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# DRP Presentation: Spectral Graph Theory

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November 20, 2024

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## 1 Expander Graphs

- Expanders are sparse highly-connected graphs.
- Constant degree ($k$ regular, or close to that)
- An expander graph on $n$ vertices has $O(n)$ edges and diameter $O(log(n))$.
- Sparsification is the problem of how can we remove edges from a graph while retaining high conductance - i.e, the process of turning non-expander graphs into expander graphs.

## 2 Spectral Graph Theory

- A graph is an adjacency matrix with real-valued entries.
- Undirected graphs have symmetric matrices, so spectral theorem applies and we have a basis of eigenvectors.
- We can encode a graph as a matrix and then analyse the spectrum (eigenvals/eigenvecs )

## 3 Eigenvalues, Connected Components, and Random Walks

- The multiplicity of the $0$ eigenvalue measures the number of connected components.
- $Lx = 0$ requires $x_i = x_j$ whenever $i, j$ in the same component, and an orthogonal set of (nonzero) vectors which all satisfy this are the indicator vectors for the connected components.
- Adjacency matrix (divided by degree) raised to powers gets you the random walk distribution.
- So if $R = D^{-1}A$, $R^k_{i,j}$ is the probability that a $k$ step walk from $i$ ends at $j$.

## 4 Laplacians

- $D$ is the degree matrix of a graph: $D_{u,u} = \deg u$, and $A$ is the adjacency matrix, with $A_{u,v} = w_{u,v}$
- Quadratic Form of a matrix: $\langle x, Ax\rangle$ or $x^T Ax$.
- $L = D - A$, so $L_{u,v} = -w_{u,v}$ if edge $u, v$ exists and $L_{u,u} = \deg u$ along the diagonals. It encodes both the degree and adjacency matrices.
- It can also be decomposed into a sum of simpler matrices $\sum_{u,v\in E} L_{u,v}$,
- $\langle x, Lx\rangle = \sum_{u,v\in E} w_{u,v}(x_u - x_v)^2$, a useful result for later.
- If we consider $\mathcal{L} = D^{-\frac{1}{2}} LD^{-\frac{1}{2}}$, we notice that all of its entries are between $-1$ and $1$. If $G$ is $d$ regular, $\mathcal{L} = \frac{1}{d}L$

## 5 Conductance

- The *conductance* of a subgraph (precisely, a subset of vertices) $S$ measures how connected $S$ is to the rest of the graph.
- $\Phi(S) = \frac{|E(S,V-S)|}{Vol(S)}$
- The conductance of the graph is just $\Phi_G = \min_{S\subset V} \Phi(S)$, it measures the conductance of the most “island-y” subgraph (most disconnected from the rest of the graph).
- If we take a subset of vertices $S$ and write it as an indicator vector $x_S$, $\Phi(S) = \frac{\langle x_s,Lx_s\rangle}{\sum_{i\in V} \deg i x_s(i)^2} = \frac{\langle x_s,Lx_s\rangle}{d\langle x_s,x_s\rangle}$.
- Numerator works out as a result of symmetry: $x_u$ and $x_v$ cancel out if both are in $S$, so only edges counted are those between $S$ and $V - S$.
- So $\Phi_G$ is the result of minimising this over all indicator vectors, i.e over $\{0, 1\}^n$.

# 6 Cheeger's Inequality

- $\langle x, \mathcal{L}x \rangle = \langle x, \sum_i \lambda_i \langle \mathcal{L}x, e_i \rangle e_i \rangle = \sum_i \lambda_i \langle \mathcal{L}x, e_i \rangle^2 \leq \lambda_1 ||x||$. So quadratic forms are bounded by minimum and maximum eigenvalues.
- $\mathcal{L}D^{\frac{1}{2}}\mathbf{1} = L\mathbf{1} = 0$ always holds, since each row sums to 0. So $D^{\frac{1}{2}}\mathbf{1}$ is always an eigenvector of $\mathcal{L}$ with eigenvalue 0.
- $\mathcal{L}$ is also invariant over the orthogonal complement of span $(\mathbf{1})$, so if we restrict ourselves to $U = \mathbf{1}^{\perp}$, $\min_{u \in U} \frac{\langle x, \mathcal{L}x \rangle}{\langle x, x \rangle} = \lambda_2$.
- So $\lambda_2 = \min_{u \in U} \frac{\langle x, Lx \rangle}{d \langle x, x \rangle}$
- Both of these are optimisation problems with the same objective, and $\lambda_2$ is a relaxed version of $\Phi_G$ (we optimise over all vectors, not just indicators).
- Cheeger's inequality formalises this similarity, stating that $2\Phi_G \geq \lambda_2 \geq \frac{\Phi_G^2}{2}$.

# 7 Cheeger's: Easy Direction

- Let $S \subseteq V$ be a set of vertices. Define our 'indicator' vector $x_S$ to have $x_s(i) = \frac{1}{|S|}$ if $i \in S$, else $x_s(i) = \frac{-1}{|V-S|}$
- $\lambda_2 \leq \frac{\langle x_s, Lx_S \rangle}{d \langle x_S, x_S \rangle} \leq 2\phi_G$.

# 8 Random Walk

- Suppose we have a $d$ regular graph. The random walk matrix is $\frac{1}{d}A$, and so $(A/d)^n$ gives the distribution of a random walk after $n$ steps.
- $\mathcal{L} = \frac{1}{d}(I - A) = \frac{1}{d}(L) = \frac{I}{d} - \frac{A}{d}$.
- Suppose $A/d = U^{-1}RU$. Then if $\lambda_{n-1}$ is small, it goes to 0 as we take powers, leaving $\lambda_n = 1$ so random walk is uniform.
- The eigenvalues of $\mathcal{L}$ are related to negatives of the eigenvalues of $A/d$. So if $\lambda_2$ of $\mathcal{L}$ is large, $\lambda_{n-1}$ of $A/d$ is small and so the random walk is uniform. We can see why that makes it a good expander.
- So if $\lambda_2$ is high, it means it is 'easy' to walk from any vertex to any other vertex in a fairly short number of steps, and that all vertices are (roughly) equally easy to get to - which matches up with our intuitions about good sparsifiers.

# 9 Paper and Results

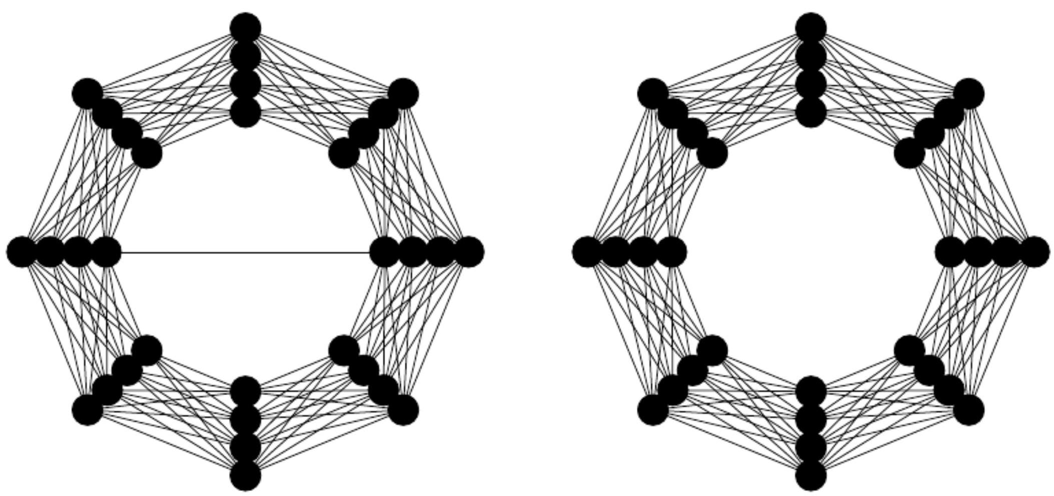
- *Spectral Sparsification of Graphs* by Daniel A. Spielman and Shang-Hua Teng
- Spectral Approximation: Quadratic forms are bounded by $\frac{1}{\sigma}\langle x, L_{\tilde{G}}x \rangle \leq \langle x, L_G x \rangle \leq \sigma \langle x, L_{\tilde{G}}x \rangle$.
- For every $G$, we can find a sparsifier $\tilde{G}$ with $(n/\epsilon^2)$ edges that is a $(1+\epsilon)$ spectral approximation of $G$.
- Their algorithm takes $(m)$ time, where $m = |E|$.

## 10 Preserving Expected Quadratic Forms

- If we select edge with probability $p_{u,v}$, and assign it weight $1/p_{u,v}$ in our sparsified graph, $\mathbb{E}[\langle x, Lx\rangle] = \mathbb{E}[\langle x, \tilde{L}x\rangle]$.
- We can see this from decomposing $L = \sum_{u,v\in E} L_{u,v}$ so $\mathbb{E}[\tilde{L}] = \sum_{u,v\in E} p_{u,v}L_{u,v}$: Dividing by $p_{u,v}$ undoes the effect of the probability of dropping it.
- Problem is now assigning $p_{u,v}$ in a way that preserves extremal quadratic forms, at least with high probability.

## 11 Spectral vs Cut Sparsifiers **ATTACH**

- Intuitively, a good sparsifier should preserve 'important' edges, such as if there is one edge which connects two otherwise-distant vertices.
- Spectral Sparsifiers are sensitive to that distance concept, while other kinds of sparsifiers like cut sparsifiers aren't.
- One way we translate that intuition is by making the probability of $\{i, j\}$ being included proportional to $\frac{1}{\min(\deg i, \deg j)}$. In fact, this is actually one of the methods the main sampling algorithm uses.



$G$: $n = 8$ sets of $k = 4$ vertices arranged in a ring and connected by complete bipartite graphs, plus one edge across.

$\widetilde{G}$: A good cut sparsifier of $G$, but a poor spectral sparsifier

## 12 Overview of Algorithm

- $p_{i,j} = \min(1, \frac{\gamma}{\min(d_i, d_j)})$.
- $\gamma$ is defined in terms of our tolerances (of quality of sparsifier and probability of success), and also inversely proportional to $\lambda_2^2$. Intuition - highly connected graphs can survive more 'aggressive' sparsification.
- Show that if $||D^{-\frac{1}{2}}(L - \tilde{L})D^{-\frac{1}{2}}||$ is small and $\lambda_2$ is large, $\tilde{G}$ is a good spectral approximation for $G$.
- Show that this sparsifier gives us $||D^{-\frac{1}{2}}(\tilde{D} - D)D^{-\frac{1}{2}}||$ and $||D^{-\frac{1}{2}}(\tilde{A} - A)D^{-\frac{1}{2}}||$ small with high probability
- So by triangle inequality $||D^{-\frac{1}{2}}(L - \tilde{L})D^{-\frac{1}{2}}||$ is small with high probability.

## 13 Small Laplacian Difference ⇒ Good Spectral Approximation (Lemma 6.2)

- If Laplacian difference $||D^{-\frac{1}{2}}(L-\tilde{L})D^{-\frac{1}{2}}|| \leq \epsilon$ and $\lambda_2(D^{-\frac{1}{2}}LD^{-\frac{1}{2}}) \geq \lambda$, $\tilde{G}$ is a $\sigma = \frac{\lambda}{\lambda-\epsilon}$ spectral approximation.
- For $d$ regular graphs, we have $\frac{1}{d}||(L-\tilde{L})|| \leq \epsilon$, and $\lambda_2(L) = d \cdot \lambda_2(\mathcal{L}) \geq d \cdot \lambda$
- We can split $x$ up into $u$ parallel to **1** and $v$ orthogonal to it and use the fact $\langle v, Lv\rangle \geq d \cdot \lambda$ (since quadforms are bounded by eigenvalues)
- In the regular case, a good intuition is that if the spectrum of $L - \tilde{L}$ has small eigenvalues ($\leq \epsilon$), then their quadratic forms behave similarly.
- As we increase $\lambda$ our bound gets tighter, reflecting that highly-connected graphs are easier to sparsify well.

## 14 $||D^{-\frac{1}{2}}(A-\tilde{A})D^{-\frac{1}{2}}||$ is Small (Lemma 6.3-4)

- Since the norm is just the highest eigenvalue, it instead bounds the trace (product of eigenvalues) of $\Delta = D^{-1}(\tilde{A}-A)$.
- Idea from earlier: Random walk matrix encodes probability of walking from $v_0$ to $v_k$. Still applies here in some form .
- For a given walk $v_0, v_1, \cdots v_k$, its probability is nonzero iff all of the necessary edges are included in $A$, so each significant sequence is analogous to a walk $v_0, \cdots, v_k$ on $A$.
- $\mathbb{E}[\Delta_{v_i,v_j}] = 0$ from the definition. $\Delta_{v_i,v_j}$ is independent of all others except $\Delta_{v_j,v_i}$. So we can split $\mathbb{E}[\Pi_{i=1}^{k}\Delta_{v_{i-1},v_i}]$ into independent pairs $E[\Delta_{v_i,v_{i+1}}\Delta_{v_{i+1},v_i}]$, so a walk has nonzero contribution only if each edge appears at least twice (i.e $\Delta_{v_i,v_{i+1}}\Delta_{v_{i+1},v_i} \neq 0$).
- They use an ingenious method to bound the number of such walks.

## 15 $||D^{-\frac{1}{2}}(D-\tilde{D})D^{-\frac{1}{2}}||$ is Probably Small (Lemma 6.7)

- The probability that $||D^{-\frac{1}{2}}(D-\tilde{D})D^{-\frac{1}{2}}|| \geq \epsilon$ is proportional to $e^{-\epsilon^2}$.
- The norm of a diagonal matrix is just it's largest entry, and $D^{-1}(\tilde{D}-D)_{i,i} = 1 - \frac{\tilde{d}_i}{d_i}$.
- $\mathbb{E}[\tilde{d}_i] = d_i$, and $\tilde{d}_i$ decomposes into a sum of independent $d_i$ indicator variables.
- So we can apply the Chernoff bound to the probability that $\tilde{d}_i - d_i > \epsilon d_i$ for a given $i$ and then use union bound to show that the probability of this occuring for any $i$ is small.

## 16 High-Conductance Subgraphs Exist (Theorem 7.1)

- We can always find a reasonably large subgraph with fairly high conductance.
- From that fact, it follows that we can partition a graph into high-conductance components, by repeatedly extracting these subgraphs.
- Let $S$ be the largest set with $\Phi_B(S) \leq \phi$, for some $B \subseteq V$. If $S$ is small with $Vol(S) = aVol(B)$ with $a \leq 1/3$, then $\Phi_{B-S} \geq \phi(\frac{1-3a}{1-a})$
- Consider the minimal-conductance subgraph $R$ of the graph $B - S$. Suppose it has conductance $\Phi_{B-S}(R) < \phi(\frac{1-3a}{1-a})$.
- Let $T = R \cup S$ so $vol(T) > vol(S)$.
- Either $\phi_B(T) \leq \phi_B(S)$ or $\phi_B(B-T) \leq \phi_B(S)$, contradiction.

## 17 Sparsifying Arbitrary Graphs (Section 8-10)

- These deal with approximating this optimal partition, and using that to sparsify arbitrary weighted and unweighted graphs.
- It's mainly extending the ideas from sections 6 and 7 in routine ways.
- In general, if we union high-conductance subgraphs, the resulting subgraph has (with high probability) higher conductance.
- So the algorithm they use calls a *Partition* algorithm to find high-conductance subgraphs, and unions the resulting subgraphs together to improve the conductance.