1 Smooth Functions on a Euclidean Space

Problem 1.1

Let $g(x) = \frac{3}{4}x^{\frac{3}{4}}$. Show that the function $h(x) = \int_0^x g(t)\,\mathrm{d}t$ is C^2 but not C^3 at $x = 0$.

Solution

$$\begin{aligned} h(x) &= \int_0^x g(t)\,\mathrm{d}t \\ &= \frac{9}{28}x^{\frac{7}{3}}, \end{aligned}$$

which is countinious at $x = 0$, thus h is C^0 at $x = 0$.
$h'(x) = g(x) = \frac{3}{4}x^{\frac{3}{4}}$ is countinious at $x = 0$, thus h is C^1 at $x = 0$.

$$h''(x) = g'(x) = x^{\frac{1}{3}},$$

which is countinious at $x = 0$, thus h is C^2 at $x = 0$.

$$h'''(x) = g''(x) = \frac{1}{3}x^{-\frac{2}{3}},$$

which is not countinious at $x = 0$, thus h is not C^3 at $x = 0$.

Problem 1.2

Let

$$f(x) = \begin{cases} e^{-\frac{1}{x}} & x > 0 \\ 0 & x \leq 0. \end{cases}$$

(a) Show by induction that for $x > 0$ and $k > 0$, the k-th derivative $f^{(k)}(x)$ is of the form $p_{2k}e^{-\frac{1}{x}}$ for some polynomial $p_{2k}(y)$ of degree $2k$ in y.
(b) Prove that f is C^∞ on $\mathbb{R}$ an that $f^{(k)}(0) = 0$ for all $k \geq 0$.

Solution

(a) Let $k = 0$, for $x > 0$, we have

$$\begin{aligned} f^{(x)} &= f(x) \\ &= e^{-\frac{1}{x}} \\ &= p_0\left(\frac{1}{x}\right)e^{-\frac{1}{x}}, \end{aligned}$$

where $p_0(y) = 1$. This is a polynomial of degree 0 in y. Thus the base case holds. Then assume the inductive hypothesis holds for $k = n$, i.e., $f^{(n)}(x) = p_{2n}\left(\frac{1}{x}\right)e^{-\frac{1}{x}}$, where $p_{2n}(y)$ is a polynomial of degree $2n$. We will show it holds for $k = n + 1$:

$$\begin{aligned}
f^{(k+1)}(x) &= \frac{\mathrm{d}}{\mathrm{d}x} f^{(k)}(x) \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left(p_{2n}\left(\frac{1}{x}\right)e^{-\frac{1}{x}}\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left(p_{2n}\left(\frac{1}{x}\right)\right)e^{-\frac{1}{x}} + p_{2n}\left(\frac{1}{x}\right)\frac{\mathrm{d}}{\mathrm{d}x}e^{-\frac{1}{x}} \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left[a_{2k}\left(\frac{1}{x}\right)^{2k} + \cdots\right]e^{-\frac{1}{x}} + \left[a_{2k}\left(\frac{1}{x}\right)^{2k} + \cdots\right]\frac{1}{x^2}e^{-\frac{1}{x}} \\
&= \left[-2ka_{2k}\left(\frac{1}{x}\right)^{2k+1} + a_{2k}\left(\frac{1}{x}\right)^{2k+2} + \cdots\right]e^{-\frac{1}{x}} \\
&= p_{2(k+1)}\left(\frac{1}{x}\right)e^{-\frac{1}{x}},
\end{aligned}$$

where $p_{2(k+1)}(y)$ is a polynomial of degree $2(k+1)$ in y. This completes the inductive step.

(b) From the result of part (a), we know that for any $k \geq 0$,

$$f^{(k)}(x) = p_{2k}\left(\frac{1}{x}\right)e^{-\frac{1}{x}},$$

where $p_{2k}(y)$ is a polynomial of degree $2k$. Then we can evaluate the limit as x approaches 0 from the right:

$$\begin{aligned}
\lim_{x \to 0^+} f((k))(x) &= \lim_{x \to 0^+} p_{2k}\left(\frac{1}{x}\right)e^{-\frac{1}{x}} \\
&= 0,
\end{aligned}$$

which implies that $f^{(k)}(0) = 0$ for all $k \geq 0$, and thus f is C^∞ on $\mathbb{R}$.

Problem 1.3

Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^n$ be open subsets. A C^∞ map $F : U \to V$ is called a *diffeomorphism* if it is bijective and has a C^∞ inverse $F^{-1} : V \to U$.

(a) Show that the function $f :]-\frac{\pi}{2}, \frac{\pi}{2}[\to \mathbb{R}$, $f(x) = \tan x$, is a diffeomorphism.

(b) Let a, b be real numbers with $a < b$. Find a linear function $h :]a, b[\to]-1, 1[$, thus proving that any two finite open intervals are diffeomorphic.

(c) The composite $f \circ h :]a, b[\to \mathbb{R}$ is then a diffeomorphism of an open interval with $\mathbb{R}$.

(d) The exponential function $\exp : \mathbb{R} \to]0, \infty[$ is a diffeomorphism. Use it to show that for any real numbers a and b, the intervals $\mathbb{R}$, $]a, \infty[$, and $]-\infty, b[$ are diffeomorphic.

Problem 1.4

Show that the map

$$f :]-\frac{\pi}{2}, \frac{\pi}{2}[n \to \mathbb{R}^n, f(x_1, ..., x_n) = (\tan x_1, ..., \tan x_n),$$

is a diffeomorphism.

Problem 1.5

Let $B(0,1)$ be the open unit disk in $\mathbb{R}^2$. To find a diffeomorphism between $B(0,1)$ and $\mathbb{R}^2$, we identify $\mathbb{R}^2$ with the xy-plane in $\mathbb{R}^3$ and introduce the lower open hemisphere

$$S : x^2 + y^2 + (z-1)^2 = 1, \quad z < 1,$$

in $\mathbb{R}^3$ as an intermediate space.

(a) The stereographic projection $g : S \to \mathbb{R}^2$ from $(0,0,1)$ is the map that sends a point $(a,b,c) \in S$ to the intersection of the line through $(0,0,1)$ and (a,b,c) with the xy -plane. Show that it is given by

$$(a,b,c) \mapsto (u,v) = \left(\frac{a}{1-c}, \frac{b}{1-c}\right), \quad c = 1 - \sqrt{1-a^2-b^2},$$

with inverse

$$(u,v) \mapsto \left(\frac{u}{\sqrt{1+u^2+v^2}}, \frac{v}{\sqrt{1+u^2+v^2}}, 1 - \frac{1}{\sqrt{1+u^2+v^2}}\right).$$

(b) Composing the maps f and g gives the map

$$h = g \circ f : B(0,1) \to \mathbb{R}^2, \quad h(a,b) = \left(\frac{a}{\sqrt{1-a^2-b^2}}, \frac{b}{\sqrt{1-a^2-b^2}}\right).$$

Find a formula for $h^{-1}(u,v) = (f^{-1} \circ g^{-1})(u,v)$ and conclude that h is a diffeomorphism of the open disk $B(0,1)$ with $\mathbb{R}^2$.

(c) Generalize part (b) to $\mathbb{R}^n$.

Problem 1.6

Prove that if $f : \mathbb{R}^2 \to \mathbb{R}$ is C^∞, then there exist C^∞ functions g_{11}, g_{12}, g_{22} on $\mathbb{R}^2$ such that

$$f(x,y) = f(0,0) + \frac{\partial f}{\partial x}(0,0)x + \frac{\partial f}{\partial y}(0,0)y + x^2 g_{11}(x,y) + xy g_{12}(x,y) + y^2 g_{22}(x,y).$$

Solution

Applying Taylor's theorem with remainder, we have

$$f(x,y) = f(0,0) + x f_1(x,y) + y f_2(x,y),$$

where $f_1(x,y) = \frac{\partial f}{\partial x}(x,y)$ and $f_2(x,y) = \frac{\partial f}{\partial y}(x,y)$。

As f is C^∞, $f_1(x,y)$ and $f_2(x,y)$ are also C^∞. we can expand $f_1(x,y)$ and $f_2(x,y)$ using Taylor's theorem with remainder around $(0,0)$:

$$\begin{aligned} f_1(x,y) &= f_1(0,0) + x f_{11}(x,y) + y f_{12}(x,y), \\ f_2(x,y) &= f_2(0,0) + x f_{21}(x,y) + y f_{22}(x,y). \end{aligned}$$

Then, we can substitute these expansions back into the expression for $f(x,y)$:

$$\begin{aligned} f(x,y) &= f(0,0) + x(f_1(0,0) + xf_{11}(x,y) + yf_{12}(x,y)) + y(f_2(0,0) + xf_{21}(x,y) + yf_{22}(x,y)) \\ &= f(0,0) + \frac{\partial f}{\partial x}(0,0)x + \frac{\partial f}{\partial y}(0,0)y + x^2 f_{11}(x,y) + 2xyf_{12}(x,y) + y^2 f_{22}(x,y). \end{aligned}$$

Then by defining $g_{11}(x,y) = f_{11}(x,y)$, $g_{12}(x,y) = 2f_{12}(x,y)$, and $g_{22}(x,y) = f_{22}(x,y)$, we get the desired result.

Problem 1.7

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a C^∞ function with $f(0,0) = \frac{\partial f}{\partial x}(0,0) = \frac{\partial f}{\partial y}(0,0) = 0$. Define

$$g(t,u) = \begin{cases} \frac{f(t,tu)}{t} & \text{for } t \neq 0 \\ 0 & \text{for } t = 0. \end{cases}$$

Prove that $g(t,u)$ is C^∞ for $(t,u) \in \mathbb{R}^2$. (Hint: Apply Problem 1.6.)

Problem 1.8

Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^3$. Show that f is a bijective C^∞ map, but that f^{-1} is not C^∞. (This example shows that a bijective C^∞ map need not have a C^∞ inverse. In complex analysis, the situation is quite different: a bijective holomorphic map $f : \mathbb{C} \to \mathbb{C}$ necessarily has a holomorphic inverse.)

2 Tangent Vectors in $\mathbb{R}^n$ as Derivations

Problem 2.1

Let X be the vector field $x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ and $f(x,y,z)$ the function $x^2 + y^2 + z^2$ on $\mathbb{R}^3$. Compute Xf.

Solution

$$\begin{aligned} Xf &= \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)(x^2 + y^2 + z^2) \\ &= 2x^2 + 2y^2 \end{aligned}$$

Problem 2.2

Define carefully addition, multiplication, and scalar multiplication in C_p^∞. Prove that addition in C_p^∞ is commutative.

Solution

Let $[f]_p, [g]_p \in C_p^\infty$. We define the addition of two equivalence classes as follows:

$$[f]_p + [g]_p = [f+g]_p,$$

where $f + g$ is the pointwise sum of the functions f and g.
The multiplication of two equivalence classes is defined as:

$$[f]_p \cdot [g]_p = [fg]_p,$$

where fg is the pointwise product of the functions f and g.
The scalar multiplication of an equivalence class by a scalar $c \in \mathbb{R}$ is defined as:

$$c[f]_p = [cf]_p,$$

where cf is the pointwise product of the function f and the scalar c.

Problem 2.3

Let D and D' be derivations at p in $\mathbb{R}^n$, and $c \in \mathbb{R}$. Prove that
(a) the sum $D + D'$ is a derivation at p.
(b) the scalar multiple cD is a derivation at p.

Solution

(a) Let $f, g \in C^{\infty(\mathbb{R}^n)}$, then we have

$$\begin{aligned}(D+D')(fg) &= D(fg) + D'(fg)\\ &= D(f)g(p) + f(p)D(g) + D'(f)g(p) + f(p)D'(g)\\ &= (D(f) + D'(f))g(p) + f(p)(D(g) + D'(g))\\ &= (D+D')(f)g(p) + f(p)(D+D')(g).\end{aligned}$$

(b)

$$\begin{aligned}(cD)(fg) &= cD(fg)\\ &= c(D(f)g(p) + f(p)D(g))\\ &= cD(f)g(p) + cf(p)D(g)\\ &= (cD)(f)g(p) + f(p)(cD)(g).\end{aligned}$$

Problem 2.4

Let A be an algebra over a field K. If D_1 and D_2 are derivations of A, show that $D_1 \circ D_2$ is not necessarily a derivation (it is if D_1 or $D_2 = 0$), but $D_1 \circ D_2 - D_2 \circ D_1$ is always a derivation of A.

Solution

Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f(x) = x$, and let $D_1 = D_2 = \frac{\mathrm{d}}{\mathrm{d}x}$. Then, for the Lebniz rule, we have

$$\begin{aligned}D_1 \circ D_2(ff) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}}{\mathrm{d}x}(x^2)\right)\\ &= \frac{\mathrm{d}}{\mathrm{d}x}(2x)\\ &= 2,\end{aligned}$$

but

$$\begin{aligned}(D_2 \circ D_1)(f)f(p) + f(p)(D_2 \circ D_1)(f) &= \mathrm{d}^2\frac{x}{\mathrm{d}x^2}p + p\mathrm{d}^2\frac{x}{\mathrm{d}x^2}\\ &= 0.\end{aligned}$$

Therefore, $D_1 \circ D_2$ is not a derivation.
Next, for $D_1 \circ D_2 - D_2 \circ D_1$, we examine the Lebniz rule:

$$\begin{aligned}
(D_1 \circ D_2 - D_2 \circ D_1)(fg) &= D_1 \circ D_2(fg) - D_2 \circ D_1(fg) \\
&= D_1[D_2(f)g(p) + f(p)D_2(g)]\text{-}D_2[D_1(f)g(p) + f(p)D_1(g)] \\
&= (D_1 \circ D_2(f)g(p) + f(p)D_1 \circ D_2(g)) \\
&-(D_2 \circ D_1(f)g(p) + f(p)D_2 \circ D_1(g)) \\
&= (D_1 \circ D_2 - D_2 \circ D_1)(f)g(p) + f(p)(D_1 \circ D_2 - D_2 \circ D_1)(g).
\end{aligned}$$

Thus, $D_1 \circ D_2 - D_2 \circ D_1$ satisfies the Leibniz rule and is a derivation.