Introduction to Quantum Mechanics

Problem 1.1

For the distribution of ages in the example in Section 1.3.1:

(a) Compute $\langle j^2 \rangle$ and $\langle j \rangle^2$

(b) Determine Δj for each j, and use Equation 1.11 to compute the standard deviation

(c) Use your results in (a) and (b) to check Equation 1.12

Problem 1.2

(a) Find the standard deviation of the distribution in Example 1.2

(b) What is the probability that a photograph, selected at random, would show a distance x more than one standard deviation away from the average?

Problem 1.3

Consider the gaussian distribution

$$\rho(x) = Ae^{-\lambda(x-a)^2}$$

where A, a, and λ are positive real constants. (The necessary integrals are inside the back cover.)

(a) Use Equation 1.16 to determine A

(b) Find $\langle x \rangle$, $\langle x^2 \rangle$, and σ

(c) Sketch the graph of $\rho(x)$

Problem 1.4

At time $t = 0$, a particle is presented by the wave function

$$\Psi(x, 0) = \begin{cases} A\frac{x}{a} & 0 \leq x \leq a \\ A\frac{b-x}{b-a} & a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

where A, a, and b are positive constants.

(a) Normalize Ψ (that is, find A, in terms of a and b)

(b) Sketch $\Psi(x, 0)$, as a function of x

(c) Where is the particle most likely to be found at $t = 0$?

(d) What is the probability of finding the particle to the left of a? Check your result in the limiting cases $b = a$ and $b = 2a$

(e) What is the expectation value of x

Solution

(a) We begin with the normalization condition:

$$
\begin{aligned}
1 &= \int_{-\infty}^{\infty} |\Psi(x,0)|^2 \, \mathrm{d}x \\
&= \int_0^a \left| A\frac{x}{a} \right|^2 \mathrm{d}x + \int_a^b \left| A\frac{b-x}{b-a} \right|^2 \mathrm{d}x \\
&= \frac{A^2}{a^2} \int_0^a x^2 \, \mathrm{d}x + \frac{A^2}{(b-a)^2} \int_a^b (b-x)^2 \, \mathrm{d}x \\
&= A^2 \left(\frac{a}{3} + \frac{b-a}{3} \right) \\
&= A^2 \frac{b}{3}
\end{aligned}
$$

Therefore, $A = \sqrt{\frac{3}{b}}$

(a) We skip the sketch as requested

(b) The particle is most likely to be found at $x = a$

(c) The probability of finding the particle to the left of a is:

$$
\begin{aligned}
P &= \int_{-\infty}^{a} |\Psi(x,0)|^2 \, \mathrm{d}x \\
&= \int_0^a \left| A\frac{x}{a} \right|^2 \mathrm{d}x \\
&= \frac{A^2}{a^2} \int_0^a x^2 \, \mathrm{d}x \\
&= A^2 \frac{a}{3} \\
&= \frac{3}{b} \frac{a}{3} \\
&= \frac{a}{b}
\end{aligned}
$$

When $b = a$, $P = 1$ When $b = 2a$, $P = \frac{1}{2}$

(a) The expectation value of x is:

$$\begin{aligned}
\langle x \rangle &= \int_{-\infty}^{\infty} x |\Psi(x,0)|^2 \, \mathrm{d}x \\
&= \int_0^a x \left| A \frac{x}{a} \right|^2 \mathrm{d}x + \int_a^b x \left| A \frac{b-x}{b-a} \right|^2 \mathrm{d}x \\
&= \frac{A^2}{a^2} \int_0^a x^3 \, \mathrm{d}x + \frac{A^2}{(b-a)^2} \int_a^b x(b-x)^2 \, \mathrm{d}x \\
&= A^2 \frac{b(b^3 - 3a^2 b + 2a^3)}{12(b-a)^2} \\
&= \frac{3}{b} \frac{b(2a+b)}{12} \\
&= \frac{2a+b}{4}
\end{aligned}$$

Problem 1.5

Consider the wave function

$$\Psi(x,t) = A e^{-\lambda\,|x|} e^{-i\omega t},$$

where A, λ, and ω are positive real constants. (We'll see in Chapter 2 for what potential (V) this wave function satisfies the Schrödinger equation.)

(a) Normalize Ψ.

(b) Determine the expectation values of x and x^2.

(c) Find the standard deviation of x. Sketch the graph of $|\Psi|^2$, as a function of x, and mark the points $\langle x \rangle + \sigma$ and $\langle x \rangle - \sigma$, to illustrate the sense in which σ represents the "spread" in x. What is the probability that the particle would be found outside this range?

Solution

(a) We begin with the normalization condition:

$$\begin{aligned}
1 &= \int_{-\infty}^{\infty} |\Psi(x,t)|^2 \, \mathrm{d}x \\
&= \int_{-\infty}^{\infty} \left| A e^{-\lambda|x|} e^{-i\omega t} \right|^2 \mathrm{d}x \\
&= A^2 \int_{-\infty}^{\infty} e^{-2\lambda|x|} \, \mathrm{d}x \\
&= 2A^2 \int_0^{\infty} e^{-2\lambda x} \, \mathrm{d}x \\
&= 2A^2 \frac{1}{2\lambda} \\
&= \frac{A^2}{\lambda}.
\end{aligned}$$

Thus, we have $A = \sqrt{\lambda}$.

(b) The expectation value of x is

$$\begin{aligned}
\langle x \rangle &= \int_{-\infty}^{\infty} x|\Psi(x,t)|^2 \, \mathrm{d}x \\
&= \int_{-\infty}^{\infty} x\left|Ae^{-\lambda|x|}e^{-i\omega t}\right|^2 \mathrm{d}x \\
&= A^2 \int_{-\infty}^{\infty} xe^{-2\lambda|x|} \, \mathrm{d}x \\
&= 0.
\end{aligned}$$

The expectation value of x^2 is

$$\begin{aligned}
\langle x^2 \rangle &= \int_{-\infty}^{\infty} x^2|\Psi(x,t)|^2 \, \mathrm{d}x \\
&= \int_{-\infty}^{\infty} x^2\left|Ae^{-\lambda|x|}e^{-i\omega t}\right|^2 \mathrm{d}x \\
&= A^2 \int_{-\infty}^{\infty} x^2e^{-2\lambda|x|} \, \mathrm{d}x \\
&= 2A^2 \int_{0}^{\infty} x^2e^{-2\lambda x} \, \mathrm{d}x \\
&= 2A^2 \frac{1}{4\lambda^3} \\
&= \frac{1}{2\lambda^2}.
\end{aligned}$$

(c) The standard deviation of x is

$$\begin{aligned}
\sigma &= \sqrt{\langle x^2 \rangle - \langle x \rangle^2} \\
&= \sqrt{\frac{1}{2\lambda^2}} \\
&= \frac{1}{\sqrt{2}\lambda}.
\end{aligned}$$

We skip the sketch as requested.

The probability that the particle would be found outside the range $[-\sigma, \sigma]$ is

$$
\begin{aligned}
P &= \int_{-\infty}^{-\sigma} |\Psi(x,t)|^2 \,\mathrm{d}x + \int_{\sigma}^{\infty} |\Psi(x,t)|^2 \,\mathrm{d}x \\
&= 2\int_{\sigma}^{\infty} |\Psi(x,t)|^2 \,\mathrm{d}x \\
&= 2A^2 \int_{\sigma}^{\infty} e^{-2\lambda x} \,\mathrm{d}x \\
&= 2A^2 \frac{1}{2\lambda} e^{-2\lambda\sigma} \\
&= e^{-2\lambda\sigma} \\
&= e^{-\sqrt{2}} \\
&\approx 0.2431.
\end{aligned}
$$

Problem 1.6

Why can't you do integration-by-parts directly on the middle expression in Equation 1.29—pull the time derivative over onto x, note that $\frac{\partial x}{\partial t} = 0$, and conclude that $\frac{\mathrm{d}\langle x\rangle}{\mathrm{d}t} = 0$?

Solution

$$
\begin{aligned}
\frac{\partial}{\partial t} x|\Psi|^2 &= x\frac{\partial}{\partial t}|\Psi|^2 + |\Psi|^2 \partial\frac{x}{\partial t} \\
&= x\frac{\partial}{\partial t}|\Psi|^2
\end{aligned}
$$

Problem 1.7

Calculate $\frac{\mathrm{d}\langle p\rangle}{\mathrm{d}t}$. Answer:

$$
\frac{\mathrm{d}\langle p\rangle}{\mathrm{d}t} = \left\langle -\frac{\partial V}{\partial x} \right\rangle.
$$

This is an instance of *Ehrenfest's theorem*, which asserts that expectation values obey the classical laws.

Solution

$$
\begin{aligned}
\frac{\mathrm{d}\langle p\rangle}{\mathrm{d}t} &= -i\hbar\frac{\partial}{\partial t}\int_{-\infty}^{\infty}\Psi^*\partial\frac{\Psi}{\partial x}\,\mathrm{d}x \\
&= -i\hbar\int_{-\infty}^{\infty}\left(\frac{\partial\Psi^*}{\partial t}\frac{\partial\Psi}{\partial x}+\Psi^*\frac{\partial}{\partial x}\frac{\partial\Psi}{\partial t}\right)\mathrm{d}x \\
&= -i\hbar\int_{-\infty}^{\infty}\left[\left(\frac{-i\hbar}{2m}\frac{\partial^2\Psi^*}{\partial x^2}+\frac{i}{\hbar}V\Psi^*\right)\frac{\partial\Psi}{\partial x}+\Psi^*\frac{\partial}{\partial x}\left(\frac{i\hbar}{2m}\frac{\partial^2\Psi}{\partial x^2}-\frac{i}{\hbar}V\Psi\right)\right]\mathrm{d}x \\
&= \frac{\hbar^2}{2m}\int_{-\infty}^{\infty}\left(\Psi^*\frac{\partial^3\Psi}{\partial x^3}-\frac{\partial^2\Psi^*}{\partial x^2}\frac{\partial\Psi}{\partial x}\right)\mathrm{d}x+\int_{-\infty}^{\infty}\left[V\Psi^*\frac{\partial\Psi}{\partial x}-\Psi^*\frac{\partial}{\partial x}(V\Psi)\right]\mathrm{d}x \\
&= \frac{\hbar^2}{2m}\left[\left(\Psi^*\frac{\partial^2\Psi}{\partial x^2}\right)\Bigg|_{-\infty}^{\infty}-\int_{-\infty}^{\infty}\frac{\partial\Psi^*}{\partial x}\frac{\partial^2\Psi}{\partial x^2}\,\mathrm{d}x-\left(\frac{\partial\Psi^*}{\partial x}\frac{\partial\Psi}{\partial x}\right)\Bigg|_{-\infty}^{\infty}+\int_{-\infty}^{\infty}\frac{\partial\Psi^*}{\partial x}\frac{\partial^2\Psi}{\partial x^2}\,\mathrm{d}x\right] \\
&\quad+\int_{-\infty}^{\infty}\left[V\Psi^*\frac{\partial\Psi}{\partial x}-\Psi^*V\frac{\partial\Psi}{\partial x}-\Psi^*\frac{\partial V}{\partial x}\Psi\right]\mathrm{d}x \\
&= \int_{-\infty}^{\infty}-\Psi^*\left[\frac{\partial V}{\partial x}\right]\Psi\,\mathrm{d}x \\
&= \left\langle-\frac{\partial V}{\partial x}\right\rangle.
\end{aligned}
$$

Problem 1.8

Suppose you add a constant V_0 to the potential energy (by "constant" I mean independent of x as well as t). In classical mechanics this doesn't change anything, but what about *quantum mechanics*? Show that the wave function picks up a time-dependent phase factor: $\exp\left(-i\frac{V_0 t}{\hbar}\right)$. What effect does this have on the expectation value of a dynamical variable?

Solution

Suppose the wave function Ψ satisfies the Schrödinger equation without the constant V_0:

$$
i\hbar\partial\frac{\Psi}{\partial t}=-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2}+V\Psi.
$$

Then, for the wave function $\Psi'=\Psi e^{-i\frac{V_0 t}{\hbar}}$, we have

$$
\begin{aligned}
i\hbar\partial\frac{\Psi'}{\partial t} &= i\hbar\frac{\partial}{\partial t}\left(\Psi e^{-i\frac{V_0 t}{\hbar}}\right) \\
&= i\hbar\left(\frac{\partial\Psi}{\partial t}e^{-i\frac{V_0 t}{\hbar}} - \frac{iV_0}{\hbar}\Psi e^{-i\frac{V_0 t}{\hbar}}\right) \\
&= \left(-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi\right)e^{-i\frac{V_0 t}{\hbar}} + V_0\Psi e^{-i\frac{V_0 t}{\hbar}} \\
&= -\frac{\hbar^2}{2m}\frac{\partial^2\Psi'}{\partial x^2} + (V + V_0)\Psi',
\end{aligned}
$$

which is the Schrödinger equation with the constant V_0. Thus, the wave function Ψ picks up a time-dependent phase factor $\exp\left(-i\frac{V_0 t}{\hbar}\right)$. The expectation value of a dynamical variable is not affected by the phase factor, since the x-independent phase factor is canceled out when taking the expectation value.