Exercise 10

Due date:

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Grade : ________

Problem 1 Score: ________. Properties of parallel transport
Let $\mathcal{M}$ be a smooth manifold. Let PT denote the parallel transport with respect to a connection ∇. Let $c : I \to \mathcal{M}$ be a smooth curve, and $t_0, t_1 \in I$ with $I \subseteq \mathbb{R}$ an open interval.
The linear map $\mathbf{PT}^c_{t_1 \leftarrow t_0} : T_{c(t_0)}\mathcal{M} \to T_{c(t_1)}\mathcal{M}$ is always invertible.
We endow $\mathcal{M}$ with a Riemannian metric $\langle \cdot, \cdot \rangle$, and let $\mathbf{PT}$ denote the parallel transport with respect to the Riemannian connection ∇.

(1) Show that the linear map $\mathbf{PT}^c_{t_1 \leftarrow t_0} : T_{c(t_0)}\mathcal{M} \to T_{c(t_1)}\mathcal{M}$ is an isometry, that is

$$\langle u, v \rangle_{c(t_0)}\rangle = \langle \mathbf{PT}^c_{t_1 \leftarrow t_0}(u), \mathbf{PT}^c_{t_1 \leftarrow t_0}(v) \rangle_{c(t_1)} \quad \forall u, v \in T_{c(t_0)}\mathcal{M}.$$

(2) Let $c : I \to \mathcal{M}$ be a geodesic of $\mathcal{M}$. Show that the velocity $c'(t)$, for $t \in I$, defines a parallel vector field along c.

(3) What can be said about $\mathbf{PT}^c_{t \leftarrow 0}$ when

(a) $v = \alpha c'(0)$ for some $\alpha \in \mathbb{R}$?

(b) v is orthogonal to $c'(0)$?

Solution: (1) Let $Z_1, Z_2 \in \mathfrak{X}(c)$ be the unique parallel vector fields along c such that $Z_1(t_0) = u$ and $Z_2(t_0) = v$, Then for any $t \in I$, $\frac{D}{dt} Z_1 = 0$ and $\frac{D}{dt} Z_2 = 0$. We definite

$$g(t) = \langle Z_1(t), Z_2(t) \rangle_{c(t)},$$

whose derivative is

$$\begin{aligned}\frac{d}{dt} g(t) &= \frac{d}{dt} \langle Z_1(t), Z_2(t) \rangle_{c(t)} \\ &= \langle \frac{D}{dt} Z_1(t), Z_2(t) \rangle_{c(t)} + \langle Z_1(t), \frac{D}{dt} Z_2(t) \rangle_{c(t)} \\ &= 0,\end{aligned}$$

which implies that $g(t)$ is a constant function. Therefore, we have

$$\langle u, v \rangle_{c(t_0)} = \langle Z_1(t_0), Z_2(t_0) \rangle_{c(t_0)} = \langle Z_1(t_1), Z_2(t_1) \rangle_{c(t_1)} = \langle \mathbf{PT}^c_{t_1 \leftarrow t_0}(u), \mathbf{PT}^c_{t_1 \leftarrow t_0}(v) \rangle_{c(t_1)}.$$

(2) Since c is a geodesic, we have $c''(t) = \frac{D}{dt} c'(t) = 0$. Therefore, $c'(t)$ is a parallel vector field along c.

(3) (a) Since $c'(t)$ is a parallel vector field along c, thus there exists a unique parallel vector field $c'(t)$ along c such that $c'(0) = v$.
Then by linearity, we have

$$\mathbf{PT}^c_{t \leftarrow 0}(v) = \mathbf{PT}^c_{t \leftarrow 0}(\alpha c'(0)) = \alpha \mathbf{PT}^c_{t \leftarrow 0}(c'(0)) = \alpha c'(t).$$

(b) Since v is orthogonal to $c'(0)$, we have $\langle v, c'(0) \rangle_{c(0)} = 0$.
Then because $\mathbf{PT}^c_{t \leftarrow 0}$ is an isometry, we have

$$\langle \mathbf{PT}^c_{t \leftarrow 0}(v), c'(t) \rangle_{c(t)} = \langle v, c'(0) \rangle_{c(0)} = 0,$$

which implies that $\mathbf{PT}^c_{t \leftarrow 0}(v)$ is orthogonal to $c'(t)$.

□

Problem 2 Score: ________. Parallel transport on the sphere
The curve $c : \mathbb{R} \to \mathbb{S}^{d-1}$ given by

$$c(t) = \cos(t)x + \sin(t)v$$

with $(x, v) \in \mathbf{T}\mathbb{S}^{d-1}$ and $\|v\| = 1$, is a geodesic of the sphere, when the sphere is seen as a Riemannian submanifold with the Riemannian connection.
Derive an expression for the parallel transport $\mathbf{PT}^c_{t \leftarrow 0}$ of a tangent vector $u \in T_{c(0)}\mathbb{S}^{d-1}$ along c.

Solution: Using the result from Problem 1, we decompose u as $u = \langle u, v\rangle v + \sum_{i=1}^{d-2}\langle u, e_i\rangle e_i$, where $\{v, e_1, \cdots, e_{d-2}\}$ is an orthonormal basis of $T_{c(0)}\mathbb{S}^{d-1}$.
As $e_i \in T_{c(0)}\mathbb{S}^{d-1}$, we have $\langle e_i, x\rangle = 0$ and $\langle e_i, v\rangle = 0$, which implies that

$$\begin{aligned}\langle e_i, c(t)\rangle &= \cos(t)\langle e_i, x\rangle + \sin(t)\langle e_i, v\rangle = 0,\\ \langle e_i, c'(t)\rangle &= -\sin(t)\langle e_i, x\rangle + \cos(t)\langle e_i, v\rangle = 0,\end{aligned}$$

for all $t \in \mathbb{R}$.
Therefore, $e_i \in T_{c(t)}\mathbb{S}^{d-1}$ for all $t \in \mathbb{R}$, and $\frac{D}{dt}e_i = 0$, which implies that $\mathbf{PT}^c_{t\leftarrow 0}e_i = e_i$.
For v, as $v = c'(0)$, we have $\mathbf{PT}^c_{t\leftarrow 0}v = c'(t)$.
Therefore, we have

$$\mathbf{PT}^c_{t\leftarrow 0}u = \langle u, v\rangle c'(t) + \sum_{i=1}^{d-2}\langle u, e_i\rangle e_i.$$

Particularly, when $d = 2$, we have $u = \langle u, v\rangle v$, and

$$\mathbf{PT}^c_{t\leftarrow 0}u = \langle u, v\rangle c'(t).$$

□

Problem 3 Score: _________. Transporters on the group of rotations
Consider the rotation group

$$\mathcal{M} = \mathrm{SO}(d) = \{X \in \mathbb{R}^{d\times d} : X^T X = I, \det(X) = 1\}$$

as a Riemannian submanifold of $\mathcal{E} = \mathbb{R}^{d\times d}$ with the usual Euclidean metric.
Recall that the tangent space at $X \in \mathcal{M}$ is given by

$$T_X\mathcal{M} = \{X\Omega : \Omega \in \mathrm{SO}(d), \Omega + \Omega^\top = 0\}.$$

Hence, we can consider the transporters $\mathbf{T}$ defined by

$$\mathbf{T}_{Y\leftarrow X} = Y\Omega$$

for $X, Y \in \mathcal{M}$ and $\Omega + \Omega^\top = 0$.
Note that if we store tangent vectors of $\mathcal{M}$ by their skew-symmetric parts, then this transporter requires no computation.

(1) Show that $\mathbf{T}_{Y\leftarrow X}(U) = YX^\top U$ for $(X, U) \in T\mathcal{M}$ and $Y \in \mathcal{M}$ and conclude that $\mathbf{T}$ is a transporter.

(2) Show that $\mathbf{T}_{Y\leftarrow X}$ is an isometry from $T_X\mathcal{M}$ to $T_Y\mathcal{M}$.

(3) Show that

$$c : \mathbb{R} \to \mathrm{SO}(d), \quad c(t) = X\exp(t\Omega)$$

is a geodesic on $\mathrm{SO}(d)$, which is such that $c(0) = X$ and $c'(0) = V := X\Omega$.

(4) Let $c : \mathbb{R} \to \mathrm{SO}(d)$ be a geodesic of $\mathrm{SO}(d)$ and $X = c(t_0), Y = c(t_1)$ for $t_1 \geq t_0 \geq 0$. Is $\mathbf{T}_{Y\leftarrow X}$ equal to the parallel transport along c from t_0 to t_1?

Solution: (1) For $(X, U) \in T\mathcal{M}$ and $Y \in \mathcal{M}$, we have $U = X\Omega$ for some skew-symmetric Ω.
Then we have

$$\begin{aligned}\mathbf{T}_{Y\leftarrow X}(U) &= YX^\top U\\ &= YX^\top X\Omega\\ &= Y\Omega\end{aligned}$$

which lies in $T_Y\mathcal{M}$.
Therefore, $\mathbf{T}$ is a transporter.

(2) For $U_1, U_2 \in T_X\mathcal{M}$, we have $U_1 = X\Omega_1$ and $U_2 = X\Omega_2$ for some skew-symmetric Ω_1 and Ω_2.
Then we have

$$\begin{aligned}\langle \mathbf{T}_{Y\leftarrow X}(U_1), \mathbf{T}_{Y\leftarrow X}(U_2)\rangle &= \langle YX^\top X\Omega_1, YX^\top X\Omega_2\rangle\\ &= \langle Y\Omega_1, Y\Omega_2\rangle\end{aligned}$$

$$
\begin{aligned}
&= \mathrm{Tr}(\Omega_1^\top Y^\top Y \Omega_2) \\
&= \mathrm{Tr}(\Omega_1^\top \Omega_2) \\
&= \langle U_1, U_2 \rangle,
\end{aligned}
$$

which shows that $\mathbf{T}_{Y \leftarrow X}$ is an isometry.

(3) Let $(X, V) \in T\mathcal{M}$, then we have $V = X\Omega$ for some skew-symmetric Ω.
We've already showed that $R_X(V) = X \exp(X^\top V)$ is a retraction of $\mathrm{SO}(d)$ at X, thus

$$
\begin{aligned}
c(t) &= X \exp(t\Omega) \\
&= X \exp(tX^\top X \Omega) \\
&= X \exp(X^\top tV) \\
&= R_X(tV),
\end{aligned}
$$

is a smooth curve. Moreover, the acceleration of c is

$$
\begin{aligned}
c''(t) &= \frac{D}{dt} c'(t) \\
&= \mathrm{Proj}_{c(t)} \frac{d}{dt} c'(t) \\
&= \mathrm{Proj}_{c(t)} (\frac{d}{dt} c(t)\Omega) \\
&= \mathrm{Proj}_{c(t)} (c(t)\Omega^2) \\
&= c(t)\Omega^2 - c(t) \frac{(\Omega^2)^\top c(t)^\top c(t) + c(t)^\top c(t) \Omega^2}{2} \\
&= c(t)\Omega^2 - c(t)\Omega^2 \\
&= 0,
\end{aligned}
$$

which implies that c is a geodesic on $\mathrm{SO}(d)$.

(4) Fix $U \in T_X\mathcal{M}$, then the transporter $\mathbf{T}_{c(t) \leftarrow X}$ is given by

$$
\mathbf{T}_{c(t) \leftarrow X}(U) = c(t) X^\top U,
$$

whose covariant derivative along $c = X \exp(t\Omega)$ is

$$
\begin{aligned}
\frac{D}{dt} \mathbf{T}_{c(t) \leftarrow X}(U) &= \frac{D}{dt} (c(t) X^\top U) \\
&= \mathrm{Proj}_{c(t)} \frac{d}{dt} (c(t) X^\top U) \\
&= \mathrm{Proj}_{c(t)} (c'(t) X^\top U) \\
&= \mathrm{Proj}_{c(t)} (c(t) \Omega X^\top U) \\
&= c(t)\Omega X^\top U - c(t) \frac{U^\top X \Omega^\top c(t)^\top c(t) + c(t)^\top c(t) \Omega X^\top U}{2} \\
&= c(t)\Omega X^\top U - c(t) \frac{U^\top X \Omega^\top + \Omega X^\top U}{2} \\
&= c(t) \frac{\Omega X^\top U - U^\top X \Omega^\top}{2}.
\end{aligned}
$$

As $U \in T_X\mathcal{M}$, we have $U = X\tilde{\Omega}$ for some skew-symmetric $\tilde{\Omega}$, thus

$$
\begin{aligned}
\frac{D}{dt} \mathbf{T}_{c(t) \leftarrow X}(U) &= c(t) \frac{\Omega X^\top X \tilde{\Omega} - \tilde{\Omega}^\top X^\top X \Omega^\top}{2} \\
&= c(t) \frac{\Omega\tilde{\Omega} - \tilde{\Omega}^\top \Omega^\top}{2},
\end{aligned}
$$

which is zero when $\Omega\tilde{\Omega}$ is skew-symmetric, but not necessarily zero in general.
Therefore, $\mathbf{T}_{c(t) \leftarrow X}$ is not equal to the parallel transport along c from t_0 to t_1.

$\square$