An Introduction to Manifold

1 Smooth Functions on a Euclidean Space

Problem 1.1

Let $g(x) = \frac{3}{4}x^{\frac{3}{4}}$. Show that the function $h(x) = \int_0^x g(t)\,\mathrm{d}t$ is C^2 but not C^3 at $x = 0$.

Solution

$$\begin{aligned} h(x) &= \int_0^x g(t)\,\mathrm{d}t \\ &= \frac{9}{28}x^{\frac{7}{3}}, \end{aligned}$$

which is countinious at $x = 0$, thus h is C^0 at $x = 0$.
$h'(x) = g(x) = \frac{3}{4}x^{\frac{3}{4}}$ is countinious at $x = 0$, thus h is C^1 at $x = 0$.

$$h''(x) = g'(x) = x^{\frac{1}{3}},$$

which is countinious at $x = 0$, thus h is C^2 at $x = 0$.

$$h'''(x) = g''(x) = \frac{1}{3}x^{-\frac{2}{3}},$$

which is not countinious at $x = 0$, thus h is not C^3 at $x = 0$.

Problem 1.2

Let

$$f(x) = \begin{cases} e^{-\frac{1}{x}} & x > 0 \\ 0 & x \le 0. \end{cases}$$

(a) Show by induction that for $x > 0$ and $k > 0$, the k-th derivative $f^{(k)}(x)$ is of the form $p_{2k}e^{-\frac{1}{x}}$ for some polynomial $p_{2k}(y)$ of degree $2k$ in y.
(b) Prove that f is C^∞ on $\mathbb{R}$ an that $f^{(k)}(0) = 0$ for all $k \ge 0$.

Solution

(a) Let $k = 0$, for $x > 0$, we have

$$\begin{aligned} f^{(x)} &= f(x) \\ &= e^{-\frac{1}{x}} \\ &= p_0\left(\frac{1}{x}\right)e^{-\frac{1}{x}}, \end{aligned}$$

where $p_0(y) = 1$. This is a polynomial of degree 0 in y. Thus the base case holds. Then assume the inductive hypothesis holds for $k = n$, i.e., $f^{(n)}(x) = p_{2n}\left(\frac{1}{x}\right)e^{-\frac{1}{x}}$, where $p_{2n}(y)$ is a polynomial of degree $2n$. We will show it holds for $k = n + 1$:

$$\begin{aligned}
f^{(k+1)}(x) &= \frac{\mathrm{d}}{\mathrm{d}x} f^{(k)}(x) \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left(p_{2n}\left(\frac{1}{x}\right)e^{-\frac{1}{x}}\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left(p_{2n}\left(\frac{1}{x}\right)\right)e^{-\frac{1}{x}} + p_{2n}\left(\frac{1}{x}\right)\frac{\mathrm{d}}{\mathrm{d}x}e^{-\frac{1}{x}} \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left[a_{2k}\left(\frac{1}{x}\right)^{2k} + \cdots\right]e^{-\frac{1}{x}} + \left[a_{2k}\left(\frac{1}{x}\right)^{2k} + \cdots\right]\frac{1}{x^2}e^{-\frac{1}{x}} \\
&= \left[-2ka_{2k}\left(\frac{1}{x}\right)^{2k+1} + a_{2k}\left(\frac{1}{x}\right)^{2k+2} + \cdots\right]e^{-\frac{1}{x}} \\
&= p_{2(k+1)}\left(\frac{1}{x}\right)e^{-\frac{1}{x}},
\end{aligned}$$

where $p_{2(k+1)}(y)$ is a polynomial of degree $2(k+1)$ in y. This completes the inductive step.

(b) From the result of part (a), we know that for any $k \geq 0$,

$$f^{(k)}(x) = p_{2k}\left(\frac{1}{x}\right)e^{-\frac{1}{x}},$$

where $p_{2k}(y)$ is a polynomial of degree $2k$. Then we can evaluate the limit as x approaches 0 from the right:

$$\begin{aligned}
\lim_{x\to 0^+} f((k))(x) &= \lim_{x\to 0^+} p_{2k}\left(\frac{1}{x}\right)e^{-\frac{1}{x}} \\
&= 0,
\end{aligned}$$

which implies that $f^{(k)}(0) = 0$ for all $k \geq 0$, and thus f is C^∞ on $\mathbb{R}$.

Problem 1.3

Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^n$ be open subsets. A C^∞ map $F : U \to V$ is called a *diffeomorphism* if it is bijective and has a C^∞ inverse $F^{-1} : V \to U$.

(a) Show that the function $f :]-\frac{\pi}{2}, \frac{\pi}{2}[\to \mathbb{R}$, $f(x) = \tan x$, is a diffeomorphism.

(b) Let a, b be real numbers with $a < b$. Find a linear function $h :]a, b[\to]-1, 1[$, thus proving that any two finite open intervals are diffeomorphic.

(c) The composite $f \circ h :]a, b[\to \mathbb{R}$ is then a diffeomorphism of an open interval with $\mathbb{R}$.

(d) The exponential function $\exp : \mathbb{R} \to]0, \infty[$ is a diffeomorphism. Use it to show that for any real numbers a and b, the intervals $\mathbb{R}$, $]a, \infty[$, and $]-\infty, b[$ are diffeomorphic.

Problem 1.4

Show that the map

$$f :]-\frac{\pi}{2}, \frac{\pi}{2}[^n \to \mathbb{R}^n, f(x_1, \cdots, x_n) = (\tan x_1, \cdots, \tan x_n),$$

is a diffeomorphism.

Problem 1.5

Let $B(0,1)$ be the open unit disk in $\mathbb{R}^2$. To find a diffeomorphism between $B(0,1)$ and $\mathbb{R}^2$, we identify $\mathbb{R}^2$ with the xy-plane in $\mathbb{R}^3$ and introduce the lower open hemisphere

$$S : x^2 + y^2 + (z-1)^2 = 1, \quad z < 1,$$

in $\mathbb{R}^3$ as an intermediate space.

(a) The stereographic projection $g : S \to \mathbb{R}^2$ from $(0,0,1)$ is the map that sends a point $(a,b,c) \in S$ to the intersection of the line through $(0,0,1)$ and (a,b,c) with the xy -plane. Show that it is given by

$$(a,b,c) \mapsto (u,v) = \left(\frac{a}{1-c}, \frac{b}{1-c}\right), \quad c = 1 - \sqrt{1-a^2-b^2},$$

with inverse

$$(u,v) \mapsto \left(\frac{u}{\sqrt{1+u^2+v^2}}, \frac{v}{\sqrt{1+u^2+v^2}}, 1 - \frac{1}{\sqrt{1+u^2+v^2}}\right).$$

(b) Composing the maps f and g gives the map

$$h = g \circ f : B(0,1) \to \mathbb{R}^2, \quad h(a,b) = \left(\frac{a}{\sqrt{1-a^2-b^2}}, \frac{b}{\sqrt{1-a^2-b^2}}\right).$$

Find a formula for $h^{-1}(u,v) = (f^{-1} \circ g^{-1})(u,v)$ and conclude that h is a diffeomorphism of the open disk $B(0,1)$ with $\mathbb{R}^2$.

(c) Generalize part (b) to $\mathbb{R}^n$.

Problem 1.6

Prove that if $f : \mathbb{R}^2 \to \mathbb{R}$ is C^∞, then there exist C^∞ functions g_{11}, g_{12}, g_{22} on $\mathbb{R}^2$ such that

$$f(x,y) = f(0,0) + \frac{\partial f}{\partial x}(0,0)x + \frac{\partial f}{\partial y}(0,0)y + x^2 g_{11}(x,y) + xy g_{12}(x,y) + y^2 g_{22}(x,y).$$

Solution

Applying Taylor's theorem with remainder, we have

$$f(x,y) = f(0,0) + x f_1(x,y) + y f_2(x,y),$$

where $f_1(x,y) = \frac{\partial f}{\partial x}(x,y)$ and $f_2(x,y) = \frac{\partial f}{\partial y}(x,y)$。

As f is C^∞, $f_1(x,y)$ and $f_2(x,y)$ are also C^∞. we can expand $f_1(x,y)$ and $f_2(x,y)$ using Taylor's theorem with remainder around $(0,0)$:

$$f_1(x,y) = f_1(0,0) + x f_{11}(x,y) + y f_{12}(x,y),$$
$$f_2(x,y) = f_2(0,0) + x f_{21}(x,y) + y f_{22}(x,y).$$

Then, we can substitute these expansions back into the expression for $f(x,y)$:

$$\begin{aligned} f(x,y) &= f(0,0) + x(f_1(0,0) + xf_{11}(x,y) + yf_{12}(x,y)) + y(f_2(0,0) + xf_{21}(x,y) + yf_{22}(x,y)) \\ &= f(0,0) + \frac{\partial f}{\partial x}(0,0)x + \frac{\partial f}{\partial y}(0,0)y + x^2 f_{11}(x,y) + 2xyf_{12}(x,y) + y^2 f_{22}(x,y). \end{aligned}$$

Then by defining $g_{11}(x,y) = f_{11}(x,y)$, $g_{12}(x,y) = 2f_{12}(x,y)$, and $g_{22}(x,y) = f_{22}(x,y)$, we get the desired result.

Problem 1.7

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a C^∞ function with $f(0,0) = \frac{\partial f}{\partial x}(0,0) = \frac{\partial f}{\partial y}(0,0) = 0$. Define

$$g(t,u) = \begin{cases} \frac{f(t,tu)}{t} & \text{for } t \neq 0 \\ 0 & \text{for } t = 0. \end{cases}$$

Prove that $g(t,u)$ is C^∞ for $(t,u) \in \mathbb{R}^2$. (Hint: Apply Problem 1.6.)

Problem 1.8

Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^3$. Show that f is a bijective C^∞ map, but that f^{-1} is not C^∞. (This example shows that a bijective C^∞ map need not have a C^∞ inverse. In complex analysis, the situation is quite different: a bijective holomorphic map $f : \mathbb{C} \to \mathbb{C}$ necessarily has a holomorphic inverse.)

2 Tangent Vectors in $\mathbb{R}^n$ as Derivations

Problem 2.1

Let X be the vector field $x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ and $f(x,y,z)$ the function $x^2 + y^2 + z^2$ on $\mathbb{R}^3$. Compute Xf.

Solution

$$\begin{aligned} Xf &= \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)(x^2 + y^2 + z^2) \\ &= 2x^2 + 2y^2 \end{aligned}$$

Problem 2.2

Define carefully addition, multiplication, and scalar multiplication in C_p^∞. Prove that addition in C_p^∞ is commutative.

Solution

Let $[f]_p, [g]_p \in C_p^\infty$. We define the addition of two equivalence classes as follows:

$$[f]_p + [g]_p = [f+g]_p,$$

where $f+g$ is the pointwise sum of the functions f and g.
The multiplication of two equivalence classes is defined as:

$$[f]_p \cdot [g]_p = [fg]_p,$$

where fg is the pointwise product of the functions f and g.
The scalar multiplication of an equivalence class by a scalar $c \in \mathbb{R}$ is defined as:

$$c[f]_p = [cf]_p,$$

where cf is the pointwise product of the function f and the scalar c.

Problem 2.3

Let D and D' be derivations at p in $\mathbb{R}^n$, and $c \in \mathbb{R}$. Prove that
(a) the sum $D + D'$ is a derivation at p.
(b) the scalar multiple cD is a derivation at p.

Solution

(a) Let $f, g \in C^{\infty(\mathbb{R}^n)}$, then we have

$$\begin{aligned}(D + D')(fg) &= D(fg) + D'(fg) \\ &= D(f)g(p) + f(p)D(g) + D'(f)g(p) + f(p)D'(g) \\ &= (D(f) + D'(f))g(p) + f(p)(D(g) + D'(g)) \\ &= (D + D')(f)g(p) + f(p)(D + D')(g).\end{aligned}$$

(b)

$$\begin{aligned}(cD)(fg) &= cD(fg) \\ &= c(D(f)g(p) + f(p)D(g)) \\ &= cD(f)g(p) + cf(p)D(g) \\ &= (cD)(f)g(p) + f(p)(cD)(g).\end{aligned}$$

Problem 2.4

Let A be an algebra over a field K. If D_1 and D_2 are derivations of A, show that $D_1 \circ D_2$ is not necessarily a derivation (it is if D_1 or $D_2 = 0$), but $D_1 \circ D_2 - D_2 \circ D_1$ is always a derivation of A.

Solution

Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f(x) = x$, and let $D_1 = D_2 = \frac{\mathrm{d}}{\mathrm{d}x}$. Then, for the Lebniz rule, we have

$$\begin{aligned}D_1 \circ D_2(ff) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}}{\mathrm{d}x}(x^2)\right) \\ &= \frac{\mathrm{d}}{\mathrm{d}x}(2x) \\ &= 2,\end{aligned}$$

but

$$\begin{aligned}(D_2 \circ D_1)(f)f(p) + f(p)(D_2 \circ D_1)(f) &= \mathrm{d}^2\frac{x}{\mathrm{d}x^2}p + p\mathrm{d}^2\frac{x}{\mathrm{d}x^2} \\ &= 0.\end{aligned}$$

Therefore, $D_1 \circ D_2$ is not a derivation.
Next, for $D_1 \circ D_2 - D_2 \circ D_1$, we examine the Lebniz rule:

$$\begin{aligned}(D_1 \circ D_2 - D_2 \circ D_1)(fg) &= D_1 \circ D_2(fg) - D_2 \circ D_1(fg) \\ &= D_1[D_2(f)g(p) + f(p)D_2(g)]\text{-}D_2[D_1(f)g(p) + f(p)D_1(g)] \\ &= (D_1 \circ D_2(f)g(p) + f(p)D_1 \circ D_2(g)) \\ &\quad -(D_2 \circ D_1(f)g(p) + f(p)D_2 \circ D_1(g)) \\ &= (D_1 \circ D_2 - D_2 \circ D_1)(f)g(p) + f(p)(D_1 \circ D_2 - D_2 \circ D_1)(g).\end{aligned}$$

Thus, $D_1 \circ D_2 - D_2 \circ D_1$ satisfies the Leibniz rule and is a derivation.

3 The Exterior Algebra of Multivectors

Problem 3.1

Let $e_1, \cdots, e_n$ be a basis for a vector space V and let $\alpha^1, \cdots, \alpha^n$ be its dual basis in V^V. Suppose $[g_{ij}] \in \mathbb{R}^{n \times n}$ is an $n \times n$ matrix. Define a bilinear function $f : V \times V \to \mathbb{R}$ by

$$f(v, w) = g_{ij} v^i w^j$$

for $v = v^i e_i$ and $w = w^j e_j$ in V. Describe f in terms of the tensor products of α^i and α^j, $1 \le i, j \le n$.

Solution

Since $v^i = \alpha^i(v)$ and $w^j = \alpha^j(w)$, we can express f as follows:

$$\begin{aligned}f(v, w) &= g_{ij} v^i w^j \\ &= g_{ij} \alpha^i(v) \alpha^j(w) \\ &= g_{ij} (\alpha^i \otimes \alpha^j)(v, w),\end{aligned}$$

then $f = g_{ij}(\alpha^i \otimes \alpha^j)$.

Problem 3.2

Let V be a vector space of dimension n and $f : V \to \mathbb{R}$ a nonzero linear functional.

(a) Show that $\dim \ker f = n - 1$. A linear subspace of V of dimension $n - 1$ is called a *hyperplane* in V.

(b) Show that a nonzero linear functional on a vector space V is determined up to a multiplicative constant by its kernel, a hyperplane in V. In other words, if f and $g : V \to \mathbb{R}$ are nonzero linear functionals and $\ker f = \ker g$, then $g = cf$ for some constant $c \in \mathbb{R}$.

Solution

(a)
$$\begin{aligned}\dim \ker f &= \dim V - \dim \operatorname{im} f \\ &= n - 1.\end{aligned}$$

(b) Let $f, g : V \to \mathbb{R}$ be nonzero linear functionals with $\ker f = \ker g$. As f is nonzero, there exists $v_0' \in V$ such that $f(v_0') = b \neq 0$. Let $v_0 = \frac{v_0'}{b}$, $f(v_0) = f\left(\frac{v_0'}{b}\right) =$

$\frac{1}{b}f(v_0') = 1$. As $\ker f = \ker g$ and $f(v_0) \neq 0$, we have $g(v_0) \neq 0$. Then, let $v \in V$, $f(v) = a$, and $w = v - av_0$. As f is linear,

$$\begin{aligned} f(w) &= f(v - av_0) \\ &= f(v) - af(v_0) \\ &= a - a \\ &= 0, \end{aligned}$$

which means $w \in \ker f$, and thus $w \in \ker g$. Then,

$$\begin{aligned} 0 &= g(w) \\ &= g(v - av_0) \\ &= g(v) - ag(v_0) \\ &= g(v) - f(v)g(v_0). \end{aligned}$$

Therefore we have that $g(v) = cf(v)$, where $c = g(v_0)$.

Problem 3.3

Let V be a vector space of dimension n with basis $e_1, \cdots, e_n$. Let $\alpha^1, \cdots, \alpha^n$ be the dual basis for $V^\vee$. Show that a basis for the space $L_k(V)$ of k-linear functions on V is $\{\alpha^{i_1} \otimes \cdots \otimes \alpha^{i_k}\}$ for all multi-indices $(i_1, \cdots, i_k)$ (not just the strictly ascending multi-indices as for $A_k(L)$). In particular, this shows that $\dim L_k(V) = n^k$.

Solution

(a) Let $T \in L_k(V)$ and $T\left(e_{i_1}, \cdots, e_{i_k}\right) = T_{i_1, \cdots, i_k}$, For the function

$$T' = T_{i_1, \cdots, i_k} \alpha^{i_1} \otimes \cdots \otimes \alpha^{i_k},$$

we have

$$\begin{aligned} T'\left(e_{j_1}, \cdots, e_{j_k}\right) &= T_{i_1, \cdots, i_k} \alpha^{i_1} \otimes \cdots \otimes \alpha^{i_k}\left(e_{j_1}, \cdots, e_{j_k}\right) \\ &= T_{i_1, \cdots, i_k} \alpha^{i_1}\left(e_{j_1}\right) \cdots \alpha^{i_k}\left(e_{j_k}\right) \\ &= T_{i_1, \cdots, i_k} \delta^{i_1}_{j_1} \cdots \delta^{i_k}_{j_k} \\ &= T_{j_1, \cdots, j_k} \\ &= T\left(e_{j_1}, \cdots, e_{j_k}\right), \end{aligned}$$

which means $T = T'$. Therefore, $\{\alpha^{i_1} \otimes \cdots \otimes \alpha^{i_k}\}$ spans $L_k(V)$.

(b) Suppose $T = 0$, then

$$
\begin{aligned}
0 &= T\left(e_{j_1}, \cdots, e_{j_k}\right) \\
&= T_{i_1, \cdots, i_k} \alpha^{i_1} \otimes \cdots \otimes \alpha^{i_k}\left(e_{j_1}, \cdots, e_{j_k}\right) \\
&= T_{i_1, \cdots, i_k} \alpha^{i_1}\left(e_{j_1}\right) \cdots \alpha^{i_k}\left(e_{j_k}\right) \\
&= T_{i_1, \cdots, i_k} \delta^{i_1}_{j_1} \cdots \delta^{i_k}_{j_k} \\
&= T_{j_1, \cdots, j_k},
\end{aligned}
$$

which means $T_{i_1, \cdots, i_k} = 0$ for all $j_1, \cdots, j_k$. Therefore $\left\{\alpha^{i_1} \otimes \cdots \otimes \alpha^{i_k}\right\}$ is linearly independent.

Thus, $\left\{\alpha^{i_1} \otimes \cdots \otimes \alpha^{i_k}\right\}$ is a basis for $L_k(V)$.

Problem 3.4

Let f be a k-tensor on a vector space V. Prove that f is alternating if and only if f changes sign whenever two successive arguments are interchanged: $f(\cdots, v_{i+1}, v_i, \cdots) = -f(\cdots, v_i, v_{i+1}, \cdots)$ for $i = 1, \cdots, k-1$.

Solution

(a) If f is alternating, then for $\sigma = (i, i+1)$,

$$
\begin{aligned}
f(\cdots, v_{i+1}, v_i, \cdots) &= f(\sigma(v_1), \cdots, \sigma(v_k)) \\
&= \operatorname{sgn}(\sigma) f(v_1, \cdots, v_k) \\
&= -f(\cdots, v_i, v_{i+1}, \cdots),
\end{aligned}
$$

which means $f(\cdots, v_{i+1}, v_i, \cdots) = -f(\cdots, v_i, v_{i+1}, \cdots)$.

(b) If $f(\cdots, v_{i+1}, v_i, \cdots) = -f(\cdots, v_i, v_{i+1}, \cdots)$, then for $\sigma = (i, i+1)$,

$$
\begin{aligned}
f(\sigma(v_1), \cdots, \sigma(v_k)) &= f(\cdots, v_{i+1}, v_i, \cdots) \\
&= -f(\cdots, v_i, v_{i+1}, \cdots) \\
&= \operatorname{sgn}(\sigma) f(v_1, \cdots, v_k),
\end{aligned}
$$

which means f is alternating.

Problem 3.5

Let f be a k-tensor on a vector space V. Prove that f is alternating if and only if $f(v_1, \cdots, v_k) = 0$ whenever two of the vectors $v_1, \cdots, v_k$ are equal.

Solution

(a) If f is alternating, and $v_i = v_j$, then for $\sigma = (i, j)$,

$$
\begin{aligned}
f\left(\cdots, \sigma(v_i), \cdots, \sigma\left(v_j\right), \cdots\right) &= f\left(\cdots, v_j, \cdots, v_i, \cdots\right) \\
&= \operatorname{sgn}(\sigma) f(v_1, \cdots, v_k) \\
&= -f\left(\cdots, v_i, \cdots, v_j, \cdots\right),
\end{aligned}
$$

which means $f(v_1, \cdots, v_k) = 0$.

(b) If $f(v_1, \cdots, v_k) = 0$ for $v_i = v_j$, then for $\sigma = (i, j)$,

$$\begin{aligned} 0 &= f(\cdots, v_j, \cdots, v_i, \cdots) \\ &= -f(\cdots, v_i, \cdots, v_j, \cdots) \\ &= \operatorname{sgn}(\sigma) f(v_1, \cdots, v_k) \\ &= f(\sigma(v_1), \cdots, \sigma(v_k)) \end{aligned}$$

which means f is alternating.

Problem 3.6

Let V be a vector space. For $a, b \in \mathbb{R}$, $f \in A_k(V)$, and $g \in A_l(V)$, show that $af \wedge bg = (ab) f \wedge g$.

Solution

$$\begin{aligned} af \wedge bg &= \frac{1}{k!l!} A(af \otimes bg) \\ &= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma) \sigma(af(v_1, \cdots, v_k) bg(v_{k+1}, \cdots, v_{k+l})) \\ &= \frac{ab}{k!l!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma) \sigma(f(v_1, \cdots, v_k) g(v_{k+1}, \cdots, v_{k+l})) \\ &= \frac{ab}{k!l!} A(f \otimes g) \\ &= (ab) f \wedge g, \end{aligned}$$

Problem 3.7

Suppose two sets of covectors on a vector space V, $\beta^1, \cdots, \beta^k$ and $\gamma^1, \cdots, \gamma^k$, are related by

$$\beta^i = a^i_j \gamma^j, \quad i = 1, \cdots, k,$$

for a $k \times k$ matrix $A = \left[a^i_j\right]$. Show that

$$\beta^1 \wedge \cdots \wedge \beta^k = (\det A) \gamma^1 \wedge \cdots \wedge \gamma^k.$$

Solution

$$\begin{aligned} \beta^1 \wedge \cdots \wedge \beta^k &= \left(a^1_{j_1} \gamma^{j_1}\right) \wedge \cdots \wedge \left(a^k_{j_k} \gamma^{j_k}\right) \\ &= a^1_{j_1} \cdots a^k_{j_k} \left(\gamma^{j_1} \wedge \cdots \wedge \gamma^{j_k}\right) \\ &= a^1_{\sigma(1)} \cdots a^k_{\sigma(k)} (\operatorname{sgn} \sigma) \left(\gamma^1 \wedge \cdots \wedge \gamma^k\right) \\ &= (\det A) \left(\gamma^1 \wedge \cdots \wedge \gamma^k\right). \end{aligned}$$

Problem 3.8

Let f be a k-covector on a vector space V. Suppose two sets of vectors $u_1, \cdots, u_k$ and $v_1, \cdots, v_k$ in V are related by

$$u_j = a_j^i v_i, \quad j = 1, \cdots, k,$$

for a $k \times k$ matrix $A = \left[a_j^i\right]$. Show that

$$f(u_1, \cdots, u_k) = (\det A) f(v_1, \cdots, v_k).$$

Solution

$$\begin{aligned} f(u_1, \cdots, u_k) &= f\left(a_1^{i_1} v_{i_1}, \cdots, a_k^{i_k} v_{i_k}\right) \\ &= a_1^{i_1} \cdots a_k^{i_k} f\left(v_{i_1}, \cdots, v_{i_k}\right) \\ &= a_1^{\sigma(1)} \cdots a_k^{\sigma(k)} (\operatorname{sgn} \sigma) f(v_1, \cdots, v_k) \\ &= (\det A) f(v_1, \cdots, v_k). \end{aligned}$$

Problem 3.9

Let V be a vector space of dimension n. Prove that if an n-covector ω vanishes on a basis $e_1, \cdots, e_n$ for V, then ω is the zero covector on V.

Solution

Let $\{e_1, \cdots, e_n\}$ be a basis for V, for $v_i = v_i^j e_j$ in V, we have

$$\begin{aligned} \omega(v_1, \cdots, v_n) &= \det\left[v_i^j\right] \omega(e_1, \cdots, e_n) \\ &= 0. \end{aligned}$$

Problem 3.10

Let $\alpha^1, \cdots, \alpha^k$ be 1-covectors on a vector space V. Show that $\alpha^1 \wedge \cdots \wedge \alpha^k \neq 0$ if and only if $\alpha^1, \cdots, \alpha^k$ are linearly independent in the dual space V^V.

Problem 3.11

Let α be a nonzero 1-covector and γ a k-covector on a finite-dimensional vector space V. Show that $\alpha \wedge \gamma = 0$ if and only if $\gamma = \alpha \wedge \beta$ for some $(k-1)$-covector β on V.

4 Differential Forms on $\mathbb{R}^n$

Problem 4.1

Let ω be the 1-form $z \, \mathrm{d}x - \mathrm{d}z$ and let X be the vector field $y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}$ on $\mathbb{R}^3$. Compute $\omega(X)$ and $\mathrm{d}\omega$.

Solution

$$\begin{aligned} \omega(X) &= (z \, \mathrm{d}x - \mathrm{d}z) \left(y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}\right) \\ &= zy \end{aligned}$$

$$\begin{aligned}
\mathrm{d}\omega &= \mathrm{d}(z\,\mathrm{d}x - \mathrm{d}z) \\
&= \frac{\partial z}{\partial z}\,\mathrm{d}z \wedge \mathrm{d}x \\
&= \mathrm{d}z \wedge \mathrm{d}x
\end{aligned}$$

Problem 4.2

At each point $p \in \mathbb{R}^3$, define a bilinear function ω_p on $T_p(\mathbb{R}^3)$ by

$$\omega_p(\boldsymbol{a}, \boldsymbol{b}) = \omega_p\left(\begin{bmatrix} a^1 \\ a^2 \\ a^3 \end{bmatrix}, \begin{bmatrix} b^1 \\ b^2 \\ b^3 \end{bmatrix}\right) = p^3 \det\begin{bmatrix} a^1 & b^1 \\ a^2 & b^2 \end{bmatrix},$$

for tangent vectors $\boldsymbol{a}, \boldsymbol{b} \in T_p(\mathbb{R}^3)$, where p^3 is the third component of $p = (p^1, p^2, p^3)$. Since ω_p is an alternating bilinear function on $T_p(\mathbb{R}^3)$, ω is a 2-form on $\mathbb{R}^3$. Write ω in terms of the standard basis $\mathrm{d}x^i \wedge \mathrm{d}x^j$ at each point.

Solution

Let $\omega = \omega_{12}\,\mathrm{d}x^1 \wedge \mathrm{d}x^2 + \omega_{13}\,\mathrm{d}x^1 \wedge \mathrm{d}x^3 + \omega_{23}\,\mathrm{d}x^2 \wedge \mathrm{d}x^3$,

$$\begin{aligned}
\omega(\boldsymbol{a}, \boldsymbol{b}) &= \omega_{12}(a^1 b^2 - a^2 b^1) + \omega_{13}(a^1 b^3 - a^3 b^1) + \omega_{23}(a^2 b^3 - a^3 b^2) \\
&= x^3 \det\begin{bmatrix} a^1 & b^1 \\ a^2 & b^2 \end{bmatrix} \\
&= x^3(a^1 b^2 - a^2 b^1),
\end{aligned}$$

then, $\omega_{12} = x^3$, $\omega_{13} = \omega_{23} = 0$. Thus, we have

$$\omega = x^3\,\mathrm{d}x^1 \wedge \mathrm{d}x^2.$$

Problem 4.3

Suppose the standard coordinates on $\mathbb{R}^2$ are called r and θ (this $\mathbb{R}^2$ is the (r, θ)-plane, not the (x, y)-plane). If $x = r\cos\theta$ and $y = r\sin\theta$, calculate $\mathrm{d}x$, $\mathrm{d}y$, and $\mathrm{d}x \wedge \mathrm{d}y$ in terms of $\mathrm{d}r$ and $\mathrm{d}\theta$.

Solution

$$\begin{aligned}
\mathrm{d}x &= \frac{\partial x}{\partial r}\,\mathrm{d}r + \frac{\partial x}{\partial \theta}\,\mathrm{d}\theta \\
&= \cos\theta\,\mathrm{d}r - r\sin\theta\,\mathrm{d}\theta,
\end{aligned}$$

$$\begin{aligned}
\mathrm{d}y &= \frac{\partial y}{\partial r}\,\mathrm{d}r + \frac{\partial y}{\partial \theta}\,\mathrm{d}\theta \\
&= \sin\theta\,\mathrm{d}r + r\cos\theta\,\mathrm{d}\theta,
\end{aligned}$$

$$\begin{aligned}
\mathrm{d}x \wedge \mathrm{d}y &= (\cos\theta\,\mathrm{d}r - r\sin\theta\,\mathrm{d}\theta) \wedge (\sin\theta\,\mathrm{d}r + r\cos\theta\,\mathrm{d}\theta) \\
&= (r\cos^2\theta + r\sin^2\theta)\,\mathrm{d}r \wedge \mathrm{d}\theta \\
&= r\,\mathrm{d}r \wedge \mathrm{d}\theta
\end{aligned}$$

Problem 4.4

Suppose the standard coordinates on $\mathbb{R}^3$ are called ρ, φ, and θ. If $x = \rho\sin\varphi\cos\theta$, $y = \rho\sin\varphi\sin\theta$, and $z = \rho\cos\varphi$, calculate $\mathrm{d}x$, $\mathrm{d}y$, $\mathrm{d}z$, and $\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z$ in terms of $\mathrm{d}\rho$, $\mathrm{d}\varphi$, and $\mathrm{d}\theta$.

Solution

$$\begin{aligned}\mathrm{d}x &= \frac{\partial x}{\partial \rho}\,\mathrm{d}\rho + \frac{\partial x}{\partial \varphi}\,\mathrm{d}\varphi + \frac{\partial x}{\partial \theta}\,\mathrm{d}\theta \\ &= \sin\varphi\cos\theta\,\mathrm{d}\rho + \rho\cos\varphi\cos\theta\,\mathrm{d}\varphi - \rho\sin\varphi\sin\theta\,\mathrm{d}\theta,\end{aligned}$$

$$\begin{aligned}\mathrm{d}y &= \frac{\partial y}{\partial \rho}\,\mathrm{d}\rho + \frac{\partial y}{\partial \varphi}\,\mathrm{d}\varphi + \frac{\partial y}{\partial \theta}\,\mathrm{d}\theta \\ &= \sin\varphi\sin\theta\,\mathrm{d}\rho + \rho\cos\varphi\sin\theta\,\mathrm{d}\varphi + \rho\sin\varphi\cos\theta\,\mathrm{d}\theta,\end{aligned}$$

$$\begin{aligned}\mathrm{d}z &= \frac{\partial z}{\partial \rho}\,\mathrm{d}\rho + \frac{\partial z}{\partial \varphi}\,\mathrm{d}\varphi + \frac{\partial z}{\partial \theta}\,\mathrm{d}\theta \\ &= \cos\varphi\,\mathrm{d}\rho - \rho\sin\varphi\,\mathrm{d}\varphi,\end{aligned}$$

$$\begin{aligned}\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z &= (\sin\varphi\cos\theta\,\mathrm{d}\rho + \rho\cos\varphi\cos\theta\,\mathrm{d}\varphi - \rho\sin\varphi\sin\theta\,\mathrm{d}\theta) \\ &\wedge (\sin\varphi\sin\theta\,\mathrm{d}\rho + \rho\cos\varphi\sin\theta\,\mathrm{d}\varphi + \rho\sin\varphi\cos\theta\,\mathrm{d}\theta) \\ &\wedge (\cos\varphi\,\mathrm{d}\rho - \rho\sin\varphi\,\mathrm{d}\varphi) \\ &= \sin\varphi\cos\theta(0 + \rho\sin\varphi\cos\theta\rho\sin\varphi)\,\mathrm{d}\rho\wedge\mathrm{d}\varphi\wedge\mathrm{d}\theta \\ &+\rho\cos\varphi\cos\theta(0 - \rho\sin\varphi\cos\theta\cos\varphi)\,\mathrm{d}\varphi\wedge\mathrm{d}\rho\wedge\mathrm{d}\theta \\ &-\rho\sin\varphi\sin\theta(-\sin\varphi\sin\theta\rho\sin\varphi - \rho\cos\varphi\sin\theta\cos\varphi)\,\mathrm{d}\theta\wedge\mathrm{d}\rho\wedge\mathrm{d}\varphi \\ &= \big[\rho^2\sin^3\varphi\cos^2\theta - \big(-\rho^2\sin\varphi\cos^2\varphi\cos^2\theta\big) \\ &+\rho^2\sin\varphi\sin^2\theta\big(\sin^2\varphi + \cos^2\varphi\big)\big]\,\mathrm{d}\rho\wedge\mathrm{d}\varphi\wedge\mathrm{d}\theta \\ &= \rho^2\sin\varphi\big(\sin^2\varphi\cos^2\theta + \cos^2\varphi\cos^2\theta + \sin^2\theta\big)\,\mathrm{d}\rho\wedge\mathrm{d}\varphi\wedge\mathrm{d}\theta \\ &= \rho^2\sin\varphi\big(\sin^2\varphi + \cos^2\varphi\big)\,\mathrm{d}\rho\wedge\mathrm{d}\varphi\wedge\mathrm{d}\theta \\ &= \rho^2\sin\varphi\,\mathrm{d}\rho\wedge\mathrm{d}\varphi\wedge\mathrm{d}\theta\end{aligned}$$

Problem 4.5

Let α be a 1-form and β a 2-form on $\mathbb{R}^3$. Then

$$\alpha = a_1\,\mathrm{d}x^1 + a_2\,\mathrm{d}x^2 + a_3\,\mathrm{d}x^3,$$

$$\beta = b_1\,\mathrm{d}x^2\wedge\mathrm{d}x^3 + b_2\,\mathrm{d}x^3\wedge\mathrm{d}x^1 + b_3\,\mathrm{d}x^1\wedge\mathrm{d}x^2.$$

Simplify the expression $\alpha\wedge\beta$ as much as possible.

Solution

$$\begin{aligned}\alpha \wedge \beta &= (a_1\,\mathrm{d}x^1 + a_2\,\mathrm{d}x^2 + a_3\,\mathrm{d}x^3) \wedge (b_1\,\mathrm{d}x^2 \wedge \mathrm{d}x^3 + b_2\,\mathrm{d}x^3 \wedge \mathrm{d}x^1 + b_3\,\mathrm{d}x^1 \wedge \mathrm{d}x^2) \\ &= (a_1 b_1)\,\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 + (a_2 b_2)\,\mathrm{d}x^2 \wedge \mathrm{d}x^3 \wedge \mathrm{d}x^1 + (a_3 b_3)\,\mathrm{d}x^3 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 \\ &= (a_1 b_1 + a_2 b_2 + a_3 b_3)\,\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3\end{aligned}$$

Problem 4.6

The correspondence between differential forms and vector fields on an open subset of $\mathbb{R}^3$ in Subsection 4.6 also makes sense pointwise. Let V be a vector space of dimension 3 with basis e_1, e_2, e_3, and dual basis $\alpha^1, \alpha^2, \alpha^3$. To a 1-covector $\alpha = a_1\alpha^1 + a_2\alpha^2 + a_3\alpha^3$ on V, we associate the vector $\boldsymbol{v}_\alpha = \langle a_1, a_2, a_3 \rangle \in \mathbb{R}^3$. To the 2-covector

$$\gamma = c_1\alpha^2 \wedge \alpha^3 + c_2\alpha^3 \wedge \alpha^1 + c_3\alpha^1 \wedge \alpha^2$$

on V, we associate the vector $\boldsymbol{v}_\gamma = \langle c_1, c_2, c_3 \rangle \in \mathbb{R}^3$. Show that under this correspondence, the wedge product of 1-covectors corresponds to the cross product of vectors in $\mathbb{R}^3$: if $\alpha = a_1\alpha^1 + a_2\alpha^2 + a_3\alpha^3$ and $\beta = b_1\alpha^1 + b_2\alpha^2 + b_3\alpha^3$, then $\boldsymbol{v}_{\alpha\wedge\beta} = \boldsymbol{v}_\alpha \times \boldsymbol{v}_\beta$.

Solution

$$\begin{aligned}\boldsymbol{v}_{\alpha\wedge\beta} &= (a_1\alpha^1 + a_2\alpha^2 + a_3\alpha^3) \wedge (b_1\alpha^1 + b_2\alpha^2 + b_3\alpha^3) \\ &= (a_2 b_3 - a_3 b_2)\alpha^2 \wedge \alpha^3 + (a_3 b_1 - a_1 b_3)\alpha^3 \wedge \alpha^1 + (a_1 b_2 - a_2 b_1)\alpha^1 \wedge \alpha^2,\end{aligned}$$

which corresponds to the vector

$$\begin{aligned}\boldsymbol{v}_{\alpha\wedge\beta} &= \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle \\ &= \boldsymbol{v}_\alpha \times \boldsymbol{v}_\beta\end{aligned}$$

Problem 4.7

Let $A = \bigoplus_{k=-\infty}^{\infty} A^k$ be a graded algebra over a field K with $A^k = 0$ for $k < 0$. Let m be an integer. A *superderivation of A of degree m* is a K-linear map $D : A \to A$ such that for all k, $D(A^k) \subset A^{k+m}$ and for all $a \in A^k$ and $b \in A^l$,

$$D(ab) = (Da)b + (-1)^{km} a(Db).$$

If D_1 and D_2 are two superderivations of A of respective degrees m_1 and m_2, define their *commutator* to be

$$[D_1, D_2] = D_1 \circ D_2 - (-1)^{m_1 m_2} D_2 \circ D_1.$$

Show that $[D_1, D_2]$ is a superderivation of degree $m_1 + m_2$. (A superderivation is said to be *even* or *odd* depending on the parity of its degree. An even superderivation is a derivation; an odd superderivation is an antiderivation.)

Solution

$$
\begin{aligned}
[D_1, D_2](ab) &= (D_1 \circ D_2 - (-1)^{m_1 m_2} D_2 \circ D_1)(ab) \\
&= (D_1 \circ D_2)(ab) - (-1)^{m_1 m_2} (D_2 \circ D_1)(ab) \\
&= D_1(D_2(ab)) - (-1)^{m_1 m_2} D_2(D_1(ab)) \\
&= D_1\big[(D_2 a)b + (-1)^{k m_2} a(D_2 b)\big] - (-1)^{m_1 m_2} D_2\big[(D_1 a)b + (-1)^{k m_1} a(D_1 b)\big] \\
&= (D_1(D_2 a))b + (-1)^{(k+m_2)m_1}(D_2 a)(D_1 b) + (-1)^{k m_2}\big[(D_1 a)(D_2 b) + (-1)^{k m_1} a(D_1(D_2 b))\big] \\
&\quad -(-1)^{m_1 m_2}\big\{(D_2(D_1 a))b + (-1)^{(k+m_1)m_2}(D_1 a)(D_2 b) + (-1)^{k m_1}\big[(D_2 a)(D_1 b) + (-1)^{k m_2} a(D_2(D_1 b)\big]\big\} \\
&= (D_1(D_2 a))b + \big[(-1)^{(k+m_2)m_1} - (-1)^{m_1 m_2 + k m_1}\big](D_2 a)(D_1 b) \\
&\quad + \big[(-1)^{k m_2} - (-1)^{m_1 m_2 + (k+m_1)m_2}\big](D_1 a)(D_2 b) + (-1)^{k m_2 + k m_1} a(D_1(D_2 b)) \\
&\quad -(-1)^{m_1 m_2}(D_2(D_1 a))b - (-1)^{m_1 m_2 + k m_1 + k m_2} a(D_2(D_1 b)) \\
&= (D_1(D_2 a))b + (-1)^{k(m_1+m_2)} a(D_1(D_2 b)) - (-1)^{m_1 m_2}(D_2(D_1 a))b - (-1)^{(k+1)(m_1+m_2)} a(D_2(D_1 b)) \\
&= [D_1(D_2 a) - (-1)^{m_1 m_2}(D_2(D_1 a))]b + (-1)^{k(m_1+m_2)} a[D_1(D_2 b) - (-1)^{m_1 m_2}(D_2(D_1 b))] \\
&= ([D_1, D_2](a))b + (-1)^{k(m_1+m_2)} a([D_1, D_2](b)),
\end{aligned}
$$

which gives us the result that $[D_1, D_2]$ is a superderivation of degree $m_1 + m_2$.