Juggling Poset for Initial and Terminal State $< 1^k >$

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What is juggling

Starting Assumptions

To formalize juggling we start with some basic assumptions:

- Time is discrete
- Balls go forwards in time
- Juggling is periodic

Basic Juggling

The juggler can only catch or throw at most one ball at a time.

Multiplex Juggling

The juggler can catch or throw at most c balls at the time. We refer to this constraint as the hand capacity constraint.

Notating juggling patterns[1]

Siteswaps

Let $t = t_1 \dots t_n$, where $t_i \geq 0$ and $1 + t_1, 2 + t_2, \dots, n + t_n$ are all distinct mod n, in other words:

$$i + t_i \not\equiv j + t_j \pmod{n} \quad \forall i \neq j$$

t_i signifies a throw thrown on beat i that will land t_i beats in the future

Diagram for the siteswap 3

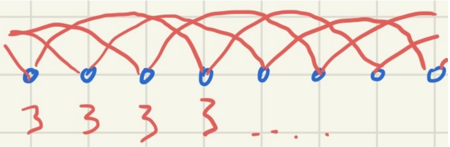
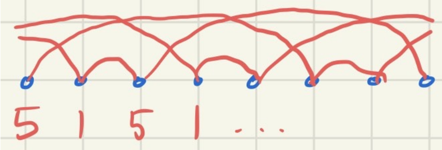


Diagram for the siteswap 51



[1] [But15]

Results for Siteswaps[2]

Theorem

If $t_1 \dots t_n$ is a valid siteswap, then we will need $b = (t_1 + \dots + t_n)/n$ balls to juggle the sequence.

Corollary

Given a valid siteswap $t_1 \dots t_n$, we have that $(t_1 + \dots + t_n)/n \in \mathbb{N}$

[2] [But15]

Proof of Corollary

Proof

By definition $1 + t_1, 2 + t_2, \dots, n + t_n$ are all distinct mod n. So we can say $1 + t_1 \pmod n, 2 + t_2 \pmod n, \dots, n + t_n \pmod n$ is some rearrangement of $0, 1, \dots, n-1$. We now have that,

$$\sum_{i=1}^{n} [i + t_i \pmod n] = \sum_{i=1}^{n} [i \pmod n] \to \sum_{i=1}^{n} (i + t_i) \equiv \sum_{i=1}^{n} i \pmod n$$

Which gives us,

$$\sum_{i=1}^{n} (i + t_i) - \sum_{i=1}^{n} i \equiv 0 \pmod n \to \sum_{i=1}^{n} t_i \equiv 0 \pmod n$$

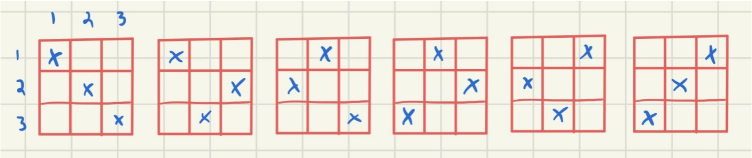
Results for siteswaps[3]

Minimal Juggling Pattern (MJP)

We say $t_1 \ldots t_n$ is a minimal juggling pattern if $t_i < n$.

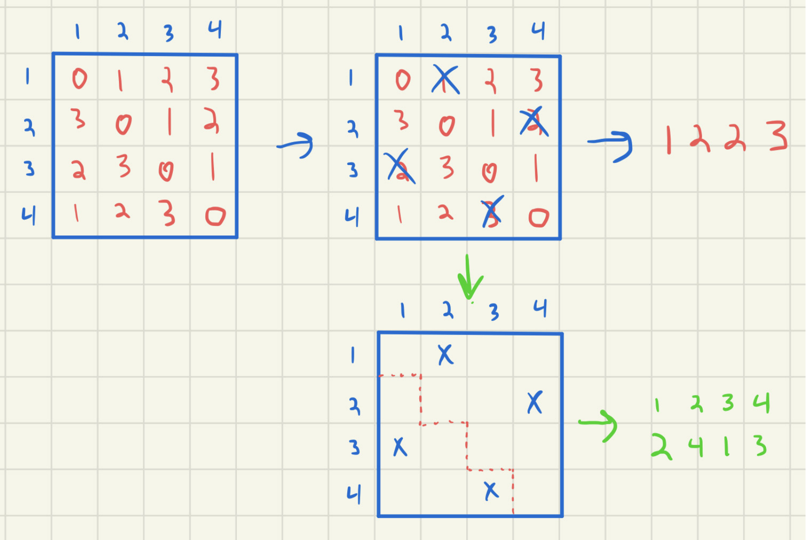
Theorem

1. There are $n!$ minimal juggling patterns (MJP's) of period n.
2. There is a bijection between MJP's of length n and $n-$rook placements on a nxn board.
3. The number of balls need for the pattern is exactly the number of rooks below the diagonal



[3] [But15]

The Bijection



Results of Siteswaps[4]

Theorem

The number of juggling patterns of period n with exactly b balls is:

$$(b+1)^n - b^n$$

Eulerian numbers

$\left\langle {n \atop k} \right\rangle = \#$ permutations in S_n with k ascents, descents, or drops. $\left\langle {n \atop k} \right\rangle$ also corresponds to the number of MJP's of length n with k balls

[4] [But15]

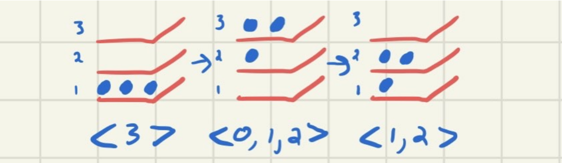
Notating Juggling Sequences[5]

Juggling States

$s = \langle s_1, \ldots s_n, \rangle \quad s_i \geq 0, \quad s_i =$ number of balls at height i.
Sequence of juggling states. Let $\mathbf{s}_1 \ldots \mathbf{s}_n$ be the individual juggling states. For $\mathbf{S} =< \mathbf{s}_1 \ldots \mathbf{s}_n >$ to be a valid juggling sequence, consecutive juggling states, $\mathbf{s}_{i-1} =< s_1, \ldots, s_h >$ and $\mathbf{s}_i$ must satisfy:

$$\mathbf{s}_i =< s_2 + b_1, s_3 + b_2, \ldots, s_h + b_{h-1}, b_h, b_{h'} >,$$

Where $b_j \geq 0$ and $\sum_{j=1}^{h'} b_j = s_1$.

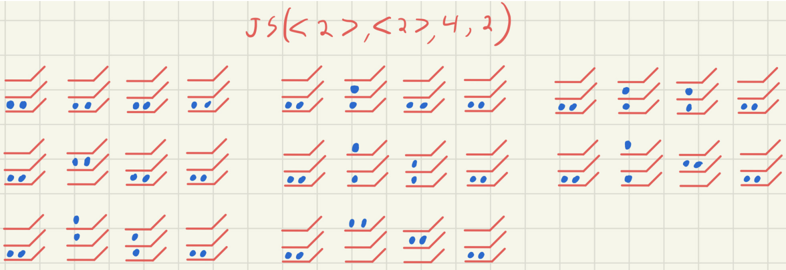


[5] [BHH+20]

More on Juggling States

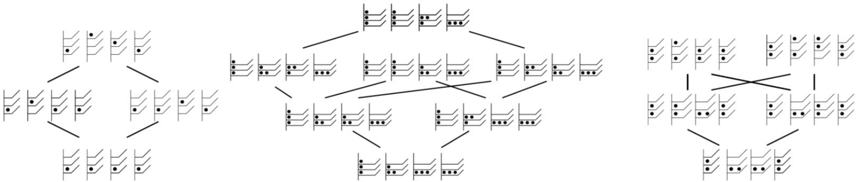
JS(a,b,n,m)

This refers to the set of Juggling sequences with an initial juggling state a, terminal state b, length n and hand capacity constraint m



The Poset[6]

Definition 6.12. Consider juggling states $\mathbf{a} = \langle a_1, \dots, a_s \rangle$ and $\mathbf{b} = \langle b_1, \dots, b_t \rangle$, satisfying $a_1 + \cdots + a_s = b_1 + \cdots + b_t$, and positive integers n and m. Let $\mathrm{PJS}(\mathbf{a}, \mathbf{b}, n, m)$ be the poset with elements $\mathrm{JS}(\mathbf{a}, \mathbf{b}, n, m)$ and cover relations $S \lessdot S'$ if S' is obtained from S by replacing both a throw at time i to height j and a throw at time $i + j$ to height k in S by one throw at time i to height $j + k$. As before, if there is no hand capacity constraint we omit m. See Figure 9 for examples.



(a) $\mathrm{PJS}(\langle 1 \rangle, \langle 1 \rangle, 3)$ (b) $\mathrm{PJS}(\langle 1,1,1 \rangle, \langle 3 \rangle, 3)$ (c) $\mathrm{PJS}(\langle 1,1 \rangle, \langle 1,1 \rangle, 3)$

FIGURE 9. Examples of juggling posets.

[6] [BHH$^+$20]

Sequence of throws

Thinking about Juggling Sequences by throws

Between juggling states balls in the air always fall, thus juggling sequences are determined by the balls in our hand and where we throw them.

To describe the throws, recall that consecutive juggling states, $\mathbf{s}_{i-1} =< s_1, \ldots, s_h >$ and $\mathbf{s}_i$ must satisfy:

$$\mathbf{s}_i =< s_2 + b_1, s_3 + b_2, \ldots, s_h + b_{h-1}, b_h, b_{h'} >,$$

where $b_j \geq 0$ and $\sum_{j=1}^{h'} b_j = s_1$. The throws between $\mathbf{s}_{i-1}$ and $\mathbf{s}_i$ would be $< b_1, b_2, \ldots, b_{h'} >$ and s_1 would be the number of throws.

Understanding the Poset

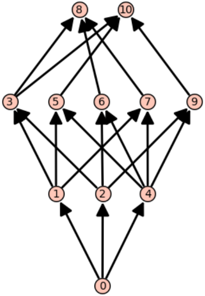
Generating the Cover Relation

A faster way of checking the relation between two juggling sequences S and S' is by comparing their throw sequences. It turns out that if there are exactly three differences then there is a cover relation.

Furthermore, let $S < S'$, then S has exactly one more throw then S'.

Poset example[7]

Figure: $\mathrm{PJS}(< 1, 1 >, < 1, 1 >, 3)$




```
0 [[1, 1], [2], [2], [1, 1]]
1 [[1, 1], [2], [1, 1], [1, 1]]
2 [[1, 1], [2], [1, 0, 1], [1, 1]]
3 [[1, 1], [2], [0, 1, 1], [1, 1]]
4 [[1, 1], [1, 1], [2], [1, 1]]
5 [[1, 1], [1, 1], [1, 1], [1, 1]]
6 [[1, 1], [1, 1], [1, 0, 1], [1, 1]]
7 [[1, 1], [1, 0, 1], [1, 1], [1, 1]]
8 [[1, 1], [1, 0, 1], [0, 1, 1], [1, 1]]
9 [[1, 1], [1, 0, 0, 1], [1, 0, 1], [1, 1]]
10 [[1, 1], [1, 0, 0, 1], [0, 1, 1], [1, 1]]
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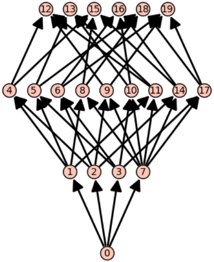
[7] [Sage]

Poset example[8]

Figure: PJS($<1,1,1>, <1,1,1>, 3$)




```
0 [[1, 1, 1], [2, 1], [3], [1, 1, 1]]
1 [[1, 1, 1], [2, 1], [2, 1], [1, 1, 1]]
2 [[1, 1, 1], [2, 1], [2, 0, 1], [1, 1, 1]]
3 [[1, 1, 1], [2, 1], [2, 0, 0, 1], [1, 1, 1]]
4 [[1, 1, 1], [2, 1], [1, 1, 1], [1, 1, 1]]
5 [[1, 1, 1], [2, 1], [1, 1, 0, 1], [1, 1, 1]]
6 [[1, 1, 1], [2, 1], [1, 0, 1, 1], [1, 1, 1]]
7 [[1, 1, 1], [1, 2], [3], [1, 1, 1]]
8 [[1, 1, 1], [1, 2], [2, 1], [1, 1, 1]]
9 [[1, 1, 1], [1, 2], [2, 0, 1], [1, 1, 1]]
10 [[1, 1, 1], [1, 2], [2, 0, 0, 1], [1, 1, 1]]
11 [[1, 1, 1], [1, 1, 1], [2, 1], [1, 1, 1]]
12 [[1, 1, 1], [1, 1, 1], [1, 1, 1], [1, 1, 1]]
13 [[1, 1, 1], [1, 1, 1], [1, 1, 0, 1], [1, 1, 1]]
14 [[1, 1, 1], [1, 1, 0, 1], [2, 0, 1], [1, 1, 1]]
15 [[1, 1, 1], [1, 1, 0, 1], [1, 1, 1], [1, 1, 1]]
16 [[1, 1, 1], [1, 1, 0, 1], [1, 0, 1, 1], [1, 1, 1]]
17 [[1, 1, 1], [1, 1, 0, 0, 1], [2, 0, 0, 1], [1, 1, 1]]
18 [[1, 1, 1], [1, 1, 0, 0, 1], [1, 1, 0, 1], [1, 1, 1]]
19 [[1, 1, 1], [1, 1, 0, 0, 1], [1, 0, 1, 1], [1, 1, 1]]
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[8] [Sage]

Results

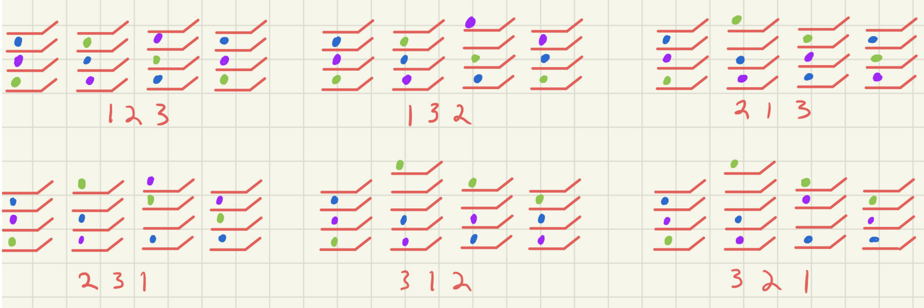
Theorem

Consider PJS(a,b,n), where $a = b =< 1^k >$, here $< 1^k >$ represents a sequence of k 1's and n is greater or equal to k.

- There exists a unique minimal element
- There exists a bijection between the maximal elements and the Symmetric Group S_k.

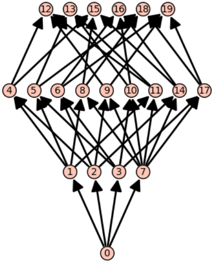
The Bijection

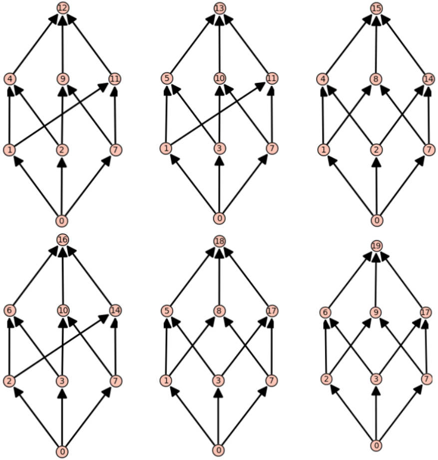
Maximal elements of $PJS(< 1,1,1 >, < 1,1,1 >, 3)$



Subposets[9]

Figure: PJS($< 1, 1, 1 >, < 1, 1, 1 >, 3$) with isomorphic subposets





9 [Sage]

Subposets

Conjecture

For $k \geq 6$, the subposets induced by the maximal elements in PJS$(< 1^k >, < 1^k >, n)$ are all isomorphic to each other. This has been tested for the cases $k < 6$.

References

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