5 Manifolds

Problem 5.1

Let A and B be points on the real line $\mathbb{R}$. Consider the set $S = (\mathbb{R} - \{0\}) \cup \{A, B\}$.

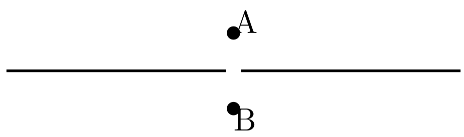

Figure 1: Real line with two origins

For any two positive real numbers c, d, define

$$I_A(-c, d) = (-c, 0) \cup \{A\} \cup (0, d)$$

Similarly for $I_B(-c, d)$, with B in place of A. Define a topology on S as follows: on $(\mathbb{R} - \{0\})$, use the subspace topology inherited from $\mathbb{R}$, with open intervals as a basis. A basis of neighborhoods at A is the collection $\{I_A(-c, d) \mid c, d > 0\}$; similarly, a basis of neighborhoods at B is $\{I_B(-c, d) \mid c, d > 0\}$.

(a) Prove that the map $h : I_A(-c, d) \to (-c, d)$ defined by

$$\begin{aligned} h(x) &= x \quad \text{for } x \in (-c, 0) \cup (0, d), \\ h(A) &= 0 \end{aligned}$$

is a homeomorphism.

(b) Prove that S is locally Euclidean and second countable, but not Hausdorff.

Solution

(a) To show that h is a homeomorphism, we need to show that it is continuous and has a continuous inverse.

(i) h is injective.
Let $x, y \in I_A(-c, d)$ such that $h(x) = h(y)$. There are two cases:
- $h(x) = h(y) = 0$, then $x = y = A$
- $h(x) = h(y) \neq 0$, then $h(x) = x$ and $h(y) = y$, so $x = y$.

(ii) h is surjective.
Let $y \in (-c, d)$. There are two cases:
- $y \neq 0$, then $y \in I_A(-c, d)$ and $h(y) = y$.
- $y = 0$, then $h(A) = 0 = y$.

(iii) h is continuous.
Let $(x, y) \subseteq (-c, d)$ be an open interval. There are two cases:
- $0 \in (x, y)$, then $h^{-1}((x, y)) = I_A(x, y) \subseteq I_A(-c, d)$ is open.
- $0 \notin (x, y)$, then $h^{-1}((x, y)) = (x, y) \subseteq I_A(-c, d)$ is open.

(iv) h^{-1} is continuous.
There are two cases:

- $A \in (x, y)$, then $h(I_A(x, y)) = (x, y) \in (-c, d)$ is open.
- $A \notin (x, y)$, then $h((x, y)) = (x, y) \in (-c, d)$ is open.

Therefore, h is a homeomorphism.

(b) (i) S is locally Euclidean.

From (a), we know that for any $x \in S$, there is a neighborhood U of x such that U is homeomorphic to an open subset of $\mathbb{R}$ with h as the homeomorphism.

(i) S is second countable.

(ii) S is not Hausdorff. Consider the points A and B. For any open set U containing A, we have $U = I_A(-a_1, a_2)$ for some $a_1, a_2 > 0$. Similarly, for any open set V containing B, we have $V = I_B(-b_1, b_2)$ for some $b_1, b_2 > 0$. Suppose $U \cap V = \emptyset$. Let $c_1 = \max(a_1, b_1)$ and $c_2 = \min(a_2, b_2)$. Then $U \cap V = (c_1, 0) \cup (0, c_2)$, which is not empty. Therefore, S is not Hausdorff.

Problem 5.2

A fundamental theorem of topology, the theorem on invariance of dimension, states that if two nonempty open sets $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ are homeomorphic, then $n = m$. Use the idea of Example 5.4 as well as the theorem on invariance of dimension to prove that the sphere with a hair in $\mathbb{R}^3$ is not locally Euclidean at q. Hence it cannot be a topological manifold.

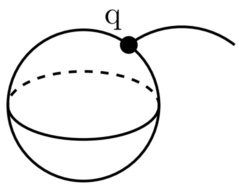

Figure 2: A sphere with a hair.

Solution

Suppose the sphere with a hair is locally Euclidean of dimension n at q. Then there is a neighborhood U of q such that U is homeomorphic to an open ball $B = B(0, \varepsilon) \subseteq \mathbb{R}^n$ with q mapping to 0. The homeomorphism $U \to B$ restricts to a homeomorphism $U \setminus \{q\} \to B \setminus \{0\}$. Since $B \setminus \{0\}$ is either connected if $n \geq 2$ or has two connected components if $n = 1$ and $U \setminus \{q\}$ has two connected components, n must be 1, i.e., U is homeomorphic to an open interval $U \subseteq \mathbb{R}$. However, a neighborhood on the sphere has dimension 2. By invarience of dimension, the sphere with a hair cannot be locally Euclidean at q.

Problem 5.3

Let S^2 be the unit sphere

$$x^2 + y^2 + z^2 = 1$$

in $\mathbb{R}^3$. Define in S^2 the six charts corresponding to the six hemispheres—the front, rear, right, left, upper, and lower hemispheres:

$$\begin{aligned}
U_1 &= \{(x,y,z) \in S^2 \mid x > 0\}, & \phi_1(x,y,z) &= (y,z), \\
U_2 &= \{(x,y,z) \in S^2 \mid x < 0\}, & \phi_2(x,y,z) &= (y,z), \\
U_3 &= \{(x,y,z) \in S^2 \mid y > 0\}, & \phi_3(x,y,z) &= (x,z), \\
U_4 &= \{(x,y,z) \in S^2 \mid y < 0\}, & \phi_4(x,y,z) &= (x,z), \\
U_5 &= \{(x,y,z) \in S^2 \mid z > 0\}, & \phi_5(x,y,z) &= (x,y), \\
U_6 &= \{(x,y,z) \in S^2 \mid z < 0\}, & \phi_6(x,y,z) &= (x,y).
\end{aligned}$$

Describe the domain $\phi_4(U_{14})$ of $\phi_1 \circ \phi_4^{-1}$ and show that $\phi_1 \circ \phi_4^{-1}$ is C^∞ on $\phi_4(U_{14})$. Do the same for $\phi_6 \circ \phi_1^{-1}$.

Solution

As $U_{14} = U_1 \cap U_4$, $\phi_4(U_{14}) = \{(x,z) \mid x > 0, x^2 + z^2 < 1\})$, and the transition map $\phi_1 \circ \phi_4^{-1}$ is given by

$$\begin{aligned}
\phi_1 \circ \phi_4^{-1}(x,z) &= \phi_1\left(x, -\sqrt{1-x^2-z^2}, z\right) \\
&= \left(-\sqrt{1-x^2-z^2}, z\right),
\end{aligned}$$

which is C^∞ on $\phi_4(U_{14})$.

Problem 5.4

Let $\{(U_\alpha, \phi_\alpha)\}$ be the maximal atlas on a manifold M. For any open set U in M and a point $p \in U$, prove the existence of a coordinate open set U_α such that $p \in U_\alpha \subseteq U$.

Solution

Let U_β be a coordinate open set such that $p \in U_\beta \subseteq U$. Then $U_\alpha = U_\beta \cap U$ is a coordinate open set such that $p \in U_\alpha \subseteq U$.

6 Smooth Maps on a Manifold

Problem 6.1

Let $\mathbb{R}$ be the real line with the differentiable structure given by the maximal atlas of the chart $(\mathbb{R}, \phi = 1 : \mathbb{R} \to \mathbb{R})$, and let $\mathbb{R}'$ be the real line with the differentiable structure given by the maximal atlas of the chart $(\mathbb{R}, \psi : \mathbb{R} \to \mathbb{R})$, where $\psi(x) = x^{\frac{1}{3}}$.

(a) Show that these two differentiable structures are distinct.
(b) Show that there is a diffeomorphism between $\mathbb{R}$ and $\mathbb{R}'$. (Hint: The identity map $\mathbb{R} \to \mathbb{R}$ is not the desired diffeomorphism; in fact, this map is not smooth.)

Solution

(a) Suppose that $\mathbb{R}$ and $\mathbb{R}'$ have the same differentiable structure. Then $F : \mathbb{R} \to \mathbb{R}'$ must be the identity map and $\psi \circ F \circ \phi$ must be a diffeomorphism. However, for $x \in \mathbb{R}$,

$$\psi \circ F \circ \phi^{-1}(x) = \psi(x) = x^{\frac{1}{3}},$$

is not C^∞ at 0. Therefore, $\mathbb{R}$ and $\mathbb{R}'$ have distinct differentiable structures.

(b) Let $F : \mathbb{R} \to \mathbb{R}'$ be the map defined by $F(x) = x^3$. For $x \in \mathbb{R}$, we have

$$\psi \circ F \circ \phi^{-1}(x) = \psi(x^3) = (x^3)^{\frac{1}{3}} = x,$$

is a diffeomorphism. Therefore, F is a diffeomorphism between $\mathbb{R}$ and $\mathbb{R}'$.

Problem 6.2

Let M and N be manifolds and let q_0 be a point in N. Prove that the inclusion map $i_{q_0} : M \to M \times N$, $i_{q_0}(p) = (p, q_0)$, is C^∞.

Solution

Let (U_α, ϕ_α) and (V_i, ψ_i) be a chart of M and N, respectively, then $(U_\alpha \times V_i, \phi_\alpha \times \psi_i)$ is a chart of $M \times N$. To show that i_{q_0} is C^∞, we have the map

$$\begin{aligned}(\phi_\alpha \times \psi_i) \circ i_{q_0} \circ \phi_\beta^{-1}(x) &= (\phi_\alpha \times \psi_i)(\phi_\alpha^{-1}, q_0) \\ &= \left(\phi_\alpha \circ \phi_\beta^{-1}(x), \psi_i(q_0)\right).\end{aligned}$$

As ϕ_α and ϕ_β are compatible and $\psi_i(q_0)$ is a constant, the map is C^∞. Therefore, i_{q_0} is C^∞.

Problem 6.3

Let V be a finite-dimensional vector space over $\mathbb{R}$, and $\mathrm{GL}(V)$ the group of all linear automorphisms of V. Relative to an ordered basis $e = (e_1, \dots, e_n)$ for V, a linear automorphism $L \in \mathrm{GL}(V)$ is represented by a matrix $\left[a^i_j\right]$ defined by

$$L(e_j) = \sum_i a^i_j e_i.$$

The map

$$\begin{aligned}\phi_e : \mathrm{GL}(V) &\to \mathrm{GL}(n, \mathbb{R}), \\ L &\mapsto \left[a^i_j\right],\end{aligned}$$

is a bijection with an open subset of $\mathbb{R}^{n\times n}$ that makes $\mathrm{GL}(V)$ into a C^∞ manifold, which we denote temporarily by $\mathrm{GL}(V)_e$. If $\mathrm{GL}(V)_u$ is the manifold structure induced from another ordered basis $u = (u_1, \dots, u_n)$ for V, show that $\mathrm{GL}(V)_e$ is the same as $\mathrm{GL}(V)_u$.

Solution

Problem 6.4

Find all points in $\mathbb{R}^3$ of which the functions $x, x^2 + y^2 + z^2 - 1, z$ can serve as a local coordinates system in a neighborhood.

Solution

For$(x, y, z) \in \mathbb{R}^3$, define $F : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$F(x,y,z) = (x, x^2+y^2+z^2-1, z).$$

The Jacobian determinant of F is

$$\begin{aligned}\frac{\partial(F^1,F^2,F^3)}{\partial(x,y,z)} &= \det\begin{bmatrix}1 & 0 & 0\\ 2x & 2y & 2z\\ 0 & 0 & 1\end{bmatrix}\\ &= 2y.\end{aligned}$$

By the inverse function theorem, F is a local coordinate system at (x,y,z) if and only if $2y \neq 0$. Therefore, the points in $\mathbb{R}^3$ of which the functions $x, x^2+y^2+z^2-1, z$ can serve as a local coordinates system in a neighborhood are all points except the $x-z$ plane.

7 Quotients

Problem 7.1

Let $f : X \to Y$ be a map of sets, and let $B \subset Y$. Prove that $f(f^{-1}(B)) = B \cap f(X)$. Therefore, if f is surjective, then $f(f^{-1}(B)) = B$.

Solution

(a) For $x \in f(f^{-1}(B))$, there exists $y \in f^{-1}(B)$ such that $x = f(y) \in f(X)$ and $y \in f^{-1}(B)$, so $f(y) \in B$. Therefore, $x \in B \cap f(X)$.

(b) For $x \in B \cap f(X)$, there exists $y \in X$ such that $x = f(y) \in B$, so $y \in f^{-1}(B)$. Therefore, $x \in f(f^{-1}(B))$.

Therefore, $f(f^{-1}(B)) = B \cap f(X)$.
If f is surjective, then $f(f^{-1}(B)) = B \cap f(X) = B \cap Y = B$.

Problem 7.2

Let H^2 be the closed upper hemisphere in the unit sphere S^2, and let $i : H^2 \to S^2$ be the inclusion map. In the notation of Example 7.13, prove that the induced map $f : H^2/\sim \to S^2/\sim$ is a homeomorphism. (Hint: Imitate Proposition 7.3.)

Solution

Problem 7.3

Deduce Theorem 7.7 from Corollary 7.8. (Hint: To prove that if $S/\sim$ is Hausdorff, then the graph R of $\sim$ is closed in $S \times S$, use the continuity of the projection map $\pi : S \to S/\sim$. To prove the reverse implication, use the openness of π.)

Solution

Suppose $\sim$ is an open equivalence relation on S, then the projection map $\pi : S \to S/\sim$ is open. Then
$S/\sim$ is Hausdorff.
$\iff$ The diagonal $\Delta = \{([x],[x]) \in (S/\sim) \times (S/\sim)\}$ is closed in $(S/\sim) \times (S/\sim)$,
$\iff$ $(\pi^{-1} \times \pi^{-1})\Delta = \{(x,y) \mid x \sim y\} = R$ is closed in $S \times S$.

Problem 7.4

Let S^n be the unit sphere centered at the origin in $\mathbb{R}^{n+1}$. Define an equivalence relation $\sim$ on S^n by identifying antipodal points:

$$x \sim y \iff x = \pm y, \quad x, y \in S^n.$$

(a) Show that $\sim$ is an open equivalence relation.
(b) Apply Theorem 7.7 and Corollary 7.8 to prove that the quotient space $S^n/\sim$ is Hausdorff, without making use of the homeomorphism $\mathbb{R}P^n \simeq S^n/\sim$.

Solution

(a) Let $U \subset S^n$ be an open set. Then $\pi^{-1}(\pi(U)) = U \cup a(U)$, where $a : S^n \to S^n, a(x) = -x$ is the antipodal map. Since a is a homeomorphism, $a(U)$ is open. Therefore, $\pi^{-1}(\pi(U))$ is open as a union of two open sets, then $\pi(U)$ is open by the definition of the quotient topology. Thus, π is an open map, i.e., $\sim$ is an open equivalence relation.
(b) The graph R of $\sim$ is

$$\begin{aligned} R &= \{(x, y) \mid x \sim y\} \\ &= \{(x, x) \in S^n \times S^n\} \cup \{(x, -x) \in S^n \times S^n\} \\ &= \Delta \cup (\mathbb{1} \times a)\Delta. \end{aligned}$$

Since S^n is Hausdorff, the diagonal Δ is closed in $S^n \times S^n$. Since $\mathbb{1} \times a : S^n \times S^n \to S^n \times S^n, (\mathbb{1} \times a)(x, y) = (x, -y)$ is a homeomorphism, $(\mathbb{1} \times a)(\Delta)$ is closed in $S^n \times S^n$. Therefore, R is closed in $S^n \times S^n$ as a union of two closed sets, then $S^n/\sim$ is Hausdorff.

Problem 7.5

Suppose a right action of a topological group G on a topological space S is continuous; this simply means that the map $S \times G \to S$ describing the action is continuous. Define two points x, y of S to be equivalent if they are in the same orbit; i.e., there is an element $g \in G$ such that $y = xg$. Let S/G be the quotient space; it is called the orbit space of the action. Prove that the projection map $\pi : S \to S/G$ is an open map. (This problem generalizes Proposition 7.14, in which $G = \mathbb{R}^\times = \mathbb{R} - \{0\}$ and $S = \mathbb{R}^{n+1} - \{0\}$. Because $\mathbb{R}^\times$ is commutative, a left $\mathbb{R}^\times$-action becomes a right $\mathbb{R}^\times$-action if scalar multiplication is written on the right.)

Solution

Let $U \subset S$ be an open set. Then

$$\pi^{-1}(\pi(U)) = \bigcup_{g \in G} Ug$$

is open in S as a union of open sets. Since π is a quotient map, $\pi(U)$ is open in S/G. Therefore, π is an open map.

Problem 7.6

Let the additive group $2\pi\mathbb{Z}$ act on $\mathbb{R}$ on the right by $x \cdot 2\pi n = x + 2\pi n$, where n is an integer. Show that the orbit space $\mathbb{R}/2\pi\mathbb{Z}$ is a smooth manifold.

Solution

(a) $\mathbb{R}/2\pi\mathbb{Z}$ is Hausdorff.
Let $[x], [y] \in \mathbb{R}/2\pi\mathbb{Z}$ be two distinct points and presentatives $x \in [x], y \in [y]$. As $\mathbb{R}$ is Hausdorff, there are open sets $U, V \subset \mathbb{R}$ such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$. Then $\pi(U)$ and $\pi(V)$ are open sets in $\mathbb{R}/2\pi\mathbb{Z}$ as π is an open map, moreover, $\pi(U) \cap \pi(V) = \emptyset$. Therefore, $\mathbb{R}/2\pi\mathbb{Z}$ is Hausdorff.

(b) $\mathbb{R}/2\pi\mathbb{Z}$ is second countable.
As π is an open map and $\mathbb{R}$ is second countable, $\mathbb{R}/2\pi\mathbb{Z}$ is second countable.

(c) $\mathbb{R}/2\pi\mathbb{Z}$ is locally Euclidean.
Define

$$\begin{aligned} \varphi_1 : \mathbb{R}/2\pi\mathbb{Z} &\to (-\pi, \pi) \\ [x] &\mapsto x \in (-\pi, \pi), \end{aligned}$$

and

$$\begin{aligned} \varphi_2 : \mathbb{R}/2\pi\mathbb{Z} &\to (0, 2\pi) \\ [x] &\mapsto x \in (0, 2\pi). \end{aligned}$$

On $\varphi_1(U_1 \cap U_2) = \varphi_1(\mathbb{R}/2\pi\mathbb{Z}) = (-\pi, \pi)$,

(i) $x \in (-\pi, 0)$

$$\begin{aligned} \varphi_2 \circ \varphi_1^{-1}(x) &= \varphi_2([x]) \\ &= x + 2\pi. \end{aligned}$$

(ii) $x \in (0, \pi)$

$$\begin{aligned} \varphi_2 \circ \varphi_1^{-1}(x) &= \varphi_2([x]) \\ &= x. \end{aligned}$$

On $\varphi_2(U_1 \cap U_2) = \varphi_2(\mathbb{R}/2\pi\mathbb{Z}) = (0, 2\pi)$,

(i) $x \in (0, \pi)$

$$\begin{aligned} \varphi_1 \circ \varphi_2^{-1}(x) &= \varphi_1([x]) \\ &= x. \end{aligned}$$

(ii) $x \in (\pi, 2\pi)$

$$\begin{aligned} \varphi_1 \circ \varphi_2^{-1}(x) &= \varphi_1([x]) \\ &= x - 2\pi. \end{aligned}$$

Therefore, $(\mathbb{R}/2\pi\mathbb{Z}, \varphi_1)$ and $(\mathbb{R}/2\pi\mathbb{Z}, \varphi_2)$ are compatible and then form a C^∞ atlas. Then, $\mathbb{R}/2\pi\mathbb{Z}$ is locally Euclidean.

In conclusion, $\mathbb{R}/2\pi\mathbb{Z}$ is a smooth manifold.

Problem 7.7

(a) Let $\{(U_\alpha, \phi_\alpha)\}_{\alpha=1}^{2}$ be the atlas of the circle S^1 in Example 5.7, and let $\bar{\phi}_\alpha$ be the map ϕ_α followed by the projection $\mathbb{R} \to \mathbb{R}/2\pi\mathbb{Z}$. On $U_1 \cap U_2 = A \sqcup B$, since ϕ_1 and ϕ_2 differ by an integer multiple of 2π, $\bar{\phi}_1 = \bar{\phi}_2$. Therefore, $\bar{\phi}_1$ and $\bar{\phi}_2$ piece together to give a well-defined map $\bar{\phi} : S^1 \to \mathbb{R}/2\pi\mathbb{Z}$. Prove that $\bar{\phi}$ is C^∞.

(b) The complex exponential $\mathbb{R} \to S^1$, $t \mapsto e^{it}$, is constant on each orbit of the action of $2\pi\mathbb{Z}$ on $\mathbb{R}$. Therefore, there is an induced map $F : \mathbb{R}/2\pi\mathbb{Z} \to S^1$, $F([t]) = e^{it}$. Prove that F is C^∞.

(c) Prove that $F : \mathbb{R}/2\pi\mathbb{Z} \to S^1$ is a diffeomorphism.

Solution

(a)

$$\begin{aligned} U_1 &= \{e^{it} \in \mathbb{C} \mid -\pi < t < \pi\}, \\ U_2 &= \{e^{it} \in \mathbb{C} \mid 0 < t < 2\pi\}, \end{aligned}$$

and

$$\begin{aligned} \phi_1(e^{it}) &= t, \quad -\pi < t < \pi, \\ \phi_2(e^{it}) &= t, \quad 0 < t < 2\pi, \end{aligned}$$

Let $\pi : \mathbb{R} \to \mathbb{R}/2\pi\mathbb{Z}$ be the projection map and $\psi_1 : \mathbb{R}/2\pi\mathbb{Z} \to V_1 = (-\pi, \pi)$ and $\psi_2 : \mathbb{R}/2\pi\mathbb{Z} \to V_2 = (0, 2\pi)$ be the coordinate maps.

$$\begin{tikzcd} S^1 \arrow[r, "\bar{\phi}"] \arrow[d, "\phi_\alpha"'] & \mathbb{R}/2\pi\mathbb{Z} \arrow[d, "\psi_\beta"] \\ \phi_\alpha(U_\alpha) \arrow[ur, "\pi"] & V_\beta \end{tikzcd}$$

(i) $$\begin{aligned} \psi_1 \circ \bar{\phi} \circ \phi_1^{-1}(x) &= \psi_1 \circ \pi(x) \\ &= x, \quad -\pi < x < \pi. \end{aligned}$$

(ii) $$\begin{aligned} \psi_2 \circ \bar{\phi} \circ \phi_1^{-1}(x) &= \psi_2 \circ \pi(x) \\ &= \begin{cases} x & 0 < x < \pi \\ x + 2\pi & -\pi < x < 0 \end{cases} \end{aligned}$$

(iii) $$\begin{aligned} \psi_1 \circ \bar{\phi} \circ \phi_2^{-1}(x) &= \psi_1 \circ \pi(x) \\ &= \begin{cases} x & 0 < x < \pi \\ x - 2\pi & \pi < x < 2\pi \end{cases} \end{aligned}$$

(iv) $$\begin{aligned} \psi_2 \circ \bar{\phi} \circ \phi_2^{-1}(x) &= \psi_2 \circ \pi(x) \\ &= x, \quad 0 < x < 2\pi. \end{aligned}$$

Then, $\psi_\beta \circ \bar{\phi} \circ \phi_\alpha^{-1}$ is C^∞. Since ψ_β and ψ_α are diffeomorphisms, $\bar{\phi}$ is C^∞.

(b)

$$\begin{tikzcd} S^1 \arrow[d, "\phi_\alpha"'] & \mathbb{R}/2\pi\mathbb{Z} \arrow[l, "F"'] \arrow[d, "\psi_\beta"] \\ \phi_\alpha(U_\alpha) & V_\beta \end{tikzcd}$$

(i)
$$\begin{aligned} \phi_1 \circ F \circ \psi_1^{-1}(x) &= \phi_1 \circ F([x]) \\ &= \phi_1(e^{ix}) \\ &= x, \quad -\pi < x < \pi. \end{aligned}$$

(ii)
$$\begin{aligned} \phi_2 \circ F \circ \psi_1^{-1}(x) &= \phi_2 \circ F([x]) \\ &= \phi_2(e^{ix}) \\ &= \begin{cases} x & 0 < x < \pi \\ x + 2\pi & -\pi < x < 0 \end{cases} \end{aligned}$$

(iii)
$$\begin{aligned} \phi_1 \circ F \circ \psi_2^{-1}(x) &= \phi_1 \circ F([x]) \\ &= \phi_1(e^{ix}) \\ &= \begin{cases} x & 0 < x < \pi \\ x - 2\pi & \pi < x < 2\pi \end{cases} \end{aligned}$$

(iv)
$$\begin{aligned} \phi_2 \circ F \circ \psi_2^{-1}(x) &= \phi_2 \circ F([x]) \\ &= \phi_2(e^{ix}) \\ &= x, \quad 0 < x < 2\pi. \end{aligned}$$

Then, $\phi_\beta \circ F \circ \psi_\alpha^{-1}$ is C^∞. Since ψ_β and ψ_α are diffeomorphisms, F is C^∞.

(c)

Problem 7.8

The Grassmannian $G(k, n)$ is the set of all k-planes through the origin in $\mathbb{R}^n$. Such a k -plane is a linear subspace of dimension k of $\mathbb{R}^n$ and has a basis consisting of k linearly independent vectors $a_1, ..., a_k$ in $\mathbb{R}^n$. It is therefore completely specified by an $n \times k$ matrix $A = [a_1 ... a_k]$ of rank k, where the rank of a matrix A, denoted by $\mathrm{rk}A$, is defined to be the number of linearly independent columns of A. This matrix is called a matrix representative of the k-plane.

Two bases $a_1, ..., a_k$ and $b_1, ..., b_k$ determine the same k-plane if there is a change-of-basis matrix $g = [g_{ij}] \in \mathrm{GL}(k, \mathbb{R})$ such that

$$b_j = \sum_i a_i g_{ij}, \quad 1 \leq i, j \leq k.$$

In matrix notation, $B = Ag$.

Let $F(k, n)$ be the set of all $n \times k$ matrices of rank k, topologized as a subspace of $\mathbb{R}^{n \times k}$, and $\sim$ the equivalence relation

$$A \sim B \quad \text{iff} \quad \text{there is a matrix } g \in \mathrm{GL}(k, \mathbb{R}) \text{ such that } B = Ag.$$

In the notation of Problem B.3, $F(k, n)$ is the set $D_{\max}$ in $\mathbb{R}^{n\times k}$ and is therefore an open subset. There is a bijection between $G(k, n)$ and the quotient space $\frac{F(k,n)}{\sim}$. We give the Grassmannian $G(k, n)$ the quotient topology on $\frac{F(k,n)}{\sim}$.

(a) Show that $\sim$ is an open equivalence relation. (Hint: Either mimic the proof of Proposition 7.14 or apply Problem 7.5.)

(b) Prove that the Grassmannian $G(k, n)$ is second countable. (Hint: Apply Corollary 7.10.)

(c) Let $S = F(k, n)$. Prove that the graph R in $S \times S$ of the equivalence relation $\sim$ is closed. (Hint: Two matrices $A = [a_1 ... a_k]$ and $B = [b_1 ... b_k]$ in $F(k, n)$ are equivalent if and only if every column of B is a linear combination of the columns of A if and only if $\mathrm{rk}[AB] \le k$ if and only if all $(k + 1) \times (k + 1)$ minors of $[AB]$ are zero.)

(d) Prove that the Grassmannian $G(k, n)$ is Hausdorff. (Hint: Mimic the proof of Proposition 7.16.)

Next we want to find a C^∞ atlas on the Grassmannian $G(k, n)$. For simplicity, we specialize to $G(2, 4)$. For any 4×2 matrix A, let A_{ij} be the 2×2 submatrix consisting of its ith row and jth row. Define

$$V_{ij} = \{A \in F(2, 4) \mid A_{ij} \text{ is nonsingular}\}.$$

Because the complement of V_{ij} in $F(2, 4)$ is defined by the vanishing of $\det A_{ij}$, we conclude that V_{ij} is an open subset of $F(2, 4)$.

(e) Prove that if $A \in V_{ij}$, then $Ag \in V_{ij}$ for any nonsingular matrix $g \in \mathrm{GL}(2, \mathbb{R})$.

(f) Define $U_{ij} = \frac{V_{ij}}{\sim}$. Since $\sim$ is an open equivalence relation, $U_{ij} = \frac{V_{ij}}{\sim}$ is an open subset of $G(2, 4)$.

For $A \in V_{12}$,

$$A \sim AA_{12}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ * & * \\ * & * \end{pmatrix} = \begin{pmatrix} I \\ A_{34}A_{12}^{-1} \end{pmatrix}.$$

This shows that the matrix representatives of a 2-plane in U_{12} have a canonical form B in which B_{12} is the identity matrix.

(e) Show that the map $\tilde{\varphi}_{12} : V_{12} \to \mathbb{R}^{2\times 2}$,

$$\tilde{\varphi}_{12}(A) = A_{34}A_{12}^{-1},$$

induces a homeomorphism $\varphi_{12} : U_{12} \to \mathbb{R}^{2\times 2}$.

(e) Define similarly homeomorphisms $\varphi_{ij} : U_{ij} \to \mathbb{R}^{2\times 2}$. Compute $\varphi_{12} \circ \varphi_{23}^{-1}$, and show that it is C^∞.

(f) Show that $\{U_{ij} \mid 1 \leq i < j \leq 4\}$ is an open cover of $G(2,4)$ and that $G(2,4)$ is a smooth manifold.

Similar consideration shows that $F(k,n)$ has an open cover $\{V_I\}$, where I is a strictly ascending multi-index $1 \leq i_1 < ... < i_k \leq n$. For $A \in F(k,n)$, let A_I be the $k \times k$ submatrix of A consisting of i_1th, ..., i_kth rows of A. Define

$$V_I = \{A \in G(k,n) \mid \det A_I \neq 0\}.$$

Next define $\tilde{\varphi}_I : V_I \to \mathbb{R}^{(n-k)\times k}$ by

$$\tilde{\varphi}_{I(A)} = \left(AA_I^{-1}\right)_{I'},$$

where $()_{I'}$ denotes the $(n-k) \times k$ submatrix obtained from the complement I' of the multi-index I. Let $U_I = \frac{V_I}{\sim}$. Then $\tilde{\varphi}$ induces a homeomorphism $\varphi : U_I \to \mathbb{R}^{(n-k)\times k}$. It is not difficult to show that $\{(U_I, \varphi_I)\}$ is a C^∞ atlas for $G(k,n)$. Therefore the Grassmannian $G(k,n)$ is a C^∞ manifold of dimension $k(n-k)$.

Problem 7.9

Show that the real projective space $\mathbb{R}P^n$ is compact.