Introduction to Quantum Mechanics

The Wave Function

1. The Schrödinger Equation

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi \tag{1}$$

2. The Statistical Interpretation

3. Probability

$$\int_{-\infty}^{\infty} \rho(x)\,\mathrm{d}x = 1 \tag{2}$$

$$\langle x\rangle = \int_{-\infty}^{\infty} x\rho(x)\,\mathrm{d}x \tag{3}$$

$$\langle f(x)\rangle = \int_{-\infty}^{\infty} f(x)\rho(x)\,\mathrm{d}x \tag{4}$$

$$\sigma^2 \equiv \langle(\Delta x)^2\rangle = \langle x^2\rangle - \langle x\rangle^2 \tag{5}$$

4. Normalization

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty} |\Psi(x,t)|^2 \,\mathrm{d}x &= \int_{-\infty}^{\infty} \frac{\partial}{\partial t} \, |\Psi(x,t)|^2 \,\mathrm{d}x \\
&= \int_{-\infty}^{\infty} \frac{\partial}{\partial t} (\Psi^*(x,t)\Psi(x,t)) \,\mathrm{d}x \\
&= \int_{-\infty}^{\infty} \left(\Psi^* \frac{\partial \Psi}{\partial t} + \frac{\partial \Psi^*}{\partial t}\Psi\right) \mathrm{d}x \\
&= \int_{-\infty}^{\infty} \left(\Psi^* \left(\frac{i\hbar}{2m}\frac{\partial^2 \Psi}{\partial x^2} - \frac{i}{\hbar} V\Psi\right) + \left(-\frac{i\hbar}{2m}\frac{\partial^2 \Psi^*}{\partial x^2} + \frac{i}{\hbar} V\Psi^*\right)\Psi\right) \mathrm{d}x \\
&= \frac{i\hbar}{2m}\int_{-\infty}^{\infty} \left(\Psi^* \frac{\partial^2 \Psi}{\partial x^2} - \frac{\partial^2 \Psi^*}{\partial x^2}\Psi\right) \mathrm{d}x \\
&= \frac{i\hbar}{2m}\left(\Psi^* \frac{\partial \Psi}{\partial x} - \frac{\partial \Psi^*}{\partial x}\Psi\right)\Big|_{-\infty}^{\infty} \\
&= 0,
\end{aligned} \tag{6}$$

as Ψ vanishes at $\pm\infty$.
So if Ψ is normalized at $t = 0$, that is

$$\int_{-\infty}^{\infty} |\Psi(x,0)|^2 \,\mathrm{d}x = 1, \tag{7}$$

then it will remain normalized for all time, that is

$$\int_{-\infty}^{\infty} |\Psi(x,t)|^2 \,\mathrm{d}x = 1. \tag{8}$$

5. Momentum

For a particle in state Ψ, the expectation value of x is

$$\begin{aligned}
\langle x \rangle &= \int_{-\infty}^{\infty} x \, |\Psi(x,t)|^2 \,\mathrm{d}x \\
&= \int_{-\infty}^{\infty} \Psi^* [x] \Psi \,\mathrm{d}x.
\end{aligned} \tag{9}$$

The expectation value of momentum is

$$
\begin{aligned}
\langle p\rangle &= m\langle v\rangle \\
&= m\frac{\mathrm{d}\langle x\rangle}{\mathrm{d}t} \\
&= m\int_{-\infty}^{\infty} x\frac{\partial}{\partial t}\,|\Psi(x,t)|^2\,\mathrm{d}x \\
&= \frac{i\hbar}{2}\int_{-\infty}^{\infty} x\frac{\partial}{\partial x}\left(\Psi^*\frac{\partial\Psi}{\partial x}-\frac{\partial\Psi^*}{\partial x}\Psi\right)\mathrm{d}x \\
&= \frac{i\hbar}{2}\left[x\left(\Psi^*\frac{\partial\Psi}{\partial x}-\frac{\partial\Psi^*}{\partial x}\Psi\right)\Big|_{-\infty}^{\infty}-\int_{-\infty}^{\infty}\left(\Psi^*\frac{\partial\Psi}{\partial x}-\frac{\partial\Psi^*}{\partial x}\Psi\right)\mathrm{d}x\right] \\
&= -\frac{i\hbar}{2}\int_{-\infty}^{\infty}\left(\Psi^*\frac{\partial\Psi}{\partial x}-\frac{\partial\Psi^*}{\partial x}\Psi\right)\mathrm{d}x \\
&= -\frac{i\hbar}{2}\left\{\int_{-\infty}^{\infty}\Psi^*\frac{\partial\Psi}{\partial x}\,\mathrm{d}x-\left[\Psi^*\Psi|_{-\infty}^{\infty}-\int_{-\infty}^{\infty}\Psi^*\frac{\partial\Psi}{\partial x}\,\mathrm{d}x\right]\right\} \\
&= -i\hbar\int_{-\infty}^{\infty}\Psi^*\frac{\partial\Psi}{\partial x}\,\mathrm{d}x \\
&= \int_{-\infty}^{\infty}\Psi^*\left[-i\hbar\frac{\partial}{\partial x}\right]\Psi\,\mathrm{d}x.
\end{aligned}
\tag{10}
$$

For any quantity $Q = Q(x,p)$, the expectation value of Q is

$$
\langle Q\rangle = \int_{-\infty}^{\infty}\Psi^* Q\left[x,-i\hbar\frac{\partial}{\partial x}\right]\Psi\,\mathrm{d}x.
\tag{11}
$$

Zum Beispiel, the expectation value of the kinetic energy $T=\frac{p^2}{2m}$ is

$$
\begin{aligned}
\langle T\rangle &= \int_{-\infty}^{\infty}\Psi^*\frac{p^2}{2m}\Psi\,\mathrm{d}x \\
&= \int_{-\infty}^{\infty}\Psi^*\frac{1}{2m}\left(-i\hbar\frac{\partial}{\partial x}\right)^2\Psi\,\mathrm{d}x \\
&= -\frac{\hbar^2}{2m}\int_{-\infty}^{\infty}\Psi^*\frac{\partial^2\Psi}{\partial x^2}\,\mathrm{d}x.
\end{aligned}
\tag{12}
$$

6. The Uncertainty Principle

The de Brogile formula is

$$
p = \frac{h}{\lambda} = \frac{2\pi\hbar}{\lambda}.
\tag{13}
$$

The Heisenberg's uncertainty principle is

$$
\sigma_x\sigma_p \geq \frac{\hbar}{2}.
\tag{14}
$$