Introduction to Quantum Mechanics

Problem 2.1

Prove the following three theorems:

(a) For normalizable solutions, the separation constant E must be real. *Hint: Write E (in Equation 2.7) as $E_0 + i\Gamma$ (with E_0 and Γ real), and show that if Equation 1.20 is to hold for all t, Γ must be zero.*

(b) The time-independent wave function $\psi(x)$ can always be taken to be real (unlike $\Psi(x,t)$, which is necessarily complex). *Hint: If $\psi(x)$ satisfies Equation 2.5, for a given E, so too does its complex conjugate, and hence also the real linear combinations $(\psi + \psi^*)$ and $i(\psi - \psi^*)$.*

(c) If $V(x)$ is an even function (that is, $V(-x) = V(x)$) then $\psi(x)$ can always be taken to be either even or odd. *Hint: If $\psi(x)$ satisfies Equation 2.5, for a given E, so too does $\psi(-x)$, and hence also the even and odd linear combinations $\psi(x) \pm \psi(-x)$.*

Solution

(a) Suppose $E = E_0 + i\Gamma$ for some real E_0 and Γ. Then the time-dependent wave function $\Psi(x,t)$ can be written as

$$\begin{aligned}\Psi(x,t) &= \psi(x)e^{-i\frac{Et}{\hbar}} \\ &= \psi(x)e^{-i\frac{(E_0+i\Gamma)t}{\hbar}} \\ &= \psi(x)e^{\frac{\Gamma t}{\hbar}}e^{-i\frac{E_0 t}{\hbar}}.\end{aligned}$$

Thus,

$$\begin{aligned}\int_{-\infty}^{\infty} |\Psi(x,t)|^2 \,\mathrm{d}x &= \int_{-\infty}^{\infty} |\psi(x)|^2 e^{\frac{2\Gamma t}{\hbar}} \,\mathrm{d}x \\ &= e^{\frac{2\Gamma t}{\hbar}} \int_{-\infty}^{\infty} |\psi(x)|^2 \,\mathrm{d}x,\end{aligned}$$

which varies with time, unless $\Gamma = 0$. Therefore, the separation constant E must be real.

(b) If $\psi(x)$ satisfies $\hat{H}\psi = E\psi$, then its complex conjugate $\psi^*(x)$ also satisfies $\hat{H}\psi^* = E\psi^*$.

If $\psi_1(x)$ and $\psi_2(x)$ are two solutions of $\hat{H}\psi = E\psi$, then any linear combination $\psi_3(x) = c_1\psi_1(x) + c_2\psi_2(x)$ is also a solution.

Thus for any complex solution $\psi(x)$, we can construct two real solutions $\psi_1(x) = \frac{1}{2}(\psi(x) + \psi^*(x))$ and $\psi_2(x) = \frac{1}{2i}(\psi(x) - \psi^*(x))$.

(c) If $\psi(x)$ satisfies $-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x)}{\mathrm{d}x^2} + V(x)\psi(x) = E\psi(x)$, then

$$
\begin{aligned}
-\frac{\hbar^2}{2m}\frac{\partial^2\psi(-x)}{\mathrm{d}(-x)^2}+V(-x)\psi(-x) &= -\frac{\hbar^2}{2m}\frac{\partial^2\psi(-x)}{\mathrm{d}x^2}+V(x)\psi(-x) \\
&= E\psi(-x),
\end{aligned}
$$

which means $\psi(-x)$ is also a solution. Thus we can construct two solutions $\psi_1(x) = \frac{1}{2}(\psi(x)+\psi(-x))$, which is even, and $\psi_2(x) = \frac{1}{2}(\psi(x)-\psi(-x))$, which is odd.

Problem 2.2

Show that E must exceed the minimum value of $V(x)$, for every normalizable solution to the time-independent Schrödinger equation. What is the classical analog to this statement? *Hint: Rewrite Equation 2.5 in the form*

$$
\frac{\partial^2\psi}{\mathrm{d}x^2} = \frac{2m}{\hbar^2}[V(x)-E]\psi.
$$

if $E < V_{\min}$, then ψ and its second derivative always have the same sign—argue that such a function cannot be normalized.

Solution

Rewrite time-independent Schrödinger equation as

$$
\frac{\partial^2\psi}{\mathrm{d}x^2} = \frac{2m}{\hbar^2}[V(x)-E]\psi.
$$

If $E < V_{\min}$, then $V(x)-E>0$ for all x. Thus ψ and its second derivative always have the same sign, which means ψ cannot be normalized.

In classical mechanics, this statement is analogous that if the total energy of a particle is less than the minimum potential energy, the particle's kinetic energy is negative, then the particle cannot exist in the system.

Problem 2.3

Show that there is no acceptable solution to the (time-independent) Schrödinger equation for the infinite square well with $E=0$ or $E<0$. (This is a special case of the general theorem in Problem 2.2, but this time do it by explicitly solving the Schrödinger equation, and showing that you cannot satisfy the boundary conditions.)

Solution

When $E=0$, the time-independent Schrödinger equation for the infinite square well becomes

$$
\frac{\partial^2\psi}{\mathrm{d}x^2} = -\frac{2mE}{\hbar^2}\psi = 0,
$$

which leads to $\psi(x)=0$, which is not normalizable.

When $E<0$, the time-independent Schrödinger equation for the infinite square well becomes

$$\frac{\partial^2 \psi}{\mathrm{d}x^2} = \kappa^2 \psi,$$

where $\kappa = \frac{\sqrt{-2mE}}{\hbar}$. The general solution to this equation is

$$\psi(x) = Ae^{\kappa x} + Be^{-\kappa x},$$

then the boundary conditions $\psi(0) = \psi(a) = 0$ lead to $A = B = 0$, which means $\psi(x) = 0$, which is not normalizable.

Problem 2.4

Calculate $\langle x \rangle$, $\langle x^2 \rangle$, $\langle p \rangle$, $\langle p^2 \rangle$, σ_x, and σ_p, for the nth stationary state of the infinite square well. Check that the uncertainty principle is satisfied. Which state comes closest to the uncertainty limit?

Solution

The expectation value of x is

$$\begin{aligned}
\langle x \rangle &= \int_0^a x \left|\psi_{n(x)}\right|^2 \mathrm{d}x \\
&= \int_0^a x \left(\sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right)\right)^2 \mathrm{d}x \\
&= \frac{2}{a} \int_0^a x \sin^2\left(\frac{n\pi x}{a}\right) \mathrm{d}x \\
&= \frac{1}{a} \left[\frac{1}{2}x^2 - \frac{a}{2n\pi} x \sin\left(\frac{2n\pi x}{a}\right) - \frac{a^2}{4n^2\pi^2} \cos\left(\frac{2n\pi x}{a}\right)\right]_0^a \\
&= \frac{a}{2}.
\end{aligned}$$

The expectation value of x^2 is

$$\begin{aligned}
\langle x^2 \rangle &= \int_0^a x^2 \left|\psi_{n(x)}\right|^2 \mathrm{d}x \\
&= \int_0^a x^2 \left(\sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right)\right)^2 \mathrm{d}x \\
&= \frac{2}{a} \int_0^a x^2 \sin^2\left(\frac{n\pi x}{a}\right) \mathrm{d}x \\
&= \frac{1}{a} \left[\frac{1}{3}x^3 - \frac{a}{2n\pi}x^2 \sin\left(\frac{2n\pi x}{a}\right) - \frac{a^2}{2n^2\pi^2} x \cos\left(\frac{2n\pi x}{a}\right) + \frac{a^3}{4n^3\pi^3} \sin\left(\frac{2n\pi x}{a}\right)\right]_0^a \\
&= \frac{1}{a}\left(\frac{a^3}{3} - \frac{a^3}{2n^2\pi^2}\right) \\
&= a^2\left(\frac{1}{3} - \frac{1}{2n^2\pi^2}\right).
\end{aligned}$$

The expectation value of p is

$$\begin{aligned}
\langle p \rangle &= m\frac{\mathrm{d}\langle x \rangle}{\mathrm{d}t} \\
&= 0
\end{aligned}$$

The expectation value of p^2 is

$$\begin{aligned}
\langle p^2 \rangle &= \int_0^a \psi_n^*(x) \left(\frac{\hbar}{i}\frac{\mathrm{d}}{\mathrm{d}x}\right)^2 \psi_n \,\mathrm{d}x \\
&= -\hbar^2 \int_0^a \psi_n^*(x) \frac{\mathrm{d}^2\psi_n}{\mathrm{d}x^2} \mathrm{d}x \\
&= -\hbar^2 \left(-\frac{2mE_n}{\hbar^2}\right) \int_0^a \left|\psi_{n(x)}\right|^2 \mathrm{d}x \\
&= 2mE_n \\
&= \frac{n^2\pi^2\hbar^2}{a^2}.
\end{aligned}$$

The standard deviation of x is

$$\begin{aligned}
\sigma_x &= \sqrt{\langle x^2 \rangle - \langle x \rangle^2} \\
&= a\sqrt{\frac{1}{12} - \frac{1}{2n^2\pi^2}}.
\end{aligned}$$

The standard deviation of p is

$$\begin{aligned}
\sigma_p &= \sqrt{\langle p^2 \rangle - \langle p \rangle^2} \\
&= \frac{n\pi\hbar}{a}.
\end{aligned}$$

The uncertainty principle is

$$\begin{aligned}
\sigma_x \sigma_p &= a\sqrt{\frac{1}{12} - \frac{1}{2n^2\pi^2}} \cdot n\pi\frac{\hbar}{a} \\
&= \frac{\hbar}{2}\sqrt{n^2\frac{\pi^2}{3} - 2} \\
&\geq \frac{\hbar}{2}\sqrt{\frac{\pi^2}{3} - 2} \\
&\geq \frac{\hbar}{2}.
\end{aligned}$$

Problem 2.5

A particle in the infinite square well has as its initial wave function an even mixture of the first two stationary states:

$$\Psi(x, 0) = A[\psi_1(x) + \psi_2(x)].$$

(a) Normalize $\Psi(x, 0)$. (That is, find A. This is very easy, if you exploit the orthonormality of ψ_1 and ψ_2. Recall that, having normalized Ψ at $t = 0$, you can rest assured that it stays normalized—if you doubt this, check it explicitly after doing part (b).)

(b) Find $\Psi(x, t)$ and $|\Psi(x, t)|^2$. Express the latter as a sinusoidal function of time, as in Example 2.1. To simplify the result, let $\omega = \pi^2 \frac{\hbar}{2ma^2}$.

(c) Compute $\langle x \rangle$. Notice that it oscillates in time. What is the angular frequency of the oscillation? What is the amplitude of the oscillation? (If your amplitude is greater than $\frac{a}{2}$, go directly to jail.)

(d) Compute $\langle p \rangle$. (As Peter Lorre would say, “Do it ze kveek vay, Johnny!”)

(e) If you measured the energy of this particle, what values might you get, and what is the probability of getting each of them? Find the expectation value of H. How does it compare with E_1 and E_2?

Solution

(a)

$$\begin{aligned}
1 &= \int_0^a |\Psi(x, 0)|^2 \, \mathrm{d}x \\
&= A^2 \int_0^a [\psi_1(x) + \psi_2(x)]^* [\psi_1(x) + \psi_2(x)] \, \mathrm{d}x \\
&= A^2 \int_0^a \left[|\psi_1(x)|^2 + |\psi_2(x)|^2 + \psi_1^*(x)\psi_2(x) + \psi_2^*(x)\psi_1(x) \right] \mathrm{d}x \\
&= 2A^2,
\end{aligned}$$

so $A = \frac{1}{\sqrt{2}}$.

(b)
$$
\begin{aligned}
\Psi(x,t) &= \frac{1}{\sqrt{2}}\left[\psi_1(x)e^{-i\frac{E_1 t}{\hbar}} + \psi_2(x)e^{-i\frac{E_2 t}{\hbar}}\right] \\
&= \frac{1}{\sqrt{2}}[\psi_1(x)e^{-i\omega t} + \psi_2(x)e^{-4i\omega t}] \\
&= \frac{1}{\sqrt{2}}\left[\sqrt{\frac{2}{a}}\sin\left(\frac{\pi x}{a}\right)e^{-i\omega t} + \sqrt{\frac{2}{a}}\sin\left(\frac{2\pi x}{a}\right)e^{-4i\omega t}\right] \\
&= \frac{1}{\sqrt{a}}e^{-i\omega t}\left[\sin\left(\frac{\pi x}{a}\right) + \sin\left(\frac{2\pi x}{a}\right)e^{-3i\omega t}\right].
\end{aligned}
$$

$$
\begin{aligned}
|\Psi(x,t)|^2 &= \frac{1}{a}\left[\sin\left(\frac{\pi x}{a}\right) + \sin\left(\frac{2\pi x}{a}\right)e^{-3i\omega t}\right]\left[\sin\left(\frac{\pi x}{a}\right) + \sin\left(\frac{2\pi x}{a}\right)e^{3i\omega t}\right] \\
&= \frac{1}{a}\left[\sin^2\left(\frac{\pi x}{a}\right) + \sin^2\left(\frac{2\pi x}{a}\right) + 2\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi x}{a}\right)\cos(3\omega t)\right].
\end{aligned}
$$

(c)
$$
\begin{aligned}
\langle x\rangle &= \int_0^a x|\Psi(x,t)|^2\,\mathrm{d}x \\
&= \frac{1}{a}\int_0^a x\left[\sin^2\left(\frac{\pi x}{a}\right) + \sin^2\left(\frac{2\pi x}{a}\right) + 2\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi x}{a}\right)\cos(3\omega t)\right]\mathrm{d}x \\
&= \frac{1}{a}\int_0^a x\left[\sin^2\left(\frac{\pi x}{a}\right) + \sin^2\left(\frac{2\pi x}{a}\right)\right]\mathrm{d}x + \frac{2}{a}\cos(3\omega t)\int_0^a x\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi x}{a}\right)\mathrm{d}x \\
&= \frac{1}{a}\left[\frac{a^2}{4} + \frac{a^2}{4}\right] + \frac{1}{a}\cos(3\omega t)\int_0^a x\left[\cos\left(\frac{\pi x}{a}\right) - \cos\left(\frac{3\pi x}{a}\right)\right]\mathrm{d}x \\
&= \frac{a}{2} + \frac{1}{a}\cos(3\omega t)\left[\frac{a}{\pi}x\sin\left(\frac{\pi x}{a}\right) + \frac{a^2}{\pi^2}\cos\left(\frac{\pi x}{a}\right) - \frac{a}{3\pi}x\sin\left(\frac{3\pi x}{a}\right) - \frac{a^2}{9\pi^2}\cos\left(\frac{3\pi x}{a}\right)\right]_0^a \\
&= \frac{a}{2} + \frac{1}{a}\cos(3\omega t)\left[-\frac{a^2}{\pi^2} - \frac{a^2}{\pi^2} + \frac{a^2}{9\pi^2} + \frac{a^2}{9\pi^2}\right] \\
&= \frac{a}{2} - \frac{16}{9\pi^2}a\cos(3\omega t) \\
&= \frac{a}{2}\left[1 - \frac{32}{9\pi^2}\cos(3\omega t)\right],
\end{aligned}
$$

where the angular frequency of the oscillation is $3\omega = \frac{3\pi^2\hbar}{2ma^2}$ and the amplitude of the oscillation is $\frac{16a}{9\pi^2} \approx 0.18a$.

(d)

$$\begin{aligned}
\langle p\rangle &= m\frac{\mathrm{d}\langle x\rangle}{\mathrm{d}t} \\
&= m\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{a}{2}\left(1-\frac{32}{9\pi^2}\cos(3\omega t)\right)\right] \\
&= \frac{16ma}{9\pi^2}3\omega\sin(3\omega t) \\
&= \frac{8\hbar}{3a}\sin(3\omega t).
\end{aligned}$$

(e) The possible values of energy are $E_1 = \frac{\pi^2\hbar^2}{2ma^2}$ and $E_2 = \frac{2\pi^2\hbar^2}{ma^2}$, with probability $\frac{1}{2}$ for each. The expectation value of H is

$$\begin{aligned}
\langle H\rangle &= \frac{1}{2}E_1 + \frac{1}{2}E_2 \\
&= \frac{1}{2}\left(\frac{\pi^2\hbar^2}{2ma^2}\right) + \frac{1}{2}\left(\frac{2\pi^2\hbar^2}{ma^2}\right) \\
&= \frac{5\pi^2\hbar^2}{4ma^2}.
\end{aligned}$$

Problem 2.6

Although the overall phase constant of the wave function is of no physical significance (it cancels out whenever you calculate a measurable quantity), the relative phase of the coefficients in Equation 2.17 does matter. For example, suppose we change the relative phase of ψ_1 and ψ_2 in Problem 2.5:

$$\Psi(x,0) = A[\psi_1(x) + e^{i\phi}\psi_2(x)],$$

where ϕ is some constant. Find $\Psi(x,t)$, $|\Psi(x,t)|^2$, and $\langle x\rangle$, and compare your results with what you got before. Study the special cases $\phi = \frac{\pi}{2}$ and $\phi = \pi$. (For a graphi.altcal exploration of this problem see the applet in footnote 9 of this chapter.)

Solution

$$\begin{aligned}
1 &= \int_0^a |\Psi(x,0)|^2\,\mathrm{d}x \\
&= A^2\int_0^a \left[|\psi_1(x)|^2 + |\psi_2(x)|^2 + e^{i\phi}\psi_1^*(x)\psi_2(x) + e^{-i\phi}\psi_2^*(x)\psi_1(x)\right]\mathrm{d}x \\
&= 2A^2,
\end{aligned}$$

so $A = \frac{1}{\sqrt{2}}$.

$$\begin{aligned}
\Psi(x,t) &= \frac{1}{\sqrt{2}}\left[\psi_1(x)e^{-i\frac{E_1 t}{\hbar}} + e^{i\phi}\psi_2(x)e^{-i\frac{E_2 t}{\hbar}}\right] \\
&= \frac{1}{\sqrt{2}}[\psi_1(x)e^{-i\omega t} + e^{i\phi}\psi_2(x)e^{-4i\omega t}] \\
&= \frac{1}{\sqrt{2}}\left[\sqrt{\frac{2}{a}}\sin\left(\frac{\pi x}{a}\right)e^{-i\omega t} + e^{i\phi}\sqrt{\frac{2}{a}}\sin\left(\frac{2\pi x}{a}\right)e^{-4i\omega t}\right] \\
&= \frac{1}{\sqrt{a}}e^{-i\omega t}\left[\sin\left(\frac{\pi x}{a}\right) + \sin\left(\frac{2\pi x}{a}\right)e^{i\phi}e^{-3i\omega t}\right].
\end{aligned}$$

$$\begin{aligned}
|\Psi(x,t)|^2 &= \frac{1}{a}\left[\sin\left(\pi\frac{x}{a}\right) + \sin\left(2\pi\frac{x}{a}\right)e^{-3i\omega t}\right]\left[\sin\left(\pi\frac{x}{a}\right) + \sin\left(2\pi\frac{x}{a}\right)e^{3i\omega t}\right] \\
&= \frac{1}{a}\left[\sin^2\left(\pi\frac{x}{a}\right) + \sin^2\left(2\pi\frac{x}{a}\right) + \sin\left(\pi\frac{x}{a}\right)\sin\left(2\pi\frac{x}{a}\right)\left(e^{i\phi}e^{-3i\omega t} + e^{-i\phi}e^{3i\omega t}\right)\right] \\
&= \frac{1}{a}\left[\sin^2\left(\pi\frac{x}{a}\right) + \sin^2\left(2\pi\frac{x}{a}\right) + 2\sin\left(\pi\frac{x}{a}\right)\sin\left(2\pi\frac{x}{a}\right)\cos(3\omega t - \phi)\right].
\end{aligned}$$

Then $\langle x\rangle = \frac{a}{2}\left[1 - \frac{32}{9\pi^2}\cos(3\omega t - \phi)\right]$.
When $\phi = \frac{\pi}{2}$, $\langle x\rangle = \frac{a}{2}\left[1 + \frac{32}{9\pi^2}\sin(3\omega t)\right]$,
When $\phi = \pi$, $\langle x\rangle = \frac{a}{2}\left[1 + \frac{32}{9\pi^2}\cos(3\omega t)\right]$.

Problem 2.7

A particle in the infinite square well has the initial wave function

$$\Psi(x,0) = \begin{cases} Ax, & 0 \le x \le \frac{a}{2} \\ A(a-x), & \frac{a}{2} \le x \le a. \end{cases}$$

(a) Sketch $\Psi(x,0)$, and determine the constant A.
(b) Find $\Psi(x,t)$.
(c) What is the probability that a measurement of the energy would yield the value E_1?
(d) Find the expectation value of the energy, using Equation 2.21.

Solution

(a)

$$\begin{aligned}
1 &= \int_0^a |\Psi(x,0)|^2 \, \mathrm{d}x \\
&= A^2 \left[\int_0^{\frac{a}{2}} x^2 \, \mathrm{d}x + \int_{\frac{a}{2}}^a (a-x)^2 \, \mathrm{d}x \right] \\
&= A^2 \left[\frac{x^3}{3}\bigg|_0^{\frac{a}{2}} - \frac{(a-x)^3}{3}\bigg|_{\frac{a}{2}}^a \right] \\
&= A^2 \left[\frac{a^3}{24} + \frac{a^3}{24} \right] \\
&= \frac{A^2 a^3}{12},
\end{aligned}$$

so $A = \frac{2\sqrt{3}}{a\sqrt{a}}$.

(b)

$$\begin{aligned}
c_n &= \int_0^a \Psi(x,0)\psi_n^*(x) \, \mathrm{d}x \\
&= A\sqrt{\frac{2}{a}} \left[\int_0^{\frac{a}{2}} x \sin\left(\frac{n\pi x}{a}\right) \mathrm{d}x + \int_{\frac{a}{2}}^a (a-x) \sin\left(\frac{n\pi x}{a}\right) \mathrm{d}x \right] \\
&= A\sqrt{\frac{2}{a}} \frac{a}{n\pi} \left[-x\cos\left(\frac{n\pi x}{a}\right)\bigg|_0^{\frac{a}{2}} + \int_0^{\frac{a}{2}} \cos\left(\frac{n\pi x}{a}\right) \mathrm{d}x \text{-} (a-x)\cos\left(\frac{n\pi x}{a}\right)\bigg|_{\frac{a}{2}}^a - \int_{\frac{a}{2}}^a \cos\left(\frac{n\pi x}{a}\right) \mathrm{d}x \right] \\
&= A\sqrt{\frac{2}{a}} \frac{a}{n\pi} \left[\frac{a}{n\pi}\sin\left(\frac{n\pi x}{a}\right)\bigg|_0^{\frac{a}{2}} - \frac{a}{n\pi}\sin\left(\frac{n\pi x}{a}\right)\bigg|_{\frac{a}{2}}^a \right] \\
&= A\sqrt{\frac{2}{a}} \frac{a}{n\pi} \left[\frac{a}{n\pi} \sin\left(\frac{n\pi}{2}\right) + \frac{a}{n\pi} \sin\left(\frac{n\pi}{2}\right) \right] \\
&= \frac{4\sqrt{6}}{n^2\pi^2} \sin\left(n\frac{\pi}{2}\right) \\
&= \begin{cases} \frac{4\sqrt{6}}{n^2\pi^2}(-1)^{\frac{n-1}{2}} & \text{odd } n \\ 0 & \text{even } n \end{cases}
\end{aligned}$$

So

$$\begin{aligned}
\Psi(x,t) &= \sum_{n=1,3,5,\ldots} c_n \psi_{n(x)} e^{-i\frac{E_n t}{\hbar}} \\
&= \sum_{n=1,3,5,\ldots} \frac{4\sqrt{6}}{n^2\pi^2}(-1)^{\frac{n-1}{2}} \sqrt{\frac{2}{a}} \sin\left(n\pi\frac{x}{a}\right) e^{-i\frac{E_n t}{\hbar}} \\
&= \frac{4\sqrt{6}}{\pi^2}\sqrt{\frac{2}{a}} \sum_{n=1,3,5,\ldots} \frac{(-1)^{\frac{n-1}{2}}}{n^2} \sin\left(n\pi\frac{x}{a}\right) e^{-i\frac{E_n t}{\hbar}},
\end{aligned}$$

where $E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$.

(c) The probability that a measurement of the energy would yield the value E_1 is

$$\begin{aligned} P(E_1) &= |c_1|^2 \\ &= \left(\frac{4\sqrt{6}}{\pi^2}\right)^2 \\ &= \frac{96}{\pi^4} \approx 0.9855 \end{aligned}$$

(d) The expectation value of the energy is

$$\begin{aligned} \langle H\rangle &= \sum_{n=1,3,5,\ldots} |c_n|^2 E_n \\ &= \sum_{n=1,3,5,\ldots} \left(\frac{4\sqrt{6}}{n^2\pi^2}\right)^2 \frac{n^2\pi^2\hbar^2}{2ma^2} \\ &= \sum_{n=1,3,5,\ldots} \frac{48\hbar^2}{n^2\pi^2 ma^2} \\ &= \frac{48\hbar^2}{\pi^2 ma^2} \sum_{n=1,3,5,\ldots} \frac{1}{n^2} \\ &= \frac{48\hbar^2}{\pi^2 ma^2}\frac{\pi^2}{8} \\ &= \frac{6\hbar^2}{ma^2}. \end{aligned}$$

Problem 2.8

A particle of mass m in the infinite square well (of width a) starts out in the state

$$\Psi(x,0) = \begin{cases} A & 0 \le x \le \frac{a}{2} \\ 0 & \frac{a}{2} < x \le a \end{cases}$$

for some constant A, so it is (at $t = 0$) equally likely to be found at any point in the left half of the well. What is the probability that a measurement of the energy (at some later time t) would yield the value $\frac{\pi^2\hbar^2}{2ma^2}$?

Solution

$$\begin{aligned} 1 &= \int_0^a |\Psi(x,0)|^2 \,\mathrm{d}x \\ &= A^2\left[\int_0^{\frac{a}{2}} \mathrm{d}x\right] \\ &= \frac{A^2 a}{2}, \end{aligned}$$

so $A = \sqrt{\frac{2}{a}}$.

$$\begin{aligned}
c_1 &= \int_0^a \Psi(x,0)\psi_1^*(x)\,\mathrm{d}x \\
&= A\sqrt{\frac{2}{a}}\int_0^{\frac{a}{2}} \sin\left(\frac{\pi x}{a}\right)\mathrm{d}x \\
&= A\sqrt{\frac{2}{a}}\left(\frac{a}{\pi}\right)\left[-\cos\left(\frac{\pi x}{a}\right)\Big|_0^{\frac{a}{2}}\right] \\
&= \frac{2}{\pi}.
\end{aligned}$$

The probability that a measurement of the energy would yield the value $\pi^2\frac{\hbar^2}{2ma^2} = E_1$ is

$$\begin{aligned}
P(E_1) &= |c_1|^2 \\
&= \left(\frac{2}{\pi}\right)^2 \\
&= \frac{4}{\pi^2} \approx 0.4053.
\end{aligned}$$

Problem 2.9

For the wave function in Example 2.2, find the expectation value of H, at time $t = 0$, the "old fashioned" way:

$$\langle H\rangle = \int \Psi(x,0)^*\hat{H}\Psi(x,0)\,\mathrm{d}x.$$

Compare the result we got in Example 2.3. Note: Because $\langle H\rangle$ is independent of time, there is no loss of generality in using $t = 0$.

Solution

$$\begin{aligned}
\hat{H}\Psi(x,0) &= \frac{-\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\Psi(x,0) \\
&= \frac{-\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2}[Ax(a-x)] \\
&= \frac{-\hbar^2}{2m}(-2A) \\
&= \frac{\hbar^2 A}{m}.
\end{aligned}$$

$$
\begin{aligned}
\langle H\rangle &= \int \Psi(x,0)^* \hat{H} \Psi(x,0)\,\mathrm{d}x \\
&= \frac{\hbar^2 A^2}{m} \int_0^a x(a-x)\,\mathrm{d}x \\
&= \frac{\hbar^2 A^2}{m} \left[a\frac{x^2}{2} - \frac{x^3}{3} \bigg|_0^a \right] \\
&= \frac{\hbar^2 A^2 a^3}{6m} \\
&= \frac{5\hbar^2}{ma^2}.
\end{aligned}
$$

Problem 2.10

(a) Construct $\psi_2(x)$.
(b) Sketch ψ_0, ψ_1, and ψ_2.
(c) Check the orthogonality of ψ_0, ψ_1, and ψ_2, by explicit integration. *Hint: If you exploit the even-ness and odd-ness of the functions, there is really only one integral left to do.*

Solution

(a)

$$
\psi_0 = \left(m\frac{\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega}{2\hbar}x^2},
$$

$$
\begin{aligned}
\hat{a}_+\psi_0 &= \frac{1}{\sqrt{2\hbar m\omega}}(-i\hat{p} + m\omega\hat{x})\psi_0 \\
&= \frac{1}{\sqrt{2\hbar m\omega}}\left(m\frac{\omega}{\pi\hbar}\right)^{\frac{1}{4}}\left(-\hbar\frac{\mathrm{d}}{\mathrm{d}x} + m\omega x\right)e^{-\frac{m\omega}{2\hbar}x^2} \\
&= \frac{1}{\sqrt{2\hbar m\omega}}\left(m\frac{\omega}{\pi\hbar}\right)^{\frac{1}{4}}\left[-\hbar\left(-\frac{m\omega}{\hbar}x\right) + m\omega x\right]e^{-\frac{m\omega}{2\hbar}x^2} \\
&= \frac{1}{\sqrt{2\hbar m\omega}}\left(m\frac{\omega}{\pi\hbar}\right)^{\frac{1}{4}} 2m\omega x e^{-\frac{m\omega}{2\hbar}x^2}.
\end{aligned}
$$

$$
\begin{aligned}
\left(\hat{a}_+\right)^2\psi_0 &= \frac{1}{\sqrt{2\hbar m\omega}}(-i\hat{p} + m\omega\hat{x})\left(\hat{a}_+\psi_0\right) \\
&= \frac{1}{2\hbar m\omega}\left(m\frac{\omega}{\pi\hbar}\right)^{\frac{1}{4}} 2m\omega\left(-\hbar\frac{\mathrm{d}}{\mathrm{d}x} + m\omega x\right)xe^{-\frac{m\omega}{2\hbar}x^2} \\
&= \frac{1}{2\hbar m\omega}\left(m\frac{\omega}{\pi\hbar}\right)^{\frac{1}{4}} 2m\omega\left[-\hbar\left(1 - \frac{m\omega}{\hbar}x^2\right) + m\omega x^2\right]e^{-\frac{m\omega}{2\hbar}x^2} \\
&= \left(m\frac{\omega}{\pi\hbar}\right)^{\frac{1}{4}}\left[\frac{2m\omega}{\hbar}x^2 - 1\right]e^{-\frac{m\omega}{2\hbar}x^2}.
\end{aligned}
$$

Therefore,

$$\begin{aligned}\psi_2 &= \frac{1}{\sqrt{2}}(\hat{a}_+)^2\psi_0\\ &= \frac{1}{\sqrt{2}}\left(m\frac{\omega}{\pi\hbar}\right)^{\frac{1}{4}}\left[\frac{2m\omega}{\hbar}x^2-1\right]e^{-\frac{m\omega}{2\hbar}x^2}.\end{aligned}$$

(b)

(c) As ψ_0 and ψ_2 are even and ψ_1 is odd, the only integral left to do is

$$\begin{aligned}\int_{-\infty}^{\infty}\psi_0^*\psi_2\,\mathrm{d}x &= \int_{-\infty}^{\infty}\left(m\frac{\omega}{\pi\hbar}\right)^{\frac{1}{4}}e^{-\frac{m\omega}{2\hbar}x^2}\frac{1}{\sqrt{2}}\left(m\frac{\omega}{\pi\hbar}\right)^{\frac{1}{4}}\left[\frac{2m\omega}{\hbar}x^2-1\right]e^{-\frac{m\omega}{2\hbar}x^2}\,\mathrm{d}x\\ &= \frac{1}{\sqrt{2}}\left(m\frac{\omega}{\pi\hbar}\right)^{\frac{1}{2}}\int_{-\infty}^{\infty}\left[\frac{2m\omega}{\hbar}x^2-1\right]e^{-\frac{m\omega}{\hbar}x^2}\,\mathrm{d}x\\ &= \frac{1}{\sqrt{2}}\left(m\frac{\omega}{\pi\hbar}\right)^{\frac{1}{2}}\left[\frac{2m\omega}{\hbar}\int_{-\infty}^{\infty}x^2e^{-\frac{m\omega}{\hbar}x^2}\,\mathrm{d}x-\int_{-\infty}^{\infty}e^{-\frac{m\omega}{\hbar}x^2}\,\mathrm{d}x\right]\\ &= \frac{1}{\sqrt{2}}\left(m\frac{\omega}{\pi\hbar}\right)^{\frac{1}{2}}\left[\frac{2m\omega}{\hbar}\frac{\hbar}{2m\omega}\sqrt{\frac{\pi\hbar}{m\omega}}-\sqrt{\frac{\pi\hbar}{m\omega}}\right]\\ &= 0.\end{aligned}$$

Problem 2.11

(a) Compute $\langle x\rangle$, $\langle p\rangle$, $\langle x^2\rangle$, and $\langle p^2\rangle$, for the states ψ_0 (Equation 2.60) and ψ_1 (Equation 2.63), by explicit integration.

(b) Check the uncertainty principle for these states.

(c) Compute $\langle T\rangle$ and $\langle V\rangle$ for these states. *(No new integration allowed!) Is their sum what you would expect?*

Solution

(a) For $\psi_0 = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}e^{-\frac{m\omega}{2\hbar}x^2}$, which is even, we have

$$\begin{aligned}\langle x\rangle &= \int_{-\infty}^{\infty}x|\psi_0(x)|^2\,\mathrm{d}x\\ &= 0,\end{aligned}$$

$$\begin{aligned}\langle p\rangle &= m\frac{\mathrm{d}}{\mathrm{d}t}\langle x\rangle\\ &= 0,\end{aligned}$$

$$\begin{aligned}
\langle x^2\rangle &= \int_{-\infty}^{\infty} x^2 |\psi_0(x)|^2 \,\mathrm{d}x \\
&= \int_{-\infty}^{\infty} x^2 \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{2}} e^{-\frac{m\omega}{\hbar}x^2} \,\mathrm{d}x \\
&= \left(m\frac{\omega}{\pi\hbar}\right)^{\frac{1}{2}} \int_{-\infty}^{\infty} x^2 e^{-\frac{m\omega}{\hbar}x^2} \,\mathrm{d}x \\
&= \left(m\frac{\omega}{\pi\hbar}\right)^{\frac{1}{2}} \left[\frac{\hbar}{2m\omega}\sqrt{\frac{\pi\hbar}{m\omega}}\right] \\
&= \frac{\hbar}{2m\omega},
\end{aligned}$$

$$\begin{aligned}
\langle p^2\rangle &= \int_{-\infty}^{\infty} \psi_0^* \left(\frac{\hbar}{i}\frac{\mathrm{d}}{\mathrm{d}x}\right)^2 \psi_0 \,\mathrm{d}x \\
&= -\hbar^2 \int_{-\infty}^{\infty} \psi_0^* \frac{\mathrm{d}^2\psi_0}{\mathrm{d}x^2} \,\mathrm{d}x \\
&= -\hbar^2 \sqrt{\frac{m\omega}{\pi\hbar}} \int_{-\infty}^{\infty} e^{-\frac{m\omega}{2\hbar}x^2} \frac{\mathrm{d}}{\mathrm{d}x}\left[-\frac{m\omega}{\hbar} x e^{-\frac{m\omega}{2\hbar}x^2}\right] \mathrm{d}x \\
&= \hbar m\omega \sqrt{\frac{m\omega}{\pi\hbar}} \int_{-\infty}^{\infty} \left(1 - \frac{m\omega}{\hbar}x^2\right) e^{-\frac{m\omega}{\hbar}x^2} \,\mathrm{d}x \\
&= \hbar m\omega \sqrt{\frac{m\omega}{\pi\hbar}} \left(\sqrt{\frac{\pi\hbar}{m\omega}} - \frac{m\omega}{\hbar}\frac{\hbar}{2m\omega}\sqrt{\frac{\pi\hbar}{m\omega}}\right) \\
&= \frac{\hbar m\omega}{2}.
\end{aligned}$$

For $\psi_1 = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \sqrt{\frac{2m\omega}{\hbar}} x e^{-\frac{m\omega}{2\hbar}x^2}$, which is odd, we have

$$\begin{aligned}
\langle x\rangle &= \int_{-\infty}^{\infty} x |\psi_1(x)|^2 \,\mathrm{d}x \\
&= 0,
\end{aligned}$$

$$\begin{aligned}
\langle p\rangle &= m\frac{\mathrm{d}}{\mathrm{d}t}\langle x\rangle \\
&= 0,
\end{aligned}$$

$$\begin{aligned}
\langle x^2\rangle &= \int_{-\infty}^{\infty} x^2 |\psi_1(x)|^2 \,\mathrm{d}x \\
&= \sqrt{\frac{m\omega}{\pi\hbar}}\frac{2m\omega}{\hbar}\int_{-\infty}^{\infty} x^4 e^{-\frac{m\omega}{\hbar}x^2}\,\mathrm{d}x \\
&= \sqrt{\frac{m\omega}{\pi\hbar}}\frac{2m\omega}{\hbar}\left[\frac{3\hbar^2}{4m^2\omega^2}\sqrt{\frac{\pi\hbar}{m\omega}}\right] \\
&= \frac{3\hbar}{2m\omega},
\end{aligned}$$

Problem 2.12

Find $\langle x\rangle$, $\langle p\rangle$, $\langle x^2\rangle$, $\langle p^2\rangle$, and $\langle T\rangle$, for the nth stationary state of the harmonic oscillator, using the method of Example 2.5. Check that the uncertainty principle is satisfied.

Solution

The expectation value of x is

$$\begin{aligned}
\langle x\rangle &= \sqrt{\frac{\hbar}{2m\omega}}\int_{-\infty}^{\infty} \psi_n^*(\hat{a}_+ + \hat{a}_-)\psi_n \,\mathrm{d}x \\
&= \sqrt{\frac{\hbar}{2m\omega}}\left[\sqrt{n+1}\int_{-\infty}^{\infty}\psi_n^*\psi_{n+1} + \sqrt{n}\int_{-\infty}^{\infty}\psi_n^*\psi_{n-1}\,\mathrm{d}x\right] \\
&= 0.
\end{aligned}$$

Thus the expectation value of p is

$$\begin{aligned}
\langle p\rangle &= \frac{\mathrm{d}}{\mathrm{d}t}\langle x\rangle \\
&= 0.
\end{aligned}$$

The expectation value of x^2 is

$$\begin{aligned}
\langle x^2\rangle &= \frac{\hbar}{2m\omega}\int_{-\infty}^{\infty}\psi_n^*(\hat{a}_+ + \hat{a}_-)^2\psi_n\,\mathrm{d}x \\
&= \frac{\hbar}{2m\omega}\int_{-\infty}^{\infty}\psi_n^*(\hat{a}_+^2 + \hat{a}_-^2 + \hat{a}_+\hat{a}_- + \hat{a}_-\hat{a}_+)\psi_n\,\mathrm{d}x \\
&= \frac{\hbar}{2m\omega}\left[\sqrt{n+1}\int_{-\infty}^{\infty}\psi_n^*\hat{a}_+\psi_{n+1} + \sqrt{n}\int_{-\infty}^{\infty}\psi_n^*\hat{a}_-\psi_{n-1}\,\mathrm{d}x + (n+n+1)\int_{-\infty}^{\infty}\psi_n^*\psi_n\,\mathrm{d}x\right] \\
&= \frac{\hbar}{2m\omega}\left[\sqrt{n+1}\sqrt{n+2}\int_{-\infty}^{\infty}\psi_n^*\psi_{n+2} + n\sqrt{n-1}\int_{-\infty}^{\infty}\psi_n^*\psi_{n-2}\,\mathrm{d}x + 2n+1\right] \\
&= \left(n+\frac{1}{2}\right)\frac{\hbar}{m\omega}.
\end{aligned}$$

The expectation value of p^2 is

$$\begin{aligned}
\langle p^2 \rangle &= -\hbar m \frac{\omega}{2} \int_{-\infty}^{\infty} \psi_n^* (\hat{a}_+ - \hat{a}_-)^2 \psi_n \, \mathrm{d}x \\
&= -\hbar m \frac{\omega}{2} \int_{-\infty}^{\infty} \psi_n^* (\hat{a}_+^2 + \hat{a}_-^2 - \hat{a}_+\hat{a}_- - \hat{a}_-\hat{a}_+) \psi_n \, \mathrm{d}x \\
&= -\hbar m \frac{\omega}{2} \left[\sqrt{n+1} \int_{-\infty}^{\infty} \psi_n^* \hat{a}_+ \psi_{n+1} + \sqrt{n} \int_{-\infty}^{\infty} \psi_n^* \hat{a}_- \psi_{n-1} \, \mathrm{d}x - (n+n+1) \int_{-\infty}^{\infty} \psi_n^* \psi_n \, \mathrm{d}x \right] \\
&= -\hbar m \frac{\omega}{2} \left[\sqrt{n+1}\sqrt{n+2} \int_{-\infty}^{\infty} \psi_n^* \psi_{n+2} + n\sqrt{n-1} \int_{-\infty}^{\infty} \psi_n^* \psi_{n-2} \, \mathrm{d}x - (2n+1) \right] \\
&= \left(n + \frac{1}{2}\right) \hbar m \omega.
\end{aligned}$$

The expectation value of T is

$$\begin{aligned}
\langle T \rangle &= \frac{1}{2m} \langle p^2 \rangle \\
&= \frac{1}{2} \left(n + \frac{1}{2}\right) \hbar \omega.
\end{aligned}$$

The standard deviation of x is

$$\begin{aligned}
\sigma_x &= \sqrt{\langle x^2 \rangle - \langle x \rangle^2} \\
&= \sqrt{\left(n + \frac{1}{2}\right) \frac{\hbar}{m\omega}}.
\end{aligned}$$

The standard deviation of p is

$$\begin{aligned}
\sigma_p &= \sqrt{\langle p^2 \rangle - \langle p \rangle^2} \\
&= \sqrt{\left(n + \frac{1}{2}\right) \hbar m \omega}.
\end{aligned}$$

The uncertainty principle is

$$\begin{aligned}
\sigma_x \sigma_p &= \sqrt{\left(n + \frac{1}{2}\right) \frac{\hbar}{m\omega}} \sqrt{\left(n + \frac{1}{2}\right) \hbar m \omega} \\
&= \left(n + \frac{1}{2}\right) \hbar \geq \frac{\hbar}{2}.
\end{aligned}$$

Problem 2.13

A particle in the harmonic oscillator potential starts out in the state

$$\Psi(x, 0) = A[3\psi_0(x) + 4\psi_1(x)].$$

(a) Find A.

(b) Construct $\Psi(x,t)$ and $|\Psi(x,t)|^2$. Don't get too excited if $|\Psi(x,t)|^2$ oscillates at exactly the classical frequency; what would it have been had I specified $\psi_2(x)$, instead of $\psi_1(x)$?

(c) Find $\langle x\rangle$ and $\langle p\rangle$. Check that Ehrenfest's theorem (Equation 1.38) holds, for this wave function.

(d) If you measured the energy of this particle, what values might you get, and with what probabilities?

Solution

(a)

$$\begin{aligned}1 &= \int_{-\infty}^{\infty} |\Psi(x,0)|^2 \,\mathrm{d}x \\ &= |A|^2(9+16),\end{aligned}$$

so $A = \frac{1}{5}$.

(b)

$$\begin{aligned}\Psi(x,t) &= \frac{1}{5}\left[3\psi_0(x)e^{-i\frac{E_0 t}{\hbar}} + 4\psi_1(x)e^{-i\frac{E_1 t}{\hbar}}\right] \\ &= \frac{1}{5}\left[3\psi_0(x)e^{-\frac{1}{2}i\omega t} + 4\psi_1(x)e^{-\frac{3}{2}i\omega t}\right].\end{aligned}$$

$$\begin{aligned}|\Psi(x,t)|^2 &= \frac{1}{25}\left[9|\psi_0(x)|^2 + 16|\psi_1(x)|^2 + 12\psi_0(x)\psi_1(x)\left(e^{i\omega t} + e^{-i\omega t}\right)\right] \\ &= \frac{1}{25}\left[9|\psi_0(x)|^2 + 16|\psi_1(x)|^2 + 24\psi_0(x)\psi_1(x)\cos(\omega t)\right].\end{aligned}$$

(c)

$$\begin{aligned}\langle x\rangle &= \int_{-\infty}^{\infty} \Psi(x,0)^* x \Psi(x,0)\,\mathrm{d}x \\ &= \frac{1}{25}\left[9\int_{-\infty}^{\infty}\psi_0 x\psi_0\,\mathrm{d}x + 16\int_{-\infty}^{\infty}\psi_1 x\psi_1\,\mathrm{d}x + 24\cos(\omega t)\int_{-\infty}^{\infty}\psi_0 x\psi_1\,\mathrm{d}x\right] \\ &= \frac{24}{25}\cos(\omega t)\int_{-\infty}^{\infty}\psi_0 x\psi_1\,\mathrm{d}x \\ &= \frac{24}{25}\cos(\omega t)\sqrt{\frac{m\omega}{\pi\hbar}}\sqrt{\frac{2m\omega}{\hbar}}\int_{-\infty}^{\infty} xe^{-\frac{m\omega}{2\hbar}x^2} xe^{-\frac{m\omega}{2\hbar}x^2}\,\mathrm{d}x \\ &= \frac{24}{25}\cos(\omega t)\sqrt{\frac{1}{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{m\omega}{\hbar}x^2}\,\mathrm{d}x \\ &= \frac{24}{25}\sqrt{\frac{\hbar}{2m\omega}}\cos(\omega t).\end{aligned}$$

Then,

$$\begin{aligned}\langle p\rangle &= m\frac{\mathrm{d}}{\mathrm{d}t}\langle x\rangle \\ &= -\frac{24}{25}\sqrt{\frac{\hbar m\omega}{2}}\sin(\omega t).\end{aligned}$$

We have

$$\frac{\mathrm{d}\langle p\rangle}{\mathrm{d}t} = -\frac{24}{25}\sqrt{\frac{\hbar m\omega}{2}}\omega\cos(\omega t),$$

and

$$\frac{\mathrm{d}V}{\mathrm{d}x} = m\omega^2 x.$$

Therefore,

$$\begin{aligned}-\langle\frac{\mathrm{d}V}{\mathrm{d}x}\rangle &= -m\omega^2\langle x\rangle \\ &= -\frac{24}{25}\sqrt{\frac{\hbar m\omega}{2}}\omega\cos(\omega t) \\ &= \frac{\mathrm{d}\langle p\rangle}{\mathrm{d}t},\end{aligned}$$

so Ehrenfest's theorem holds.

(d) We can get the energy $E_0 = \frac{\hbar\omega}{2}$ with probability $\frac{9}{25}$ and the energy $E_1 = \frac{3\hbar\omega}{2}$ with probability $\frac{16}{25}$.