Equipo 7

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Obtención de Esperanza y Varianza utilizando la Función Generadora de Momentos

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1. Variables Aleatorias Discretas

1.1. Bernoulli: $X \sim Bernoulli(p)$

Función Generadora de Momentos

$$M_X(x) = p\mathrm{e}^t - p + 1$$

- Primera Derivada

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left[p\mathrm{e}^t - p + 1\right] &= p \cdot \frac{\mathrm{d}}{\mathrm{d}t}\left[\mathrm{e}^t\right] + \frac{\mathrm{d}}{\mathrm{d}t}\left[-p\right] + \frac{\mathrm{d}}{\mathrm{d}t}\left[1\right] \\
&= p\mathrm{e}^t + 0 + 0 \\
&= p\mathrm{e}^t
\end{aligned}$$

- Segunda Derivada

$$\begin{aligned}
\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left[p\mathrm{e}^t - p + 1\right] &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}}{\mathrm{d}t}\left[p\mathrm{e}^t - p + 1\right]\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}p\mathrm{e}^t \\
&= p \cdot \frac{\mathrm{d}}{\mathrm{d}t}\left[\mathrm{e}^t\right] \\
&= p\mathrm{e}^t
\end{aligned}$$

Esperanza
Usando la primera derivada de la función generadora de momentos con $t = 0$

$$\mathbb{E}(X) = pe^t\Big|_{t=0} = pe^{(0)} = p(1) = p$$

Varianza
Usando la segunda derivada de la función generadora de momentos con $t = 0$

$$\begin{aligned}
Var(X) &= \mathbb{E}(X^2) - \mathbb{E}(X)^2 \\
&= ne^t\Big|_{t=0} - (p))^2 \\
&= pe^{(0)} - (pe^{(0)})^2 \\
&= p \cdot (1) - (p \cdot (1))^2 \\
&= p - p^2 \\
&= p(1 - p)
\end{aligned}$$

1.2. Binomial: $X \sim Bin(n,p)$

Función Generadora de Momentos

$$M_X(t) = (1-p+pe^t)^n$$

- Primera Derivada

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[(1-p+pe^t)^n\right] = n\left(pe^t-p+1\right)^{n-1}\cdot\frac{d}{dt}\left[pe^t-p+1\right]$$

$$= n\left(pe^t-p+1\right)^{n-1}\left(p\cdot\frac{d}{dt}[e^t]+\frac{d}{dt}[-p]+\frac{d}{dt}[1]\right)$$

$$= n\left(pe^t-p+1\right)^{n-1}\cdot\left(pe^t+0+0\right)$$

$$= npe^t\left(pe^t-p+1\right)^{n-1}$$

- Segunda Derivada

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left[(1-p+pe^t)^n\right] = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}}{\mathrm{d}t}(1-p+pe^t)^n\right)$$

$$= \frac{\mathrm{d}}{\mathrm{d}t}\left(npe^t\left(pe^t-p+1\right)^{n-1}\right)$$

$$= np\left(\frac{d}{dt}[e^t]\cdot\left(pe^t-p+1\right)^{n-1}+e^t\cdot\frac{d}{dt}\left[\left(pe^t-p+1\right)^{n-1}\right]\right)$$

$$= np\left(e^t\left(pe^t-p+1\right)^{n-1}+e^t(n-1)(pe^t-p+1)^{n-2}\left(p\cdot\frac{d}{dt}[e^t]+0+0\right)\right)$$

$$= np\Big((n-1)p(pe^t-p+1)^{n+2}e^{2t}+e^t(pe^t-p+1)^{n-1}\Big)$$

$$= npe^t\left(pe^t-p+1\right)^{n-2}\left(npe^t-p+1\right)$$

Esperanza

Usando la primera derivada de la función generadora de momentos con $t = 0$

$$
\begin{aligned}
\mathbb{E}(X) &= npe^t \left(pe^t - p + 1\right)^{n-1}\Big|_{t=0} \\
&= np(p - p + 1)^{n-1} \\
&= np
\end{aligned}
$$

Varianza

Usando la segunda derivada de la función generadora de momentos con $t = 0$

$$
\begin{aligned}
Var(X) &= \mathbb{E}(X^2) - \mathbb{E}(X)^2 \\
&= npe^t \left(pe^t - p + 1\right)^{n-2} \left(npe^t - p + 1\right)\Big|_{t=0} - (np)^2 \\
&= np(p - p + 1)^{n-2}(np - p + 1) - (np)^2 \\
&= np(np - p + 1) - (np)^2 \\
&= np(1 - p)
\end{aligned}
$$

1.3. Poisson: $X \sim Poission(\lambda)$

Función Generadora de Momentos

$$M_X(t) = e^{\lambda(e^t-1)}$$

- Primera Derivada

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[e^{\lambda(e^t-1)}\right] = e^{\lambda(e^t-1)} \cdot \frac{d}{dt}\left[\lambda(e^t-1)\right]$$

$$= e^{\lambda(e^t-1)} \cdot \lambda(e^t)$$

$$= \lambda e^{\lambda(e^t-1)+t}$$

- Segunda Derivada

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left[e^{\lambda(e^t-1)}\right] = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}}{\mathrm{d}t}\left[e^{\lambda(e^t-1)}\right]\right)$$

$$= \frac{\mathrm{d}}{\mathrm{d}t}\left(\lambda e^{\lambda(e^t-1)+t}\right)$$

$$= \lambda e^{\lambda(e^t-1)+t} \cdot \frac{d}{dt}\left[\lambda(e^t-1)+t\right]$$

$$= \lambda e^{\lambda(e^t-1)+t}(\lambda(e^t+0)+1)$$

$$= \lambda(\lambda e^t+1)e^{\lambda(e^t-1)+t}$$

Esperanza

Usando la primera derivada de la función generadora de momentos con $t = 0$

$$\mathbb{E}(X) = \left.\lambda e^{\lambda(e^t-1)+t}\right|_{t=0}$$

$$= \lambda e^{\lambda(1-1)+0}$$

$$= \lambda$$

Varianza

Usando la segunda derivada de la función generadora de momentos con $t = 0$

$$
\begin{aligned}
Var(X) &= \mathbb{E}(X^2) - \mathbb{E}(X)^2 \\
&= \lambda(\lambda e^t + 1)e^{\lambda(e^t-1)+t}\Big|_{t=0} - (\lambda)^2 \\
&= \lambda(\lambda + 1)e^{\lambda(1-1)+0} - (\lambda)^2 \\
&= \lambda(\lambda + 1) - \lambda^2 \\
&= \lambda^2 + \lambda - \lambda^2 \\
&= \lambda
\end{aligned}
$$

1.4. Geométrica: $X \sim Geom(p)$

Función Generadora de Momentos

$$M_X(t) = \frac{pe^t}{1-(1-p)e^t}$$

- Primera Derivada

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{pe^t}{1-(1-p)e^t}\right] &= \frac{p\left(\mathrm{e}^t\left(1-(1-p)\,\mathrm{e}^t\right)-(p-1)\,\mathrm{e}^{2t}\right)}{\left(1-(1-p)\,\mathrm{e}^t\right)^2} \\
&= \frac{(1-p)\,p\mathrm{e}^{2t}}{\left(1-(1-p)\,\mathrm{e}^t\right)^2} + \frac{p\mathrm{e}^t}{1-(1-p)\,\mathrm{e}^t} \\
&= \frac{p\mathrm{e}^t}{\left((p-1)\,\mathrm{e}^t+1\right)^2}
\end{aligned}$$

- Segunda Derivada

$$\begin{aligned}
\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left[\frac{pe^t}{1-(1-p)e^t}\right] &= \frac{p\left(\mathrm{e}^t\left((p-1)\,\mathrm{e}^t+1\right)^2-2\,(p-1)\left((p-1)\,\mathrm{e}^t+1\right)\mathrm{e}^{2t}\right)}{\left((p-1)\,\mathrm{e}^t+1\right)^4} \\
&= \frac{p\mathrm{e}^t}{\left((p-1)\,\mathrm{e}^t+1\right)^2} - \frac{2\,(p-1)\,p\mathrm{e}^{2t}}{\left((p-1)\,\mathrm{e}^t+1\right)^3} \\
&= -\frac{p\mathrm{e}^t\left((p-1)\,\mathrm{e}^t-1\right)}{\left((p-1)\,\mathrm{e}^t+1\right)^3}
\end{aligned}$$

Esperanza

Usando la primera derivada de la función generadora de momentos con $t = 0$

$$\begin{aligned}
\mathbb{E}(X) &= \frac{p\mathrm{e}^0}{\left((p-1)\,\mathrm{e}^0+1\right)^2} \\
&= \frac{p}{p^2} \\
&= \frac{1}{p}
\end{aligned}$$

Varianza

Usando la segunda derivada de la función generadora de momentos con $t = 0$:

$$\begin{aligned}
Var(X) &= \mathbb{E}(X^2) - \mathbb{E}(X)^2 \\
&= -\frac{p\mathrm{e}^0\left((p-1)\,\mathrm{e}^0-1\right)}{\left((p-1)\,\mathrm{e}^0+1\right)^3} - \frac{1}{p^2} \\
&= -\frac{p(p-2)}{p^3} - \frac{1}{p^2} \\
&= \frac{-p+2-1}{p^2} \\
&= \frac{1-p}{p^2}
\end{aligned}$$

1.5. Uniforme Discreta: $X \sim Unif(1, \ldots, n)$

Función Generadora de Momentos

$$M_X(t) = \frac{e^t(1-e^{nt})}{n(1-e^t)}$$

Esta la fórmula condensada pero nosotros utilizaremos la fórmula desarrollada previa a la condensada:

$$M_X(x) = \frac{1}{n}\left(e^t + e^{2t} + e^{3t} + e^{4t} + \ldots + e^{nt}\right)$$

- Primera Derivada

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{n}\left(e^t + e^{2t} + e^{3t} + e^{4t} + \ldots + e^{nt}\right)\right] = \frac{1}{n}\left(e^t + 2e^{2t} + 3e^{3t} + 4e^{4t} + \ldots + ne^{nt}\right)$$

- Segunda Derivada

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left[\frac{1}{n}\left(e^t + e^{2t} + e^{3t} + e^{4t} + \ldots + e^{nt}\right)\right] &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{n}\left(e^t + e^{2t} + e^{3t} + e^{4t} + \ldots + e^{nt}\right)\right)\\ &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{n}\left(e^t + 2e^{2t} + 3e^{3t} + 4e^{4t} + \ldots + ne^{nt}\right)\\ &= \frac{1}{n}\left(e^t + 2^2e^{2t} + 3^2e^{3t} + 4^2e^{4t} + \ldots + n^2e^{nt}\right)\end{aligned}$$

Esperanza

Usando la primera derivada de la función generadora de momentos con $t = 0$

$$\begin{aligned}\mathbb{E}(X) &= \frac{1}{n}\left(e^t + 2e^{2t} + 3e^{3t} + 4e^{4t} + \ldots + ne^{nt}\right)\Big|_{t=0}\\ &= \frac{1}{n}\left(e^0 + 2e^{2\cdot 0} + 3e^{3\cdot 0} + 4e^{4\cdot 0} + \ldots + ne^{n\cdot 0}\right)\\ &= \frac{1}{n}\left(e^0 + 2e^0 + 3e^0 + 4e^0 + \ldots + ne^0\right)\\ &= \frac{1}{n}(1 + 2 + 3 + 4 + \ldots + n)\\ &= \frac{1}{n}\left(\frac{n(n+1)}{2}\right)\\ &= \frac{(n+1)}{2}\end{aligned}$$

Varianza

Usando la segunda derivada de la función generadora de momentos con $t = 0$

$$
\begin{aligned}
Var(X) &= \mathbb{E}(X^2) - \mathbb{E}(X)^2 \\
&= \frac{1}{n}\left(e^t + 2^2e^{2t} + 3^2e^{3t} + 4^2e^{4t} + \ldots + n^2e^{nt}\right)\Big|_{t=0} - \left(\frac{(n+1)}{2}\right)^2 \\
&= \frac{1}{n}\left(e^0 + 2^2e^{2\cdot 0} + 3^2e^{3\cdot 0} + 4^2e^{4\cdot 0} + \ldots + n^2e^{n\cdot 0}\right) - \left(\frac{(n+1)}{2}\right)^2 \\
&= \frac{1}{n}\left(e^0 + 2^2e^0 + 3^2e^0 + 4^2e^0 + \ldots + n^2e^0\right) - \left(\frac{(n+1)}{2}\right)^2 \\
&= \frac{1}{n}\left(1 + 2^2 + 3^2 + 4^2 + \ldots + n^2\right) - \left(\frac{(n+1)}{2}\right)^2 \\
&= \frac{1}{n}\left(\frac{n(n+1)(2n+1)}{6}\right) - \left(\frac{(n+1)}{2}\right)^2 \\
&= \frac{(n+1)(2n+1)}{6} - \left(\frac{(n+1)}{2}\right)^2 \\
&= \frac{(n+1)(2n+1)}{6} - \frac{(n+1)^2}{4} \\
&= \frac{4(n+1)(2n+1) - 6(n+1)^2}{24} \\
&= \frac{8n^2 + 12n + 4 - 6n^2 - 12n - 6}{24} \\
&= \frac{2n^2 - 2}{24} \\
&= \frac{2\left(n^2 - 1\right)}{24} \\
&= \frac{n^2 - 1}{12}
\end{aligned}
$$

1.6. Binomial Negativa: $X \sim BN(r,p)$

Función Generadora de Momentos

$$M_X(t) = \left(\frac{p}{1-(1-p)e^t}\right)^r$$

- Primera Derivada

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left[\left(\frac{p}{1-(1-p)e^t}\right)^r\right] &= r\left(\frac{p}{(1-(1-p)e^t)}\right)^{r-1}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{p}{1-(1-p)e^t}\right) \\
&= -\frac{rp\left(\frac{p}{1-(1-p)e^t}\right)^{r-1}((p-1)e^t)}{(1-(1-p)e^t)^2} \\
&= -\frac{(p-1)\,pr\mathrm{e}^t\cdot\left(\frac{p}{1-(1-p)\mathrm{e}^t}\right)^{r-1}}{\left(1-(1-p)\,\mathrm{e}^t\right)^2} \\
&= \frac{(1-p)\,r\mathrm{e}^t\cdot\left(\frac{p}{1-(1-p)\mathrm{e}^t}\right)^{r}}{1-(1-p)\,\mathrm{e}^t}
\end{aligned}$$

- Segunda Derivada (ya evaluada en cero)

$$\begin{aligned}
\mathbb{E}(X^2) = &\frac{(1-p)r\left(\left((p-1)\,e^0+1\right)\left(e^0\cdot\left(\frac{p}{1-(1-p)e^0}\right)^r - \frac{(p-1)pr\cdot\left(\frac{p}{1-(1-p)e^0}\right)^{r-1}e^{2(0)}}{(1-(1-p)e^0)^2}\right)\right)}{\left((p-1)\,e^0+1\right)^2} \\
&- \frac{(1-p)r\,(p-1)\cdot\left(\frac{p}{1-(1-p)e^0}\right)^r e^{2(0)}}{\left((p-1)\,e^0+1\right)^2} \\
&= \frac{(1-p)r(p-p^2r+pr)}{p^3}
\end{aligned}$$

Esperanza

Usando la primera derivada de la función generadora de momentos con $t=0$

$$\begin{aligned}
\mathbb{E}(X) &= \frac{(1-p)\,r\mathrm{e}^0\cdot\left(\frac{p}{1-(1-p)\mathrm{e}^0}\right)^r}{1-(1-p)\,\mathrm{e}^0} \\
&= \frac{(1-p)r\left(\frac{p}{1-(1-p)}\right)^r}{1-(1-p)} \\
&= \frac{r(1-p)}{p}
\end{aligned}$$

Varianza

Usando la segunda derivada de la función generadora de momentos con $t=0$:

$$
\begin{aligned}
Var(X) &= \mathbb{E}(X^2) - \mathbb{E}(X)^2 \\
&= \frac{(1-p)r(p - p^2 r + pr)}{p^3} - \frac{r^2(1-p)^2}{p^2} \\
&= \frac{p^2(1-p)r(p - p^2 r + pr) - p^3 r^2 (1-p)^2}{p^5} \\
&= \frac{p^3(1-p)r}{p^5} \\
&= \frac{(1-p)r}{p^2}
\end{aligned}
$$

2. Variables Aleatorias Continuas

2.1. Uniforme Continua: $X \sim Unif(a,b)$

Función Generadora de Momentos

$$M_X(t) = \frac{e^{tb} - e^{ta}}{t(b-a)}$$

- Primera Derivada

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{e^{tb} - e^{ta}}{t(b-a)}\right] &= \frac{(\frac{\mathrm{d}}{\mathrm{d}t}(e^{bt}) - \frac{\mathrm{d}}{\mathrm{d}t}(e^{at}))t - (e^{bt} - e^{at})}{(b-a)t^2} \\
&= \frac{-e^{bt} + e^{at} + (e^{bt}b - e^{at}a)t}{(b-a)t^2} \\
&= \frac{t\left(b\mathrm{e}^{bt} - a\mathrm{e}^{at}\right) - \mathrm{e}^{bt} + \mathrm{e}^{at}}{(b-a)\,t^2} \\
&= \frac{(bt-1)\,\mathrm{e}^{bt} + (1-at)\,\mathrm{e}^{at}}{(b-a)\,t^2}
\end{aligned}$$

- Segunda Derivada

$$\begin{aligned}
\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left[\frac{e^{tb} - e^{ta}}{t(b-a)}\right] &= \frac{b^2e^{bt} - a^2e^{at}}{t(b-a)} - \frac{be^{bt} - ae^{at}}{t^2(b-a)} - \left(\frac{be^{bt} - ae^{at}}{t^2(b-a)} - \frac{2(e^{bt} - e^{at})}{t^3(b-a)}\right) \\
&= \frac{t^2b^2e^{bt} - t^2a^2e^{at} - 2tbe^{bt} - 2tae^{at} + 2e^{bt} - 2e^{at}}{t^3(b-a)} \\
&= \frac{e^{bt}(t^2b^2 - 2tb + 2) - e^{at}(t^2a^2 - 2ta + 2)}{t^3(b-a)}
\end{aligned}$$

Esperanza

Usando la primera derivada de la función generadora de momentos, aplicamos el límite cuando t tiende a cero y usamos la regla de L'Hopital:

$$\begin{aligned}
\mathbb{E}(X) &= \lim_{t\to 0} \frac{(bt-1)\,\mathrm{e}^{bt} + (1-at)\,\mathrm{e}^{at}}{(b-a)\,t^2} \\
&= \lim_{t\to 0} \frac{t(b^2e^{bt} - a^2e^{at})}{2t(b-a)} \\
&= \lim_{t\to 0} \frac{b^2e^{bt} - a^2e^{at}}{2(b-a)} \\
&= \frac{b^2 - a^2}{2(b-a)} \\
&= \frac{a+b}{2}
\end{aligned}$$

Varianza

Usando la segunda derivada de la función generadora de momentos, aplicamos el límite cuando t tiende a cero y aplicamos la regla de L'Hopital:

$$\begin{aligned}
\mathbb{E}(X^2) &= \lim_{t\to 0} \frac{be^t(t^2b^2 - 2tb + 2) + e^{bt}(2tb^2 - 2b) - ae^{at}(t^2a^2 - 2ta + 2) - e^{at}(2ta^2 - 2a)}{3t^2(b-a)} \\
&= \lim_{t\to 0} \frac{e^{bt}(t^2b^3 - 2tb^2 + 2b + 2tb^2 - 2b) - e^{at}(t^2a^3 - 2ta^2 + 2a + 2ta^2 - 2a)}{3t^2(b-a)} \\
&= \lim_{t\to 0} \frac{e^{bt}t^2b^3 - e^{at}t^2a^3}{3t^2(b-a)} \\
&= \lim_{t\to 0} \frac{e^{bt}b^3 - e^{at}a^3}{3(b-a)} \\
&= \frac{b^3 - a^3}{3(b-a)} \\
&= \frac{(b-a)(a^2 + ab + b^2)}{3(b-a)} \\
&= \frac{a^2 + ab + b^2}{3}
\end{aligned}$$

Así podemos obtener la varianza:

$$\begin{aligned}
Var(X) &= \mathbb{E}(X^2) - \mathbb{E}(X)^2 \\
&= \frac{a^2 + ab + b^2}{3} - \frac{(b-a)^2}{4} \\
&= \frac{4a^2 + 4ab + 4b^2 - 3a^2 - 6ab - 3b^2}{12} \\
&= \frac{a^2 - 2ab + b^2}{12} \\
&= \frac{(b-a)^2}{12}
\end{aligned}$$

2.2. Normal: $X \sim Norm(\mu, \sigma^2)$

Función Generadora de Momentos

$$M_X(x) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$$

- Primera Derivada

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left[\mathrm{e}^{\frac{\sigma^2t^2}{2}+\mu t}\right] &= \mathrm{e}^{\frac{\sigma^2t^2}{2}+\mu t}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\sigma^2t^2}{2}+\mu t\right]\\
&= \mathrm{e}^{\frac{\sigma^2t^2}{2}+\mu t}\left(\frac{\sigma^2}{2}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\left[t^2\right]+\mu\cdot\frac{\mathrm{d}}{\mathrm{d}t}\left[t\right]\right)\\
&= \mathrm{e}^{\frac{\sigma^2t^2}{2}+\mu t}\left(\frac{2t\sigma^2}{2}+\mu\cdot 1\right)\\
&= \left(\sigma^2t+\mu\right)\mathrm{e}^{\frac{\sigma^2t^2}{2}+\mu t}
\end{aligned}$$

- Segunda Derivada

$$\begin{aligned}
\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left[\mathrm{e}^{\frac{\sigma^2t^2}{2}+\mu t}\right] &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{\frac{\sigma^2t^2}{2}+\mu t}\right)\\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(\left(\sigma^2t+\mu\right)\mathrm{e}^{\frac{\sigma^2t^2}{2}+\mu t}\right)\\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left[\sigma^2t+\mu\right]\cdot\mathrm{e}^{\frac{\sigma^2t^2}{2}+\mu t}+\left(\sigma^2t+\mu\right)\cdot\frac{\mathrm{d}}{\mathrm{d}t}\left[\mathrm{e}^{\frac{\sigma^2t^2}{2}+\mu t}\right]\\
&= \left(\sigma^2\cdot\frac{\mathrm{d}}{\mathrm{d}t}\left[t\right]+\frac{\mathrm{d}}{\mathrm{d}t}\left[\mu\right]\right)\mathrm{e}^{\frac{\sigma^2t^2}{2}+\mu t}+\left(\sigma^2t+\mu\right)\mathrm{e}^{\frac{\sigma^2t^2}{2}+\mu t}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\sigma^2t^2}{2}+\mu t\right]\\
&= \left(\sigma^2\cdot 1+0\right)\mathrm{e}^{\frac{\sigma^2t^2}{2}+\mu t}+\left(\sigma^2t+\mu\right)\mathrm{e}^{\frac{\sigma^2t^2}{2}+\mu t}\left(\frac{\sigma^2}{2}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\left[t^2\right]+\mu\cdot\frac{\mathrm{d}}{\mathrm{d}t}\left[t\right]\right)\\
&= \left(\sigma^2t+\mu\right)\mathrm{e}^{\frac{\sigma^2t^2}{2}+\mu t}\left(\frac{2t\sigma^2}{2}+\mu\cdot 1\right)+\sigma^2\mathrm{e}^{\frac{\sigma^2t^2}{2}+\mu t}\\
&= \left(\sigma^2t+\mu\right)^2\mathrm{e}^{\frac{\sigma^2t^2}{2}+\mu t}+\sigma^2\mathrm{e}^{\frac{\sigma^2t^2}{2}+\mu t}
\end{aligned}$$

Esperanza

Usando la primera derivada de la función generadora de momentos con $t = 0$

$$\begin{aligned}
\mathbb{E}(X) &= \left(\sigma^2\cdot 0+\mu\right)\mathrm{e}^{\frac{\sigma^2 0^2}{2}+\mu\cdot 0}\\
&= (0+\mu)\,\mathrm{e}^{\frac{0}{2}+0}\\
&= (\mu)\,\mathrm{e}^{0}\\
&= (\mu)\cdot 1\\
&= \mu
\end{aligned}$$

Varianza

Usando la segunda derivada de la función generadora de momentos con $t = 0$

$$\begin{aligned}
Var(X) &= \mathbb{E}(X^2) - \mathbb{E}(X)^2 \\
&= \left[\left(\sigma^2 \cdot 0 + \mu\right)^2 \mathrm{e}^{\frac{\sigma^2 0^2}{2} + \mu \cdot 0} + \sigma^2 \mathrm{e}^{\frac{\sigma^2 0^2}{2} + \mu \cdot 0}\right] - \mu^2 \\
&= \left[(0 + \mu)^2 \mathrm{e}^{\frac{0}{2} + 0} + \sigma^2 \mathrm{e}^{\frac{0}{2} + 0}\right] - \mu^2 \\
&= \left[(\mu)^2 \mathrm{e}^0 + \sigma^2 \mathrm{e}^0\right] - \mu^2 \\
&= \left[(\mu)^2 \cdot 1 + \sigma^2 \cdot 1\right] - \mu^2 \\
&= \left[\mu^2 + \sigma^2\right] - \mu^2 \\
&= \mu^2 + \sigma^2 - \mu^2 \\
&= \sigma^2
\end{aligned}$$

2.3. Exponencial: $X \sim Exp(\lambda)$

Función Generadora de Momentos

$$M_X(x) = \frac{\lambda}{\lambda - t}$$

- Primera Derivada

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\lambda}{\lambda - t}\right] &= \lambda \cdot \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{\lambda - t}\right] \\
&= -\lambda \cdot \frac{\frac{\mathrm{d}}{\mathrm{d}t}\left[\lambda - t\right]}{(\lambda - t)^2} \\
&= -\frac{\lambda\left(\frac{\mathrm{d}}{\mathrm{d}t}\left[\lambda\right] - \frac{\mathrm{d}}{\mathrm{d}t}\left[t\right]\right)}{(\lambda - t)^2} \\
&= -\frac{\lambda\left(0 - 1\right)}{(\lambda - t)^2} \\
&= \frac{\lambda}{(\lambda - t)^2}
\end{aligned}$$

- Segunda Derivada

$$\begin{aligned}
\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left[\frac{\lambda}{\lambda - t}\right] &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\lambda}{\lambda - t}\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\lambda}{(\lambda - t)^2} \\
&= \lambda \cdot \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{(\lambda - t)^2}\right] \\
&= \lambda\,(-2)\,(\lambda - t)^{-3} \cdot \frac{\mathrm{d}}{\mathrm{d}t}\left[\lambda - t\right] \\
&= -\frac{2\lambda\left(\frac{\mathrm{d}}{\mathrm{d}t}\left[\lambda\right] - \frac{\mathrm{d}}{\mathrm{d}t}\left[t\right]\right)}{(\lambda - t)^3} \\
&= -\frac{2\lambda\,(0 - 1)}{(\lambda - t)^3} \\
&= \frac{2\lambda}{(\lambda - t)^3}
\end{aligned}$$

Esperanza

Usando la primera derivada de la función generadora de momentos con $t = 0$

$$\begin{aligned}
\mathbb{E}(X) &= \left.\frac{\lambda}{(\lambda - t)^2}\right|_{t=0} \\
&= \frac{\lambda}{(\lambda - 0)^2} \\
&= \frac{\lambda}{(\lambda)^2} \\
&= \frac{\lambda}{\lambda^2} \\
&= \frac{1}{\lambda}
\end{aligned}$$

Varianza

Usando la segunda derivada de la función generadora de momentos con $t = 0$

$$\begin{aligned}
Var(X) &= \mathbb{E}(X^2) - \mathbb{E}(X)^2 \\
&= \left.\frac{2\lambda}{(\lambda - t)^3}\right|_{t=0} - \left(\frac{1}{\lambda}\right)^2 \\
&= \frac{2\lambda}{(\lambda - 0)^3} - \frac{1}{\lambda^2} \\
&= \frac{2\lambda}{(\lambda - 0)^3} - \frac{1}{\lambda^2} \\
&= \frac{2\lambda}{\lambda^3} - \frac{1}{\lambda^2} \\
&= \frac{2}{\lambda^2} - \frac{1}{\lambda^2} \\
&= \frac{1}{\lambda^2}
\end{aligned}$$

2.4. Gamma: $X \sim \Gamma(k, \theta)$

Función Generadora de Momentos

$$M_X(t) = (1 - \theta t)^{-k}$$

- Primera Derivada

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left[(1-\theta t)^{-k}\right] &= (-k)\,(1-\theta t)^{-k-1} \cdot \frac{\mathrm{d}}{\mathrm{d}t}\,[1-\theta t] \\
&= (-k)\,(1-\theta t)^{-k-1}\left(\frac{\mathrm{d}}{\mathrm{d}t}\,[1] - \theta \cdot \frac{\mathrm{d}}{\mathrm{d}t}\,[t]\right) \\
&= (-k)\,(1-\theta t)^{-k-1}\,(0 - \theta \cdot 1) \\
&= k\theta\,(1-\theta t)^{-k-1}
\end{aligned}$$

- Segunda Derivada

$$\begin{aligned}
\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left[(1-\theta t)^{-k}\right] &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}}{\mathrm{d}t}(1-\theta t)^{-k}\right) \\
&= k\theta\,(1-\theta t)^{-k-1} \\
&= k\theta \cdot \frac{\mathrm{d}}{\mathrm{d}t}\left[(1-\theta t)^{-k-1}\right] \\
&= k\theta\,(-k-1)\,(1-\theta t)^{-k-2} \cdot \frac{\mathrm{d}}{\mathrm{d}t}\,[1-\theta t] \\
&= k\theta\,(-k-1)\,(1-\theta t)^{-k-2}\left(\frac{\mathrm{d}}{\mathrm{d}t}\,[1] - \theta \cdot \frac{\mathrm{d}}{\mathrm{d}t}\,[t]\right) \\
&= k\theta\,(-k-1)\,(1-\theta t)^{-k-2}\,(0 - \theta \cdot 1) \\
&= -\,(-k-1)\,k\theta^2\,(1-\theta t)^{-k-2}
\end{aligned}$$

Esperanza

Usando la primera derivada de la función generadora de momentos con $t = 0$

$$\begin{aligned}
\mathbb{E}(X) &= \left. k\theta\,(1-\theta t)^{-k-1} \right|_{t=0} \\
&= k\theta\,(1-\theta \cdot 0)^{k-1} \\
&= k\theta\,(1)^{k-1} \\
&= k\theta \cdot 1 \\
&= k\theta
\end{aligned}$$

Varianza

Usando la segunda derivada de la función generadora de momentos con $t = 0$

$$\begin{aligned}
Var(X) &= \mathbb{E}(X^2) - \mathbb{E}(X)^2 \\
&= -\left(-k-1\right) k\theta^2 \left(1-\theta t\right)^{-k-2}\Big|_{t=0} - \left(k\theta\right)^2 \\
&= -\left(-k-1\right) k\theta^2 \left(1-\theta \cdot 0\right)^{-k-2} - k^2\theta^2 \\
&= -\left(-k-1\right) k\theta^2 \left(1\right)^{-k-2} - k^2\theta^2 \\
&= -\left(-k-1\right) k\theta^2 \cdot 1 - k^2\theta^2 \\
&= k^2\theta^2 + k\theta^2 - k^2\theta^2 \\
&= k\theta^2
\end{aligned}$$