An Introduction to Manifold

Chapter 1 Euclidean Spaces

1. Smooth Functions on a Euclidean Space

The calculus of C^∞ functions will be our primary tool for studying higher-dimensional manifolds.

1.1. C^∞ Analytic Functions

Let $p = (p^1, \cdots, p^n)$ be a point in an open subset $U \subset \mathbb{R}^n$.

Definition 1.1 Let k be a non-negative integer. A real-valued function $f : U \to \mathbb{R}$ is said to be C^k at p if its partial derivatives

$$\frac{\partial^j f}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}}$$

of all orders $j \leq k$ exist and are continuous at p.
The function $f : U \to \mathbb{R}$ is C^∞ at p if it is C^k at p for all $k \geq 0$.
A vector-valued function $f : U \to \mathbb{R}^m$ is said to be C^k at p if all of its components $f^1, \cdots, f^m$ are C^k at p.
$f : U \to \mathbb{R}$ is said to be C^k on U if it is C^k at every point $p \in U$.
The set of all C^∞ functions on U is denoted by $C^\infty(U)$ or $\mathcal{F}(U)$.

The function $f : U \to \mathbb{R}$ is real-analytic at p if in some neighborhood of p, it is equal to its Taylor series at p.
A real-analytic function is necessarily C^∞, but the converse is not true.

1.2. Taylor's Theorem with Remainder

Definition 1.2 A subset $S \subseteq \mathbb{R}^n$ is **star-shaped** with respect to a point $p \in S$ if for every point $x \in S$, the line segment from p to x lies entirely in S.

Lemma 1.3 Let $f \in C^\infty(U)$, where $U \subset \mathbb{R}^n$ is an open subset, star-shaped with respect to a point $p \in U$. Then there are functions $g_1(x), \cdots, g_n(x) \in C^\infty(U)$ such that

$$f(x) = f(p) + (x^i - p^i)g_i(x), \quad g_i(x) = \frac{\partial f}{\partial x^i}(p).$$

If f is a C^∞ function on an open subset U containing p, then there is an $\varepsilon > 0$ such that

$$p \in B(p, \varepsilon) \subset U,$$

where $B(p, \varepsilon) = \{x \in \mathbb{R}^n : \|x - p\| < \varepsilon\}$ is the open ball of radius ε centered at p, which is clearly star-shaped with respect to p.

2. Tangent Vevtors in $\mathbb{R}^n$ as Derivations

In this section, we will find a characterization of tangent vectors in $\mathbb{R}^n$ that will generalize to manifolds.

2.1. The Directional Derivative

To distinguish between points and vectors, we write a point in $\mathbb{R}^n$ as $p = (p^1, \cdots, p^n)$ and a vector in the tangent space at p, denoted by $T_p\mathbb{R}^n$, as

$$v = \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix} \text{ or } v = \langle v^1, \cdots, v^n \rangle.$$

We usually denote the standard basis of $\mathbb{R}^n$ by $e_1, \cdots, e_n$, then $v = v^i e_i$ for some $v^i \in \mathbb{R}$. The line through $p = (p^1, \cdots, p^n)$ in the direction of $v = (v^1, \cdots, v^n)$ has parametrization

$$c(t) = (p^1 + tv^1, \cdots, p^n + tv^n).$$

Definition 2.1 If f is C^∞ in a neighborhood of p in $\mathbb{R}^n$, the **directional derivative** of f at p in the direction of v is defined as the limit

$$\begin{aligned} D_v f &= \lim_{t \to 0} \frac{f(c(t)) - f(c(0))}{t} \\ &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f(c(t)) \\ &= \frac{\mathrm{d}c^i}{\mathrm{d}t}(0) \frac{\partial f}{\partial x^i}(p) \\ &= v^i \frac{\partial f}{\partial x^i}(p). \end{aligned}$$

We write

$$D_v = v^i \left.\frac{\partial}{\partial x^i}\right|_p$$

for the map from a function f to its directional derivative $D_v f$.

The association $v \to D_v$ offers a way to characterize tangent vectors as a certain operators on C^∞ functions.

2.2. Germs of Functions

Definition 2.2 A **relation** on a set S is a subset R of $S \times S$. Given $x, y \in S$, we write $x \sim y$ if and only if $(x, y) \in R$.
A relation R is an **equivalence relation** if it satisfies the following properties for all $x, y, z \in S$:
(i) **Reflexivity:** $x \sim x$,
(ii) **Symmetry:** If $x \sim y$, then $y \sim x$,
(iii) **Transitivity:** If $x \sim y$ and $y \sim z$, then $x \sim z$.

Consider the set of all pairs (f, U) where U is a neighborhood of p and $f : U \to \mathbb{R}$ is a C^∞ function. We say that (f, U) is **equivalent** to (g, V) if there exists a neighborhood $W \subseteq (U \cap V)$ such that $f|_W = g|_W$.

Definition 2.3 The **germ** of f at p is the equivalence class of the pair (f, U).
We write $C_p^\infty(\mathbb{R}^n)$, or simply C_p^∞, for the set of all germs of C^∞ functions on $\mathbb{R}^n$ at p.

Definition 2.4 An **algebra** over a field K is a vector space A over K with a multiplication map

$$\mu : A \times A \to A,$$

usually written $\mu(a, b) = a \cdot b$, that satisfies the following properties for all $a, b, c \in A$ and $r \in K$:
(i) **Associativity:** $(a \cdot b) \cdot c = a \cdot (b \cdot c)$,
(ii) **Distributivity:** $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(a + b) \cdot c = a \cdot c + b \cdot c$,
(iii) **Homogeneity:** $r(a \cdot b) = (ra) \cdot b = a \cdot (rb)$.

Usually we write the multiplication as simply ab instead of $a \cdot b$.

Definition 2.5 A map $L : V \to W$ between two vector spaces over the field K is said to be a **linear map** or a **linear operator** if for all $u, v \in V$ and $r \in K$:
(i) $L(u + v) = L(u) + L(v)$,
(ii) $L(ru) = rL(u)$.

To emphasize the scalars are in the field K, such a map is said to be ***K*-linear**.

Definition 2.6 If A and A' are algebras over a field K, an **algebra homomorphism** is a linear map $L : A \to A'$ that preserves the algebra multiplication: $L(ab) = L(a)L(b)$ for all $a, b \in A$.

The addition and multiplication of functions induce corresponding operations on C_p^∞, making it into an algebra over $\mathbb{R}$.

2.3. Derivations at a Point

For each tangent vector $v \in T_p\mathbb{R}^n$, the directional derivative at p gives a map

$$D_v : C_p^\infty \to \mathbb{R}.$$

Definition 2.7 A linear map $D : C_p^\infty \to \mathbb{R}$ is called a **derivation** at p or a **point derivation** if it satisfies the Leibniz rule:

$$D(fg) = D(f)g(p) + f(p)D(g)$$

Denote the set of all derivations at p by $\mathcal{D}_p(\mathbb{R}^n)$, which is a vector space over $\mathbb{R}$.

Obviously, the directional derivatives at p are all derivations at p, so there is a map

$$\begin{aligned} \phi : T_p(\mathbb{R}^n) &\to \mathcal{D}_p(\mathbb{R}^n), \\ v &\mapsto D_v = v^i \frac{\partial}{\partial x^i}\bigg|_p . \end{aligned}$$

Since D_v is clearly linear in v, ϕ is a linear map of vector spaces.

Lemma 2.8 If D is a point-derivation of C_p^∞, then $D(c) = 0$ for any constant function c.

Proof. By $\mathbb{R}$-linearity, $D(c) = cD(1)$. By the Leibniz rule, we have

$$\begin{aligned} D(1) &= D(1 \cdot 1) \\ &= D(1) \cdot 1(p) + 1(p) \cdot D(1) \\ &= 2D(1), \end{aligned}$$

which implies that $D(1) = 0$, and therefore $D(c) = cD(1) = c \cdot 0 = 0$. □

Lemma 2.9 The map $\phi : T_p(\mathbb{R}^n) \to \mathcal{D}_p(\mathbb{R}^n)$ is an isomorphism of vector spaces.

Proof. To show that ϕ is injective, suppose $\phi(v) = D_v = 0$ for some $v \in T_p(\mathbb{R}^n)$. For the coordinate functions x^j, we have

$$\begin{aligned} 0 = D_v x^j &= v^i \frac{\partial x^j}{\partial x^i}\bigg|_p \\ &= v^i \delta_i^j \\ &= v^j, \end{aligned}$$

which implies that $v = 0$. Thus, ϕ is injective.
To show that ϕ is surjective, let $D \in \mathcal{D}_p(\mathbb{R}^n)$ and let (f, V) be a representative of a germ in C_p^∞. We may assume V is an open ball, hence star-shaped. From Taylor's theorem with remainder, we have

$$f(x) = f(p) + (x^i - p^i)g_i(x), \quad g_i(p) = \frac{\partial f}{\partial x^i}(p).$$

Applying D to both sides, we get

$$\begin{aligned} D(f(x)) &= D[f(p)] + D[(x^i - p^i)g_i(x)] \\ &= (Dx^i)g_i(p) + (p^i - p^i)Dg_i(x) \\ &= (Dx^i)g_i(p) \\ &= (Dx^i)\frac{\partial f}{\partial x^i}(p), \end{aligned}$$

which gives $D = D_v$ for $v = \langle Dx^1, \cdots, Dx^n \rangle$. Thus, ϕ is surjective. $\square$

Under this vector space isomorphism $T_p(\mathbb{R}^n) \simeq \mathcal{D}_p(\mathbb{R}^n)$, we can identify tangent vectors with derivations at p, and the standard basis $e_1, \cdots, e_n$ of $T_p(\mathbb{R}^n)$ with the set $\frac{\partial}{\partial x^1}\big|_p, \cdots, \frac{\partial}{\partial x^n}\big|_p$ of partial derivatives,

$$\begin{aligned} v &= \langle v^1, \cdots, v^n \rangle \\ &= v^i e_i \\ &= v^i \frac{\partial}{\partial x^i}\bigg|_p . \end{aligned}$$

2.4. Vector Fields

Definition 2.10 A **vector field** on an open subset $U \subseteq \mathbb{R}^n$ is a function that assigns to each point $p \in U$ a tangent vector $X_p \in T_p(\mathbb{R}^n)$. Since $T_p(\mathbb{R}^n)$ has basis $\frac{\partial}{\partial x^i}\big|_p$, we can write

$$X_p = a^i(p) \frac{\partial}{\partial x^i}\bigg|_p, \quad a^i(p) \in \mathbb{R}.$$

Omitting p, we can write

$$X = a^i \frac{\partial}{\partial x^i} \quad \leftrightarrow \quad \begin{bmatrix} a^1 \\ \vdots \\ a^n \end{bmatrix},$$

where a^i are functions on U. We say that X is C^∞ on U if all the coefficient functions a^i are C^∞ on U.
The set of all C^∞ vector fields on U is denoted by $\mathfrak{X}(U)$.

Definition 2.11 If R is a commutative ring with identity, a (left) R-**module** is an abelian group A with a scalar multiplication

$$\mu : R \times A \to A,$$

usually written $\mu(r, a) = ra$, such that for all $r, s \in R$ and $a, b \in A$,
(i) **Associativity:** $(rs)a = r(sa)$,
(ii) **Identity:** $1a = a$,
(iii) **Distributivity:** $r(a + b) = ra + rb$ and $(r + s)a = ra + sa$.

$\mathfrak{X}(U)$ is a module over the ring $C^\infty(U)$ with the multiplication defined pointwise:

$$(fX)_p = f(p)X_p,$$

for $f \in C^\infty(U), X \in \mathfrak{X}(U), p \in U$.

Definition 2.12 Let A and A' be R-modules. An R-**module homomorphism** from A to A' is a map $f : A \to A'$ that preserves both the addition and the scalar multiplication: for all $a, b \in A$ and $r \in R$,
(i) $f(a + b) = f(a) + f(b)$,

(ii) $f(ra) = rf(a)$.

2.5. Vector Fields as Derivations

If $X \in \mathfrak{X}(U)$ and $f \in C^\infty(U)$, we can define a new function Xf by

$$(Xf)(p) = X_p f \quad \text{for all } p \in U.$$

Writing $X = a^i \frac{\partial}{\partial x^i}$, we have

$$(Xf)(p) = a^i(p)\frac{\partial f}{\partial x^i}(p),$$

or

$$Xf = a^i \frac{\partial f}{\partial x^i},$$

which is a C^∞ function on U. Thus, a C^∞ vector field X induces an $\mathbb{R}$-linear map

$$\begin{aligned} X : C^\infty(U) &\to C^\infty(U), \\ f &\mapsto Xf. \end{aligned}$$

$X(fg)$ satisfies the Leibniz rule:

$$X(fg) = (Xf)g + f(Xg).$$

Definition 2.13 If A is an algebra over a field K, a **derivation** on A is a K-linear map $D : A \to A$ that satisfies the Leibniz rule:

$$D(ab) = (Da)b + a(Db) \quad \text{for all } a, b \in A.$$

The set of all derivations on A is closed under addition and scalar multiplication and forms a vector space, denoted by $\mathrm{Der}(A)$.

We therefore have a map

$$\begin{aligned} \varphi : \mathfrak{X}(U) &\to \mathrm{Der}(C^\infty(U)), \\ X &\mapsto (f \mapsto Xf), \end{aligned}$$

which is an isomorphism of vector spaces, just as the map $\phi : T_p(\mathbb{R}^n) \to \mathcal{D}_p(\mathbb{R}^n)$.