Chapter 1

The Wave Function

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1 The Schrödinger Equation

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi \tag{1}$$

2 The Statistical Interpretation

3 Probability

$$\int_{-\infty}^{\infty} \rho(x)\ \mathrm{d}x = 1 \tag{2}$$

$$\langle x\rangle = \int_{-\infty}^{\infty} x\rho(x)\ \mathrm{d}x \tag{3}$$

$$\langle f(x)\rangle = \int_{-\infty}^{\infty} f(x)\rho(x)\ \mathrm{d}x \tag{4}$$

$$\sigma^2 \equiv \langle(\Delta x)^2\rangle = \langle x^2\rangle - \langle x\rangle^2 \tag{5}$$

4 Normalization

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty} |\Psi(x,t)|^2\ \mathrm{d}x &= \int_{-\infty}^{\infty} \frac{\partial}{\partial t}|\Psi(x,t)|^2\ \mathrm{d}x && (6)\\
&= \int_{-\infty}^{\infty} \frac{\partial}{\partial t}\left(\Psi^*(x,t)\Psi(x,t)\right)\ \mathrm{d}x && (7)\\
&= \int_{-\infty}^{\infty} \left(\Psi^*\frac{\partial\Psi}{\partial t} + \frac{\partial\Psi^*}{\partial t}\Psi\right)\ \mathrm{d}x && (8)\\
&= \int_{-\infty}^{\infty} \left(\Psi^*(\frac{i\hbar}{2m}\frac{\partial^2\Psi}{\partial x^2} - \frac{i}{\hbar}V\Psi) + (-\frac{i\hbar}{2m}\frac{\partial^2\Psi^*}{\partial x^2} + \frac{i}{\hbar}V\Psi^*)\Psi\right)\ \mathrm{d}x && (9)\\
&= \frac{i\hbar}{2m}\int_{-\infty}^{\infty} \left(\Psi^*\frac{\partial^2\Psi}{\partial x^2} - \frac{\partial^2\Psi^*}{\partial x^2}\Psi\right)\ \mathrm{d}x && (10)\\
&= \frac{i\hbar}{2m}\left(\Psi^*\frac{\partial\Psi}{\partial x} - \frac{\partial\Psi^*}{\partial x}\Psi\right)\Bigg|_{-\infty}^{\infty} && (11)\\
&= 0, && (12)
\end{aligned}$$

as Ψ vanishes at $\pm\infty$.
So if Ψ is normalized at $t = 0$, that is

$$\int_{-\infty}^{\infty} |\Psi(x,0)|^2\ \mathrm{d}x = 1, \tag{13}$$

then it will remain normalized for all time, that is

$$\int_{-\infty}^{\infty} |\Psi(x,t)|^2\ \mathrm{d}x = 1. \tag{14}$$

5 Momentum

For a particle in state Ψ, the expectation value of x is

$$\begin{aligned}\langle x\rangle &= \int_{-\infty}^{\infty} x\,|\Psi(x,t)|^2 \,\mathrm{d}x &(15)\\ &= \int_{-\infty}^{\infty} \Psi^*[x]\Psi \,\mathrm{d}x. &(16)\end{aligned}$$

The expectation value of momentum is

$$\begin{aligned}\langle p\rangle &= m\langle v\rangle &(17)\\
&= m\frac{\mathrm{d}\langle x\rangle}{\mathrm{d}t} &(18)\\
&= m\int_{-\infty}^{\infty} x\frac{\partial}{\partial t}\,|\Psi(x,t)|^2 \,\mathrm{d}x &(19)\\
&= \frac{i\hbar}{2}\int_{-\infty}^{\infty} x\frac{\partial}{\partial x}\left(\Psi^*\frac{\partial\Psi}{\partial x}-\frac{\partial\Psi^*}{\partial x}\Psi\right)\mathrm{d}x &(20)\\
&= \frac{i\hbar}{2}\left[x\left(\Psi^*\frac{\partial\Psi}{\partial x}-\frac{\partial\Psi^*}{\partial x}\Psi\right)\bigg|_{-\infty}^{\infty}-\int_{-\infty}^{\infty}\left(\Psi^*\frac{\partial\Psi}{\partial x}-\frac{\partial\Psi^*}{\partial x}\Psi\right)\mathrm{d}x\right] &(21)\\
&= -\frac{i\hbar}{2}\int_{-\infty}^{\infty}\left(\Psi^*\frac{\partial\Psi}{\partial x}-\frac{\partial\Psi^*}{\partial x}\Psi\right)\mathrm{d}x &(22)\\
&= -\frac{i\hbar}{2}\left\{\int_{-\infty}^{\infty}\Psi^*\frac{\partial\Psi}{\partial x}\,\mathrm{d}x-\left[\Psi^*\Psi\bigg|_{-\infty}^{\infty}-\int_{-\infty}^{\infty}\Psi^*\frac{\partial\Psi}{\partial x}\,\mathrm{d}x\right]\right\} &(23)\\
&= -i\hbar\int_{-\infty}^{\infty}\Psi^*\frac{\partial\Psi}{\partial x}\,\mathrm{d}x &(24)\\
&= \int_{-\infty}^{\infty}\Psi^*\left[-i\hbar\frac{\partial}{\partial x}\right]\Psi\,\mathrm{d}x. &(25)\end{aligned}$$

For any quantity $Q=Q(x,p)$, the expectation value of Q is

$$\langle Q\rangle = \int_{-\infty}^{\infty}\Psi^* Q\left[x,-i\hbar\frac{\partial}{\partial x}\right]\Psi\,\mathrm{d}x. \tag{26}$$

Zum Beispiel, the expectation value of the kinetic energy $T=\frac{p^2}{2m}$ is

$$\begin{aligned}\langle T\rangle &= \int_{-\infty}^{\infty}\Psi^*\frac{p^2}{2m}\Psi\,\mathrm{d}x &(27)\\
&= \int_{-\infty}^{\infty}\Psi^*\frac{1}{2m}\left(-i\hbar\frac{\partial}{\partial x}\right)^2\Psi\,\mathrm{d}x &(28)\\
&= -\frac{\hbar^2}{2m}\int_{-\infty}^{\infty}\Psi^*\frac{\partial^2\Psi}{\partial x^2}\,\mathrm{d}x. &(29)\end{aligned}$$

6 The Uncertainty Principle

The de Brogile formula is

$$p=\frac{h}{\lambda}=\frac{2\pi\hbar}{\lambda}. \tag{30}$$

The Heisenberg's uncertainty principle is

$$\sigma_x \sigma_p \geq \frac{\hbar}{2}. \tag{31}$$