induced map $\bar{f} : S/\sim \to Y$ is continuous if and only if the map $f : S \to Y$ is continuous.

Proof.
(i) ($\Rightarrow$) If $\bar{f}$ is continuous, then $f = \bar{f} \circ \pi$ is continuous because π is continuous.
(ii) ($\Leftarrow$) Suppose f is continuous. Let $V \subset Y$ be open. Then $f^{-1}(V) = \pi^{-1}\big(\bar{f}^{-1}(V)\big)$ is open in S, so $\bar{f}^{-1}(V)$ is open in $S/\sim$ by the definition of the quotient topology. Since V is arbitrary, $\bar{f}$ is continuous. □

This proposition gives a useful criterion for checking whether a function $\bar{f}$ on a quotient space $S/\sim$ is continuous: simply lift the function $\bar{f}$ to $f := \bar{f} \circ \pi$ on S and check whether f is continuous.

7.3. Identification of a Subset to a Point

Definition 7.4 Let $A \subset S$ be a subset of a topological space S, we can define a relation $\sim$ on S by

$$x \sim x \quad \forall x \in S \tag{78}$$

and

$$x \sim y \quad \forall x, y \in A. \tag{79}$$

We say the quotient space $S/\sim$ is obetained from S by **identifying A to a point**.

Example 7.5 Let $I = [0, 1]$ and $I/\sim$ the quotient space obtained from I by identifying the two endpoints $\{0, 1\}$ to a point. Denote by S^1 the unit circle in the complex plane. The function $f : I \to S^1, f(x) = e^{2\pi i x}$, assumes the same value at 0 and 1, and so induces a function $\bar{f} : I/\sim \to S^1$.

Proposition 7.6 The function $\bar{f} : I/\sim \to S^1$ is a homeomorphism.

7.4. A Necessary Condition for a Hausdorff Quotient

The quotient construction does not in general preserve the Hausdorff property or second countability. Indeed, since every singleton set in a Hausdorff space is closed, if $\pi : S \to S/\sim$ is the projection and the quotient $S/\sim$ is Hausdorff, then for any $p \in S$, its image $\{\pi(p)\}$ is closed in $S/\sim$. By the continuity of π, the inverse image $\pi^{-1}(\{\pi(p)\}) = [p]$ is closed in S. This gives a necessary condition for a quotient space to be Hausdorff.

Proposition 7.7 If the quotient space $S/\sim$ is Hausdorff, then the equivalence class $[p]$ of any point $p \in S$ is closed in S.

Example 7.8 Define an equivalence relation $\sim$ on $\mathbb{R}$ by identifying the open interval $(0, \infty)$ to a point. Then the quotient space $\mathbb{R}/\sim$ is not Hausdorff because the equivalence class $(0, \infty)$ of $\sim$ in $\mathbb{R}$ corresponding to the point $(0, \infty) \in \mathbb{R}/\sim$ is not closed in $\mathbb{R}$.

7.5. Open Equivalence Relations

Definition 7.9 A map $f : X \to Y$ of topological spaces is **open** if for every open set $U \subset X$, the image $f(U)$ is open in Y.

Definition 7.10 An equivalence relation $\sim$ on a topological space S is **open** if the projection map $\pi : S \to S/\sim$ is open.
Equivalently, $\sim$ is open if for every open set $U \subset S$, the set

$$\pi^{-1}(\pi(U)) = \bigcup_{x \in U} [x] \tag{80}$$

is open in S.

Example 7.11 The projection map ro a quotient space is in general not open. For example, let $\sim$ be the equivalence relation on the real line $\mathbb{R}$ that identifies the two points 1 and -1 to a point. The projection map $\pi : \mathbb{R} \to \mathbb{R}/\sim$ is not open.

Solution. Let $V = (-2, 0) \subset \mathbb{R}$ be an open set in $\mathbb{R}$. Then

$$\pi^{-1}(\pi(V)) = (-2, 0) \cup \{1\}, \tag{81}$$

which is not open in $\mathbb{R}$. Thus, π is not open.

Definition 7.12 Given an equivalence relation $\sim$ on S, the **graph** of $\sim$ is the subset $R \subset S \times S$ defined by

$$R = \{(x, y) \in S \times S \mid x \sim y\} \tag{82}$$

Theorem 7.13 Suppose $\sim$ is an open equivalence relation on a topological space S. Then the quotient space $S/\sim$ is Hausdorff if and only if the graph R of $\sim$ is closed in $S \times S$.

Proof. There is a sequence of equivalent statements:
R is closed in $S \times S$
$\iff$ $(S \times S) - R$ is open in $S \times S$
$\iff$ For every $(x, y) \in S \times S$, there is a basic open set $U \times V$ containing (x, y) such that $(U \times V) \cap R = \emptyset$
$\iff$ For every pair $x \nsim y$ in S, there exists neighborhoods U of x and V of y such that no element of U is equivalent to an element of V
$\iff$ For any two points $[x] \neq [y]$ in $S/\sim$, there exists neighborhoods U of x and V of y in S such that $\pi(U) \cap \pi(V) = \emptyset$ in $S/\sim$
Since π is open, $\pi(U)$ and $\pi(V)$ are disjoint open sets in $S/\sim$ containing $[x]$ and $[y]$, respectively. Therefore, $S/\sim$ is Hausdorff.
Conversely, suppose $S/\sim$ is Hausdorff. Let $[x] \neq [y]$ in $S/\sim$. Then there exist disjoint open sets $A, B \subset S/\sim$ such that $[x] \in A$ and $[y] \in B$. By the surjectivity of π, we have $A = \pi(\pi^{-1}(A))$ and $B = \pi(\pi^{-1}(B))$. Let $U = \pi^{-1}(A)$ and $V = \pi^{-1}(B)$. Then $x \in U, y \in V$, and $A = \pi(U), B = \pi(V)$ are disjoint open sets in $S/\sim$. □

If the equivalence relation $\sim$ is equality, then the quotient space $S/\sim$ is S itself and the graph R of $\sim$ is simply the diagonal

$$\Delta = \{(x, x) \in S \times S\}, \tag{83}$$

where Theorem 7.13 becomes the following well-known characterization of a Hausdorff space by its diagonal.

Corollary 7.14 A topological space S is Hausdorff if and only if the diagonal $\Delta = \{(x, x) \mid x \in S\}$ is closed in $S \times S$.

Theorem 7.15 Let $\sim$ be an open equivalence relation on a topological space S with projection $\pi : S \to S/\sim$. If $\mathcal{B} = \{B_\alpha\}$ is a basis for S, then its image $\{\pi(B_\alpha)\}$ under π is a basis for $S/\sim$.

Proof. Since π is an open map, $\{\pi(B_\alpha)\}$ is a collection of open sets in $S/\sim$. Let W be an open set in $S/\sim$ and $[x] \in W, x \in S$. Then $x \in \pi^{-1}(W)$. Since $\pi^{-1}(W)$ is open in S, there exists a basic open set $B \in \mathcal{B}$ such that

$$x \in B \subset \pi^{-1}(W). \tag{84}$$

Then

$$[x] = \pi(x) \in \pi(B) \subset W, \tag{85}$$

which proves that $\{\pi(B_\alpha)\}$ is a basis for $S/\sim$. □

Corollary 7.16 If $\sim$ is an open equivalence relation on a second-countable space S, then the quotient space $S/\sim$ is second countable.

7.6. Real Projective Space

Definition 7.17 The **real projective space** $\mathbb{R}P^n$ is the quotient space of $\mathbb{R}^{n+1} - \{0\}$ by the equivalence relation $\sim$ defined by

$$x \sim y \iff y = tx \text{ for some nonzero real number } t. \tag{86}$$

The **homogeneous coordinates** of a point $(a^0, \cdots, a^n) \in \mathbb{R}^{n+1} - \{0\}$ are the equivalence class $[a^0, \cdots, a^n]$.

Geometrically, two nonzero points in $\mathbb{R}^{n+1}$ are equivalent if and only if they lie on the same line through the origin, so $\mathbb{R}P^n$ can be interpreted as the set of all lines through the origin in $\mathbb{R}^{n+1}$. Each line through the origin in $\mathbb{R}^{n+1}$ meets the unit sphere S^n in a pair of antipodal points, and conversely, a pair of antipodal points on S^n determines a unique line through the origin. This suggests that we define an equivalence relation $\sim$ on S^n by identifying antipodal points:

$$x \sim y \iff x = \pm y, \quad x, y \in S^n, \tag{87}$$

which gives a bijection $\mathbb{R}P^n \leftrightarrow S^n/\sim$.

Example 7.18 (Real projective space as a quotient of a sphere) For $x = (x^1, \cdots, x^n) \in \mathbb{R}^n$, let $\|x\| = \sqrt{\sum_i (x^i)^2}$ be the module of x. Prove that the map $f : \mathbb{R}^{n+1} - \{0\} \to S^n$ given by

$$f(x) = \frac{x}{\|x\|} \tag{88}$$

introduces a homeomorphism $\bar{f} : \mathbb{R}P^n \to S^n/\sim$.(Hint: Find an inverse map

$$\bar{g} : S^n/\sim \to \mathbb{R}P^n \tag{89}$$

and show that both $\bar{f}$ and $\bar{g}$ are continuous.)

Solution. Let $\sim_1$ be the equivalence relation on $\mathbb{R}^{n+1} - \{0\}$ defined by

$$x \sim y \iff y = tx \text{ for some nonzero real number } t, \tag{90}$$

and the projection map $\pi_1 : \mathbb{R}^{n+1} - \{0\} \to \mathbb{R}P^n$.
Let $\sim_2$ be the equivalence relation on S^n defined by

$$x \sim y \iff x = \pm y, \quad x, y \in S^n, \tag{91}$$

and the projection map $\pi_2 : S^n \to S^n/\sim_2$.

(i) $\bar{f}$ is continuous.
For $x \sim_1 y$ in $\mathbb{R}^{n+1} - \{0\}$, i.e., $y = tx$ for some nonzero real number t, we have

$$\begin{aligned} f(y) &= \frac{y}{\|y\|} \\ &= \frac{tx}{\|tx\|} \\ &= \frac{tx}{|t|\|x\|} \\ &= \mathrm{sgn(t)}\ f(x) \\ &= \pm f(x), \end{aligned} \tag{92}$$

which means $f(y) \sim_2 f(x)$ in S^n. Then the map

$$\pi_2 \circ f : \mathbb{R}^{n+1} - \{0\} \to S^n/\sim_2 \tag{93}$$

is constant on the equivalence classes of $\sim_1$ of $\mathbb{R}^{n+1} - \{0\}$, so it induces a map

$$\bar{f} : \mathbb{R}P^n \to S^n/\sim_2, \quad \bar{f}\Big([x]_{\sim_1}\Big) = \pi_2 \circ f(x), \tag{94}$$

which is continuous since f and π_2 are continuous.

(ii) $\bar{g}$ is continuous.
For $q \in S^n \subset \mathbb{R}^{n+1} - \{0\}$ with inclusion map

$$i : S^n \to \mathbb{R}^{n+1} - \{0\}, \quad i(q) = q, \tag{95}$$

define

$$\bar{g} : S^n/\sim_2 \to \mathbb{R}P^n, \quad \bar{g}\Big([q]_{\sim_2}\Big) = [q]_{\sim_1}. \tag{96}$$

For $p \sim_2 q$ in S^n, i.e., $p = \pm q$, we have

$$\begin{aligned} \bar{g}\Big([p]_{\sim_2}\Big) &= [p]_{\sim_1} \\ &= [\pm q]_{\sim_1} \\ &= [q]_{\sim_1}, \end{aligned} \tag{97}$$

so $\bar{g}$ is well defined. Since

$$\bar{g} \circ \pi_2 : S^n \to \mathbb{R}P^n, \quad \bar{g} \circ \pi_2(q) = [q]_{\sim_1} = \pi_1 \circ i(q), \tag{98}$$

then

$$\bar{g} \circ \pi_2 = \pi_1 \circ i, \tag{99}$$

which shows that $\bar{g}$ is continuous.

(iii) $\bar{f}$ is bijective

(a) $\bar{f}$ is surjective.
For any$[q]_{\sim_2}$ in $S^n / \sim_2$, a presentative $q \in S^n \subset \mathbb{R}^{n+1} - \{0\}$,

$$\begin{aligned} \bar{f}\Big([q]_{\sim_1}\Big) &= \pi_2 \circ f(q) \\ &= \left[\frac{q}{\|q\|}\right]_{\sim_2} \\ &= [q]_{\sim_2}. \end{aligned} \tag{100}$$

(b) $\bar{f}$ is injective.
Suppose

$$\bar{f}\Big([x]_{\sim_1}\Big) = \bar{f}\Big([y]_{\sim_1}\Big), \quad x, y \in \mathbb{R}^{n+1} - \{0\}. \tag{101}$$

Then

$$\begin{aligned} \left[\frac{x}{\|x\|}\right]_{\sim_2} &= \left[\frac{y}{\|y\|}\right]_{\sim_2} \\ \frac{y}{\|y\|} &= \pm \frac{x}{\|x\|} \\ y &= \pm \frac{\|y\|}{\|x\|} x, \end{aligned} \tag{102}$$

which means $y \sim_1 x$ in $\mathbb{R}^{n+1} - \{0\}$, i.e., $[y]_{\sim_1} = [x]_{\sim_1}$.

(iv) $\bar{g}$ is bijective

(a) $\bar{g}$ is surjective.
For any $[x]_{\sim_1}$ in $\mathbb{R}P^n$, a presentative $x \in \mathbb{R}^{n+1} - \{0\}$,

$$\begin{aligned} \bar{g}\left(\left[\frac{x}{\|x\|}\right]_{\sim_2}\right) &= \left[\frac{x}{\|x\|}\right]_{\sim_1} \\ &= [x]_{\sim_1}. \end{aligned} \tag{103}$$

(b) $\bar{g}$ is injective.
Suppose

$$\bar{g}\Big([x]_{\sim_2}\Big) = \bar{g}\Big([y]_{\sim_2}\Big), \quad x, y \in S^n. \tag{104}$$

Then

$$[x]_{\sim_1} = [y]_{\sim_1} \tag{105}$$

$$x = ty \text{ for some nonzero real number } t, \tag{105}$$

which means $x = \pm y$, since $x, y \in S^n$. Thus, $[x]_{\sim_2} = [y]_{\sim_2}$.

(v) $\bar{f}$ and $\bar{g}$ are mutually inverse maps.

(a) For $[x]_{\sim_1} \in \mathbb{R}P^n$,

$$\begin{aligned}\bar{g} \circ \bar{f}\Big([x]_{\sim_1}\Big) &= \bar{g}(\pi_2 \circ f(x)) \\ &= \bar{g}\left(\left[\frac{x}{\|x\|}\right]_{\sim_2}\right) \\ &= \left[\frac{x}{\|x\|}\right]_{\sim_1} \\ &= [x]_{\sim_1}.\end{aligned} \tag{106}$$

(b) For $[q]_{\sim_2} \in S^n/\sim_2$,

$$\begin{aligned}\bar{f} \circ \bar{g}\Big([q]_{\sim_2}\Big) &= \bar{f}\Big([q]_{\sim_1}\Big) \\ &= \left[\frac{q}{\|q\|}\right]_{\sim_2} \\ &= [q]_{\sim_2}.\end{aligned} \tag{107}$$

Example 7.19 (The real projective line $\mathbb{R}P^1$) Each line through the origin in $\mathbb{R}^2$ meets the unit circle S^1 in a pair of antipodal points. As we've proved, $\mathbb{R}P^n$ is homeomorphic to $S^n/\sim$, which is in turn homeomorphic to the closed upper semicircle with the two endpoints identified. Thus, $\mathbb{R}P^1$ is homeomorphic to the circle S^1.

Example 7.20 (The real projective plane $\mathbb{R}P^2$) We've shown that there is a homeomorphism

$$\mathbb{R}P^2 \simeq S^2/\sim. \tag{108}$$

Let H^2 be the closed upper hemisphere

$$H^2 = \big\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1, z \geq 0\big\}. \tag{109}$$

and let D^2 be the closed disk

$$D^2 = \big\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\big\}. \tag{110}$$

These two spaces are homeomorphic to each other via the continuous map

$$\begin{aligned}&\varphi : H^2 \to D^2, \\ &\varphi(x, y, z) = (x, y),\end{aligned} \tag{111}$$

and its inverse

$$\begin{aligned}&\psi : D^2 \to H^2, \\ &\psi(x, y) = \Big(x, y, \sqrt{1 - x^2 - y^2}\Big).\end{aligned} \tag{112}$$

On H^2, define an equivalence relation $\sim$ by identifying the antipodal points on the equator:

$$(x, y, 0) \sim (-x, -y, 0), \quad x^2 + y^2 = 1. \tag{113}$$

On D^2, define an equivalence relation $\sim$ by identifying the antipodal points on the boundary:

$$(x, y) \sim (-x, -y), \quad x^2 + y^2 = 1. \tag{114}$$

Then φ and ψ induce homeomorphisms

$$\bar{\varphi} : H^2/\sim \to D^2/\sim, \quad \bar{\psi} : D^2/\sim \to H^2/\sim . \tag{115}$$

There is a homeomorphism between $S^2/\sim$ and $H^2/\sim$.
In summary, there is a sequence of homeomorphisms

$$\mathbb{R}P^2 \xrightarrow{\sim} S^2/\sim \xrightarrow{\sim} H^2/\sim \xrightarrow{\sim} D^2/\sim . \tag{116}$$

Proposition 7.21 The equivalence relation $\sim$ on $\mathbb{R}^{n+1} - \{0\}$ in the definition of $\mathbb{R}P^n$ is open.

Proof. For any open set $U \subset \mathbb{R}^{n+1} - \{0\}$, the image $\pi(U)$ is open in $\mathbb{R}P^n$ if and only if

$$\pi^{-1}(\pi(U)) = \bigcup_{t \in \mathbb{R}^\times} tU \tag{117}$$

is open in $\mathbb{R}^{n+1} - \{0\}$. Since tU is open in $\mathbb{R}^{n+1} - \{0\}$ for any nonzero real number t, the union $\bigcup_{t \in \mathbb{R}^\times} tU$ is open in $\mathbb{R}^{n+1} - \{0\}$. Thus, $\sim$ is open. □

Corollary 7.22 The real projective space $\mathbb{R}P^n$ is second countable.

Proposition 7.23 The real projective space $\mathbb{R}P^n$ is Hausdorff.

Proof. Let $S = \mathbb{R}^{n+1} - \{0\}$ and consider the set

$$R = \{(x, y) \in S \times S \mid y = tx \text{ for some } t \in \mathbb{R}^\times\}. \tag{118}$$

As R is closed in $S \times S$, since $\sim$ is open, the quotient space $\mathbb{R}P^n$ is Hausdorff by Theorem 7.13. □

7.7. The Standard C^∞ Atlas on a Real Projective Space

Let $[a^0, \cdots, a^n]$ be homogeneous coordinates on $\mathbb{R}P^n$. Although a^0 is not a well-defined function on $\mathbb{R}P^n$, the condition $a^0 \neq 0$ is independent of the choice of a representative for $[a^0, \cdots, a^n]$. Hence, the condition $a^0 \neq 0$ makes sense on $\mathbb{R}P^n$, and we may define

$$U_0 := \{[a^0, \cdots, a^n] \in \mathbb{R}P^n \mid a^0 \neq 0\}. \tag{119}$$

Similarly, for each $i = 1, \cdots, n$, we define

$$U_i := \{[a^0, \cdots, a^n] \in \mathbb{R}P^n \mid a^i \neq 0\}. \tag{120}$$

Define

$$\begin{aligned}\phi_0 : U_0 &\to \mathbb{R}^n \\ [a^0, \cdots, a^n] &\mapsto \left(\frac{a^1}{a^0}, \cdots, \frac{a^n}{a^0}\right),\end{aligned} \tag{121}$$

which has a continuous inverse

$$(b^1, \cdots, b^n) \mapsto [1, b^1, \cdots, b^n] \tag{122}$$

and is therefore a homeomorphism. Similarly, there are homeomorphisms

$$\begin{aligned}\phi_i : U_i &\to \mathbb{R}^n \\ [a^0, \cdots, a^n] &\mapsto \left(\frac{a^0}{a^i}, \cdots, \frac{\hat{a^i}}{a^i}, \cdots, \frac{a^n}{a^i}\right),\end{aligned} \tag{123}$$

where the caret sign $\widehat{}$ over $\frac{a^i}{a^i}$ means that entry is to be omitted. This proves that $\mathbb{R}P^n$ is locally Euclidean with (U_i, ϕ_i) as charts. On the intersection $U_0 \cap U_1$, there are two coordinate systems

$$\begin{tikzcd} & {[a^0, a^1, a^2, \cdots, a^n]} \arrow[dl, "\phi_0"'] \arrow[dr, "\phi_1"] & \\ \left(\frac{a^1}{a^0}, \frac{a^2}{a^0}, \cdots, \frac{a^n}{a^0}\right) & & \left(\frac{a^0}{a^1}, \frac{a^2}{a^1}, \cdots, \frac{a^n}{a^1}\right) \end{tikzcd}$$

We will refer the coordinate functions on U_0 as $x^1, \cdots, x^n$, and $y^1, \cdots, y^n$ on U_1. On U_0,

$$x^i = \frac{a^i}{a^0}, \quad i = 1, \cdots, n, \tag{124}$$

and on U_1,

$$y^1 = \frac{a^0}{a^1}, \quad y^i = \frac{a^i}{a^1}, i = 2, \cdots, n. \tag{125}$$

Then on $U_0 \cap U_1$,

$$y^1 = \frac{1}{x^1}, \quad y^i = \frac{x^i}{x^1}, i = 2, \cdots, n, \tag{126}$$

so

$$\phi_1 \circ \phi_0^{-1}(x) = \left(\frac{1}{x^1}, \frac{x^2}{x^1}, \cdots, \frac{x^n}{x^1}\right), \tag{127}$$

and

$$\phi_0 \circ \phi_1^{-1}(y) = \left(\frac{1}{y^1}, \frac{y^2}{y^1}, \cdots, \frac{y^n}{y^1}\right), \tag{128}$$

which are both smooth because $x^1 \neq 0$ on $\phi_0(U_0 \cap U_1)$ and $y^1 \neq 0$ on $\phi_1(U_0 \cap U_1)$, then (U_0, ϕ_0) and (U_1, ϕ_1) are compatible. On any other intersection $U_i \cap U_j$, an analogous formula holds. Therefore, the collection$\{(U_i, \phi_i)\}_{i=0,\cdots,n}$ is a C^∞ atlas for $\mathbb{R}P^n$, called the **standard atlas**. This concludes the proof that $\mathbb{R}P^n$ is a manifold of dimension n.