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A deterministic global optimization algorithm

Ying Ji *, Ke-Cun Zhang, Shao-Jian Qu

Faculty of Science, Xi'an Jiaotong University, Xi'an 710049, PR China

Abstract

In this paper, we consider a deterministic global optimization algorithm for solving a general linear sum of ratios (LFP). First, an equivalent optimization problem (LFP1) of LFP is derived by exploiting the characteristics of the constraints of LFP. By a new linearizing method the linearization relaxation function of the objective function of LFP1 is derived, then the linear relaxation programming (RLP) of LFP1 is constructed and the proposed branch and bound algorithm is convergent to the global minimum through the successive refinement of the linear relaxation of the feasible region of the objection function and the solutions of a series of RLP. And finally the numerical experiments are given to illustrate the feasibility of the proposed algorithm.

Keywords: General linear sum of ratios; Linearization relaxation; Branch and bound algorithm

1. Introduction

In this paper, we shall be concerned with a deterministic global optimization algorithm of the linear sum of ratios programming problem of the following form:

$$LFP: \begin{cases} \min \quad f(x) = \sum_{j=1}^{p} h_j(x) = \sum_{j=1}^{p} \dfrac{c_{j0} + c_j^{\mathrm{T}} x}{d_{j0} + d_j^{\mathrm{T}} x}, \\ \text{s.t.} \quad Ax \leqslant b, \ x \in R^n, \end{cases} \tag{1}$$

where $A \in R^{m\times n}, b \in R^m, c_{j0}$ and d_{j0} are all arbitrary real number and $c_j, d_j \in R^n, c_{j0} + c_j^{\mathrm{T}} x \geqslant 0$ and $d_{j0} + d_j^{\mathrm{T}} x \neq 0$, for any $x \in \{x | Ax \leqslant b\}$, $j = 1, \ldots, p$.

The linear sum of ratios problem is a special class optimization among fractional programming. It was worth study both because it frequently appears in application and because many other nonlinear problems can be transformed into this form. From a research point view, these problems pose significant theoretical and optimization problems, i.e., they are known to generally possess multiple local optima that are not global optima. So it is necessary to put forward good method.

* Corresponding author.
E-mail address: jiying_1981@126.com (Y. Ji).

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During the past years, various algorithms have been proposed for solving special cases of problem LFP, which are intended only for the sum of linear ratios problem with the assumption that $c_{j0} + c_j^T x \geqslant 0$ and $d_{j0} + d_j^T x > 0$, for any $x \in \{x|Ax \leqslant b\}$. For instance, in 1962, Charnes and Cooper [1] put forward a efficient elementary simplex method with $p = 1$ in LFP. Based on the work of [1], as $p = 2$, Konno [4] proposed one similar parametric elementary simplex method which can be used to solve large scale problem. A heuristic algorithm [3] is proposed when $p = 3$. Above algorithms will be impossible when $p \leqslant 4$. Recently, Kuno [2] proposed a new method to solve the maximization of linear sum of ratios, which used a concave function to upper estimate the optimal value of the original problem. A global optimization method was considered by Benson [5] by introducing a parameter, then the global optimal solution can be derived using convex optimization method and univariate research.

The purpose of this paper is to introduce a new deterministic global algorithm to solve the problem LFP. The main feature of this algorithm, (1) A equivalent optimization problem LFP1 is derived by exploiting the characteristics of the constraints. (2) A new linearizing method is proposed to linearize the objective function of LFP1. (3) The linear relaxation of LFP1 is obtained which is easier in computation than the method in [2] and does not generate new variables and constraints compared with the method in [5]. This allows it to be naturally incorporated into a branch-and-bound scheme. (4) The proposed branch and bound algorithm is convergent to the global minimum through the successive refinement of the linear relaxation of the feasible region of the objection function and the solutions of a series of RLP. Finally, the numerical tests show the efficiency of our method.

The paper is organized in the following way. We start by deriving the equivalent problem LFP1 of LFP and the linear relaxation LFP1. In Section 3, the new algorithm is introduced and convergence results are discussed. Section 4 reports the computational results obtained by solving some example. Finally concluding remarks are given.

2. The linear relaxation programming

Firstly, we solve the following $2n$ optimization problem:

$$\underline{x}_i = \begin{cases} \min & x_i, \\ \text{s.t.} & Ax \leqslant b, \end{cases} \tag{2}$$

$$\bar{x}_i = \begin{cases} \max & x_i, \\ \text{s.t.} & Ax \leqslant b, \end{cases} \tag{3}$$

where $i = 1, \ldots, n$. Then we can get the initial interval vector $X^0 = \{x : \underline{x}_i \leqslant x_i \leqslant \bar{x}_i, i = 1, \ldots, n\}$. Then the problem LFP can be rewritten in the following way:

$$LFP1(X^0) : \begin{cases} \min & f(x) = \sum_{j=1}^{p} h_j(x) = \sum_{j=1}^{p} \dfrac{c_{j0} + c_j^T x}{d_{j0} + d_j^T x}, \\ \text{s.t.} & Ax \leqslant b, \\ & x \in X^0. \end{cases} \tag{4}$$

Theorem 1. *Problem* LFP *and* LFP1 *have the same global optimal solution.*

Proof. Obviously if x is feasible to problem LFP, then $x \in X^0$, i.e., x is feasible to problem LFP1. On the other hand, if x is feasible to problem LFP1, then $Ax \leqslant b$, i.e., x is feasible to problem LFP. So they have the same feasible region, then conclusion follows. □

From Theorem 1, notice that, in order to globally solve problem LFP, we can globally solve problem LFP1 instead. In the following the main work is to globally solve the problem LFP1.

For convenience in exposition, in the following we assume that $X^k = [\underline{x}^k, \bar{x}^k]$ represents either the initial bounds on the variables of the problem, or modified bounds as defined for some partitioned subproblem in a branch-and-bound scheme. The linear relaxation of LFP1 can be realized by underestimating the function $f(x)$ with a linear function $f^l(x)$. All the details of this linearization technique for generating relaxations will be given in the following Theorems.

Given any $X = [\underline{x}, \bar{x}] \subset X^0 \subset R^{2p+N}$ and $\forall x = (x_n)_{(2p+N)\times 1} \in X$, the following notations are introduced:

$$f^l(x) = \sum_{j=1}^{p}(c_{j0} + c_j^{\mathrm{T}}x)\min\left\{\frac{1}{d_j^l}, \frac{1}{d_j^u}\right\}, \tag{5}$$

$$f^u(x) = \sum_{j=1}^{p}(c_{j0} + c_j^{\mathrm{T}}x)\max\left\{\frac{1}{d_j^l}, \frac{1}{d_j^u}\right\}, \tag{6}$$

$$d_j^l = \min_{x\in X}(d_{j0} + d_j^{\mathrm{T}}x);\, d_j^u = \max_{x\in X}(d_{j0} + d_j^{\mathrm{T}}x). \tag{7}$$

Theorem 2. *Assume $d_{j0} + d_j^{\mathrm{T}}x \neq 0$, for any $x \in X$, consider the value d_j^l and d_j^u defined as (7)*, we have $d_j^l d_j^u > 0$.

Proof. Obviously, there must be $x^1 \in X$ and $x^2 \in X$, such that $d_{j0} + d_j^{\mathrm{T}}x^1 = d_j^l$ and $d_{j0} + d_j^{\mathrm{T}}x^2 = d_j^u$. If $d_j^l d_j^u < 0$, then the following statement is true: $d_j^l < 0$ and $d_j^u > 0$, i.e., $d_{j0} + d_j^{\mathrm{T}}x^1 < 0$ and $d_{j0} + d_j^{\mathrm{T}}x^2 > 0$. Then there must be one positive number $\sigma \in (0,1)$, such that $d_{j0} + d_j^{\mathrm{T}}(x^1 + \sigma(x^2 - x^1)) = 0$, where $x^1 + \sigma(x^2 - x^1) \in X$, which contradicts $d_{j0} + d_j^{\mathrm{T}}x \neq 0$, for any $x \in X$. So the conclusion is valid. $\square$

Theorem 3. *Consider the functions $f_{(x)}$, $f^l(x)$ and $f^u(x)$ for any $x \in X$. Then the following two statements are valid*:

(i) *The functions $f^l(x)$, and $f^u(x)$ satisfy*:

$$f^l(x) \leqslant f(x) \leqslant f^u(x). \tag{8}$$

(ii) *The maximal errors of bounding $f(x)$ using $f^l(x)$ and $f^u(x)$ satisfy*

$$\lim_{\|\bar{x}-\underline{x}\|\to 0} \mathrm{Error}_{\max}^L = \lim_{\|\bar{x}-\underline{x}\|\to 0} \mathrm{Error}_{\max}^U = 0, \tag{9}$$

where

$$\mathrm{Error}_{\max}^L = \max_{x\in X} f(x) - f^l(x), \quad \mathrm{Error}_{\max}^U = \max_{x\in X} f^u(x) - f(x).$$

Proof

(i) From the construction method, (i) is direct.

(ii) Since

$$\begin{aligned} f(x) - f^l(x) &\leqslant \max_{x\in X}(c_{j0} + c_j^{\mathrm{T}}x)\left[\max\left\{\frac{1}{d_j^l}, \frac{1}{d_j^u}\right\} - \min\left\{\frac{1}{d_j^l}, \frac{1}{d_j^u}\right\}\right] \\ &= \max_{x\in X}(c_{j0} + c_j^{\mathrm{T}}x)\left|\frac{1}{d_j^u} - \frac{1}{d_j^l}\right|. \end{aligned} \tag{10}$$

We have $\frac{1}{d_j^u} = \frac{1}{d_j^l}$, as $\|\bar{x} - \underline{x}\| \to 0$. Therefore $\lim_{\|\bar{x}-\underline{x}\|\to 0}\mathrm{Error}_{\max}^L = 0$.

The same as the above proof, we also can get that $\lim_{\|\bar{x}-\underline{x}\|\to 0}\mathrm{Error}_{\max}^U = 0$. $\square$

Remark 1. By means of Theorem 3, we can give the linear relaxation of LFP1. Let $X^k = [\underline{x}^k, \bar{x}^k] \subset X^0$, consequently we construct the the corresponding approximation relaxation linear programming (RLP) of LFP1 in X^k as follows:

$$\mathrm{RLP}(X^k):\begin{cases}\min & f^l(x),\\ \text{s.t.} & Ax\leqslant b,\ x\in X^k.\end{cases}\tag{11}$$

Theorem 4. *The feasible solution of* RLP(Ω) *is also feasible to* LSP1(Ω) *and the optimal value of* RLP(Ω) *is less than that of* LSP1(Ω).

Proof. It is obviously from above discussions. □

Denote Vol[p] as the optimal value of the problem (P), from above theorem, for any $\overline{\Omega}\subseteq\Omega$, then the optimal values of LSP1($\overline{\Omega}$) and RLP($\overline{\Omega}$) satisfy: Vol[RLP1($\overline{\Omega}$)] $\leqslant$ Vol[GGP($\overline{\Omega}$)]. But from Theorem 3 RLP($\overline{\Omega}$) and LSP($\overline{\Omega}$) have the same optimal solution as $\overline{\Omega}\to 0$.

3. The algorithm and its convergence

In this section, a deterministic global optimization algorithm is given. By solving a sequence of linear relaxation programming RLP, the upper bound and lower bound of the optimal value of LSP1 will become better, finally the global optimization solution is obtained. In kth iteration, let L_k denotes the set made of the active set i.e., the subrectangle which possibly covers the global optimization solution, and Q_k denotes its corresponding index set, i.e., $\forall q\in Q_k$, there must be a subrectangle Ω^q satisfy $\Omega^q\subseteq\Omega\in L_k$. For any Ω^q, solve the linear programming RLP (Ω^q), we obtain its optimal value $\mathrm{LB}_q=\mathrm{Vol}[\mathrm{RLP}\ (\Omega^q)]$, then $\mathrm{LB}(k)=\min\{\mathrm{LB}_q{:}q\in Q_k\}$ is a lower bound of the optimal value of LSP1(Ω). $\forall\ q\in Q_k$, if the optimal solution of RLP(Ω^q) is feasible to LSP1(Ω) then update the upper bound Vol* of LSP(Ω) is necessary. Now a active node $\Omega^{q(k)}$ is chosen which satisfy $\mathrm{LB}(s)=\mathrm{LB}_{q(k)}=\{LB_q|q\in Q_k\}$, then $\Omega^{q(k)}$ is partitioning two parts and the linear programming over the two parts is solved respectively, the procedure is repeated until the stop rule is satisfied.

3.1. The algorithm statements

Step 1: *Initialization*. Give tolerance $\epsilon>0$, set $k=1$, $Q_k=\{1\}$, $q(k)=1$, $\Omega^1=\Omega$, $\mathrm{Vol}^*=\infty$. Solve RLP(Ω^1) and obtain its solution $\hat{x}$, let $x_1=\hat{x}\in\Omega^1$ and $\mathrm{LB}_1=\mathrm{Vol}[\mathrm{RLP}(\Omega^1)]$. If $\hat{x}$ is feasible to LSP1(Ω), then let $\mathrm{Vol}^*=\Phi_0(\hat{x})$. If $\mathrm{Vol}^*-LB_1<\epsilon$, then stop; otherwise, goto step 2.

Step 2: *Branching*. For the chosen subrectangle $\Omega^{q(k)}=\{x:\underline{x}_i^{q(k)}\leqslant x_i\leqslant\bar{x}_i^{q(k)},\forall i\in\{1,\ldots,n\}\}$, suppose $\hat{x}$ is the solution of RLP($\Omega^{q(k)}$). Let $p=\mathrm{argmax}\{\underline{x}_i^{q(k)}-\bar{x}_i^{q(k)}:i\{1,\ldots,n\}\}$, then partition $\Omega^{q(k)}$ into the following two parts $\Omega^{q(k)\cdot r}(r=1,2)$, where $\Omega^{q(k)\cdot 1}=\{x:\underline{x}_p^{q(k)}\leqslant x_p\leqslant\frac{\underline{x}_p^{q(k)}+\bar{x}_p^{q(k)}}{2},\underline{x}_i^{q(k)}\leqslant x_i\leqslant\bar{x}_i^{q(k)},\forall i\in\{1,\ldots,n\},i\neq p\}$ and $\Omega^{q(k)\cdot 2}=\{x:\frac{\underline{x}_p^{q(k)}+\bar{x}_p^{q(k)}}{2}\leqslant x_p\leqslant\bar{x}_p^{q(k)},\ \underline{x}_i^{q(k)}\leqslant x_i\leqslant\bar{x}_i^{q(k)},\forall i\in\{1,\ldots,n\},i\neq p\}$, and $Q_k=Q_k\backslash\{q(k)\}\ \cup\backslash\{q(k)\cdot r,r=1,2\}$. Compute $f^{l(\Omega^{q(k).r})}(\hat{x}),r=1,2,\ r=1,2$.

If $f^{l(\Omega^{q(k).r})}(\hat{x})>Vol^*$, or there is $t\in\{1,\ldots,m\}$, such that $(A\hat{x})_t>b_t$, then $L_k=L_k\backslash\ \{\Omega^{q(k)\cdot r}\}$ and $Q_k=Q_k\backslash\ \{q(k)\cdot r\}$, $r=1,2$. If $\Omega^{q(k)\cdot r}$ $(r=1,2)$ are all delete, then goto Step 4; Otherwise goto Step 3.

Step 3: *Bounding*. Solve RLP ($\Omega^{q(k)\cdot w}$), where ($w=1$, or $w=2$, or $w=1,2$) denote $x^{\Omega^{q(k)\cdot w}}$ as its optimal solution and $LB_{\Omega^{q(k)\cdot w}}=\mathrm{Vol}[\mathrm{RLP}(\Omega^{q(k)\cdot w})]$ as its optimal value. If $x^{\Omega^{q(k)\cdot w}}$ is feasible to LSP, then let $x_k=x^{\Omega^{q(k)\cdot w}}$ and $\mathrm{Vol}^*=\min\{\mathrm{Vol}^*,f(x_k)\}$.

Step 4: *Judging*. Let $Q_{k+1}=Q_k-\{q\in Q_k$: $\mathrm{Vol}^*-\mathrm{LB}_q\leqslant\epsilon\}$, and if $Q_{k+1}=\emptyset$, then stop with Vol* as the optimal value of LSP1(Ω) and $x_{\bar{\kappa}}(\bar{\kappa}\in\kappa^0)$ as the optimal solution, where $\kappa^0=\{\kappa$: $f_0(x_\kappa)=\mathrm{Vol}^*$, $\kappa=1,\ldots,k\}$; Otherwise, set $k{:}=k+1$ and choose $q(k)$ which satisfies $\mathrm{LB}_{q(k)}=\min\{\mathrm{LB}_q$: $q\in Q_k\}$, goto Step 2.

3.2. The convergence and the numerical test

Based on above discussions the following convergence theorem is derived.

Theorem 5. *Assume there is at least one global optimization solution of* LFP1, *then the algorithm either stops finitely with the global optimization solution of* LFP1, *or generates one infinite sequence which limits points must be the global optimization solution of* LFP1.

Proof. It is obviously the algorithm stops finitely with the global solution. Now assume one infinite sequence is generated, then one infinite active nodes $\{\Omega^{q(k)}\}$ is also generated, and the iteration $k \in K$ is also infinite. According to the algorithm

$$V[\mathrm{LFP1}(\Omega)] \geqslant \mathrm{LB}(k) = \mathrm{LB}_{q(k)} = V[\mathrm{RLP}(\Omega^{q(k)})] = f^{l(\Omega^{q(k)})}(x^{q(k)}), \ \forall\, k \in K, \tag{12}$$

where for any active node $q(k)$, $k \in K, x^{q(k)}$ is the optimal solution of $\mathrm{RLP}(\Omega^q)$. $\Omega^{q(k)} = \{x : \underline{x}^{q(k)} \leqslant x \leqslant \bar{x}^{q(k)}\}$, without loss generality, suppose $(x^*, \underline{x}^*, \bar{x}^*)$ $(k \to \infty)$ is the limit of arbitrary convergent subsequence $\{x^{q(k)}, \underline{x}^{q(k)}, \bar{x}^{q(k)}\}$. Let $\Omega^* = \{x : \underline{x}^* \leqslant x \leqslant \bar{x}^*\}$, Now we will prove x^* is the global optimization solution of $\mathrm{LSP1}(\Omega)$.

For k is sufficient large, from the branching scheme of the algorithm for any $i \in \{1, \ldots, n\}$, $\underline{x}_i^* = \bar{x}_i^*$, i.e., $x_i^* = \underline{x}_i^* = \bar{x}_i^*$. According to Theorem 3(ii),

$$f(x^*) = f^{l(\Omega^*)}(x^*) = \lim_{k \to \infty, k \in K} f^{l(\Omega^{q(k)})}(x^{q(k)}), \tag{13}$$

Since $x^{q(k)}$ is feasible to $\mathrm{RLP}(\Omega^{q(k)})$ and $\underline{x}^* \leqslant x^* \leqslant \bar{x}^*$, so $Ax^*P \leqslant b$. So x^* is feasible to $\mathrm{LSP1}(\Omega)$. So from (13) and (14), when $k \to \infty$,

$$V[\mathrm{LFP1}(\Omega)] \geqslant f^{l(\Omega^*)}(x^*) = f(x^*). \tag{14}$$

So $f(x^*) = V[\mathrm{LFP1}(\Omega)]$, i.e., x^* is the global optimization solution of $V[\mathrm{LFP1}(\Omega)]$. □

4. Numerical test

In the computation, the tolerance $\epsilon = 10^{-6}$ and the elementary simplex method is chosen to solve the linear programming. Examples 1 and 2 are chosen from [6] and the last one is constructed by ourselves. The results are summarized in the following table. In the table, the notations has been used for row headers: x^*: the global optimization solution, V^*: the global objective function value, IN: the number of the iteration.

Example 1

$$\begin{cases} \min & -\left(\frac{4x_1+3x_2+3x_3+50}{3x_2+3x_3+50} + \frac{3x_1+4x_3+50}{4x_1+4x_2+5x_3+50} + \frac{x_1+2x_2+5x_3+50}{x_1+5x_2+5x_3+50} + \frac{x_1+2x_2+4x_3+50}{5x_2+4x_3+50}\right), \\ \text{s.t.} & 2x_1 + x_2 + 5x_3 \leqslant 10, \\ & x_1 + 6x_2 + 3x_3 \leqslant 10, \\ & 5x_1 + 9x_2 + 2x_3 \leqslant 10, \\ & 9x_1 + 7x_2 + 3x_3 \leqslant 10, \\ & x_1, x_2, x_3 \geqslant 0. \end{cases}$$

Example 2

$$\begin{cases} \min & -\left(\frac{3x_1+5x_2+3x_3+50}{3x_1+4x_2+5x_3+50} + \frac{3x_1+4x_2+50}{4x_1+3x_2+2x_3+50} + \frac{4x_1+2x_2+4x_3+50}{5x_1+4x_2+3x_3+50}\right), \\ \text{s.t.} & 6x_1 + 3x_2 + 3x_3 \leqslant 10, \\ & 10x_1 + 3x_2 + 8x_3 \leqslant 10, \\ & x_1, x_2, x_3 \geqslant 0. \end{cases}$$

Example 3

$$\begin{cases} \min & \frac{x_1+3x_2+2}{4x_1+x_2+3} + \frac{4x_1+3x_2+1}{x_1+x_2+4}, \\ \text{s.t.} & -(x_1+x_2) \leqslant -1, \\ & x_1, x_2 \geqslant 0. \end{cases}$$

Example	x^*	V^*	IN
1	(1.0715,0,0)	−4.087412	17
2	(0,0.33329,0)	−3.000042	30
3	(1.0,0)	1.428571	10

According to Theorem 5, [6] and the algorithm, the computational results demonstrate the efficiency of our method.

5. The conclusion

A global convergent algorithm for solving general linear sum of ratios problem LSP is considered in this paper. Firstly, initial box constraints is derived by solving $2n$ linear programming problem, then one equivalent linear sum of ratios problem LSP1 is obtained. Secondly, one new linearizing method is proposed to linearize the objective function of LSP1. Thirdly, based on this linearizing method the linear relaxation programming RLP of LSP1 is constructed, the proposed branch and bound algorithm is convergent to the global minimum through the successive refinement of the linear relaxation of the feasible region of the objection function and the solutions of a series of RLP. And finally the numerical experiments are given to illustrate the feasibility of the proposed algorithm.

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