Chapter1

Due date:

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Grade : ________

Problem 1 Score: ________. For the distribution of ages in the example in Section 1.3.1:

(a) Compute $\langle j^2 \rangle$ and $\langle j \rangle^2$.

(b) Determine Δj for each j, and use Equation 1.11 to compute the standard deviation.

(c) Use your results in (a) and (b) to check Equation 1.12.

Problem 2 Score: ________. (a) Find the standard deviation of the distribution in Example 1.2.

(b) What is the probability that a photograph, selected at random, would show a distance x more than one standard deviation away from the average?

Problem 3 Score: ________. Consider the gaussian distribution

$$\rho(x) = Ae^{-\lambda(x-a)^2},$$

where A, a, and λ are positive real constants. (The necessary integrals are inside the back cover.)

(a) Use Equation 1.16 to determine A.

(b) Find $\langle x \rangle$, $\langle x^2 \rangle$, and σ.

(c) Sketch the graph of $\rho(x)$.

Problem 4 Score: ________. At time $t = 0$, a particle is presented by the wave function

$$\Psi(x,0) = \begin{cases} A(x/a), & 0 \le x \le a, \\ A(b-x)/(b-a), & a \le x \le b, \\ 0, & \text{otherwise}, \end{cases}$$

where A, a, and b are positive constants.

(a) Normalize Ψ (that is, find A, in terms of a and b).

(b) Sketch $\Psi(x,0)$, as a function of x.

(c) Where is the particle most likely to be found at $t = 0$?

(d) What is the probability of finding the particle to the left of a? Check your result in the limiting cases $b = a$ and $b = 2a$.

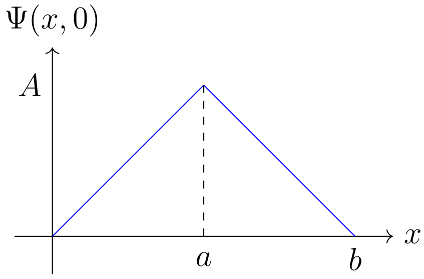
(e) What is the expectation value of x.

Solution: (a)

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} |\Psi(x,0)|^2 \, \mathrm{d}x \\ &= \int_0^a |A(x/a)|^2 \, \mathrm{d}x + \int_a^b |A(b-x)/(b-a)|^2 \, \mathrm{d}x \\ &= \frac{A^2}{a^2} \int_0^a x^2 \, \mathrm{d}x + \frac{A^2}{(b-a)^2} \int_a^b (b-x)^2 \, \mathrm{d}x \\ &= A^2 \left(\frac{a}{3} + \frac{b-a}{3} \right) \\ &= A^2 \frac{b}{3}. \end{aligned}$$

Thus, we have $A = \sqrt{\frac{3}{b}}$.

(b) The sketch of $\Psi(x,0)$ is shown below.



(c) The particle is most likely to be found at $x = a$.

(d) The probability of finding the particle to the left of a is

$$\begin{aligned} P &= \int_{-\infty}^{a} |\Psi(x,0)|^2 \, \mathrm{d}x \\ &= \int_{0}^{a} |A(x/a)|^2 \, \mathrm{d}x \\ &= \frac{A^2}{a^2} \int_{0}^{a} x^2 \, \mathrm{d}x \\ &= A^2 \frac{a}{3} \\ &= \frac{3}{b}\frac{a}{3} \\ &= \frac{a}{b}. \end{aligned}$$

When $b = a$, we have $P = 1$. When $b = 2a$, we have $P = 1/2$.

(e) The expectation value of x is

$$\begin{aligned} \langle x \rangle &= \int_{-\infty}^{\infty} x|\Psi(x,0)|^2 \, \mathrm{d}x \\ &= \int_{0}^{a} x|A(x/a)|^2 \, \mathrm{d}x + \int_{a}^{b} x|A(b-x)/(b-a)|^2 \, \mathrm{d}x \\ &= \frac{A^2}{a^2} \int_{0}^{a} x^3 \, \mathrm{d}x + \frac{A^2}{(b-a)^2} \int_{a}^{b} x(b-x)^2 \, \mathrm{d}x \\ &= A^2 \frac{b(b^3 - 3a^2 b + 2a^3)}{12(b-a)^2} \\ &= \frac{3}{b}\frac{b(2a+b)}{12} \\ &= \frac{2a+b}{4}. \end{aligned}$$

□

Problem 5 Score: _________. Consider the wave function

$$\Psi(x,t) = A e^{-\lambda |x|} e^{-i\omega t},$$

, where A, λ, and ω are positive real constants. (We'll see in Chapter 2 for what potential (V) this wave function satisfies the Schrödinger equation.)

(a) Normalize Ψ.

(b) Determine the expectation values of x and x^2.

(c) Find the standard deviation of x. Sketch the graph of $|\Psi|^2$, as a function of x, and mark the points $\langle x\rangle + \sigma$ and $\langle x\rangle - \sigma$, to illustrate the sense in which σ represents the "spread" in x. What is the probability that the particle would be found outside this range?

Solution: (a)

$$\begin{aligned}
1 &= \int_{-\infty}^{\infty} |\Psi(x,t)|^2 \,\mathrm{d}x \\
&= \int_{-\infty}^{\infty} |Ae^{-\lambda|x|}e^{-i\omega t}|^2 \,\mathrm{d}x \\
&= A^2 \int_{-\infty}^{\infty} e^{-2\lambda|x|} \,\mathrm{d}x \\
&= 2A^2 \int_{0}^{\infty} e^{-2\lambda x} \,\mathrm{d}x \\
&= 2A^2 \frac{1}{2\lambda} \\
&= \frac{A^2}{\lambda}.
\end{aligned}$$

Thus, we have $A = \sqrt{\lambda}$.

(b) The expectation value of x is

$$\begin{aligned}
\langle x\rangle &= \int_{-\infty}^{\infty} x|\Psi(x,t)|^2 \,\mathrm{d}x \\
&= \int_{-\infty}^{\infty} x|Ae^{-\lambda|x|}e^{-i\omega t}|^2 \,\mathrm{d}x \\
&= A^2 \int_{-\infty}^{\infty} xe^{-2\lambda|x|} \,\mathrm{d}x \\
&= 0.
\end{aligned}$$

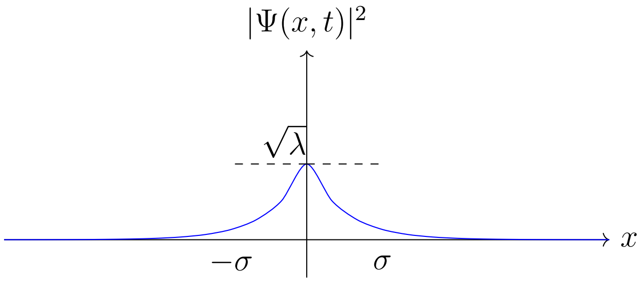
The expectation value of x^2 is

$$\begin{aligned}
\langle x^2\rangle &= \int_{-\infty}^{\infty} x^2|\Psi(x,t)|^2 \,\mathrm{d}x \\
&= \int_{-\infty}^{\infty} x^2|Ae^{-\lambda|x|}e^{-i\omega t}|^2 \,\mathrm{d}x \\
&= A^2 \int_{-\infty}^{\infty} x^2e^{-2\lambda|x|} \,\mathrm{d}x \\
&= 2A^2 \int_{0}^{\infty} x^2e^{-2\lambda x} \,\mathrm{d}x \\
&= 2A^2 \frac{1}{4\lambda^3} \\
&= \frac{1}{2\lambda^2}.
\end{aligned}$$

(c) The standard deviation of x is

$$\begin{aligned}\sigma &= \sqrt{\langle x^2\rangle - \langle x\rangle^2} \\ &= \sqrt{\frac{1}{2\lambda^2}} \\ &= \frac{1}{\sqrt{2}\lambda}.\end{aligned}$$

The graph of $|\Psi|^2$ is shown below.



The probability that the particle would be found outside the range $[-\sigma, \sigma]$ is

$$\begin{aligned}P &= \int_{-\infty}^{-\sigma} |\Psi(x,t)|^2 \, \mathrm{d}x + \int_{\sigma}^{\infty} |\Psi(x,t)|^2 \, \mathrm{d}x \\ &= 2\int_{\sigma}^{\infty} |\Psi(x,t)|^2 \, \mathrm{d}x \\ &= 2A^2 \int_{\sigma}^{\infty} e^{-2\lambda x} \, \mathrm{d}x \\ &= 2A^2 \frac{1}{2\lambda} e^{-2\lambda\sigma} \\ &= e^{-2\lambda\sigma} \\ &= e^{-\sqrt{2}} \\ &\approx 0.2431.\end{aligned}$$

□

Problem 6 Score: ________. Why can't you do integration-by-parts directly on the middle expression in Equation 1.29—pull the time derivative over onto x, note that $\partial x/\partial t = 0$, and conclude that $d\langle x\rangle/dt = 0$?

Solution:

$$\frac{\mathrm{d}}{\mathrm{d}t} x|\Psi|^2 = x\frac{\partial}{\partial t}|\Psi|^2 + |\Psi|^2\frac{\partial x}{\partial t} \tag{1}$$

$$= x\frac{\partial}{\partial t}|\Psi|^2 \tag{2}$$

□

Problem 7 Score: ________. Calculate $d\langle p\rangle/dt$. Answer:

$$\frac{\mathrm{d}\langle p\rangle}{\mathrm{d}t} = \left\langle -\frac{\partial V}{\partial x}\right\rangle.$$

This is an instance of *Ehrenfest's theorem*, which asserts that expectation values obey the classical laws.

Solution:

$$
\begin{aligned}
\frac{\mathrm{d}\langle p\rangle}{\mathrm{d}t} &= -i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty}\Psi^*\frac{\partial\Psi}{\partial x}\ \mathrm{d}x && (3)\\
&= -i\hbar\int_{-\infty}^{\infty}\left(\frac{\partial\Psi^*}{\partial t}\frac{\partial\Psi}{\partial x}+\Psi^*\frac{\partial}{\partial x}\frac{\partial\Psi}{\partial t}\right)\mathrm{d}x && (4)\\
&= -i\hbar\int_{-\infty}^{\infty}\left[\left(-\frac{i\hbar}{2m}\frac{\partial^2\Psi^*}{\partial x^2}+\frac{i}{\hbar}V\Psi^*\right)\frac{\partial\Psi}{\partial x}+\Psi^*\frac{\partial}{\partial x}\left(\frac{i\hbar}{2m}\frac{\partial^2\Psi}{\partial x^2}-\frac{i}{\hbar}V\Psi\right)\right]\mathrm{d}x && (5)\\
&= \frac{\hbar^2}{2m}\int_{-\infty}^{\infty}\left(\Psi^*\frac{\partial^3\Psi}{\partial x^3}-\frac{\partial^2\Psi^*}{\partial x^2}\frac{\partial\Psi}{\partial x}\right)\mathrm{d}x+\int_{-\infty}^{\infty}\left[V\Psi^*\frac{\partial\Psi}{\partial x}-\Psi^*\frac{\partial}{\partial x}(V\Psi)\right]\mathrm{d}x && (6)\\
&= \frac{\hbar^2}{2m}\left[\left(\Psi^*\frac{\partial^2\Psi}{\partial x^2}\right)\Bigg|_{-\infty}^{\infty}-\int_{-\infty}^{\infty}\frac{\partial\Psi^*}{\partial x}\frac{\partial^2\Psi}{\partial x^2}\ \mathrm{d}x-\left(\frac{\partial\Psi^*}{\partial x}\frac{\partial\Psi}{\partial x}\right)\Bigg|_{-\infty}^{\infty}+\int_{-\infty}^{\infty}\frac{\partial\Psi^*}{\partial x}\frac{\partial^2\Psi}{\partial x^2}\right]\mathrm{d}x && (7)\\
&+ \int_{-\infty}^{\infty}\left[V\Psi^*\frac{\partial\Psi}{\partial x}-\Psi^*V\frac{\partial\Psi}{\partial x}-\Psi^*\frac{\partial V}{\partial x}\Psi\right]\mathrm{d}x && (8)\\
&= \int_{-\infty}^{\infty}-\Psi^*\left[\frac{\partial V}{\partial x}\right]\Psi\ \mathrm{d}x && (9)\\
&= \left\langle-\frac{\partial V}{\partial x}\right\rangle. && (10)
\end{aligned}
$$

□

Problem 8 Score: _________. Problem 1.8

Suppose you add a constant V_0 to the potential energy (by "constant" I mean independent of x as well as t). In classical mechanics this doesn't change anything, but what about *quantum mechanics*? Show that the wave function picks up a time-dependent phase factor: $\exp(-iV_0t/\hbar)$. What effect does this have on the expectation value of a dynamical variable?

Solution: Suppose the wave function Ψ satisfies the Schrödinger equation without the constant V_0:

$$i\hbar\frac{\partial\Psi}{\partial t}=-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2}+V\Psi. \tag{11}$$

Then, for the wave function $\Psi'=\Psi e^{-iV_0t/\hbar}$, we have

$$
\begin{aligned}
i\hbar\frac{\partial\Psi'}{\partial t} &= i\hbar\frac{\partial}{\partial t}(\Psi e^{-iV_0t/\hbar}) && (12)\\
&= i\hbar\left(\frac{\partial\Psi}{\partial t}e^{-iV_0t/\hbar}-\frac{iV_0}{\hbar}\Psi e^{-iV_0t/\hbar}\right) && (13)\\
&= \left(-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2}+V\Psi\right)e^{-iV_0t/\hbar}+V_0\Psi e^{-iV_0t/\hbar} && (14)\\
&= -\frac{\hbar^2}{2m}\frac{\partial^2\Psi'}{\partial x^2}+(V+V_0)\Psi', && (15)
\end{aligned}
$$

which is the Schrödinger equation with the constant V_0. Thus, the wave function Ψ picks up a time-dependent phase factor $\exp(-iV_0t/\hbar)$. The expectation value of a dynamical variable is not affected by the phase factor, since the x-independent phase factor is canceled out when taking the expectation value. □