Exercise 6

Due date:

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Grade : ________

Problem 1 Score: ________. Riemannian connection on the sphere
For a smooth manifold $\mathcal{M}$, let $\mathfrak{F}(\mathcal{M})$ denote the vector space of smooth real-valued functions on $\mathcal{M}$ and $\mathfrak{X}(\mathcal{M})$ denote the vector space of smooth vector fields on $\mathcal{M}$.
Let $\mathcal{M} = \mathbb{S}^{d-1}$ be embedded in $\mathcal{E} = \mathbb{R}^d$. Let Proj_x denote the orthogonal projector from $\mathcal{E}$ to $T_x\mathcal{M}$. Defeine a map ∇ as follows: for all $(x, u) \in T\mathcal{M}$ and $V \in \mathfrak{X}(\mathcal{M})$,

$$\nabla_u V = \mathrm{Proj}_x \left(D\bar{V}(x)[u] \right),$$

where $\bar{V}$ is any smooth extension of V.

(1) Show that ∇ satisfies the Smoothness property of connections.
Hint: Use the fact that if $U, V \in \mathfrak{X}(\mathcal{M})$ with smooth extensions $\bar{U}, \bar{V}$ defined on some open set $\mathcal{M} \subseteq \mathcal{U} \subseteq \mathcal{E}$, then $D\bar{V}(x)[\bar{U}(x)]$ is smooth.

(2) Show that ∇ satisfies the "Linearity in u" property of connections.

(3) Show that ∇ satisfies the "Linearity in V" property of connections.

(4) Show that ∇ satisfies the "Leibniz rule" property of connections.

(5) Conclude that ∇ is a connection on $\mathcal{M}$.

(6) Show that ∇ satisfies the "Compatibility with the metric" property of Riemannian connections.

(7) Show that ∇ satisfies the "Symmetry" property of Riemannian connections in following steps:

  (a) Show that if $\bar{U}$ and $\bar{f}$ are smooth extensions of U and f respectively, then $\bar{U}\bar{f}$ is a smooth extension of Uf. Conclude that

$$\overline{[U, V]f} := \bar{U}(\bar{V}\bar{f}) - \bar{V}(\bar{U}\bar{f})$$

  is a smooth extension of $[U, V]f$.

  (b) Using symmetry of the "Euclidean" Hessian, show that

$$(\overline{[U, V]f})(x) = D\bar{f}(x) \left[D\bar{V}(x)[\bar{U}(x)] - D\bar{U}(x)[\bar{V}(x)] \right]$$

  (c) Show that $D\bar{V}(x)[\bar{U}(x)] - D\bar{U}(x)[\bar{V}(x)]$ is tangent to $\mathcal{M}$ at x.

  (d) Show that ∇ satisfies the "Symmetry" property of Riemannian connections.

  (e) Conclude that ∇ is a Riemannian connection on $\mathcal{M}$.

(8) Give an expression for the Riemannian Hessian of $f : \mathcal{M} \to \mathbb{R}$ in terms of the Euclidean gradient and Hessian of a smooth extension $\bar{f}$ of f.

Solution: (1) As $\mathcal{M} = \mathbb{S}^{d-1}$, we have $\mathrm{Proj}_x(u) = (I_d - xx^\top)u$. Thus for $U, V \in \mathfrak{X}(\mathcal{M})$ with smooth extensions $\bar{U}, \bar{V}$ defined on some open set $\mathcal{M} \subseteq \mathcal{U} \subseteq \mathcal{E}$,

$$\begin{aligned} \nabla_U V(x) &= \mathrm{Proj}_x \left(D\bar{V}(x)[U(x)] \right) \\ &= \mathrm{Proj}_x \left(D\bar{V}(x)[\bar{U}(x)] \right) \\ &= (I_d - xx^\top) D\bar{V}(x)[\bar{U}(x)] \end{aligned}$$

is a smooth extension of $\nabla_U V$. Thus ∇ satisfies the Smoothness property of connections.

(2) For $V \in \mathfrak{X}(\mathcal{M})$ and $u, w \in T_x\mathcal{M}$ and $a, b \in \mathbb{R}$, we have

$$\begin{aligned} \nabla_{au+bw} V(x) &= \mathrm{Proj}_x \left(D\bar{V}(x)[(au + bw)] \right) \\ &= \mathrm{Proj}_x \left(aD\bar{V}(x)[u] + bD\bar{V}(x)[w] \right) \\ &= a\mathrm{Proj}_x \left(D\bar{V}(x)[u] \right) + b\mathrm{Proj}_x \left(D\bar{V}(x)[w] \right) \\ &= a\nabla_u V(x) + b\nabla_w V(x) \end{aligned}$$

due to the linearity of the differential and the projection.

(3) For $W, V \in \mathfrak{X}(\mathcal{M})$ and $u \in T_x\mathcal{M}$ and $a, b \in \mathbb{R}$, we have

$$\begin{aligned}\nabla_u(aV + bW)(x) &= \text{Proj}_x\left(D(\overline{aV + bW})(x)[u]\right)\\ &= \text{Proj}_x\left(aD\bar{V}(x)[u] + bD\bar{W}(x)[u]\right)\\ &= a\text{Proj}_x\left(D\bar{V}(x)[u]\right) + b\text{Proj}_x\left(D\bar{W}(x)[u]\right)\\ &= a\nabla_u V(x) + b\nabla_u W(x)\end{aligned}$$

due to the linearity of the differential and the projection.

(4) For $V \in \mathfrak{X}(\mathcal{M})$ and $f \in \mathfrak{F}(\mathcal{M})$ and $u \in T_x\mathcal{M}$, we have

$$\begin{aligned}\nabla_u(fV)(x) &= \text{Proj}_x\left(D(\overline{fV})(x)[u]\right)\\ &= \text{Proj}_x\left(D(\bar{f}\bar{V})(x)[u]\right)\\ &= \text{Proj}_x\left(D\bar{f}(x)[u]\bar{V}(x) + \bar{f}(x)D\bar{V}(x)[u]\right)\\ &= D\bar{f}(x)[u]\text{Proj}_x\left(\bar{V}(x)\right) + \bar{f}(x)\text{Proj}_x\left(D\bar{V}(x)[u]\right)\\ &= D\bar{f}(x)[u]V(x) + f(x)\nabla_u V(x)\end{aligned}$$

(5) As ∇ satisfies all the properties of connections, we conclude that ∇ is a connection on $\mathcal{M}$.

$$\nabla : T\mathcal{M} \times \mathfrak{X}(\mathcal{M}) \to T\mathcal{M} \quad (u, V) \mapsto \text{Proj}_x\left(D\bar{V}(x)[u]\right)$$

(6) For $U, V, W \in \mathfrak{X}(\mathcal{M})$, we have

$$\begin{aligned}U\langle V, W\rangle &= \bar{U}\langle\bar{V}, \bar{W}\rangle|_{\mathcal{M}}\\ &= \left(\langle\nabla_{\bar{U}}\bar{V}, \bar{W}\rangle + \langle\bar{V}, \nabla_{\bar{U}}\bar{W}\rangle\right)|_{\mathcal{M}}.\end{aligned}$$

For $x \in \mathcal{M}$, we have

$$\begin{aligned}\langle\nabla_{\bar{U}}\bar{V}, \bar{W}\rangle(x) &= \langle(\nabla_{\bar{U}}\bar{V})(x), \text{Proj}_x(W(x))\rangle\\ &= \langle\text{Proj}_x\left((\nabla_{\bar{U}}\bar{V})(x)\right), W(x)\rangle_x\\ &= \langle\nabla_U V, W\rangle(x)\end{aligned}$$

Similarly, we have $\langle\bar{V}, \nabla_{\bar{U}}\bar{W}\rangle(x) = \langle V, \nabla_U W\rangle(x)$.
Thus, we have

$$U\langle V, W\rangle = \langle\nabla_U V, W\rangle + \langle V, \nabla_U W\rangle.$$

(7) (a) Let $\bar{U}$ and $\bar{f}$ be smooth extensions of U and f on some open set $\mathcal{M} \subseteq \mathcal{U} \subseteq \mathcal{E}$. For $x \in \mathcal{M}$, we have

$$\begin{aligned}(Uf)(x) &= Df(x)[U(x)]\\ &= D\bar{f}(x)[\bar{U}(x)]|_{\mathcal{M}}\\ &= (\bar{U}\bar{f})(x)|_{\mathcal{M}}\end{aligned}$$

and for $\bar{V}$ a smooth extension of $V \in \S(\mathcal{M})$, we have

$$\begin{aligned}U(Vf)(x) - V(Uf) &= \bar{U}(\bar{V}\bar{f})(x) - \bar{V}(\bar{U}\bar{f})(x)\\ &= \overline{[U, V]}f(x)\end{aligned}$$

(b) Let $\bar{U}, \bar{V}$ be smooth extensions of $U, V \in \mathfrak{X}(\mathcal{M})$ and $\bar{f}$ be a smooth extension of $f \in \mathfrak{F}(\mathcal{M})$ on some open set $\mathcal{M} \subseteq \mathcal{U} \subseteq \mathcal{E}$.
For $x \in \mathcal{U}$, as in the Euclidean space, we have

$$\begin{aligned}\bar{U}(\bar{V}\bar{f})(x) &= \bar{U}(D\bar{f}(x)[\bar{V}(x)])(x)\\ &= \bar{U}\langle grad\bar{f}(x), \bar{V}(x)\rangle(x)\\ &= D\langle grad\bar{f}, \bar{V}\rangle(x)[\bar{U}(x)]\\ &= \langle D(grad\bar{f})(x)[\bar{U}(x)], \bar{V}(x)\rangle + \langle grad\bar{f}(x), D\bar{V}(x)[\bar{U}(x)]\rangle\end{aligned}$$

$$= \langle Hess\bar{f}(x)[\bar{U}(x)], \bar{V}(x)\rangle + D\bar{f}(x)[D\bar{V}(x)[\bar{U}(x)]]$$

Similarly,

$$\bar{V}(\bar{U}\bar{f})(x) = \langle Hess\bar{f}(x)[\bar{V}(x)], \bar{U}(x)\rangle + D\bar{f}(x)[D\bar{U}(x)[\bar{V}(x)]]$$

Thus we have

$$\begin{aligned}(\overline{[U,V]}\bar{f})(x) &= \bar{U}(\bar{V}\bar{f})(x) - \bar{V}(\bar{U}\bar{f})(x) \\ &= \langle Hess\bar{f}(x)[\bar{U}(x)], \bar{V}(x)\rangle + D\bar{f}(x)[D\bar{V}(x)[\bar{U}(x)]] \\ &- \langle Hess\bar{f}(x)[\bar{V}(x)], \bar{U}(x)\rangle + D\bar{f}(x)[D\bar{U}(x)[\bar{V}(x)]] \\ &= D\bar{f}(x)\left[D\bar{V}(x)[\bar{U}(x)] - D\bar{U}(x)[\bar{V}(x)]\right]\end{aligned}$$

(c) Let $\bar{f}(x) = x^\top x - 1$ be a smooth extension of the local defining function f. For $x \in \mathcal{M}$ and $V, W \in \mathcal{X}(\mathcal{M})$, we have

$$\begin{aligned}\bar{V}\bar{f}(x) &= D\bar{f}(x)[\bar{V}(x)] \\ &= Df(x)[V(x)] \\ &= 0\end{aligned}$$

Thus,

$$\begin{aligned}\bar{U}(\bar{V}\bar{f})(x) &= D(\bar{V}\bar{f})(x)[\bar{U}(x)] \\ &= D(Vf)(x)[U(x)] \\ &= 0.\end{aligned}$$

Similiarly, we have $\bar{V}(\bar{U}\bar{f})(x) = 0$.
Thus,

$$\overline{[U,V]}\bar{f}(x) = \bar{U}(\bar{V}\bar{f})(x) - \bar{V}(\bar{U}\bar{f})(x) = 0$$

which means $\overline{[U,V]} \in \ker(f(x)) = T_x\mathcal{M}$.

(d) Let $\bar{U}, \bar{V}$ be smooth extensions of $U, V \in \mathfrak{X}(\mathcal{M})$ on some open set $\mathcal{M} \subseteq \mathcal{U} \subseteq \mathcal{E}$, and $f \in \mathfrak{F}(\mathcal{M})$.

$$\begin{aligned}(\nabla_U V - \nabla_V U) &= \mathrm{Proj}_x\left(D\bar{V}(x)[U(x)]\right) - \mathrm{Proj}_x\left(D\bar{U}(x)[V(x)]\right) \\ &= \mathrm{Proj}_x\left(D\bar{V}(x)[\bar{U}(x)]\right) - \mathrm{Proj}_x\left(D\bar{U}(x)[\bar{V}(x)]\right) \\ &= D\bar{V}(x)[\bar{U}(x)] - D\bar{U}(x)[\bar{V}(x)]\end{aligned}$$

and we have

$$\begin{aligned}[U,V]f(x) &= \overline{[U,V]}\bar{f}(x) \\ &= D\bar{f}(x)\left[D\bar{V}(x)[\bar{U}(x)] - D\bar{U}(x)[\bar{V}(x)]\right] \\ &= Df(x)\left[(\nabla_U V - \nabla_V U)(x)\right] \\ &= (\nabla_U V - \nabla_V U)f(x)\end{aligned}$$

which means $[U,V]f = (\nabla_U V - \nabla_V U)f$.

(e) As ∇ satisfies all the properties of Riemannian connections, we conclude that ∇ is a Riemannian connection on $\mathcal{M}$.

(8) For $(x,u) \in T\mathcal{M}$

$$\begin{aligned}\mathrm{Hess}f(x)[u] &= (\nabla_u \mathrm{grad}f)(x) \\ &= \mathrm{Proj}_x\left(D\overline{\mathrm{grad}}f(x)[u]\right) \\ &= \mathrm{Proj}_x\left(D(\mathrm{grad}\bar{f}(x) - \langle x, \mathrm{grad}\bar{f}(x)\rangle x)[u]\right) \\ &= \mathrm{Proj}_x\left(\mathrm{Hess}\bar{f}(x)[u] - \left[\langle u, \mathrm{grad}\bar{f}(x)\rangle + \langle x, \mathrm{Hess}\bar{f}(x)[u]\rangle\right]x - \langle x, \mathrm{grad}\bar{f}(x)\rangle u\right) \\ &= \mathrm{Proj}_x\left(\mathrm{Hess}\bar{f}(x)[u]\right) - \langle x, \mathrm{grad}\bar{f}(x)\rangle u\end{aligned}$$

$\square$

Problem 2 Score: _________. The Lie bracket is nonzero
For $\mathcal{M} = \mathbb{R}^2$, consider the vector fields $U(x) = (1, 0), V(x) = (0, x_1x_2)$, and the function $f(x) = x_2$. Show that $[U, V]f = f$.

Solution: For $x = (x_1, x_2) \in \mathbb{R}^2$, we have

$$\begin{aligned} Uf(x) &= Df(x)[U(x)] \\ &= Df(x)[(1,0)] \\ &= 0, \end{aligned}$$

and

$$\begin{aligned} Vf(x) &= Df(x)[V(x)] \\ &= Df(x)[(0, x_1x_2)] \\ &= x_1x_2. \end{aligned}$$

Then

$$\begin{aligned} U(Vf)(x) &= D(Vf)(x)[U(x)] \\ &= D(x_1x_2)[(1,0)] \\ &= x_2, \end{aligned}$$

and

$$\begin{aligned} V(Uf)(x) &= D(Uf)(x)[V(x)] \\ &= D(0)[(0, x_1x_2)] \\ &= 0. \end{aligned}$$

Thus

$$\begin{aligned} [U,V]f(x) &= U(Vf)(x) - V(Uf)(x) \\ &= x_2 - 0 \\ &= x_2 \\ &= f(x). \end{aligned}$$

Therefore, $[U, V]f = f$. $\square$