IP 1: knapsack

a) Our IP consists of the following:

Decision variables are: x_{shampoo}, $x_{\text{conditioner}}$, x_{shampoo}, $x_{\text{toothpaste}}$, x_{perfume}, x_{nyquil}, $x_{\text{lip balm}}$, where $x \in \{0, 1\}$.

The constraint is:

$$\sum_i x_i v_i \leq 1 \tag{1}$$

where v_i is the volume for item i.

The objective function is

$$\max \sum_i x_i u_i \tag{2}$$

where u_i is the utility for item i.

b) If we do not want to take more than 3 items, the following constraint needs to be added to 1

$$\sum_i x_i \leq 3$$

c) If we take conditioner, we also want to take shampoo. Therefore, we need to add an auxiliary variable to model this relationship. We define an auxiliary variable $C \in \{0, 1\}$ and M which is arbitrarily large, take in this case 100. The additional constraints are:

$$x_{\text{shampoo}} \geq x_{\text{conditioner}}$$

When conditioner is zero, it does not matter whether shampoo is zero or one, the inequality always holds. When conditioner is one, shampoo needs to be one for the inequality to hold.

d) If we take perfume, we cannot take lip balm. To model this constraint, we add an additional auxiliary variable $P \in \{0, 1\}$. The additional constraints are:

$$x_{\text{perfume}} x_{\text{lip balm}} \leq 0$$

We cannot have both perfume and lip balm, therefore the product can never be one.

e) If shampoo, nyquil or lip balm are brought, then at least two of those must be taken. We have to add an auxiliary variable $S \in \{0, 1\}$ to model this constraint. The additional constraints are then:

$$x_{\text{perfume}} + x_{\text{lip balm}} + x_{\text{nyquil}} \leq 0 + MS x_{\text{perfume}} + x_{\text{lip balm}} + x_{\text{nyquil}} \geq 2 - M(1 - S)$$

Like this, either the sum of all is 0 or 2+. Therefore, when we have one of the items we will take at least two.

IP 2: Facility location

a) Our IP consists of the following:

The decision variables are:

- location variables x_{NY}, x_{LA}, x_{Chicago}, x_{Atlanta}, where $x \in \{0, 1\}$
- variables for shipping amounts $s_{l,r}$ for $l \in \{\text{NY, LA, Chicago, Atlanta}\}$ and $r \in \{\text{r1, r2, r3}\}$.

For simplicity, we will call the set of locations loc and the set of regions reg from now on.

The constraints are:

$$\begin{aligned} \sum_{l \in loc} x_l s_{l,r1} &\geq 80 \\ \sum_{l \in loc} x_l s_{l,r2} &\geq 70 \\ \sum_{l \in loc} x_l s_{l,r3} &\geq 40 \\ \forall l \in loc \sum_{r \in reg} s_{l,r} &\leq 100 \end{aligned} \tag{3}$$

The objective is to minimize cost:

$$\min \sum_{l \in loc} \left(x_l c_l + \sum_{r \in reg} s_{l,r} c_{l,r} \right), \tag{4}$$

where c_l is the cost of having that location and $c_{l,r}$ is the cost of shipping an item from location l to region r.

b) If the New York warehouse is opened, the Los Angeles warehouse must be opened as well. We add the following constraint to 3

$$x_{\text{LA}} \geq x_{\text{NY}}$$

If we open our New York warehouse, the LA warehouse needs to be opened for the inequality to hold true. If the New York warehouse is not opened, the LA warehouse can be opened or closed.

c) If at most 2 warehouses can be opened, we need the following constraint

$$\sum_{l \in loc} x_l \leq 2$$

d) Either the Atlanta or the Los Angeles warehouses must be opened. We assume that this means both cannot be opened simultaneously. Therefore we add the constraints:

$$x_{\text{Atlanta}} + x_{\text{Los Angeles}} \leq 1$$
$$x_{\text{Atlanta}} + x_{\text{Los Angeles}} \geq 1$$

To ensure that the sum of both variables is exactly 1, which means that only 1 is opened, we need to include both inequalities.

e) A constraint for all shipping to only be 20 or more units. We need a the following constraint $\forall r \in reg \forall l \in l$

$$s_{l,r}^2 \geq 20 s_{l,r}$$

If the shipping amount is zero, the inequality will hold true. If the shipping amount is greater than zero, the inequality will hold true if and only if the shipping amount is 20+. Therefore, this inequality accurately covers the constraint.

However, this is not a linear constraint and will therefore complicate the IP further. Instead, we could use an auxiliary variable for each combination of region and city $a_{l,r} \in \{0, 1\}$. Then our constraints are $\forall l \in loc$, $\forall r \in reg$:

$$s_{l,r} \geq 20 - M a_{l,r}$$
$$s_{l,r} \leq 0 + M(1 - a_{l,r})$$

Alternatively, we could require each operating warehouse to ship at least 20 units to each region. However, this would be a binding constraint and change our optimal solution compared to the other two options. The constraints would then be, $\forall l \in loc$, $\forall r \in reg$:

$$s_{l,r} >= 20 * x_l$$

For the modelling of this question we have chosen the second option, since it provides the constraint we are looking for with the tools provided in class. Code for all options has been provided.

f) If the New York warehouse ships to region 1, no other warehouses can do so. For this, we need an auxiliary variable $Y \in \{0, 1\}$. The additional constraints are:

$$x_{N,1} \leq 0 + M(1 - Y)$$
$$x_{\text{Atlanta},1} + x_{\text{Chicago},1} + x_{\text{LA},1} \leq 0 + MY$$

We define Y to be positive if the shipping amounts are also positive and Y is zero if the shipping amounts are zero. For this, we need the following constraints $\forall r \in reg \forall l \in l$:

$$Y \leq s_{r,l}$$
$$Y s_{r,l} \geq s_{r,l}$$

The first constraint ensures that when shipping is zero the auxiliary variable is also zero and the second constraint ensures that when shipping is positive, the auxiliary variable is one.

g) For the Jupyter notebook, see the other submitted document.