Chapter 1

Euclidean Spaces

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Grade : ________

1 Smooth Functions on a Euclidean Space

Problem 1.1 Score: ________. Let $g(x) = \frac{3}{4}x^{4/3}$. Show that the function $h(x) = \int_0^x g(t)\,dt$ is C^2 but not C^3 at $x = 0$.

Solution:

$$\begin{aligned} h(x) &= \int_0^x g(t)\ \mathrm{d}t \\ &= \frac{9}{28}x^{7/3}, \end{aligned}$$

which is continuous at $x = 0$, thus h is C^0 at $x = 0$.
$h'(x) = g(x) = \frac{3}{4}x^{4/3}$ is continuous at $x = 0$ and hence h is C^1 at $x = 0$.

$$h''(x) = g'(x) = \frac{3}{4} \cdot \frac{4}{3}x^{1/3} = x^{1/3},$$

which is also continuous at $x = 0$, hence h is C^2 at $x = 0$.

$$h'''(x) = \frac{1}{3}x^{-2/3},$$

which is not continuous at $x = 0$ because as $x \to 0$, $h'''(x) \to \infty$. Therefore, h is not C^3 at $x = 0$.
To summarize, we have shown that h is C^0, C^1, and C^2 at $x = 0$, but not C^3. Therefore, h is a function that is C^2 but not C^3 at $x = 0$. $\square$

Problem 1.2 Score: ________. Let $f(x)$ be the function on $\mathbb{R}$ defined by

$$f(x) = \begin{cases} e^{-1/x} & \text{if } x > 0, \\ 0 & \text{if } x \le 0. \end{cases}$$

(a) Show by induction that for $x > 0$ and $k \ge 0$, the kth derivative $f^{(k)}(x)$ is of the form $p_{2k}(1/x)e^{-1/x}$ for some polynomial $p_{2k}(y)$ of degree $2k$ in y.

(b) Prove that f is C^∞ on $\mathbb{R}$ and that $f^{(k)}(0) = 0$ for all $k \ge 0$.

Solution: (a) Let $k = 0$. Then, we have $f(x) = e^{-1/x}$ for $x > 0$, and $f^{(0)}(x) = e^{-1/x} = p_0(1/x)e^{-1/x}$, where $p_0(y) = 1$.
Now, assume that for some $k \ge 0$, the kth derivative $f^{(k)}(x)$ is of the form $p_{2k}(\frac{1}{x})e^{-1/x}$ for some polynomial $p_{2k}(y)$ of degree $2k$.

$$\begin{aligned} f^{(k+1)}(x) &= \frac{d}{dx}f^{(k)}(x) \\ &= \frac{d}{dx}\left(p_{2k}(\frac{1}{x})e^{-1/x}\right) \\ &= \frac{d}{dx}p_{2k}(\frac{1}{x})e^{-1/x} + p_{2k}(\frac{1}{x}) \cdot \frac{d}{dx}e^{-1/x} \\ &= \frac{d}{dx}\left[a_{2k}\left(\frac{1}{x}\right)^{2k} + \cdots\right]e^{-1/x} + \left[a_{2k}\left(\frac{1}{x}\right)^{2k} + \cdots\right]\frac{1}{x^2}e^{-1/x} \\ &= \left(-2ka_{2k}\left(\frac{1}{x}\right)^{2k+1} + \cdots\right)e^{-1/x} + \left(a_{2k}\left(\frac{1}{x}\right)^{2k+2} + \cdots\right)\frac{1}{x^2}e^{-1/x} \end{aligned}$$

$$= \left(a_{2k} \left(\frac{1}{x} \right)^{2k+2} - 2ka_{2k} \left(\frac{1}{x} \right)^{2k+1} + \cdots \right) e^{-1/x}$$
$$= p_{2(k+1)}(\frac{1}{x})e^{-1/x},$$

where $p_{2(k+1)}(y)$ is a polynomial of degree $2(k+1)$. This completes the induction step.

(b) From the result of part (a), we know that for any $k \geq 0$, the kth derivative of f at $x > 0$ is given by

$$f^{(k)}(x) = p_{2k}(\frac{1}{x})e^{-1/x}.$$

Then, we can evaluate the limit of $f^{(k)}(x)$ as x approaches 0 from the right:

$$\begin{aligned}\lim_{x\to 0^+} f^{(k)}(x) &= \lim_{x\to 0^+} p_{2k}(\frac{1}{x})e^{-1/x} \\ &= \lim_{x\to 0^+} p_{2k}(+\infty)e^{-\infty} \\ &= 0,\end{aligned}$$

which implies that $f^{(k)}(x)$ is continuous at $x = 0$ for all $k \geq 0$, i.e., f is C^∞ at $x = 0$, and $f^{(k)} = 0$ for all $k \geq 0$ at $x = 0$.

□

Problem 1.3 Score: _________. Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^n$ be open subsets. A C^∞ map $F : U \to V$ is called a diffeomorphism if it is bijective and has a C^∞ inverse $F^{-1} : V \to U$.

(a) Show that the function $f :]-\pi/2, \pi/2[\to \mathbb{R}$, $f(x) = \tan x$, is a diffeomorphism.

(b) Let a, b be real numbers with $a < b$. Find a linear function $h :]a, b[\to]-1, 1[$, thus proving that any two finite open intervals are diffeomorphic.

(c) The composite $f \circ h :]a, b[\to \mathbb{R}$ is then a diffeomorphism of an open interval with $\mathbb{R}$.

(d) The exponential function $\exp : \mathbb{R} \to]0, \infty[$ is a diffeomorphism. Use it to show that for any real numbers a and b, the intervals $\mathbb{R}$, $]a, \infty[$, and $]-\infty, b[$ are diffeomorphic.

Problem 1.4 Score: _________. Show that the map

$$f : \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[^n \to \mathbb{R}^n,\ f(x_1, \ldots, x_n) = (\tan x_1, \ldots, \tan x_n),$$

is a diffeomorphism.

Problem 1.5 Score: _________. Let $B(0, 1)$ be the open unit disk in $\mathbb{R}^2$. To find a diffeomorphism between $B(0, 1)$ and $\mathbb{R}^2$, we identify $\mathbb{R}^2$ with the xy-plane in $\mathbb{R}^3$ and introduce the lower open hemisphere

$$S : x^2 + y^2 + (z - 1)^2 = 1, \quad z < 1,$$

in $\mathbb{R}^3$ as an intermediate space.

(a) The stereographic projection $g : S \to \mathbb{R}^2$ from $(0, 0, 1)$ is the map that sends a point $(a, b, c) \in S$ to the intersection of the line through $(0, 0, 1)$ and (a, b, c) with the xy-plane. Show that it is given by

$$(a, b, c) \mapsto (u, v) = \left(\frac{a}{1 - c}, \frac{b}{1 - c} \right), \quad c = 1 - \sqrt{1 - a^2 - b^2},$$

with inverse

$$(u, v) \mapsto \left(\frac{u}{\sqrt{1 + u^2 + v^2}}, \frac{v}{\sqrt{1 + u^2 + v^2}}, 1 - \frac{1}{\sqrt{1 + u^2 + v^2}} \right).$$

(b) Composing the two maps f and g gives the map

$$h = g \circ f : B(0,1) \to \mathbb{R}^2, \quad h(a,b) = \left(\frac{a}{\sqrt{1-a^2-b^2}}, \frac{b}{\sqrt{1-a^2-b^2}} \right).$$

Find a formula for $h^{-1}(u,v) = (f^{-1} \circ g^{-1})(u,v)$ and conclude that h is a diffeomorphism of the open disk $B(0,1)$ with $\mathbb{R}^2$.

(c) Generalize part (b) to $\mathbb{R}^n$.

Problem 1.6 Score: _________. Prove that if $f : \mathbb{R}^2 \to \mathbb{R}$ is C^∞, then there exist C^∞ functions g_{11}, g_{12}, g_{22} on $\mathbb{R}^2$ such that

$$f(x,y) = f(0,0) + \frac{\partial f}{\partial x}(0,0)x + \frac{\partial f}{\partial y}(0,0)y + x^2 g_{11}(x,y) + xy g_{12}(x,y) + y^2 g_{22}(x,y).$$

Solution: Applying Taylor's theorem with remainder, we have

$$f(x,y) = f(0,0) + x f_1(x,y) + y f_2(x,y),$$

where $f_1(x,y) = \frac{\partial f}{\partial x}(x,y)$ and $f_2(x,y) = \frac{\partial f}{\partial y}(x,y)$.
As f is C^∞, both $f_1(x,y)$ and $f_2(x,y)$ are also C^∞ functions. We can expand $f_1(x,y)$ and $f_2(x,y)$ using Taylor's theorem around $(0,0)$ as follows:

$$\begin{aligned} f_1(x,y) &= f_1(0,0) + x f_{11}(x,y) + y f_{12}(x,y), \\ f_2(x,y) &= f_2(0,0) + x f_{21}(x,y) + y f_{22}(x,y). \end{aligned}$$

Then, we can substitute these expansions back into the expression for $f(x,y)$ to obtain:

$$\begin{aligned} f(x,y) &= f(0,0) + x\left(f_1(0,0) + x f_{11}(x,y) + y f_{12}(x,y)\right) + y\left(f_2(0,0) + x f_{21}(x,y) + y f_{22}(x,y)\right) \\ &= f(0,0) + \frac{\partial f}{\partial x}(0,0)x + \frac{\partial f}{\partial y}(0,0)y + x^2 f_{11}(x,y) + 2xy f_{12}(x,y) + y^2 f_{22}(x,y). \end{aligned}$$

Then by defining $g_{11}(x,y) = f_{11}(x,y)$, $g_{12}(x,y) = 2f_{12}(x,y)$, and $g_{22}(x,y) = f_{22}(x,y)$, we get the desired result. □

Problem 1.7 Score: _________. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a C^∞ function with $f(0,0) = \frac{\partial f}{\partial x}(0,0) = \frac{\partial f}{\partial y}(0,0) = 0$. Define

$$g(t,u) = \begin{cases} \frac{f(t,tu)}{t} & \text{for } t \neq 0, \\ 0 & \text{for } t = 0. \end{cases}$$

Prove that $g(t,u)$ is C^∞ for $(t,u) \in \mathbb{R}^2$. (Hint: Apply Problem 1.6.)

Problem 1.8 Score: _________. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^3$. Show that f is a bijective C^∞ map, but that f^{-1} is not C^∞. (This example shows that a bijective C^∞ map need not have a C^∞ inverse. In complex analysis, the situation is quite different: a bijective holomorphic map $f : \mathbb{C} \to \mathbb{C}$ necessarily has a holomorphic inverse.)

2 Tangent Vectors in $\mathbb{R}^n$ as Derivations

Problem 2.1 Score: ________. Let X be the vector field $x\partial/\partial x + y\partial/\partial y$ and $f(x,y,z)$ the function $x^2+y^2+z^2$ on $\mathbb{R}^3$. Compute Xf.

Solution:

$$\begin{aligned} Xf &= \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)(x^2+y^2+z^2) \\ &= 2x^2+2y^2 \end{aligned}$$

□

Problem 2.2 Score: ________. Define carefully addition, multiplication, and scalar multiplication in C_p^∞. Prove that addition in C_p^∞ is commutative.

Solution: Let $[f]_p, [g]_p \in C_p^\infty$. We define the addition of two equivalence classes as follows:

$$[f]_p + [g]_p = [f+g]_p,$$

where $f+g$ is the pointwise sum of the functions f and g.
The multiplication of two equivalence classes is defined as:

$$[f]_p \cdot [g]_p = [fg]_p,$$

where fg is the pointwise product of the functions f and g.
The scalar multiplication of an equivalence class by a scalar $c \in \mathbb{R}$ is defined as:

$$c[f]_p = [cf]_p,$$

where cf is the pointwise product of the function f and the scalar c. □

Problem 2.3 Score: ________. Let D and D' be derivations at p in $\mathbb{R}^n$, and $c \in \mathbb{R}$. Prove that

(a) the sum $D + D'$ is a derivation at p.

(b) the scalar multiple cD is a derivation at p.

Solution: (a) Let $f, g \in C^\infty(\mathbb{R}^n)$, then we have

$$\begin{aligned} (D+D')(fg) &= D(fg) + D'(fg) \\ &= D(f)g(p) + f(p)D(g) + D'(f)g(p) + f(p)D'(g) \\ &= (D(f)+D'(f))g(p) + f(p)(D(g)+D'(g)) \\ &= (D+D')(f)g(p) + f(p)(D+D')(g). \end{aligned}$$

(b)

$$\begin{aligned} (cD)(fg) &= cD(fg) \\ &= c(D(f)g(p) + f(p)D(g)) \\ &= cD(f)g(p) + cf(p)D(g) \\ &= (cD)(f)g(p) + f(p)(cD)(g). \end{aligned}$$

□

Problem 2.4 Score: _________. Let A be an algebra over a field K. If D_1 and D_2 are derivations of A, show that $D_1 \circ D_2$ is not necessarily a derivation (it is if D_1 or $D_2 = 0$), but $D_1 \circ D_2 - D_2 \circ D_1$ is always a derivation of A.

Solution: Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f(x) = x$, and let $D_1 = D_2 = \frac{\mathrm{d}}{\mathrm{d}x}$. Then, for the Lebniz rule, we have

$$\begin{aligned} D_1 \circ D_2(ff) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}}{\mathrm{d}x}(x^2)\right) \\ &= \frac{\mathrm{d}}{\mathrm{d}x}(2x) \\ &= 2, \end{aligned}$$

but

$$\begin{aligned} (D_2 \circ D_1)(f)f(p) + f(p)(D_2 \circ D_1)(f) &= \frac{\mathrm{d}^2x}{\mathrm{d}x^2}p + p\frac{\mathrm{d}^2x}{\mathrm{d}x^2} \\ &= 0. \end{aligned}$$

Therefore, $D_1 \circ D_2$ is not a derivation. Next, for $D_1 \circ D_2 - D_2 \circ D_1$, we examine the Lebniz rule:

$$\begin{aligned} (D_1 \circ D_2 - D_2 \circ D_1)(fg) &= D_1 \circ D_2(fg) - D_2 \circ D_1(fg) \\ &= D_1[D_2(f)g(p) + f(p)D_2(g)] - D_2[D_1(f)g(p) + f(p)D_1(g)] \\ &= (D_1 \circ D_2(f)g(p) + f(p)D_1 \circ D_2(g)) - (D_2 \circ D_1(f)g(p) + f(p)D_2 \circ D_1(g)) \\ &= (D_1 \circ D_2 - D_2 \circ D_1)(f)g(p) + f(p)(D_1 \circ D_2 - D_2 \circ D_1)(g). \end{aligned}$$

$\square$