An Introduction to Manifold

Chapter 1 Euclidean Spaces

1. Smooth Functions on a Euclidean Space

The calculus of C^∞ functions will be our primary tool for studying higher-dimensional manifolds.

1.1. C^∞ Analytic Functions

Let $p = (p^1, \cdots, p^n)$ be a point in an open subset $U \subset \mathbb{R}^n$.

Definition 1.1 Let k be a non-negative integer. A real-valued function $f : U \to \mathbb{R}$ is said to be C^k at p if its partial derivatives

$$\frac{\partial^j f}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}} \tag{1}$$

of all orders $j \leq k$ exist and are continuous at p.
The function $f : U \to \mathbb{R}$ is C^∞ at p if it is C^k at p for all $k \geq 0$.
A vector-valued function $f : U \to \mathbb{R}^m$ is said to be C^k at p if all of its components $f^1, \cdots, f^m$ are C^k at p.
$f : U \to \mathbb{R}$ is said to be C^k on U if it is C^k at every point $p \in U$.
The set of all C^∞ functions on U is denoted by $C^\infty(U)$ or $\mathcal{F}(U)$.

The function $f : U \to \mathbb{R}$ is real-analytic at p if in some neighborhood of p, it is equal to its Taylor series at p.
A real-analytic function is necessarily C^∞, but the converse is not true.

1.2. Taylor's Theorem with Remainder

Definition 1.2 A subset $S \subseteq \mathbb{R}^n$ is **star-shaped** with respect to a point $p \in S$ if for every point $x \in S$, the line segment from p to x lies entirely in S.

Lemma 1.3 Let $f \in C^\infty(U)$, where $U \subset \mathbb{R}^n$ is an open subset, star-shaped with respect to a point $p \in U$. Then there are functions $g_1(x), \cdots, g_n(x) \in C^\infty(U)$ such that

$$f(x) = f(p) + (x^i - p^i) g_i(x), \quad g_i(x) = \frac{\partial f}{\partial x^i}(p). \tag{2}$$

If f is a C^∞ function on an open subset U containing p, then there is an $\varepsilon > 0$ such that

$$p \in B(p, \varepsilon) \subset U, \tag{3}$$

where $B(p, \varepsilon) = \{x \in \mathbb{R}^n : \|x - p\| < \varepsilon\}$ is the open ball of radius ε centered at p, which is clearly star-shaped with respect to p.

2. Tangent Vevtors in $\mathbb{R}^n$ as Derivations

In this section, we will find a characterization of tangent vectors in $\mathbb{R}^n$ that will generalize to manifolds.

2.1. The Directional Derivative

To distinguish between points and vectors, we write a point in $\mathbb{R}^n$ as $p = (p^1, \cdots, p^n)$ and a vector in the tangent space at p, denoted by $T_p\mathbb{R}^n$, as

$$v = \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix} \text{ or } v = \langle v^1, \cdots, v^n \rangle. \tag{4}$$

We usually denote the standard basis of $\mathbb{R}^n$ by $e_1, \cdots, e_n$, then $v = v^i e_i$ for some $v^i \in \mathbb{R}$.
The line through $p = (p^1, \cdots, p^n)$ in the direction of $v = (v^1, \cdots, v^n)$ has parametrization

$$c(t) = (p^1 + tv^1, \cdots, p^n + tv^n). \tag{5}$$

Definition 2.1 If f is C^∞ in a neighborhood of p in $\mathbb{R}^n$, the **directional derivative** of f at p in the direction of v is defined as the limit

$$\begin{aligned} D_v f &= \lim_{t \to 0} \frac{f(c(t)) - f(c(0))}{t} \\ &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f(c(t)) \\ &= \frac{\mathrm{d}c^i}{\mathrm{d}t}(0) \frac{\partial f}{\partial x^i}(p) \\ &= v^i \frac{\partial f}{\partial x^i}(p). \end{aligned} \tag{6}$$

We write

$$D_v = v^i \left.\frac{\partial}{\partial x^i}\right|_p \tag{7}$$

for the map from a function f to its directional derivative $D_v f$.

The association $v \to D_v$ offers a way to characterize tangent vectors as a certain operators on C^∞ functions.

2.2. Germs of Functions

Definition 2.2 A **relation** on a set S is a subset R of $S \times S$. Given $x, y \in S$, we write $x \sim y$ if and only if $(x, y) \in R$.
A relation R is an **equivalence relation** if it satisfies the following properties for all $x, y, z \in S$:
(i) **Reflexivity:** $x \sim x$,
(ii) **Symmetry:** If $x \sim y$, then $y \sim x$,
(iii) **Transitivity:** If $x \sim y$ and $y \sim z$, then $x \sim z$.

Consider the set of all pairs (f, U) where U is a neighborhood of p and $f : U \to \mathbb{R}$ is a C^∞ function. We say that (f, U) is **equivalent** to (g, V) if there exists a neighborhood $W \subseteq (U \cap V)$ such that $f|_W = g|_W$.

Definition 2.3 The **germ** of f at p is the equivalence class of the pair (f, U).
We write $C_p^\infty(\mathbb{R}^n)$, or simply C_p^∞, for the set of all germs of C^∞ functions on $\mathbb{R}^n$ at p.

Definition 2.4 An **algebra** over a field K is a vector space A over K with a multiplication map

$$\mu : A \times A \to A, \tag{8}$$

usually written $\mu(a, b) = a \cdot b$, that satisfies the following properties for all $a, b, c \in A$ and $r \in K$:
(i) **Associativity:** $(a \cdot b) \cdot c = a \cdot (b \cdot c)$,
(ii) **Distributivity:** $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(a + b) \cdot c = a \cdot c + b \cdot c$,
(iii) **Homogeneity:** $r(a \cdot b) = (ra) \cdot b = a \cdot (rb)$.

Usually we write the multiplication as simply ab instead of $a \cdot b$.

Definition 2.5 A map $L : V \to W$ between two vector spaces over the field K is said to be a **linear map** or a **linear operator** if for all $u, v \in V$ and $r \in K$:
(i) $L(u + v) = L(u) + L(v)$,
(ii) $L(ru) = rL(u)$.

To emphasize the scalars are in the field K, such a map is said to be ***K*-linear**.

Definition 2.6 If A and A' are algebras over a field K, an **algebra homomorphism** is a linear map $L : A \to A'$ that preserves the algebra multiplication: $L(ab) = L(a)L(b)$ for all $a, b \in A$.

The addition and multiplication of functions induce corresponding operations on C_p^∞, making it into an algebra over $\mathbb{R}$.

2.3. Derivations at a Point

For each tangent vector $v \in T_p\mathbb{R}^n$, the directional derivative at p gives a map

$$D_v : C_p^\infty \to \mathbb{R}. \tag{9}$$

Definition 2.7 A linear map $D : C_p^\infty \to \mathbb{R}$ is called a **derivation** at p or a **point derivation** if it satisfies the Leibniz rule:

$$D(fg) = D(f)g(p) + f(p)D(g) \tag{10}$$

Denote the set of all derivations at p by $\mathcal{D}_p(\mathbb{R}^n)$, which is a vector space over $\mathbb{R}$.

Obviously, the directional derivatives at p are all derivations at p, so there is a map

$$\begin{aligned} \phi : T_p(\mathbb{R}^n) &\to \mathcal{D}_p(\mathbb{R}^n), \\ v &\mapsto D_v = v^i \left.\frac{\partial}{\partial x^i}\right|_p . \end{aligned} \tag{11}$$

Since D_v is clearly linear in v, ϕ is a linear map of vector spaces.

Lemma 2.8 If D is a point-derivation of C_p^∞, then $D(c) = 0$ for any constant function c.

Proof. By $\mathbb{R}$-linearity, $D(c) = cD(1)$. By the Leibniz rule, we have

$$\begin{aligned} D(1) &= D(1 \cdot 1) \\ &= D(1) \cdot 1(p) + 1(p) \cdot D(1) \\ &= 2D(1), \end{aligned} \tag{12}$$

which implies that $D(1) = 0$, and therefore $D(c) = cD(1) = c \cdot 0 = 0$. □

Lemma 2.9 The map $\phi : T_p(\mathbb{R}^n) \to \mathcal{D}_p(\mathbb{R}^n)$ is an isomorphism of vector spaces.

Proof. To show that ϕ is injective, suppose $\phi(v) = D_v = 0$ for some $v \in T_p(\mathbb{R}^n)$. For the coordinate functions x^j, we have

$$\begin{aligned} 0 = D_v x^j = v^i \frac{\partial x^j}{\partial x^i}\bigg|_p \\ = v^i \delta_i^j \\ = v^j, \end{aligned} \tag{13}$$

which implies that $v = 0$. Thus, ϕ is injective.
To show that ϕ is surjective, let $D \in \mathcal{D}_p(\mathbb{R}^n)$ and let (f, V) be a representative of a germ in C_p^∞. We may assume V is an open ball, hence star-shaped. From Taylor's theorem with remainder, we have

$$f(x) = f(p) + (x^i - p^i) g_i(x), \quad g_i(p) = \frac{\partial f}{\partial x^i}(p). \tag{14}$$

Applying D to both sides, we get

$$\begin{aligned} D(f(x)) &= D[f(p)] + D[(x^i - p^i) g_i(x)] \\ &= (Dx^i) g_i(p) + (p^i - p^i) D g_i(x) \\ &= (Dx^i) g_i(p) \\ &= (Dx^i) \frac{\partial f}{\partial x^i}(p), \end{aligned} \tag{15}$$

which gives $D = D_v$ for $v = \langle Dx^1, \cdots, Dx^n \rangle$. Thus, ϕ is surjective. □

Under this vector space isomorphism $T_p(\mathbb{R}^n) \simeq \mathcal{D}_p(\mathbb{R}^n)$, we can identify tangent vectors with derivations at p, and the standard basis $e_1, \cdots, e_n$ of $T_p(\mathbb{R}^n)$ with the set $\frac{\partial}{\partial x^1}\big|_p, \cdots, \frac{\partial}{\partial x^n}\big|_p$ of partial derivatives,

$$\begin{aligned} v &= \langle v^1, \cdots, v^n \rangle \\ &= v^i e_i \\ &= v^i \frac{\partial}{\partial x^i}\bigg|_p . \end{aligned} \tag{16}$$

2.4. Vector Fields

Definition 2.10 A **vector field** on an open subset $U \subseteq \mathbb{R}^n$ is a function that assigns to each point $p \in U$ a tangent vector $X_p \in T_p(\mathbb{R}^n)$. Since $T_p(\mathbb{R}^n)$ has basis $\frac{\partial}{\partial x^i}\big|_p$, we can write

$$X_p = a^i(p) \frac{\partial}{\partial x^i}\bigg|_p, \quad a^i(p) \in \mathbb{R}. \tag{17}$$

Omitting p, we can write

$$X = a^i \frac{\partial}{\partial x^i} \quad \leftrightarrow \quad \begin{bmatrix} a^1 \\ \vdots \\ a^n \end{bmatrix}, \tag{18}$$

where a^i are functions on U. We say that X is C^∞ on U if all the coefficient functions a^i are C^∞ on U.
The set of all C^∞ vector fields on U is denoted by $\mathfrak{X}(U)$.

Definition 2.11 If R is a commutative ring with identity, a (left) R-**module** is an abelian group A with a scalar multiplication

$$\mu : R \times A \to A, \tag{19}$$

usually written $\mu(r, a) = ra$, such that for all $r, s \in R$ and $a, b \in A$,
(i) **Associativity:** $(rs)a = r(sa)$,
(ii) **Identity:** $1a = a$,
(iii) **Distributivity:** $r(a + b) = ra + rb$ and $(r + s)a = ra + sa$.

$\mathfrak{X}(U)$ is a module over the ring $C^\infty(U)$ with the multiplication defined pointwise:

$$(fX)_p = f(p)X_p, \tag{20}$$

for $f \in C^\infty(U), X \in \mathfrak{X}(U), p \in U$.

Definition 2.12 Let A and A' be R-modules. An R-**module homomorphism** from A to A' is a map $f : A \to A'$ that preserves both the addition and the scalar multiplication: for all $a, b \in A$ and $r \in R$,
(i) $f(a + b) = f(a) + f(b)$,
(ii) $f(ra) = rf(a)$.

2.5. Vector Fields as Derivations

If $X \in \mathfrak{X}(U)$ and $f \in C^\infty(U)$, we can define a new function Xf by

$$(Xf)(p) = X_p f \quad \text{for all } p \in U. \tag{21}$$

Writing $X = a^i \frac{\partial}{\partial x^i}$, we have

$$(Xf)(p) = a^i(p)\frac{\partial f}{\partial x^i}(p), \tag{22}$$

or

$$Xf = a^i \frac{\partial f}{\partial x^i}, \tag{23}$$

which is a C^∞ function on U. Thus, a C^∞ vector field X induces an $\mathbb{R}$-linear map

$$\begin{aligned} X : C^\infty(U) &\to C^\infty(U), \\ f &\mapsto Xf. \end{aligned} \tag{24}$$

$X(fg)$ satisfies the Leibniz rule:

$$X(fg) = (Xf)g + f(Xg). \tag{25}$$

Definition 2.13 If A is an algebra over a field K, a **derivation** on A is a K-linear map $D : A \to A$ that satisfies the Leibniz rule:

$$D(ab) = (Da)b + a(Db) \quad \text{for all } a, b \in A. \tag{26}$$

The set of all derivations on A is closed under addition and scalar multiplication and forms a vector space, denoted by $\operatorname{Der}(A)$.

We therefore have a map

$$\begin{aligned} \varphi : \mathfrak{X}(U) &\to \operatorname{Der}(C^\infty(U)), \\ X &\mapsto (f \mapsto Xf), \end{aligned} \tag{27}$$

which is an isomorphism of vector spaces, just as the map $\phi : T_p(\mathbb{R}^n) \to \mathcal{D}_p(\mathbb{R}^n)$.

3. The Exterior Algebra of Multicovectors

3.1. Dual Spaces

Definition 3.1 If V and W are real vector spaces, we denote by $\operatorname{Hom}(V, W)$ the vector space of all linear maps $f : V \to W$.
The **dual space** $V^\vee$ of V is the vector space of all the real-valued linear functions on V:

$$V^\vee = \operatorname{Hom}(V, \mathbb{R}). \tag{28}$$

The elements of $V^\vee$ are called **covectors** or **1-covectors** on V.

In the rest of this section, assume V to be a *finite-dimensional* vector space. Let $e_1, \cdots, e_n$ be a basis of V. Then every $v \in V$ is uniquely a linear combination $v = v^i e_i$ with $v^i \in \mathbb{R}$. Let $\alpha^i : V \to \mathbb{R}$ be the linear function that picks out the ith coordinate, $\alpha^i(v) = v^i$. Note that

$$\alpha^i(e_j) = \delta^i_j. \tag{29}$$

Proposition 3.2 The functions $\alpha^1, \cdots, \alpha^n$ form a basis of $V^\vee$.

Proof. Let $f \in V^\vee$ and $v = v^i e_i \in V$, then

$$\begin{aligned} f(v) &= v^i f(e_i) \\ &= f(e_i)\alpha^i(v), \end{aligned} \tag{30}$$

which means $f = f(e_i)\alpha^i$, i.e., $\alpha^1, \cdots, \alpha^n$ span $V^\vee$.
Suppose $c_i \alpha^i = 0$ for some $c_i \in \mathbb{R}$. Applying both sides to e_j gives

$$\begin{aligned} 0 &= c_i \alpha^i(e_j) \\ &= c_i \delta^i_j \\ &= c_j, \end{aligned} \tag{31}$$

which means $\alpha^1, \cdots, \alpha^n$ are linear independent. □

The basis $\alpha^1, \cdots, \alpha^n$ of $V^\vee$ is said to be *dual* to the basis $e_1, \cdots, e_n$ of V.

3.2. Permutations

Definition 3.3 Fix a positive integer k. A **permutation** of a set $A = \{1, \cdots, k\}$ is a bijection $\sigma : A \to A$. σ can be thought of as a reordering of the list $1, \cdots, k$ from $1, \cdots, k$ to $\sigma(1), \cdots, \sigma(k)$.
A simple way to describe a permutation is by its matrix

$$M(\sigma) = \begin{bmatrix} 1 & \cdots & k \\ \sigma(1) & \cdots & \sigma(k) \end{bmatrix}. \tag{32}$$

The **cyclic permutation**, $(a_1 \cdots a_r)$ where a_i are distinct, is the permutation σ such that $\sigma(a_1) = a_2, \cdots, \sigma(a_{r-1}) = a_r, \sigma(a_r) = a(1)$ and fixes all other elements of A. A cyclic permutation $(a_1, \cdots, a_r)$ is called a **cycle of length r** or a **r-cycle**.
A **transposition** is a 2-cycle, i.e., a cycle of the form $(a_1\ a_2)$ that interchanges a_1 and a_2 and fixes all other elements of A.
Two cycles $(a_1 \cdots a_r)$ and $(b_1 \cdots b_s)$ are **disjoint** if $a_i \neq b_j$ for all i and j.
The **product** $\tau\sigma$ of two permutations σ and τ of A is the composition $\tau \circ \sigma$.

Any permutation can be written as a product of disjoint cycles $(a_1 \cdots a_r)(b_1 \cdots b_s) \cdots$.

Definition 3.4 Let S_k be the set of all permutations of the set $\{1, \cdots, k\}$. A permutation is **even** or **odd** if it can be expressed as a product of an even or odd number of transpositions, respectively.
The **sign** of a permutation $\sigma \in S_k$ is defined as

$$\operatorname{sgn}(\sigma) = \begin{cases} 1, & \text{if } \sigma \text{ is even} \\ -1, & \text{if } \sigma \text{ is odd.} \end{cases} \tag{33}$$

Clearly, $\operatorname{sgn}(\sigma\tau) = \operatorname{sgn}(\sigma) \operatorname{sgn}(\tau)$ for all $\sigma, \tau \in S_k$.

Generally, the r-cycle can be decomposed into $r - 1$ transpositions:

$$(a_1 \cdots a_r) = (a_1\ a_r)(a_1\ a_{r-1}) \cdots (a_1\ a_2), \tag{34}$$

which means that an r-cycle is even if r is odd and odd if r is even. Thus one way to compute the sign of a permutation is to decompose it into a product of disjoint cycles and count the number of even-length cycles.

Definition 3.5 An **inversion** of a permutation σ is an ordered pair $(\sigma(i), \sigma(j))$ such that $i < j$ but $\sigma(i) > \sigma(j)$.

The second way to compute the sign of a permutation is to count the number of inversions.

Proposition 3.6 A permutation σ can be written as a product of as many transpositions as the number of inversions it has, so σ is even if and only if it has an even number of inversions.

3.3. Multilinear Functions

Definition 3.7 Denote by $V^k = V \times \cdots \times V$ the Cartesian product of k copies of a real vector space V. A function $f : V^k \to \mathbb{R}$ is called ***k*-linear** if it is linear in each of its k arguments:

$$f(\cdots, av + bw, \cdots) = af(\cdots, v, \cdots) + bf(\cdots, w, \cdots) \tag{35}$$

for all $a, b \in \mathbb{R}$ and $v, w \in V$. Instead of 2-linear and 3-linear, it is customary to say **bilinear** and **trilinear**, respectively.
A k-linear function on V is also called a ***k*-tensor** on V. We denote the vector space of all k-tensors on V by $L_k(V)$, k is called the **degree** of the tensor f.

Example 3.8
(i) The dot product $f(v, w) = v \cdot w$ on $\mathbb{R}^n$ is bilinear.
(ii) The determinant $f(v_1, \cdots, v_n) = \det[v_1 \cdots v_n]$ on $\mathbb{R}^n$ is n-linear.

Definition 3.9 A k-linear function $f : V^k \to \mathbb{R}$ is **symmetric** if

$$f\left(v_{\sigma(1)}, \cdots, v_{\sigma(k)}\right) = f(v_1, \cdots, v_k) \tag{36}$$

for all permutations $\sigma \in S_k$.
A k-linear function $f : V^k \to \mathbb{R}$ is **alternating** if

$$f\left(v_{\sigma(1)}, \cdots, v_{\sigma(k)}\right) = [\operatorname{sgn}(\sigma)] f(v_1, \cdots, v_k) \tag{37}$$

for all permutations $\sigma \in S_k$.

Example 3.10
(i) The dot product $f(v, w) = v \cdot w$ on $\mathbb{R}^n$ is symmetric.
(ii) The determinant $f(v_1, \cdots, v_n) = \det[v_1 \cdots v_n]$ on $\mathbb{R}^n$ is alternating.
(iii) The cross product $f(v, w) = v \times w$ on $\mathbb{R}^3$ is alternating.

We are escpecially interested in the space $A_k(V)$ of all alternating k-linear functions on V for $k > 0$. They are also called **alternating *k*-tensors**, ***k*-covectors**, or **multicovectors of degree *k*** on V.

Definition 3.11 The vector space of all alternating k-linear functions on V is denoted by $A_k(V)$, the elements of $A_k(V)$ are also called **alternating *k*-tensors**, ***k*-covectors**, or **multicovectors of degree *k*** on V.
For $k = 0$, we define a 0-covector to be a constant, so $A_0(V) = \mathbb{R}$.
For $k = 1$, a 1-covector is simply a covector.

3.4. The Permutation Action on Multilinear Functions

Definition 3.12 If $f \in L_k(V)$ and $\sigma \in S_k$ is a permutation, we define a new k-linear function σf by

$$(\sigma f)(v_1, \cdots, v_k) = f\left(v_{\sigma(1)}, \cdots, v_{\sigma(k)}\right). \tag{38}$$

Thus f is symmetric if and only if $\sigma f = f$ for all $\sigma \in S_k$, and f is alternating if and only if $\sigma f = [\operatorname{sgn}(\sigma)]f$ for all $\sigma \in S_k$.

When there is only one argument, the permutation group S_1 is the identity group and a 1-linear function is both symmetric and alternating. In particular,

$$A_1(V) = L_1(V) = V^\vee. \tag{39}$$

Lemma 3.13 If $\sigma, \tau \in S_k$ and $f \in L_k(V)$, then $\tau(\sigma f) = (\tau\sigma)f$.

Proof. For $v_1, \cdots, v_k \in V$, we have

$$\begin{aligned}(\tau(\sigma f))(v_1, \cdots, v_k) &= (\sigma f)\left(v_{\tau(1)}, \cdots, v_{\tau(k)}\right) \\ &= f\left(v_{\sigma(\tau(1))}, \cdots, v_{\sigma(\tau(k))}\right) \\ &= f\left(v_{(\tau\sigma)(1)}, \cdots, v_{(\tau\sigma)(k)}\right) \\ &= (\tau\sigma)f(v_1, \cdots, v_k).\end{aligned} \tag{40}$$

□

Definition 3.14 If G is a group and X is a set, a map

$$\begin{aligned} G \times X &\to X, \\ (\sigma, x) &\mapsto \sigma \cdot x \end{aligned} \tag{41}$$

is called a **left action** of G on X if for all $\sigma, \tau \in G$ and $x \in X$,
(i) $e \cdot x = x$, where e is the identity element of G,
(ii) $\tau \cdot (\sigma \cdot x) = (\tau\sigma) \cdot x$.

The **orbit** of an element $x \in X$ is the set

$$Gx := \{\sigma \cdot x \in X \mid \sigma \in G\} \tag{42}$$

A **right action** of G on X is defined similarly: it is a map

$$\begin{aligned} X \times G &\to X, \\ (x, \sigma) &\mapsto x \cdot \sigma \end{aligned} \tag{43}$$

such that for all $\sigma, \tau \in G$ and $x \in X$,
(i) $x \cdot e = x$,
(ii) $(x \cdot \tau) \cdot \sigma = x \cdot (\tau\sigma)$.

In this terminology, we have defined a left action of S_k on $L_k(V)$.

3.5. The Symmetrizing and Alternating Operators

Definition 3.15 Given any k-linear function f on V, we can make a symmetric k-linear function Sf by

$$(Sf)(v_1, \cdots, v_k) = \sum_{\sigma \in S_k} f\left(v_{\sigma(1)}, \cdots, v_{\sigma(k)}\right) \tag{44}$$

or, in our new sharthand, the **symmetric operator** S is defined by

$$Sf = \sum_{\sigma \in S_k} \sigma f. \tag{45}$$

Similarly, the **alternating operator** A is defined by

$$Af = \sum_{\sigma \in S_k} [\mathrm{sgn}(\sigma)] \sigma f. \tag{46}$$

Proposition 3.16 If f is a k-linear function on V, then
(i) Sf is symmetric,
(ii) Af is alternating.

Proof.
(i) For $\tau \in S_k$, we have

$$\begin{aligned}(\tau Sf) &= \sum_{\sigma \in S_k} \tau(\sigma f) \\ &= \sum_{\sigma \in S_k} (\tau\sigma) f \\ &= Sf,\end{aligned} \tag{47}$$

which means Sf is symmetric.
(i) For $\tau \in S_k$, we have

$$\begin{aligned}(\tau Af) &= \sum_{\sigma \in S_k} [\mathrm{sgn}(\sigma)] \tau(\sigma f) \\ &= \sum_{\sigma \in S_k} [\mathrm{sgn}(\sigma)] (\tau\sigma) f \\ &= [\mathrm{sgn}(\tau)] \sum_{\sigma \in S_k} [\mathrm{sgn}(\tau\sigma)] (\tau\sigma) f \\ &= [\mathrm{sgn}(\tau)] Af,\end{aligned} \tag{48}$$

which means Af is alternating. □

Lemma 3.17 If $f \in A_k(V)$, then $Af = (k!)f$.

Proof. Since $f \in A_k(V)$, we have $\sigma f = [\mathrm{sgn}(\sigma)] f$ for all $\sigma \in S_k$. Thus,

$$\begin{aligned}Af &= \sum_{\sigma \in S_k} [\mathrm{sgn}(\sigma)] \sigma f \\ &= \sum_{\sigma \in S_k} [\mathrm{sgn}(\sigma)][\mathrm{sgn}(\sigma)] f \\ &= \sum_{\sigma \in S_k} f \\ &= (k!) f.\end{aligned} \tag{49}$$

□

3.6. The Tensor Product

Definition 3.18 Let $f \in L_k(V)$ and $g \in L_l(V)$. The **tensor product** of f and g is the $k + l$-linear function $f \otimes g$ defined by

$$(f \otimes g)(v_1, \cdots, v_k, v_{k+1}, \cdots, v_{k+l}) = f(v_1, \cdots, v_k) g(v_{k+1}, \cdots, v_{k+l}). \tag{50}$$

Example 3.19 *Bilinear maps.* Let $e_1, \cdots, e_n$ be a basis of V, $\alpha^1, \cdots, \alpha^n$ the dual basis of $V^\vee$, and

$$\langle\ ,\ \rangle : V \times V \to \mathbb{R} \tag{51}$$

a bilinear map on V. Set $g_{ij} = \langle e_i, e_j \rangle \in \mathbb{R}$. If $v = v^i e_i$ and $w = w^i e_i$, with $v^i = \alpha^i(v)$, $w^i = \alpha^i(w)$ and the bilinearity, we can express $\langle\ ,\ \rangle$ in terms of the tensor product:

$$\begin{aligned} \langle v, w \rangle &= v^i w^j \langle e_i, e_j \rangle \\ &= \alpha^i(v) \alpha^j(w) g_{ij} \\ &= g_{ij} (\alpha^i \otimes \alpha^j)(v, w). \end{aligned} \tag{52}$$

Hence, $\langle\ ,\ \rangle = g_{ij}(\alpha^i \otimes \alpha^j)$. This notation is often used to describe an inner product on V.

Proposition 3.20 The tensor product is associative: $(f \otimes g) \otimes h = f \otimes (g \otimes h)$ for multilinear functions f, g, h on V.

Proof. For $f \in L_k(V)$, $g \in L_l(V)$, and $h \in L_m(V)$, we have

$$\begin{aligned} [(f \otimes g) \otimes h](v_1, \cdots, v_{k+l+m}) &= (f \otimes g)(v_1, \cdots, v_{k+l}) h(v_{k+l+1}, \cdots, v_{k+l+m}) \\ &= f(v_1, \cdots, v_k) g(v_{k+1}, \cdots, v_{k+l}) h(v_{k+l+1}, \cdots, v_{k+l+m}) \\ &= f(v_1, \cdots, v_k)(g \otimes h)(v_{k+1}, \cdots, v_{k+l+m}) \\ &= [f \otimes (g \otimes h)](v_1, \cdots, v_{k+l+m}), \end{aligned} \tag{53}$$

which means $(f \otimes g) \otimes h = f \otimes (g \otimes h)$. □

3.7. The Wedge Product

Definition 3.21 For $f \in A_k(V)$ and $g \in A_l(V)$, the **wedge product** or **exterior product** of f and g is the $(k + l)$-linear function $f \wedge g$ defined by

$$(f \wedge g) = \frac{1}{k!l!} A(f \otimes g), \tag{54}$$

or explicitly,

$$(f \wedge g)(v_1, \cdots, v_{k+l}) = \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} [\operatorname{sgn}(\sigma)] f\left(v_{\sigma(1)}, \cdots, v_{\sigma(k)}\right) g\left(v_{\sigma(k+1)}, \cdots, v_{\sigma(k+l)}\right) \tag{55}$$

By Proposition 3.16, the wedge product $f \wedge g \in A_{k+l}(V)$ When $k = 0$, the element $f \in A_0(V)$ is a constant c, (55) gives

$$\begin{aligned}(c \wedge g)(v_1, \cdots, v_l) &= \frac{1}{0!l!} \sum_{\sigma \in S_l} [\operatorname{sgn}(\sigma)] c g\left(v_{\sigma(1)}, \cdots, v_{\sigma(l)}\right) \\ &= \frac{c}{l!} \sum_{\sigma \in S_l} [\operatorname{sgn}(\sigma)] g\left(v_{\sigma(1)}, \cdots, v_{\sigma(l)}\right) \\ &= c g(v_1, \cdots, v_l),\end{aligned} \tag{56}$$

which means $(c \wedge g) = cg$, is a scalar multiplication.

Example 3.22 For $f \in A_2(V)$ and $g \in A_1(V)$,

$$\begin{aligned}A(f \otimes g) = {} & f(v_1, v_2) g(v_3) - f(v_1, v_3) g(v_2) - f(v_2, v_1) g(v_3) \\ & + f(v_2, v_3) g(v_1) + f(v_3, v_1) g(v_2) - f(v_3, v_2) g(v_1),\end{aligned} \tag{57}$$

where $f(v_1, v_2) g(v_3) = -f(v_2, v_1) g(v_3)$ and so on.
Therefore, dividing by 2, we have

$$(f \wedge g)(v_1, v_2, v_3) = f(v_1, v_2) g(v_3) - f(v_1, v_3) g(v_2) + f(v_2, v_3) g(v_1). \tag{58}$$

One way to avoid redundancy in the definition of $f \wedge g$ is to stipulate that in the sum (55), $\sigma(1), \cdots, \sigma(k)$ be in ascending order and $\sigma(k+1), \cdots, \sigma(k+l)$ be in ascending order.

Definition 3.23 A permutation $\sigma \in S_{k+l}$ is called a (k, l)-**shuffle** if

$$\sigma(1) < \cdots < \sigma(k) \text{ and } \sigma(k+1) < \cdots < \sigma(k+l). \tag{59}$$

Then (55) can be rewritten asy

$$(f \wedge g)(v_1, \cdots, v_{k+l}) = \sum_{\substack{(k,l)-\text{shuffles} \\ \sigma}} [\operatorname{sgn}(\sigma)] f\left(v_{\sigma(1)}, \cdots, v_{\sigma(k)}\right) g\left(v_{\sigma(k+1)}, \cdots, v_{\sigma(k+l)}\right), \tag{60}$$

which is a sum of $\binom{k+l}{k}$ terms, instead of $(k+l)!$ terms.

Example 3.24 For $f, g \in A_2(V)$,

$$\begin{aligned}(f \wedge g)(v_1, v_2, v_3, v_4) = {} & f(v_1, v_2) g(v_3, v_4) - f(v_1, v_3) g(v_2, v_4) + f(v_1, v_4) g(v_2, v_3) \\ & + f(v_2, v_3) g(v_1, v_4) - f(v_2, v_4) g(v_1, v_3) + f(v_3, v_4) g(v_1, v_2)\end{aligned} \tag{61}$$

3.8. Anticommutative of the Wedge Product

Proposition 3.25 The wedge product is **anticommutative**: if $f \in A_k(V)$ and $g \in A_l(V)$, then

$$f \wedge g = (-1)^{kl} g \wedge f. \tag{62}$$

Proof. Define $\tau \in S_{k+l}$ to be the permutation

$$\tau = \begin{bmatrix} 1 & \cdots & l & l+1 & \cdots & l+k \\ k+1 & \cdots & k+l & 1 & \cdots & k \end{bmatrix}. \tag{63}$$

Then

$$\begin{aligned}\sigma(1) = \sigma\tau(l+1), \cdots, \sigma(k) = \sigma\tau(l+k),\\ \sigma(k+1) = \sigma\tau(1), \cdots, \sigma(k+l) = \sigma\tau(l).\end{aligned} \tag{64}$$

For any $v_1, \cdots, v_{k+l} \in V$, we have

$$\begin{aligned}A(f \otimes g)(v_1, \cdots, v_{k+l}) &= \sum_{\sigma \in S_{k+l}} [\operatorname{sgn}(\sigma)] f\left(v_{\sigma(1)}, \cdots, v_{\sigma(k)}\right) g\left(v_{\sigma(k+1)}, \cdots, v_{\sigma(k+l)}\right)\\ &= \sum_{\sigma \in S_{k+l}} [\operatorname{sgn}(\sigma)] f\left(v_{\sigma\tau(l+1)}, \cdots, v_{\sigma\tau(l+k)}\right) g\left(v_{\sigma\tau(1)}, \cdots, v_{\sigma\tau(l)}\right)\\ &= \operatorname{sgn}(\tau) \sum_{\sigma \in S_{k+l}} [\operatorname{sgn}(\sigma\tau)] g\left(v_{\sigma\tau(1)}, \cdots, v_{\sigma\tau(l)}\right) f\left(v_{\sigma\tau(l+1)}, \cdots, v_{\sigma\tau(l+k)}\right)\\ &= \operatorname{sgn}(\tau) A(g \otimes f)(v_1, \cdots, v_{k+l}),\end{aligned} \tag{65}$$

which means

$$A(f \otimes g) = [\operatorname{sgn}(\tau)] A(g \otimes f). \tag{66}$$

Dividing by $k!l!$, we have

$$(f \wedge g) = [\operatorname{sgn}(\tau)](g \wedge f). \tag{67}$$

For every $i \in [k+1, k+l], j \in [1, k]$, (i, j) is an inversion of τ, so $[\operatorname{sgn}(\tau)] = (-1)^{kl}$, and therefore

$$f \wedge g = (-1)^{kl} g \wedge f. \tag{68}$$

□

Corollary 3.26 If $f \in A_k(V)$ with odd k, then $f \wedge f = 0$.

Proof. By the anticommutative property of the wedge product, we have

$$f \wedge f = (-1)^{k^2} f \wedge f = -f \wedge f, \tag{69}$$

which implies that $f \wedge f = 0$. □

3.9. Associativity of the Wedge Product

Lemma 3.27 Suppose $f \in L_k(V)$ and $g \in L_l(V)$, then

(i) $A(A(f) \otimes g) = k! A(f \otimes g)$,

(ii) $A(f \otimes A(g)) = l! A(f \otimes g)$.

Proof.

(i) By definition,

$$\begin{aligned}A(A(f) \otimes g) &= \sum_{\sigma \in S_{k+l}} [\operatorname{sgn}(\sigma)] \sigma\left(\left[\sum_{\tau \in S_k} [\operatorname{sgn}(\tau)] \tau f\right] \otimes g\right)\\ &= \sum_{\sigma \in S_{k+l}} \sum_{\tau \in S_k} [\operatorname{sgn}(\sigma)][\operatorname{sgn}(\tau)] \sigma\tau f \otimes g.\end{aligned} \tag{70}$$

For each $\mu \in S_{k+l}$ and each $\tau \in S_k$, there is a unique $\sigma = \mu\tau^{-1} \in S_{k+l}$ such that $\mu = \sigma\tau$. Then (70) can be rewritten as

$$\begin{aligned} A(A(f) \otimes g) &= k! \sum_{\mu \in S_{k+l}} [\operatorname{sgn}(\mu)]\mu f \otimes g \\ &= k! A(f \otimes g). \end{aligned} \tag{71}$$

(ii) It can be shown similarly that

$$A(f \otimes A(g)) = l! A(f \otimes g). \tag{72}$$

□

Proposition 3.28 If $f \in A_k(V), g \in A_l(V)$ and $h \in A_m(V)$, then

$$(f \wedge g) \wedge h = f \wedge (g \wedge h) \tag{73}$$

Proof. By definition,

$$\begin{aligned} (f \wedge g) \wedge h &= \frac{1}{(k+l)!m!} A((f \wedge g) \otimes h) \\ &= \frac{1}{(k+l)!m!} \frac{1}{k!l!} A(A(f \otimes g) \otimes h) \\ &= \frac{(k+l)!}{(k+l)!m!k!l!} A((f \otimes g) \otimes h) \\ &= \frac{1}{k!l!m!} A((f \otimes g) \otimes h). \end{aligned} \tag{74}$$

Similarly,

$$\begin{aligned} f \wedge (g \wedge h) &= \frac{1}{k!(l+m)!} \frac{1}{l!m!} A(f \otimes (g \otimes h)) \\ &= \frac{1}{k!l!m!} A(f \otimes (g \otimes h)). \end{aligned} \tag{75}$$

Since $(f \otimes g) \otimes h = f \otimes (g \otimes h)$, we have

$$(f \wedge g) \wedge h = f \wedge (g \wedge h). \tag{76}$$

□

By associativity, we can omit parentheses and simply write $f \wedge g \wedge h$.

Corollary 3.29 For $f_i \in A_{d_i}(V)$,

$$f_1 \wedge \cdots \wedge f_r = \frac{1}{(d_1)! \cdots (d_r)!} A(f_1 \otimes \cdots \otimes f_r). \tag{77}$$

Proposition 3.30 If $\alpha^1, \cdots, \alpha^k \in V^\vee$ and $v_1, \cdots, v_k \in V$, then

$$(\alpha^1 \wedge \cdots \wedge \alpha^k)(v_1, \cdots, v_k) = \det\left[\alpha^i(v_j)\right], \tag{78}$$

where $\left[\alpha^i(v_j)\right]$ is the matrix whose (i, j)th entry is $\alpha^i(v_j)$.

Proof. By Corollary 3.29, we have

$$\begin{aligned}\left(\alpha^1\wedge\cdots\wedge\alpha^k\right)(v_1,\cdots,v_k)&=A\left(\alpha^1\otimes\cdots\otimes\alpha^k\right)(v_1,\cdots,v_k)\\&=\sum_{\sigma\in S_k}[\operatorname{sgn}(\sigma)]\alpha^1\left(v_{\sigma(1)}\right)\cdots\alpha^k\left(v_{\sigma(k)}\right)\\&=\det\left[\alpha^i\left(v_j\right)\right]\end{aligned}\tag{79}$$

□

Definition 3.31 An algebra A over a field K is said to be **graded** if it can be written as a direct sum $A=\bigoplus_{k=0}^{\infty}A^k$ over K such that the multiplication map sends $A^k\times A^l$ into A^{k+l}. The notation $A=\bigoplus_{k=0}^{\infty}A^k$ means that each nonzero element of A can be written uniquely as a finite sum

$$a=a_{i_1}+\cdots+a_{i_m},\tag{80}$$

where $a_{i_j}\neq 0\in A^{i_j}$.
A graded algebra $A=\bigoplus_{k=0}^{\infty}A^k$ is called **anticommutative** or **graded commutative** if for all $a\in A^k$ and $b\in A^l$,

$$ab=(-1)^{kl}ba.\tag{81}$$

A **homomorphism** of graded algebras is an algebra homomorphism that preserves the degree.

Example 3.32 The polynomial algebra $A=\mathbb{R}[x,y]$ is graded by degree; A^k consists of all homogeneous polynomials of total degree k in x and y.

Definition 3.33 For a finite-dimensional vector space V, say of dimension n, the **exterior algebra** or **Grassmann algebra** of multivectors on V is the graded algebra

$$A_*(V)=\bigoplus_{k=0}^{\infty}A_k(V)=\bigoplus_{k=0}^{n}A_k(V),\tag{82}$$

with the wedge product as multiplication.

3.10. A Basis for k-Covectors

Let $e_1,\cdots,e_n$ be a basis for V and $\alpha^1,\cdots,\alpha^n$ be the dual basis for $V^\vee$. Introduce the multi-index notation

$$I=(i_1,\cdots,i_k)\tag{83}$$

and write e_I for $\left(e_{i_1},\cdots,e_{i_k}\right)$ and α^I for $\left(\alpha^{i_1},\cdots,\alpha^{i_k}\right)$.
A k-linear function f on V is completely determined by its values on all k-tuples $\left(e_{i_1},\cdots,e_{i_k}\right)$. If f is alternating, then it is completely determined by its values on $\left(e_{i_1},\cdots,e_{i_k}\right)$ with $1\leq i_1<\cdots<i_k\leq n$; that is, it suffices to consider e_I with I in strictly ascending order.

Lemma 3.34 Let $e_1, \cdots, e_n$ be a basis for V and $\alpha^1, \cdots, \alpha^n$ be the dual basis for $V^\vee$. If $I = (1 \le i_1 < \cdots < i_k \le n)$ and $J = (1 \le j_1 < \cdots < j_k \le n)$ are two strictly ascending multi-indices of length k, then

$$\alpha^I(e_J) = \delta^I_J = \begin{cases} 1 \text{ for } I = J \\ 0 \text{ for } I \ne J. \end{cases} \tag{84}$$

Proof. By Proposition 3.30,

$$\alpha^I(e_J) = \det\left[\alpha^i(e_j)\right]_{i \in I, j \in J}. \tag{85}$$

If $I = J$, $\left[\alpha^i(e_j)\right]$ is the identity matrix, so $\alpha^I(e_J) = 1$.
If $I \ne J$, we compare them term by term until th terms differ:

$$i_1 = j_1, \cdots, i_{l-1} = j_{l-1}, i_l \ne j_l, \cdots. \tag{86}$$

Without loss of generality, we can assume $i_l < j_l$. Then $i_l \ne j_1, \cdots, j_{l-1}$, and $i_l \ne j_{l+1}, \cdots, j_k$, so the l-th row of $\left[\alpha^i(e_j)\right]$ will be all zeros. Thus, $\alpha^I(e_J) = 0$. □

Proposition 3.35 The alternating k-linear function $\alpha^I, I = (i_1 < \cdots < i_k)$, form a basis for $A_k(V)$.

Proof. To show linear independence, suppose $c_I \alpha^I = 0$ for some $c_I \in \mathbb{R}$. Applying both sides to e_J gives

$$\begin{aligned} 0 &= c_I \alpha^I(e_J) \\ &= c_I \delta^I_J \\ &= c_J, \end{aligned} \tag{87}$$

which means $c_J = 0$ for all J, so α^I are linearly independent.
To show that they span $A_k(V)$, let $f \in A_k(V)$ and $g = f(e_I)\alpha^I$. Then

$$\begin{aligned} g(e_J) &= f(e_I)\alpha^I(e_J) \\ &= f(e_I)\delta^I_J \\ &= f(e_J), \end{aligned} \tag{88}$$

which means $f = g = f(e_I)\alpha^I$, so f is a linear combination of α^I. Thus, α^I span $A_k(V)$. □

Corollary 3.36 If V is n-dimensional, then the dimension of $A_k(V)$ is $\binom{n}{k}$.

Corollary 3.37 If $k > \dim V$, then $A_k(V) = 0$.

4. Differential Forms on $\mathbb{R}^n$

Differential forms extend Grassmann's exterior algebra from the tangent space at a point to an entire manifold.
In this section, we will study differential forms on an open set of $\mathbb{R}^n$.

4.1. Differential 1-forms and the Differential of a Function

Definition 4.1 The **cotangent space** to $\mathbb{R}^n$ at p, denoted by $T_p^*(\mathbb{R}^n)$, is defined to be the dual space $\left(T_p\mathbb{R}^n\right)^\vee$ of the tangent space $T_p\mathbb{R}^n$.

Definition 4.2 In parallel with the definition of a vector field, a **covector field** or a **differential 1-form** on an open set $U \subset \mathbb{R}^n$ is a function ω that assigns to each point $p \in U$ a covector $\omega_p \in T_p^*(\mathbb{R}^n)$,

$$\begin{aligned} \omega : U &\to \bigcup_{p \in U} T_p^*(\mathbb{R}^n), \\ p &\to \omega_p \in T_p^*(\mathbb{R}^n). \end{aligned} \tag{89}$$

Note that in the union $\bigcup_{p\in U} T_p^*(\mathbb{R}^n)$, the sets $T_p^*(\mathbb{R}^n)$ are disjoint. We call a differential 1-form a **1-form** for short.

Definition 4.3 For any $f \in C^\infty(U)$, the **differential** of f is the 1-form $\mathrm{d}f$ defined, for $p \in U$ and $X_p \in T_pU$, by

$$(\mathrm{d}f)_p\left(X_p\right) = X_p f. \tag{90}$$

The directional derivative sets a bilinear pairing

$$\begin{aligned} T_p(\mathbb{R}^n) \times C_p^\infty(\mathbb{R}^n) &\to \mathbb{R}, \\ \left(X_p, f\right) &\mapsto \langle X_p, f\rangle = X_p f. \end{aligned} \tag{91}$$

One may think of a tangent vector as a function on the second argument of the pairing: $\langle X_p, \cdot\rangle$, then the differential can be thought of as a function on the first argument of the pairing:

$$(\mathrm{d}f)_p = \langle \cdot, f\rangle, \tag{92}$$

which is also written as $\mathrm{d}f\ |_p$.

Proposition 4.4 If $\left\{x^1, \cdots, x^n\right\}$ are the coordinates of $\mathbb{R}^n$, then at each point $p \in \mathbb{R}^n$, $\left\{\left(\mathrm{d}x^1\right)_p, \cdots, \left(\mathrm{d}x^n\right)_p\right\}$ is the basis for $T_p^*(\mathbb{R}^n)$ dual to the basis $\left\{\frac{\partial}{\partial x^1}\Big|_p, \cdots, \frac{\partial}{\partial x^n}\Big|_p\right\}$ of $T_p(\mathbb{R}^n)$.

Proof. By definition,

$$\begin{aligned} \left(\mathrm{d}x^i\right)_p\left(\left.\frac{\partial}{\partial x^j}\right|_p\right) &= \left.\frac{\partial}{\partial x^j}\right|_p x^i \\ &= \left.\frac{\partial x^i}{\partial x^j}\right|_p \\ &= \delta^i_j. \end{aligned} \tag{93}$$

□

If ω is a 1-form on an open set $U \subset \mathbb{R}^n$, then by Proposition 4.4, at each point $p \in U$, ω can be expressed as

$$\omega_p = \omega_i(p)\left(\mathrm{d}x^i\right)_p, \tag{94}$$

for some $\omega_i(p) \in \mathbb{R}$. As p varies over U, the coefficients ω_i become functions on U. Thus, we can write

$$\omega = \omega_i\,\mathrm{d}x^i. \tag{95}$$

A covector field ω is said to be C^∞ on U if the coefficients ω_i are all C^∞ functions on U.

Proposition 4.5 If $f \in C^\infty(U)$, then

$$\mathrm{d}f = \frac{\partial f}{\partial x^i}\,\mathrm{d}x^i. \tag{96}$$

Proof. By Proposition 4.4, we have

$$\mathrm{d}f = (\mathrm{d}f)_i\,\mathrm{d}x^i, \tag{97}$$

applying both sides to $\frac{\partial}{\partial x^j}$ gives

$$\begin{aligned}\mathrm{d}f\left(\frac{\partial}{\partial x^j}\right) &= (\mathrm{d}f)_i\left(\mathrm{d}x^i\right)\left(\frac{\partial}{\partial x^j}\right)\\ &= (\mathrm{d}f)_i\frac{\partial x^i}{\partial x^j}\\ &= (\mathrm{d}f)_i\delta^i_j\\ &= (\mathrm{d}f)_j.\end{aligned} \tag{98}$$

Therefore, we have

$$\begin{aligned}\mathrm{d}f &= \mathrm{d}f\left(\frac{\partial}{\partial x^j}\right)\mathrm{d}x^j\\ &= \frac{\partial f}{\partial x^j}\,\mathrm{d}x^j\end{aligned} \tag{99}$$

□

This also shows that if f is a C^∞ function on U, then $\mathrm{d}f$ is a C^∞ 1-form on U.

4.2. Differential k-Forms

Definition 4.6 Generally, a **differential form ω of degree k** or **k-form** on an open set $U \subset \mathbb{R}^n$ is a function that assigns to each point $p \in U$ an alternating k-linear function $\omega_p \in A_k(T_p\mathbb{R}^n)$.

By Proposition 3.35, a basis for $A_k(T_p\mathbb{R}^n)$ is

$$\mathrm{d}x^I_p = \mathrm{d}x^{i_1}_p \wedge \cdots \wedge \mathrm{d}x^{i_k}_p, \quad 1 \le i_1 < \cdots < i_k \le n. \tag{100}$$

Therefore, at each point $p \in U$, ω_p is a linear combination

$$\omega_p = \omega_I(p)\,\mathrm{d}x_p^I, \quad 1 \le i_1 < \cdots < i_k \le n, \tag{101}$$

and a k-form ω on U can be expressed as

$$\omega = \omega_I\,\mathrm{d}x^I, \tag{102}$$

with function coefficients $\omega_I : U \to \mathbb{R}$. We say that a k-form ω is C^∞ on U if the coefficients $\omega_I \in C^\infty(U)$.

Denote by $\Omega^k(U)$ the vector space of all C^∞ k-forms on U. A 0-form assigns to each point $p \in U$ an element of $A_0(T_p\mathbb{R}^n) = \mathbb{R}$. Thus a 0-form is simply a C^∞ function on U, so $\Omega^0(U) = C^\infty(U)$.

There are no nonzero k-forms on $\mathbb{R}^n$ for $k > n$. This is because when $k > n$, in $\mathrm{d}x^I$ at least two of the 1-forms $\mathrm{d}x^{i_\alpha}$ will be the same, forcing $\mathrm{d}x^I = 0$.

Definition 4.7 The **wedge product** of a k form ω and an l-form τ on an open set U is defined pointwise:

$$(\omega \wedge \tau)_p = \omega_p \wedge \tau_p \in A_{k+l}(T_p\mathbb{R}^n). \tag{103}$$

In terms of coordinates, if $\omega = \omega_I\,\mathrm{d}x^I$ and $\tau = \tau_J\,\mathrm{d}x^J$, then

$$\omega \wedge \tau = \omega_I \tau_J\,\mathrm{d}x^I \wedge \mathrm{d}x^J, \tag{104}$$

where if I and J are not disjoint, then $\mathrm{d}x^I \wedge \mathrm{d}x^J = 0$. Hence, the sum is actually over disjoint I and J.

This also shows that the wedge product of two C^∞ forms is C^∞. So the wedge product is a bilinear map

$$\wedge : \Omega^k(U) \times \Omega^l(U) \to \Omega^{k+l}(U), \tag{105}$$

which is associative and anticommutative.

In case one of the factors has degree 0, say $k = 0$, the wedge product

$$\wedge : \Omega^0(U) \times \Omega^l(U) \to \Omega^l(U) \tag{106}$$

is the pointwise multiplication of a C^∞ l-form by a C^∞ function:

$$(f \wedge \tau)_p = f(p) \wedge \tau_p = f(p)\tau_p. \tag{107}$$

Example 4.8 Let x, y, z be the coordinates on $\mathbb{R}^3$. The C^∞ 1-form on $\mathbb{R}^3$ is given by

$$f\,\mathrm{d}x + g\,\mathrm{d}y + h\,\mathrm{d}z, \tag{108}$$

where $f, g, h \in C^\infty(\mathbb{R}^3)$ are functions. The C^∞ 2-form is given by

$$f\,\mathrm{d}y \wedge \mathrm{d}z + g\,\mathrm{d}x \wedge \mathrm{d}z + h\,\mathrm{d}x \wedge \mathrm{d}y, \tag{109}$$

and the C^∞ 3-form is given by

$$f\,\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z. \tag{110}$$

Example 4.9 Let x^1, x^2, x^3, x^4 be the coordinates on $\mathbb{R}^4$ and p a point in $\mathbb{R}^4$. A basis for $A_3(T_p\mathbb{R}^4)$ is

$$\begin{aligned} \{ &\mathrm{d}x_p^1 \wedge \mathrm{d}x_p^2 \wedge \mathrm{d}x_p^3, \\ &\mathrm{d}x_p^1 \wedge \mathrm{d}x_p^2 \wedge \mathrm{d}x_p^4, \\ &\mathrm{d}x_p^1 \wedge \mathrm{d}x_p^3 \wedge \mathrm{d}x_p^4, \\ &\mathrm{d}x_p^2 \wedge \mathrm{d}x_p^3 \wedge \mathrm{d}x_p^4 \} \end{aligned} \tag{111}$$

With the wedge product as multiplication and the degree of a form as the grading, the direct sum $\Omega^*(U) = \bigoplus_{k=0}^n \Omega^k(U)$ becomes an anticommutative graded algebra over $\mathbb{R}$. Since one can multiply C^∞ k-forms by C^∞ functions, the set $\Omega^k(U)$ of C^∞ k-forms on U is both a vector space over $\mathbb{R}$ and a module over $C^\infty(U)$, then $\Omega^*(U) = \bigoplus_{k=0}^n \Omega^k(U)$ is also a module over $C^\infty(U)$.

4.3. Differential Forms as Multilinear Functions on Vector Fields

For $\omega \in \Omega^1(U)$ and $X \in \mathfrak{X}(U)$, we define a function $\omega(X)$ on U by

$$\omega(X)(p) = \omega_p(X_p), \tag{112}$$

or written in coordinates,

$$\omega = \omega_i \,\mathrm{d}x^i, \quad X = X^i \frac{\partial}{\partial x^i}, \text{for } \omega_i, X^i \in C^\infty(U), \tag{113}$$

So,

$$\begin{aligned} \omega(X) &= \omega_i \,\mathrm{d}x^i \left(X^j \frac{\partial}{\partial x^j} \right) \\ &= \omega_i X^j \frac{\partial x^i}{\partial x^j} \\ &= \omega_i X^j \delta^i_j \\ &= \omega_i X^i, \end{aligned} \tag{114}$$

which is C^∞ on U. Thus, a C^∞ 1-form on U gives eise to a map from $\mathfrak{X}(U)$ to $C^\infty(U)$.

Proposition 4.10 The map ω is linear over the ring $C^\infty(U)$: for $f \in C^\infty(U)$,

$$\omega(fX) = f\omega(X). \tag{115}$$

Proof. By definition,

$$\begin{aligned} (\omega(fX))_p &= \omega_p((fX)_p) \\ &= \omega_p(f(p)X_p) \\ &= f(p)\omega_p(X_p) \\ &= (f\omega(X))_p. \end{aligned} \tag{116}$$

□

Let $\mathscr{F}(U) = C^\infty(U)$, a 1-form ω on U gives rise to an $\mathscr{F}(U)$-linear map $\mathfrak{X}(U) \to C^\infty(U)$. Similarly, a k-form ω on U gives rise to a k-linear map over $\mathscr{F}(U)$,

$$\begin{aligned} \mathfrak{X}(U) \times \cdots \times \mathfrak{X}(U) &\to \mathscr{F}(U), \\ (X_1, \cdots, X_k) &\mapsto \omega(X_1, \cdots, X_k). \end{aligned} \tag{117}$$

Example 4.11 Let $\omega \in \Omega^2(\mathbb{R}^3)$ and $\tau \in \Omega^1(\mathbb{R}^3)$. If $X, Y, Z \in \mathfrak{X}(M)$, then

$$(\omega \wedge \tau)(X, Y, Z) = \omega(X, Y)\tau(Z) + \omega(Y, Z)\tau(X) - \omega(X, Z)\tau(Y) \tag{118}$$

4.4. The Exterior Derivative

Definition 4.12

(i) The **exterior derivative** of $f \in \Omega^0(U) = C^\infty(U)$ is the 1-form $\mathrm{d}f$ defined, by Proposition 4.5, by

$$\mathrm{d}f = \frac{\partial f}{\partial x^i}\,\mathrm{d}x^i. \tag{119}$$

(ii) For $k \geq 1$, if $\omega = \omega_I\,\mathrm{d}x^I \in \Omega^k(U)$, the **exterior derivative** of ω is the $(k+1)$-form $\mathrm{d}\omega$ defined by

$$\begin{aligned} \mathrm{d}\omega &= \mathrm{d}\omega_I\,\mathrm{d}x^I \\ &= \left(\frac{\partial \omega_I}{\partial x^i}\,\mathrm{d}x^i\right) \wedge \mathrm{d}x^I \in \Omega^{k+1}(U) \end{aligned} \tag{120}$$

Example 4.13 Let $\omega = f\,\mathrm{d}x + g\,\mathrm{d}y \in \mathbb{R}^2$, where $f, g \in C^\infty(\mathbb{R}^2)$. With simplified notation, $f_x = \frac{\partial f}{\partial x}$, then

$$\begin{aligned} \mathrm{d}\omega &= \mathrm{d}f \wedge \mathrm{d}x + \mathrm{d}g \wedge \mathrm{d}y \\ &= \left(f_x\,\mathrm{d}x + f_y\,\mathrm{d}y\right) \wedge \mathrm{d}x + \left(g_x\,\mathrm{d}x + g_y\,\mathrm{d}y\right) \wedge \mathrm{d}y \\ &= \left(g_x - f_y\right)\mathrm{d}x \wedge \mathrm{d}y \end{aligned} \tag{121}$$

Definition 4.14 Let $A = \bigoplus_{k=0}^\infty A^k$ be a graded algebra over a field K. An **antiderivation** of the graded algebra A is a K-linear map $D : A \to A$ such that for $a \in A^k$ and $b \in A^l$,

$$D(ab) = (Da)b + (-1)^k a Db. \tag{122}$$

If there is an integer m such that the antiderivation D sends A^k to A^{k+m} for all k, then we say that it is an antiderivation of **degree m**. By defining $A_k = 0$ for $k < 0$, the grading of the graded algebra A can be extended to negative integers, and the degree m of an antiderivation D can be negative. (An example of an antiderivation of degree -1 is interior multiplication.)

Proposition 4.15

(i) The **exterior differentiation** $\mathrm{d} : \Omega^*(U) \to \Omega^*(U)$ is an antiderivation of degree 1:

$$\mathrm{d}(\omega \wedge \tau) = (\mathrm{d}\omega) \wedge \tau + (-1)^{\deg \omega} \omega \wedge (\mathrm{d}\tau). \tag{123}$$

(ii) $\mathrm{d}^2 = 0$.

(iii) If $f \in C^\infty(U)$ and $X \in \mathfrak{X}(U)$, then

$$(\mathrm{d}f)(X) = Xf. \tag{124}$$

Proof.

(i) For $\omega = \omega_I \,\mathrm{d}x^I$ and $\tau = \tau_J \,\mathrm{d}x^J$, we have

$$\begin{aligned}
\mathrm{d}(\omega \wedge \tau) &= \mathrm{d}(\omega_I \tau_J \,\mathrm{d}x^I \wedge \mathrm{d}x^J) \\
&= \frac{\partial(\omega_I \tau_J)}{\partial x^i} \,\mathrm{d}x^i \wedge \mathrm{d}x^I \wedge \mathrm{d}x^J \\
&= \frac{\partial \omega_I}{\partial x^i} \tau_J \,\mathrm{d}x^i \wedge \mathrm{d}x^I \wedge \mathrm{d}x^J + \omega_I \frac{\partial \tau_J}{\partial x^i} \,\mathrm{d}x^i \wedge \mathrm{d}x^I \wedge \mathrm{d}x^J \\
&= \frac{\partial \omega_I}{\partial x^i} \,\mathrm{d}x^i \wedge \mathrm{d}x^I \wedge \tau_J \,\mathrm{d}x^J + (-1)^{\deg \omega} \omega_I \,\mathrm{d}x^I \wedge \frac{\partial \tau_J}{\partial x^i} \,\mathrm{d}x^i \wedge \mathrm{d}x^J \\
&= (\mathrm{d}\omega) \wedge \tau + (-1)^{\deg \omega} \omega \wedge (\mathrm{d}\tau).
\end{aligned} \tag{125}$$

(ii) For $\omega = \omega_I \,\mathrm{d}x^I$, we have

$$\begin{aligned}
\mathrm{d}^2\omega &= \mathrm{d}(\mathrm{d}\omega) \\
&= \mathrm{d}\left(\frac{\partial \omega_I}{\partial x^i} \,\mathrm{d}x^i \wedge \mathrm{d}x^I \right) \\
&= \frac{\partial^2 \omega_I}{\partial x^j \partial x^i} \,\mathrm{d}x^j \wedge \mathrm{d}x^i \wedge \mathrm{d}x^I,
\end{aligned} \tag{126}$$

where if $i = j$, then $\mathrm{d}x^j \wedge \mathrm{d}x^i = 0$; if $i \neq j$, then $\frac{\partial \omega_I}{\partial x^j \partial x^i} = \frac{\partial \omega_I}{\partial x^i \partial x^j}$, so

$$\begin{aligned}
\mathrm{d}^2\omega &= \frac{\partial^2 \omega_I}{\partial x^j \partial x^i} \,\mathrm{d}x^j \wedge \mathrm{d}x^i \wedge \mathrm{d}x^I \\
&= \frac{\partial^2 \omega_I}{\partial x^i \partial x^j} \,\mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^I \\
&= -\frac{\partial^2 \omega_I}{\partial x^j \partial x^i} \,\mathrm{d}x^j \wedge \mathrm{d}x^i \wedge \mathrm{d}x^I,
\end{aligned} \tag{127}$$

which means $\mathrm{d}^2\omega = 0$.

(iii) This is just the definition of Xf. □

Proposition 4.16 Proposition 4.15 uniquely characterizes exterior differentiation on an open set $U \subset \mathbb{R}^n$, i.e., if $D : \Omega^*(U) \to \Omega^*(U)$ satisfies Proposition 4.15, then $D = \mathrm{d}$.

Proof. From Proposition 4.15 (ii), $D \,\mathrm{d}x^i = DDx^i = 0$, then

$$D(\mathrm{d}x^I) = D(\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}) = 0. \tag{128}$$

Finally, for $\omega = f \,\mathrm{d}x^I$,

$$\begin{aligned}D(\omega) &= D(f\,\mathrm{d}x^I)\\ &= (Df)\wedge \mathrm{d}x^I + fD(\mathrm{d}x^I)\\ &= (\mathrm{d}f)\wedge \mathrm{d}x^I\\ &= \mathrm{d}(f\,\mathrm{d}x^I)\\ &= \mathrm{d}\omega,\end{aligned} \tag{129}$$

which means $D = \mathrm{d}$ on $\Omega^*(U)$. □

4.5. Closed Forms and Exact Forms

Definition 4.17 A k-form ω is said to be **closed** if $\mathrm{d}\omega = 0$, and **exact** if there exists a $(k-1)$-form τ such that $\omega = \mathrm{d}\tau$. Since $\mathrm{d}(\mathrm{d}\tau) = 0$, every exact form is closed.

Example 4.18 The 1-form $\omega = \frac{1}{x^2+y^2}(-y\,\mathrm{d}x + x\,\mathrm{d}y)$ on $\mathbb{R}^2 - \{(0,0)\}$ is closed:

$$\begin{aligned}\mathrm{d}\omega &= \frac{\partial}{\partial y}\left(-\frac{y}{x^2+y^2}\right)\mathrm{d}y\wedge\mathrm{d}x + \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right)\mathrm{d}x\wedge\mathrm{d}y\\ &= \left(\frac{x^2+y^2-2x^2}{(x^2+y^2)^2} + \frac{x^2+y^2-2y^2}{(x^2+y^2)^2}\right)\mathrm{d}x\wedge\mathrm{d}y\\ &= 0.\end{aligned} \tag{130}$$

Definition 4.19 A collection of vector spaces $\{V^k\}_{k=0}^{\infty}$ with linear maps $d_k : V^k \to V^{k+1}$ such that $d_{k+1}\circ d_k = 0$ is called a **differential complex** or a **cochain complex**. For any open set $U \subset \mathbb{R}^n$, the exterior derivative d makes $\Omega^*(U)$ into a differential complex, called the **de Rham complex** of U:

$$0 \to \Omega^0(U) \xrightarrow{\mathrm{d}} \Omega^1(U) \xrightarrow{\mathrm{d}} \Omega^2(U) \xrightarrow{\mathrm{d}} \cdots. \tag{131}$$

The closed forms are the elements of the kernel of d and the exact forms are the elements of the image of d.

4.6. Applications to Vector calculus

A 1-form with vector fields on U can be identified via

$$P\,\mathrm{d}x + Q\,\mathrm{d}y + R\,\mathrm{d}z \longleftrightarrow \begin{bmatrix}P\\Q\\R\end{bmatrix}. \tag{132}$$

A 2-form with vector fields on U can be identified via

$$P\,\mathrm{d}y\wedge\mathrm{d}z + Q\,\mathrm{d}z\wedge\mathrm{d}x + R\,\mathrm{d}x\wedge\mathrm{d}y \longleftrightarrow \begin{bmatrix}P\\Q\\R\end{bmatrix}. \tag{133}$$

A 3-form with vector fields on U can be identified via

$$f\,\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z \longleftrightarrow f. \tag{134}$$

In terms of these identifications, the exterior derivative of a 0-form is

$$\mathrm{d}f = \frac{\partial f}{\partial x}\,\mathrm{d}x + \frac{\partial f}{\partial y}\,\mathrm{d}y + \frac{\partial f}{\partial z}\,\mathrm{d}z \longleftrightarrow \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix} = \text{grad } f; \tag{135}$$

the exterior derivative of a 1-form is

$$\begin{aligned} &\mathrm{d}(P\,\mathrm{d}x + Q\,\mathrm{d}y + R\,\mathrm{d}z) \\ &\quad = \left(R_y - Q_z\right)\mathrm{d}y \wedge \mathrm{d}z + \left(P_z - R_x\right)\mathrm{d}z \wedge \mathrm{d}x + \left(Q_x - P_y\right)\mathrm{d}x \wedge \mathrm{d}y \\ &\quad \longleftrightarrow \begin{bmatrix} R_y - Q_z \\ P_z - R_x \\ Q_x - P_y \end{bmatrix} = \text{curl } \begin{bmatrix} P \\ Q \\ R \end{bmatrix}; \end{aligned} \tag{136}$$

the exterior derivative of a 2-form is

$$\begin{aligned} &\mathrm{d}(P\,\mathrm{d}y \wedge \mathrm{d}z + Q\,\mathrm{d}z \wedge \mathrm{d}x + R\,\mathrm{d}x \wedge \mathrm{d}y) \\ &\quad = \left(P_x + Q_y + R_z\right)\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \\ &\quad \longleftrightarrow P_x + Q_y + R_z = \text{div } \begin{bmatrix} P \\ Q \\ R \end{bmatrix}. \end{aligned} \tag{137}$$

In summary,

$$\begin{array}{ccccccc} \Omega^0(U) & \xrightarrow{\mathrm{d}} & \Omega^1(U) & \xrightarrow{\mathrm{d}} & \Omega^2(U) & \xrightarrow{\mathrm{d}} & \Omega^3(U) \\ \simeq\downarrow & & \simeq\downarrow & & \simeq\downarrow & & \simeq\downarrow \\ \mathcal{F}(U) & \xrightarrow[\text{grad}]{} & \mathfrak{X}(U) & \xrightarrow[\text{curl}]{} & \mathfrak{X}(U) & \xrightarrow[\text{div}]{} & \mathcal{F}(U). \end{array} \tag{138}$$

Proposition 4.20

(i) $\text{curl }(\text{grad } f) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

(ii) $\text{div }\left(\text{curl } \begin{bmatrix} P \\ Q \\ R \end{bmatrix}\right) = 0$.

(iii) On $\mathbb{R}^3$, a vector fielf F is the gradient of some scalar function f if and only if curl $F = 0$, i.e., a 1-form is exact if and only if it is closed on $\mathbb{R}^3$.

Whether Proposition 4.20 (iii) is true for a region U depends only on the topology of U.

Definition 4.21 The quatient vector space

$$H^k(U) = \frac{\{\text{closed } k\text{-forms on } U\}}{\{\text{exact } k\text{-forms on } U\}} \tag{139}$$

measures the failure of closed forms to be exact, and is called the ***k*-th de Rham cohomology** of U.

Lemma 4.22 (Poincaré lemma) For $k \geq 1$, every closed k-form on $\mathbb{R}^n$ is exact, leading to the vanishing of $H^k(\mathbb{R}^n)$.

4.7. Convention on Subscripts and Superscripts