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2019 Mathematical Contest in Modeling(MCM) Summary Sheet

Fictional Dragons in Real World

Abstract

The dragon is a mysterious and formidable creature widely known by people. Three dragons appearing in *Game of Throne* and *A Song of Ice and Fire* are people's favorite. For a long time, we people do not even know whether the dragon exists. Through our model, we can analyze the requirements for the dragon's living detaily.

To determine the growth of the dragon, we suppose the dragon is geometrical similar as time goes by. We use Richards equation to simulate, which presents a clear 'S' curve varying with time. This curve could be divided into three parts, including initial growth stage, exponential growth stage and steady growth stage. The turning points to divide these three stages are at the age of 19 and 131. The fastest growing speed occurs when the dragon is 52 years old.

To calculate the energy expenditure for the dragon, we suppose that the dragon will not fly and breathe fire unless predating. First we yield the conversion efficiency of the dragon's preies that net energy takes over 57% of the whole absorbing energy. Second, we divide the net energy into five parts, which are: the energy for basic metabolish, the energy for growth, the energy for flight, the energy for breathing fire and energy for recovery of trauma. These five parts are derived by the chemical formula of dragon's respiration, the derivative of 'S' trend curve, the emprical formula between speed consuming minimum power and the dragon's weight, the SFM model of the heating transferation and the coefficient describing the posibility of trauma. Since the energy of flying and breathing fire are coupling with the total energy expenditure by the flying distance and the number of preies, we develop an algorithm to calculate the total energy expenditure. This energy also presents a 'S' curve as time goes by. Finally, we answer the question of 'how much area is required to the three dragons', we use the method solving 'Cattle grazing problem' for reference.

After that, we build a model analyzing the manifestation of the dragon under different climate condition. The total energy expenditure and the distribution of the preies will both be affected by this factor. We use the density data of sheep and cattle in Australia to simulate. Our results shows that the higher temperature, the less energy the dragon consumes. The higher the humidity, the lower the energy the dragon need. And the influence of temperature on the dragon is greater than humidity.

At last, our model analyze the interaction between the dragon and environment. Cattle are most affected, followed by sheep, and hares have the least impact, and we can draw a conclusion that the dragon has a greater impact on the prey with high energy and dense distribution.

Sensitivity analysis shows that the trauma coefficient , the ratio of dragon's bone and muscle and the sheep's density are not sensitive to the total energy expenditure. For the first two factors, the energy for growth and recovery of trauma only take a low proportion of the total energy expenditure. For the last factor, this will only changes the energy for flying a little with flying distance's varying, which also takes a low proportion of the total energy.

Keywords: the dragon, growth curve, the SFM model, Maximum entropy method

Contents

1 Introduction

1.1 Background

In the famous fantasy drama television series *Game of Thrones* and its original novel series *A Song of Ice and Fire , Daenerys Targaryen* is one of the most popular characters. *The New York Times* cites her as one of the author's finest creatures, named as "Mother of Dragons", for she has three dragons: *Drogon, Rhaegal,* and *Viserion.* We appreciate *Daenerys Stormborn*, and are willing to see her conquering the Westeros with her dragons. But there is another question catching our eyesight now. What if these three fictional dragons are alive today?

Dragons are the creatures standing on the top of the food chain. Their existence will have a great impact on the ecological environment. It will consume quantities of resources to secure their survival. In order to make the existence of the dragon more possible, we need to figure out these processes.

1.2 Restatement of the Problem

To find out how the dragon will live in in a real world. We need to analyze the following questions:

- How will the dragon grow?
- How will the dragon interact with the ecological environment?
- Will the dragon perform differently in different environments? If true, what is the discrepancy?

In order to answer these questions, we take some of the scientific research results in the field of biology today into consideration.

1.3 Literature Review

The growth curve model is a model for studying the laws of a certain biological indicator's change varying with time. So far, scientists have used nonlinear curves to fit the growth trajectory related to animals. In 1838, *Verhulst* proposed the Logistic growth curve equation[1]. In 1938, *Von Bertalanffy* corrected the Logistic growth curve to accommodate the metabolic laws of the organism[2], and he used this for fish weight studies successfully. In 1959, *F.J. Richards* extended the Bertalanffy growth curve, and proposed the Richards growth curve[3].

Two general approaches have been in use to determine the distance to specified level of heat flux hazard from fires (Raj, 1977; Raj 1979; Considine, 1984; Moorehouse & Pritchard, 1982; McGrattan, et al., 2000). These are the so-called Solid Flame Model and the Point Source Model[4].

Species Distribution Models (SDMs) mainly use species distribution data and environmental data to estimate the niche of a species according to a specific algorithm. And project it into the landscape to indicate the species' preference for habitat in probabilistic. The results can be interpreted as the probability of occurrence of species, habitat suitability or species richness, etc.. In the 1970s, Nix et al first used SDMs to predict the spatial distribution of species[5]. After the 1990s, with the rapid developments of GIS technology and easy refining of remote sensing data related to climate, sea surface, land surface, greatly enhance the application of SDMs, a large number of species distribution models and software emerged.

2 Assumptions and Notations

2.1 Assumptions and Justifications

- **There is no human intervention in the ecosystem we discuss.** We only discuss the dragons' livelihood without human, because human's behavior can totally affect the whole objective laws of nature.

- **The dragons will not fly except for predation.** The dragons' flight may consume a lot of energy, and the purposeless flight is not propitious to survival. Thus we assume that the dragon will not fly in addition to predating.
- **The energy of the dragon's fire is provided with food.** In the real world, we do not consider the elements of magic. According to the law of conservation of energy, it is impossible for a dragon to create the energy for fire.
- **The dragons are carnivores.** The dragons are ferocious, and only eat meat containing sufficient energy.
- **The three dragons are far away from each other.** They will not influence each other's predating by the competiton effect.

2.2 Notations

Some notations used in this paper are listed in Tab. 2.1.

Table 2.1: Notations

Notation	Definition	Unit
	Variable	
E_n	total energy for expenditure	kJ
E_m	the energy for basic metabolish	kJ
E_g	the energy for growing in t_d	kJ
E_f	the energy for flying	kJ
E_b	the energy for breathing fire	kJ
E_t	the energy of recovery of trauma	kJ
E_v	the cost energy for flying	J/(kg • hour)
E_a	the energy for absorbing all preys	J
m_d	the weight of dragon	kg
dm_d	the gain of dragon's weight in t_d	kg
dm_b	the gain of dragon's bone weight in t_d	kg
dm_m	the gain of dragon's muscle weight in t_d	kg
ϕ_t	the heat flux producing by the dragon	kg
ϕ_l	the heat flux transfer to the side latheral area	kg
ϕ_b	the heat flux transfer to prey	kg
V_E	the consuming oxygen's volume	ml/(kg·min)
dV_b	the gain of the bone volume	m^3
dV_m	the gain of the muscle volume	m^3
T_e	the temperature of environment	°C
v_d	the flying speed refering to minimum power	m/s
n	the number of preys eaten by the dragon	none
	Constant	
E_p	the energy producing by 1kg protein	kJ/kg
E_{bone}	the energy for growing 1kg bone	kJ/kg
E_o	the energy producing by 1mol O_2	kJ/mol
E_{pi}	the energy of 1kg prey i	kcal/kg
ρ_o	the air density of oxygen at 1atm	kg/m^3
ρ_b	the desity of dragon's bone	kg/m^3
ρ_m	the desity of dragon's muscle	kg/m^3
ρ_{pi}	the distribution denstity of prey i	$/km^2$
m_{pi}	the weight of the prey i	kg
η	the absorbing coeffcient for prey	none
ξ	the trauma coefficient	none
c_{pi}	the heat capacity of the prey i	kJ/(kg·°C)
T_r	the reference temperature	°C
T_p	the temperature of prey able to eat	°C

3 Model Establishment

3.1 The Model of Dragon Growing

Organism has a common feature on it's growing speed, which can be discribed as Slow-Fast-Slow. Fig. 3.1 Clarifies the growth process of the organism.

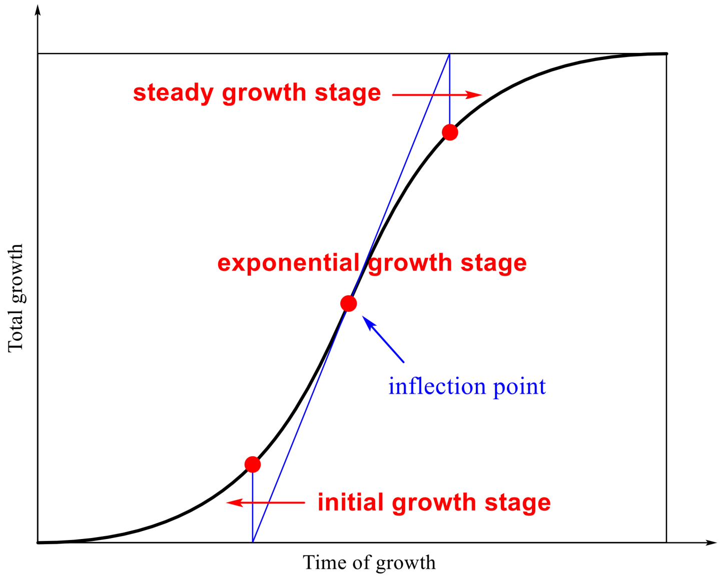


Figure 3.1: Sigmoid Form Curve of Growth Process

As we can see, the total growth of the organism shows a 'S' trend that includes three stages: the initial growing stage, the exponential growing stage and the steady growing stage. These three stages are determined by two crossover point, which the tangent line of the inflection point intersects with $y = 0$ and $y = A$.

Richards equation can simulate these three stages by four parameters with boilogical meanings: the maximum specific growth rate μ_m, the lag time λ, the asymptotic value A and the shape parameter v, which can be written as below[6]:

$$y = A\left\{1 + v \cdot \exp(1+v) \cdot \exp\left[\frac{\mu_m}{A} \cdot (1+v)^{\left(1+\frac{1}{v}\right)} \cdot (\lambda - t)\right]\right\}^{\left(-\frac{1}{v}\right)} \tag{3.1}$$

In order to simplify the problem, We suppose the dragon's shape satisfying geometrically similar with time going by[7]. This can be shown as:

$$S = kV^{\frac{2}{3}} \tag{3.2}$$

S is the surface area of the dragon, V is the volume of the dragon and k is a constant. If Richards equation satisfies Eq. 3.2 , we could yield $v = -\frac{1}{3}$, and Eq. 3.1 only has three parameters that need being determined, which transforms Bertaianffy equation. Considering the requirements in the problem that dragon's weight is 10kg when they were hatched and $30 - 40$ kg after a year's growing, so we need another requirement. We tend to determine this by estimating A, which represents the maximum weight of the dragon.

Since the dragon does not exist in the real life, we use the data of real creatures to analogize the weight of adult dragons. Tyrannosaurus Rex is a good choice whose shape is very similar to the dragon. After investigation, we found that the largest T. rex weighs about 14.85 tons. The latest T. rex was found in Montana, USA, in 1987. It is the most complete specimen found, and it has a skull about five feet long, which is equal to about 1.5 m[8]. For the dragon, we refer to the description in *Game of Thrones*, '....'s head is as large as a carriage'. We set the carriage to have length of 4 m. Then, according to geometrical similarity, we can derive:

$$\frac{14.85 \times 10^3}{A} = \frac{1.5^3}{4^3}$$

We yield $A = 281.6$ tons.

3.2 The Model of Dragon's Energy Expenditures

After clarifying how the dragon grows, we need to determine the energy that the dragon needs in daily life. Accoarding to the research results in bioenergetics, we divide the individual biological energy flow into several categories, which is shown in Fig. 3.2.

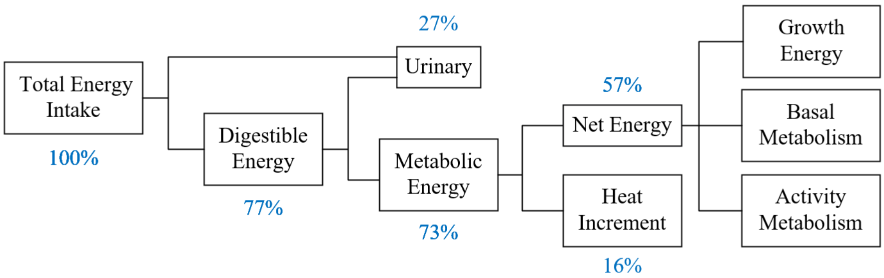


Figure 3.2: Flow Chart of Energy Distribution

To estimate the total energy intake of a dragon, we need to determine net energy. By our analysis, net energy can be divided into five parts, which are: energy for basic metabolish, energy for growth, energy for flight, energy for breathing fire and energy for recovery of trauma. As we can see them located in Tab. 3.1.

Table 3.1: Energy Cost Activities We Need to Analyze

Energy cost activity	category	Influencing factor
Energy for basic metabolish	Basal Metabolism	weight
Energy for growth	Growth Energy	time
Energy for flight	Activity Metabolism	location of food
Energy for breathing fire	Activity Metabolism	amount of food
Energy for recovery of trauma	Activity Metabolism	constant

From the consuming side, the total energy E_n for the dragon can be written as Eq. 3.3 below:

$$E_n = E_m + E_g + E_f + E_b + E_t \tag{3.3}$$

Next we will determine each five part of E_n.

3.2.1 Energy for basic metabolish

Animals produce energy by breathing while they live on earth, which is called cellular respiration. Even though an animal remains static in a certain place, they need the energy for basic metabolish E_m. We can calculate E_m by the chemical equation below:

$$C_6H_{12}O_6 + 6H_2O + 6O_2 \rightarrow 6CO_2 + 12H_2O + energy \tag{3.4}$$

If we could calculate the oxygen consumption, then we can get the energy releasing by the cellular respiration.

Oxygen consumption V_{o_2} increases progressively less with the increase in species body mass. V_{o_2}/kg is larger in small animals than in large animals, for small mammals have a larger body surface area/body mass ratio than heavier mammals. Dragon is a huge organism, so they may have less V_{o_2}/kg than the human.

Let V_E to be V_{o_2}/kg/min. By referring to the literature, we can get that V_E for an adult is 150 ml/kg/min, and we also can get that V_E for the mouse that is 600 ml/kg/min, almost four times of an adult[9]. Because the scale comparism between a dragon and an aldult is similar to that between an adult and a mouse, we can estimate V_E for dragon is 37.5 ml/kg/min.

$$E_m = \frac{\rho_o m_d V_E t}{6\mu_o} E_o \tag{3.5}$$

ρ_o means the air density of oxygen, m_d means the weight of the dragon, μ_o represents the relative molecular mass of oxygen, and E_o means the energy provided by 1 mol oxygen through Eq. 3.4 to compose ATP.

3.2.2 Energy for growth

The dragon's body mostly consists of bone and muscle, so the energy for dragon growing can be estimated by the bone growth and muscle growth. As we assume that the dragon's shape satisfying geometrically similar varying with time, the aixal growth rate for these two conponents remains the same but becomes different in radial direction. This process can be shown by Fig. 3.3.

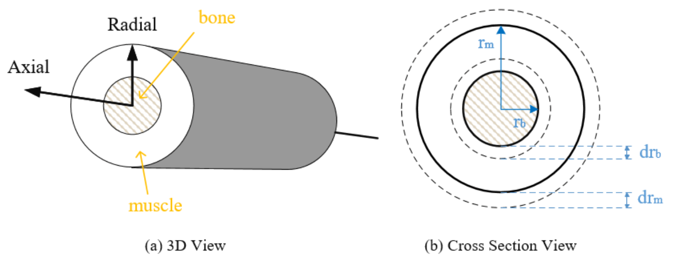


Figure 3.3: Sketch of Muscle and Bone

Let r_b be the external radius of bone, r_m be the external radius of muscle, so growth of bone volume dV_b, muscle volume dV_m, total volumn dV in time period t_d have the relationship:

$$\frac{dV_b}{dV_m} = \frac{2\pi r_b \cdot dr_b \cdot l}{2\pi r_m \cdot dr_m \cdot l} \tag{3.6}$$

$$dV = dV_b + dV_m \tag{3.7}$$

In order to keep the ratio of bone and muscle in cross section remaining the same, we can derive:

$$\frac{\pi r_b^2}{\pi r_m^2} = \frac{\pi (r_b + dr_b)^2}{\pi (r_m + dr_m)^2} \approx \frac{\pi \left(r_b^2 + 2r_b dr_b\right)}{\pi (r_m^2 + 2r_m dr_m)} = \frac{2\pi r_b dr_b}{2\pi r_m dr_m} \tag{3.8}$$

Combining Eq. 3.6 and Eq. 3.8, we can derive:

$$\frac{dV_b}{dV_m} = \frac{r_b^2}{r_m^2} \tag{3.9}$$

Let dm_b be the bone's weight increment, dm_m be the muscle's weight increment. Suppose the dm_m is contributed by protein, dm_b is contributed by protein partialy. E_p is the energy/kg protein contained, E_{bone} is the energy/kg bone contained. By referring to the literature, we can get bone consists of 10%-30% protein[10]. Other parts are water and bone mineral which do not contain energy. So we set $E_{bone} = 0.2E_p$:

$$E_{gm} = \rho_m dV_m E_p \tag{3.10}$$

$$E_{gb} = \rho_b dV_b E_{bone} \tag{3.11}$$

While ρ_m and ρ_b are the density of muscle and bone, dm_d is the mass increment of dragon in time period t_d. and they also satisfy:

$$dm_d = \rho_m dV_m + \rho_b dV_b = \left(\rho_m + \rho_b \frac{r_b^2}{r_m^2}\right)\left(\frac{r_b^2}{r_b^2 + r_m^2}\right) dV \tag{3.12}$$

$$E_g = E_{gm} + E_{gb} = \left(\rho_m \frac{r_m^2}{r_m^2 + r_b^2} E_p + \rho_b \frac{r_b^2}{r_m^2 + r_b^2} E_{bone}\right) dV \tag{3.13}$$

We set $r_b = 0.8 r_m$. Through Eq. 3.12, we can get dV . Then we could derive the energy of growth by Eq. 3.13.

3.2.3 Energy for flight

Dragons are aggresive, so they do not have any natural enemy. We suppose that drangons always fly in the speed by consuming minimum power, because they will not escape. Accroding to the literature, we can get the relationship between dragon's flying speed v_d and its weight:

$$v_d = 5.70 m_d^{0.16} \tag{3.14}$$

Let L_d be the total distance that the dragon fly to the prey, E_v be the cost energy /kg/hour which can be estimated by the literature. We can get the energy for flight below:

$$E_f = m_d \frac{L_d}{v_d} E_v \tag{3.15}$$

3.2.4 Energy for breathing fire

Accroding to our assumption, the dragon need to roast preys, for they have to eat them. The breathing fire can be depicted by flame. For analysis, we cannot and do not need the whole temperature field's distribution accurately, but using the Solid Flame Model (SFM) to simplify the flame[4].

In the SFM, we suppose the prey's shap is similar to the dragon's mouth, so the shape of the fire is chosen to be a circular cylinder, which can be depicted by the Fig. 3.4.

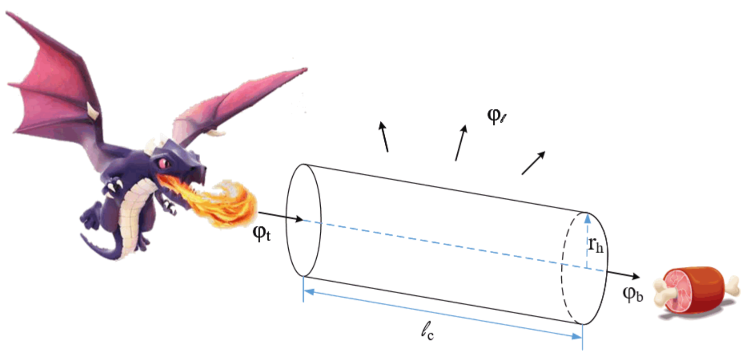


Figure 3.4: Sketch of Heat Transfer

ϕ_b is the heat flux at the bottom of cylinder, which outputs to the prey. ϕ_l is the heat flux transferring to the side lateral area of the cylinder, which represents the lost. ϕ_t is the heat flux inputted by dragon. r_h is the radius of the dragon's mouth, l_c is the length of the cylinder.

We set ϕ_l and ϕ_b are propotional to the tranferring area, and heat transfer ratio q be a constant, which can be determined:

$$\frac{\phi_l}{2\pi r_h l_c} = \frac{\phi_b}{\pi r_h^2} = q \tag{3.16}$$

By referring to Eq. 3.16, we can get the equality which shows the conservation of the heat energy:

$$\phi_t = \phi_l + \phi_b = \phi_b \left(1 + \frac{2l_c}{r_h}\right) \tag{3.17}$$

We set the heat capacity of the prey as c_p, prey's weight as m_p , heating time as t_p, environment temperature as T_e. We also set $l_c = 4r_h$, for this will keep almost the same while the dragon grows. The prey's temperature needs to be appropriate for the dragon to eat, and we set this as T_p. So we can derive:

$$\phi_b t_p = c_p m_p (T_p - T_e) \tag{3.18}$$

We let n be the number of the preys. Accroding to the Eq. 3.163.173.18, we can get the energy for breathing fire E_b:

$$E_b = \phi_t t_p = n c_p m_p \left(1 + \frac{2l_c}{r_h}\right)(T_p - T_e) \tag{3.19}$$

3.2.5 Energy for recovery of trauma

As we metioned above, the dragons are aggresive, so they seldom get hurt from the preys. We suppose this part of energy is propotional to the energy for basic metabolish, and the coeficient for the ratio ξ is small. So the energy for recovery of trauma can be written as:

$$E_t = \xi E_m \tag{3.20}$$

3.2.6 Algorithm for calculating the total energy expenditure

Eq. 3.3 determines E_n from the consuming side. In the aspect of absorbing side, We can calculate the net energy by Eq. 3.21 and Eq. 3.22 below:

$$E_n = 0.57\eta E_a \tag{3.21}$$

$$E_a = \sum_i n_i m_{pi} E_{pi} \tag{3.22}$$

Not all parts of the preys can be eaten by the dragon. η is the efficiency which indicates the propotion of preys that the dragon eats. E_a is the absorbed energy and E_q is the energy per kilograms for the preys. From Fig. 3.2, we can get E_n is 0.57 times of the absorbed energy.

From Eq. 3.15, we can get that E_f is proportional to the flying distance. From Eq 3.19, we can know that E_b is proportional to the number of preys. So these two parts of energy are associated to a single prey's energy and the preys' distribution, and this will set the total energy expenditure uncertain. In order to calculate the total energy expenditure, we develop an algorithm and keep the category of the preys and their distribution type fixed. We also find the population density of these preys in Australia online [11, 12] for a better simulation.

Our algorithm supposes that the dragon will find the nearest prey relating to its position, because this corresponds the lowest energy expenditure. Our algorithm is shown in 1:

Algorithm 1 Hunting Algorithm

Step 1: Set a maximum of age age_{max} to be simulated. For the age of dragon age from 0 to $age_{max} - 1$, do the following steps:

Step 2: Initialize a square area according the preset side length a, and generate preys(cattles and sheep) according the area of the square and the density of each kind of prey using random distribution. Initialize the number of preys caught by the dragon $n_{cattles} = n_{sheep} = 0$.

Step 3: Initialize the position of dragon (x_d, y_d) in the square area.

Step 4: Set the time period for predating $period$ =2 days. Calculate the growing speed μ and weight $weight$ at age age. Then calculate E_m, E_g, E_t, and initialize $E_f = E_b = 0$. Initialize absorbed energy $E_a = 0$ and $E_n = E_m + E_g + E_t$.

Step 5: While $0.57\eta E_a < E_n$, do the following steps:

Step 6: Search the nearest prey (x^*, y^*) in the square area.

$$L(x, y) = \sqrt{(x - x_d)^2 + (y - y_d)^2}$$
$$(x^*, y^*) = \underset{(x,y)}{\arg\min}\, L(x, y)$$

Step 7: Eat the prey at position (x^*, y^*) and update $n_{cattles}$or n_{sheep}, and update $(x_d, y_d) = (x^*, y^*)$. Calculate the energy for flying E_f and energy for breathing fire E_b, and update the energy consumed $E_n = E_n + E_f + E_b$. Update the energy absorbed $E_a = E_a + E_{prey}$where the E_{prey} is the energy of the prey eaten.

We can calculate the energy expenditure E_n for each age through this algorithm, and get the curve of E_n versus age. Then we can estimate the minimun area for the living of three dragons. This is similar to the famous 'Cattle grazing problem' proposed by Newton. Dragons are eating the preys and the preys will reproduce, and they would reach a transient balance.

Let B_i be the adding number of prey type i by considering breeding among time peroid t_b, t_d be the time period for the dragon's predation. Then the absorbing energy producing by prey type i in peroid t_b in supported area $S(age)$ can be written as:

$$v_i = \frac{B_i}{2\frac{t_b}{t_d}} \cdot \eta_i m_{pi} E_{pi} \rho_{pi} S(age) \tag{3.23}$$

In Eq. 3.23 , the formula dividing 2 means that each newborn prey is breeded by two mature preies. Also we can get Eq. 3.24by transient energy balance:

$$E_n(age) = \sum_i v_i \tag{3.24}$$

So we can derive S by Eq. 3.25:

$$S(age) = 2\frac{t_b}{t_d} \cdot \frac{E_n(age)}{\sum_i B_i \eta_i m_{pi} E_{pi} \rho_{pi}} \tag{3.25}$$

3.3 The model of the climate condition

Climate condition changes may have several influence on our model, mostly changing the temperature and the humidity of environment. We tend to analyze this influence for two parts: the influence on the dragon itself and the influence on the preys's distribution.

3.3.1 The influence on the dragon

After referring to the literature, we find that the higher environment temperature causes the more energy consuming for basic metabolish and we suppose that the tendency of this phemomenon is similar to a bird. Then we can get Eq. 3.26 by averaging the consumption in daytime and at night:

$$\frac{E_m(T_e)}{E_m(T_r)} = \frac{V_E(T_e)}{V_E(T_r)} = 1.82 - 0.033T_e \tag{3.26}$$

T_r is the reference temperature, which was defined to be 25 ℃. $E_m(T_e)$ is the energy consuming for basic metabolish at T_e while $E_m(T_r)$ is that at T_r .

Obviously, the temperature of environment will also influence E_b by affecting T_e. From Eq. 3.18 we can see: while T_e is lower, E_b becomes greater.

There will be no obvious influence on dragon while the humidity of environment changes, which agrees with our basic knowledge.

3.3.2 The influence on the preies's distribution

Different climates such as a arid region, a warm temperate region, and an arctic region, also affect the distribution of the dragon's food. We still take the Australia continent as an example. For Australia, according to website [13], we can find the states with different temperature and humidity. According to website [11, 12], we could determine the number of cattles and sheep in different states.

We use data of Queensland (Qld) and New South Wales (NSW) to compare different temperature conditions. We use Western Australia (WA) and South Australia (SA) to compare different humidity conditions. The detailed data is shown in the Tab. 3.2.

Table 3.2: Statistics for Australian States

State	Qld	NSW	WA	SA
Relative Temperature	low	high	$\times$	$\times$
Relative Humidity	$\times$	$\times$	high	low
Area ($\times 10^4$ km^2)	172.72	80.16	252.55	104.35
Number of Sheep (million)	2.1	26.9	14.2	11.5
Number of Cattle (million)	11.1	5.3	2.1	1.1
Density of Sheep (/km^2)	1.21	33.56	5.62	11.02
Density of Cattle (/km^2)	6.43	6.61	0.83	1.05

It can be seen from Tab. 3.2 that different climates affect the distribution density of sheep and cattle, thus changing the input parameters of our models.

3.4 Model of Interaction with Environment

In this part, we tend to determine which type of preys will be affected mostly by the dragon. We define three preys in our model, which are different in distribution density and meat energy. The detailed data of them is shown in Tab. 3.3.

Table 3.3: Detailed Data of Three Preies

Preies	Distribution Density (/km^2)	Meat Energy (kcal/100g)
Sheep	9.4514 (high)	118 (middle)
Cattle	3.4130 (middle)	125 (high)
Hare	2.426 (low)	102 (low)

(According to biological principle, it is impossible for a species to contain high individual energy as well as high distribution density.)

To determine the ecological impact of the dragon, we can input the species in Tab. 3.3 into our model, and statistic out the changes in distribution of each species.

4 Results

4.1 Results of the Model of Dragon Growing

4.1.1 Determination of λ and μ_m

According to Eq. 3.1, the dragon's weight y is a function of time t. As $A = 281.6$ tons is a constant, there are only two undetermined parameters: λ and μ_m. By using the constraints given by the problem description:

$$\begin{cases} y(0) = 10 \\ y(1) = b \qquad 30 \leq b \leq 40 \end{cases}$$

we solve the value of λ and $\mu_m$using SciPy's scipy.optimize.fsolve function. Because the result obtained by SciPy is related to the given initial value, we traverse all the initial values using step size 10 from 1 to 1000 for λ and μ_m, and then substitute the obtained solution into the original equation to measure its accuracy, and finally find the optimal solution. According to the range of b, we take three special values 30, 35, and 40 to solve the parameters, and the final result is shown in Tab. 4.1.

Table 4.1: Solving results λ and μ_m

$y(1)$	λ	μ_m
30	26.31	1895.42
35	22.42	2224.25
40	19.76	2523.83

From Tab. 4.1, we can conclude that when A and $y(0)$ is a constant, the larger $y(1)$, the smaller λ, because the faster the growth rate, the shorter the lag time; the larger $y(1)$, the larger μ_m, because the dragon can grow bigger at a short time, and then it will grow faster when bigger, which forms a virtuous cycle.

4.1.2 Determination of Growth Stages

Once λ and μ_m are determined, we can draw the tangent of y at the inflection point, which can be written as follows:

$$y = \mu_m(t - \lambda)$$

Then we can derive the three essential points and stages of the growth curve by

- the intersection of a vertical line which passes the intersection of the tangent and the horizontal line $y = 0$ and the original y-curve,
- the intersection of the tangent and the original y-curve,
- and the intersection of a vertical line which passes the intersection of the tangent and the horizontal line $y = A$ and the original y-curve.

When $b = 40$, the results of our solution are shown in Fig. 4.1.

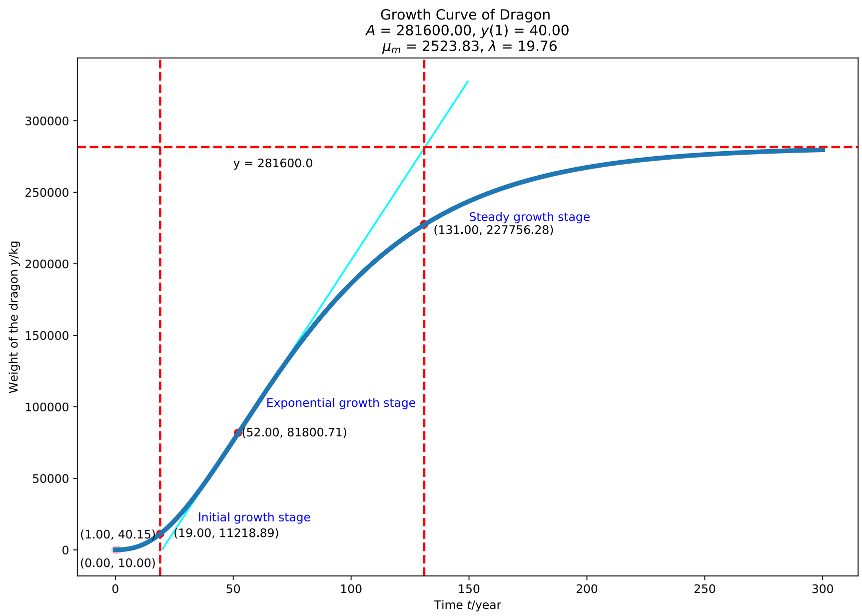


Figure 4.1: Groth curve of a dragon when $y(1) = 40$

Form Fig. 4.1, we can conclude that, when $y(1) = 40$, the dragon's asymptotic weight value A can be reached after nearly 280 years, which is a long time. The periods of the three stages of a dragon are:

1. Initial growth stage: 0 - 19 years old
2. Exponential growth stage: 19 - 131 years old
3. Steady growth stage: > 131 years old

The time when the dragon reaches maximum growth rate μ_m is around 52 years old, and it weights 81,800.71 kg.

The predating path of the dragon is shown in Fig. 4.2.

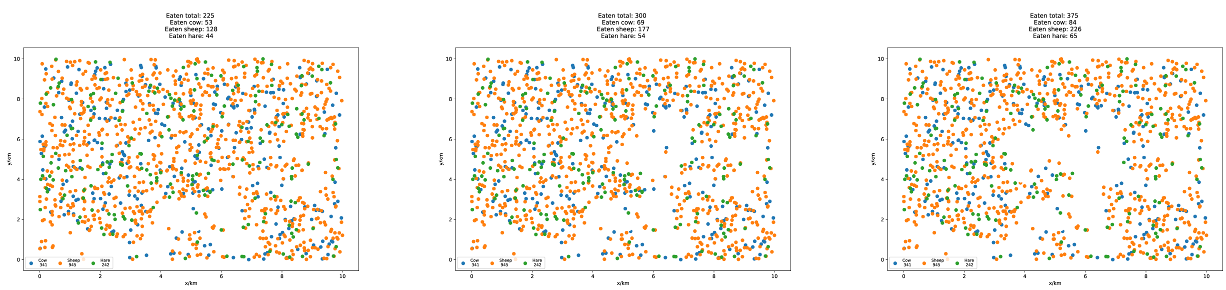


Figure 4.2: Predating Path of Dragon

4.2 Results of the Model of Dragon's Energy Expenditures

In our implementation of Algorithm 1, we keep the preset parameters unchanged, run and output the results for further analysis.

4.2.1 Preset Parameters

- We simulate the age of dragon form 0 to 500 years old.
- The dragon hunts once every 2 days.
- We assume $y(1) = 40$ and make use of the result of Fig. 4.1. If using other value of $y(1)$, the results will be similar.
- The side of the square area is empirically set to 10 meters to meet the dragon's hunting demand.
- The dragon starts hunting at position $(0, 0)$, which is the left-bottom of the square.
- Values of constant in Subsection 2.2 we use are shown in Tab.

Table 4.2: Values of constant

Notation	Definition	Value
E_p	the energy producing by 1kg protein	17,130 kJ/kg
E_{bone}	the energy for growing 1kg bone	3,426 kJ/kg
E_o	the energy producing by 1mol O_2	193.5 kJ/mol
ρ_o	the air density of oxygen at 1atm	1.429 kg/m^3
ρ_b	the desity of dragon's bone	1,230 kg/m^3
ρ_{p0}	the distribution denstity of cattle	9.4514 /km^2
ρ_{p1}	the distribution denstity of sheep	3.4130 /km^2
ρ_m	the desity of dragon's muscle	1,120 kg/m^3
E_{p0}	the energy of 1kg cow	1,250 kcal/kg
E_{p1}	the energy of 1kg sheep	1,180kcal/kg
m_{p0}	the weight of a cow	753 kg
m_{p1}	the weight of a sheep	87.5 kg
η	the absorbing coeffcient for prey	0.7
ξ	the trauma coefficient	0.57
c_{p0}	the heat capacity of cattle	1.47 kJ/(kg·°C)
c_{p1}	the heat capacity of sheep	1.50 kJ/(kg·°C)
T_r	the reference temperature	25 °C
T_e	the temperature of environment	25 °C
T_p	the temperature of prey able to eat	80 °C

4.2.2 Simulation Results

By running Algorithm 1, we get the output as Fig. 4.3, Fig. 4.4 and Fig. 4.5.

Energy expenditures of a dragon and preys eaten in two days vary with age is shown in Fig. 4.3. From Fig. 4.3, we can see that:

- The energy a dragon need is also S-shaped. This is because E_m and E_t increase as y increases.
- The curve is steeper than the growth curve Fig. 4.1 at the inflection age (52 years old). This is because E_g increases as μ increases, not y, the former increases faster than the latter.

- A mature dragon needs 470 cattles and 178 sheep, which is 140, 000 million calories of energy, per two days.

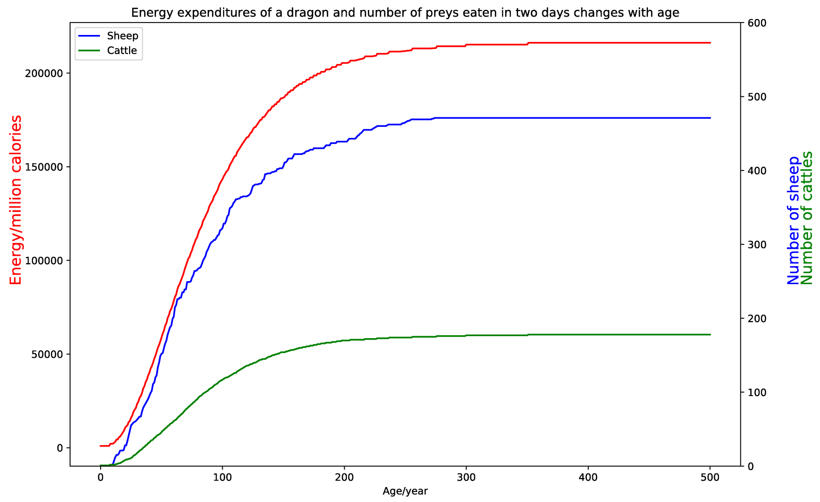


Figure 4.3: Energy expenditures of a dragon and preys eaten in two days$-age$ curve

Minmum area required to support the three dragons vary with age is shown in Fig. 4.4. As for minmum area required to support the three dragons, accoarding to Eq. 3.25, we can determine the minmum area to support a dragon. Then the minmum area of three dragons is three times of $S(age)$ obtained by Eq. 3.25, which is shown in Fig. 4.4. We can see that:

- The trajectory is the same as that of the Fig. 4.3. This is because the area required is proportional to the preys eaten.
- We need at least about 14, 000 km^2 area to support three mature dragons.

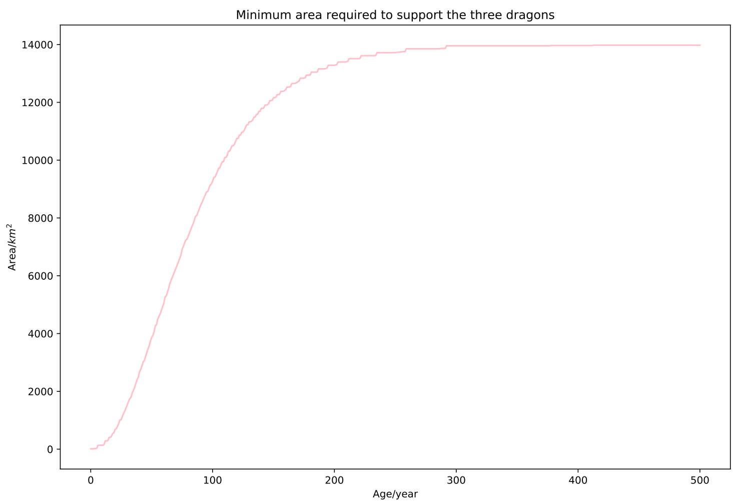


Figure 4.4: Minmum area required to support the three dragons$-age$ curve

Energy percentage of a dragon in different stages is shown in Fig 4.5. Through the Fig. 4.5, we could get the information below:

- The energy for basic metabolish E_m takes the most percentage of the total energy expenditure E_n during the whole life of the dragon, while the energy of growth, the energy of flying and the energy of recovery from trauma only take a low proportion. Because the dragon's growth causes an increase on its weight, which will make E_m become greater.
- The energy for Breathing fire E_b first takes a high precentage of E_n when the dragon is young. After it beomes mature, the percentage of E_b decreases a little and then become stable, for the reason that E_b is proportional to the number of predated preys n.

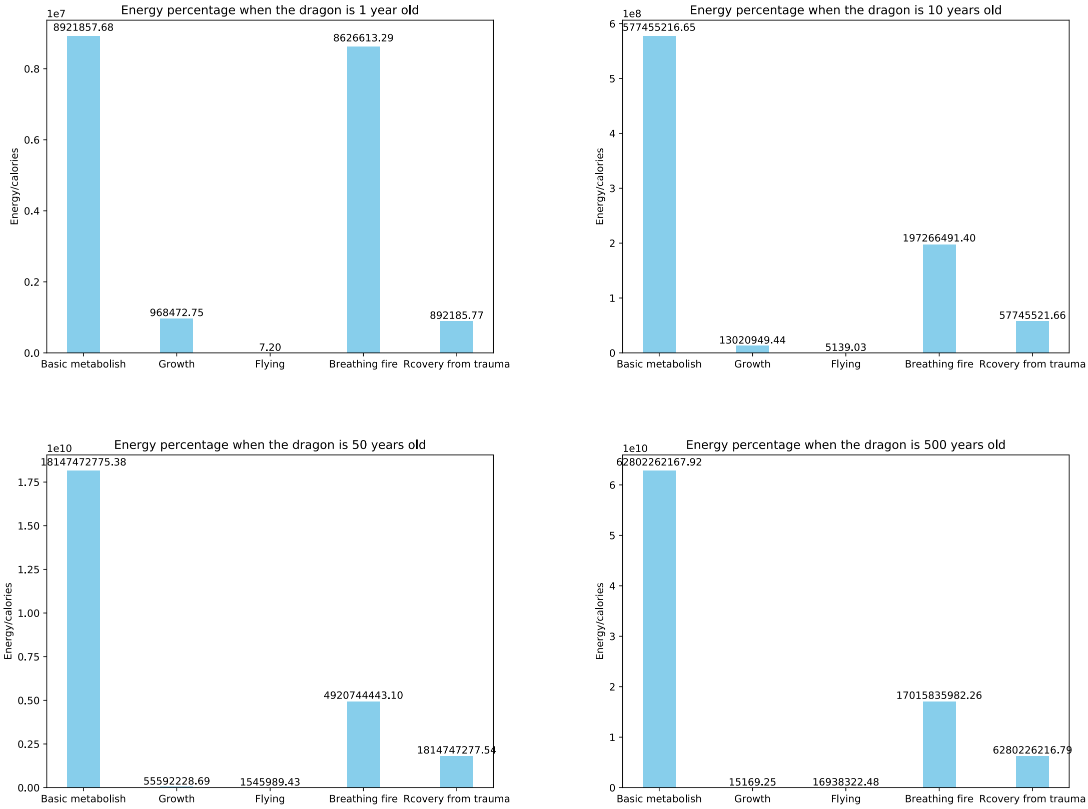
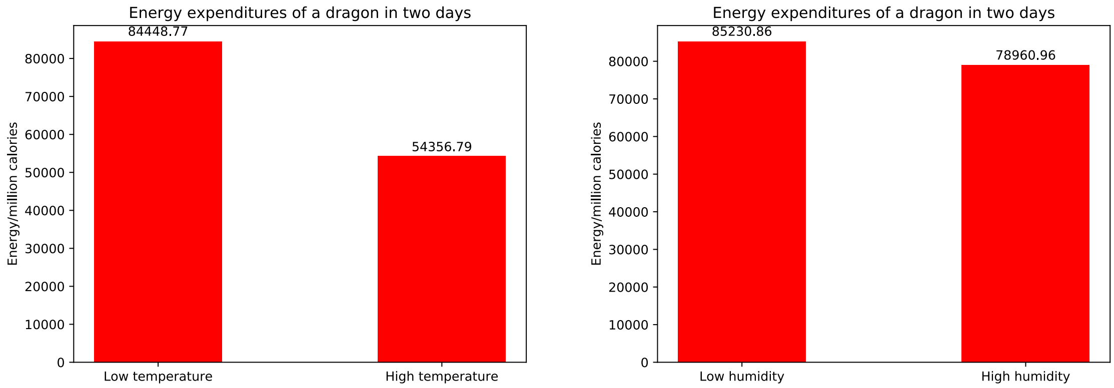


Figure 4.5: Energy percentage of a dragon$-age$ histgram

4.3 Results of the model of the climate condition

We fix the dragon's age at 280 years old. Using the data of sheep and cattle density in Qld, NSW, WA and SA. The simulation results are shown in Fig. 4.6.



(a) Energy expenditures of a dragon in two days—T_e histgram (b)Energy expenditures of a dragon in two days—Humidity histgram

Figure 4.6: Simulation Results of the Model of the Climate Condition

We can conclude that:

- The higher T_e, the less energy the dragon consumes. This is because according to Eq. 3.19 and Eq. 3.26, the higher T_e, the less $E_b$and E_m. And other energy expenditure components do not change with T_e, so the energy required by the dragon is reduced. At the same time, the higher T_e, the higher the density of preys. Thus, the dragon can obtain a prey with less flight distance. So the results come cout.
- The higher the humidity, the lower the energy the dragon need. This is because prey is often densely distributed in areas with high humidity, making it easier for dragons to predate.
- The influence of temperature on the dragon is greater than humidity.

4.4 Results of the Model of Interaction with Environment

We preset the age of the dragon to 100 years old, and then run step 2 - 7 of the Algorithm 1, drawing the distribution changes with dragon's hunting contrast without dragon's hunting as Fig. 4.7.

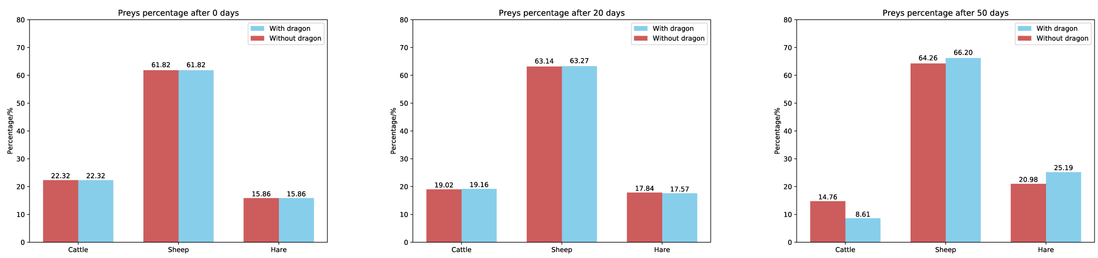


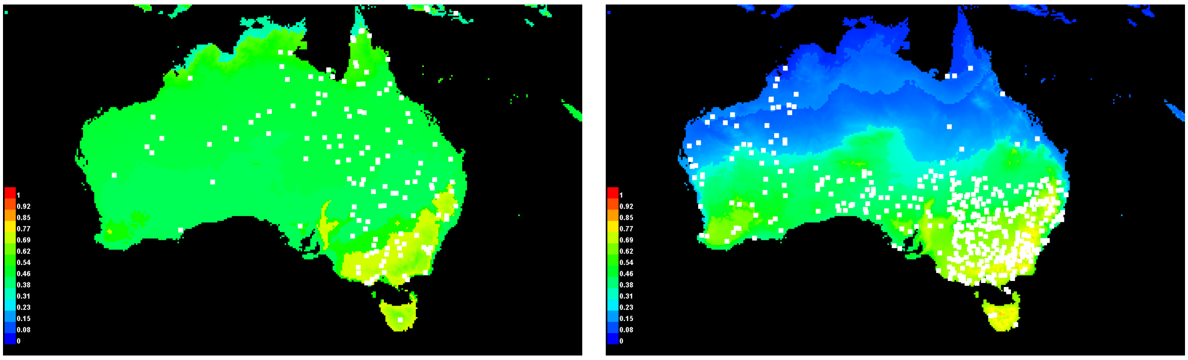
Figure 4.7: Preys percentage changes when the dragon is 100 years old—age histgram

From the figure, we can conclude that, with dragon hunting, the proportion of cattle is smaller than that of normal breeding, and other animals are opposite. And the increase in the proportion of hare is greater than that of sheep. In short, cattle are most affected, followed by sheep, and hares have the least impact. Therefore, we can draw general conclusions: the dragon has a great impact on the prey with high energy and dense distribution, and has little impact on prey with low energy and sparse distribution.

5 Model Expansion

Our previous model is based on the assumption that species are subject to random distribution. Under this assumption, we ignore the differences in the environment. However, in the real world, the environment of a region cannot be the same everywhere.

In order to get closer to the real situation of species distribution, we use Maxent (Maximum entropy)[14] to model the species geographic distributions. In maximum entropy density estimation, we input the species distribution samples data (species name, longitude, latitude)[11, 12] of cattles and sheep and climate data [15] in Australia, and then the true distribution of a species is represented as a probability distribution π over the set X of sites in the study area. Thus, π assigns a non-negative value to every site x and the values $\pi(x)$ sum to one. The weather constraints are expressed in terms of simple functions of the environmental variables, called features. The mean of each feature is required to be close (within some error bounds) to the empirical average over the union of the presence sites. As a result, the species appropriate survival map in Australia is shown in the Fig. 5.1.



(a) Cattles distribution model (b) Sheep distribution model

Figure 5.1: Maxent species distribution modelling result for cattles and sheep in Australia
Warmer colors indicating areas high probability of suitable conditions for the species. White dots show the presence locations used for training.

It can be reasonably assumed that the more suitable the species to survive, the greater the distribution density of the species. So we use the species appropriate survival map to represent the distribution density of the species. Replacing the random distribution with this distribution density will make our model more realistic.

6 Sensitivity Analysis

6.1 parameters for calculating the net energy

From the Tab. 6.16.2, we can see that parameter ξ, $\frac{r_b}{r_m}$ will not influence the net energy E_n a lot. So these parameters are not sensitive to the results, for the energy of growth E_g and recovery from trauma E_t only take a low proportion of E_n.

Table 6.1: The turbulence of ξ

$\Delta\xi$	-8%	-6%	-4%	-2%	0%	2%	4%	6%	8%
ΔE_n	-0.729%	-0.543%	-0.292%	-0.146%	0	0.251%	0.397%	0.543%	0.784%

Table 6.2: The turbulence of $\frac{r_b}{r_m}$

$\frac{r_b}{r_m}$	-8%	-6%	-4%	-2%	0%	2%	4%	6%	8%
$\Delta E_n(\times 10^{-4})$	3.715 %	2.685 %	1.726 %	8.331 %	0	-7.779%	-1.505 %	-2.185 %	-2.823 %

6.2 parameters for model of interaction with environment

From the Tab. 6.3 , we can get that the hare's density and energy are not sensitive to the final distribution, because the varying density only causes the change of the energy of flying E_f through the flying distance, which also takes a little part of E_n.

Table 6.3: The turbulence of density

Density	-8%	-6%	-4%	-2%	0%	2%	4%	6%	8%
ΔE_n	0.0693	0.0018	-0.0458	-0.0099	0	0.0423	0.0277	0.0257	0.1470

7 Conclusion

7.1 Our Conclusion

Through our model, we derived that the dragon's growth curve is S-shaped. It takes 280 years for a dragon to reach the upper limit of weight, which is 281.6 tons. The dragon is in initial growth stage before the age of 19. After the age of 131, the dragon enter steady growth stage. The age from 19 to 131 is exponential growth stage. The dragon grew fastest at the age of 52.

By analyzing the energy composition required by a dragon, our model figured out that a dragon's energy expenditures consist of five categories, which are: Energy for basic metabolish, Energy for growth, Energy for flight, Energy for breathing fire and Energy for recovery of trauma. The curve of the energy a gragon need vary with time is also S-shaped. A mature dragon require about 140000 million calories per two days, which consist of a community of 470 cattles and 178 sheep. The energy for Breathing fire takes a high precentage of the total energy expenditure when the dragon is young. The energy for basic metabolish takes the most percentage of the total energy expenditure during the whole life of the dragon. And it needs at least about 14000 km^2 area to support three mature dragons.

As for the influence of climate conditions, we consider both temperature and humidity. Our results shows that the higher temperature, the less energy the dragon consumes. The higher the humidity, the lower the energy the dragon need. And the influence of temperature on the dragon is greater than humidity.

To dertermine what is the ecological impact of a dragon, we simulated a ecosystem with three different types of prey. The results reveal the dragon has a great impact on the prey with high energy and dense distribution, and has little impact on prey with low energy and sparse distribution.

In order to get closer to the real situation of species distribution, we use Maxent to model the species geographic distributions. As a result, we derived the species appropriate survival map in Australia.

Besides, if we suppose the dragon breed through sexual multiplication, it will not be stable for three dragons to make up of a population. By referring to the literature, we discover that the number of individual that consists of a stably population will be 500. Accroding to our model, it will take approximately 2.33×10^6 km^2 area for these dragons to live, almost 30 of Australia. In summary, the dragon can be alive on the earth.

7.2 Strengths and Weaknesses

7.2.1 Strength

- Our model simplify the whole predating and growing process of the dragon. It is easy for us to calculate the total energy expenditure, which will make the analysis clearer. For example, our model simplify the analysis of the energy for flying and recovery of trauma, and the results exactly prove that these two parts takes a small proportion.

- Our model use some convincing data in reality and achievements in the literature, which will make the result be more realistic and in accrodence with our basic experience. For example, the energy for breathing fire will increase while the dragon is in frigid zone comparing to the tropic zone.

7.2.2 Weakness

- While comparing the manifestation between different climate, our model chooses the data of climate and the distribution of sheep and cattle in Australia to simulate. Nevertheless, due to the lackness of data, the maximum discrepancy in temperature or humidity betwen two states in Australia that we get are not prominent enough. If the dispcrepancy is more distinctive, results will be more meaningful.

- As we determine the minimum area needed for growing three dragons, we suppose that there are only sheep and cattle for the dragons to predate. However, the preies' distribution will be more complex, and competition effect among the dagons and other predators will exist. Our model cannot analyze this situation.

7.3 Future Work

- We could consider the catagories of preies will change with district with different climate. For example, the dragon can predate polar bears and morses in Arctic while penguins in Antarctic, but these animals do not exist in the tropical area.

- We could consider the different predating difficulties provided by the environment. For example, the landform is grass in temperate zone causing a easier predating, while that is forest result in a hurdle for predating because of predator's hiding.

- We could extend our model ot the condition of predating on the sea.

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Appendix A

Growth Curve for $y(1) = 30$ and $y(10) = 35$

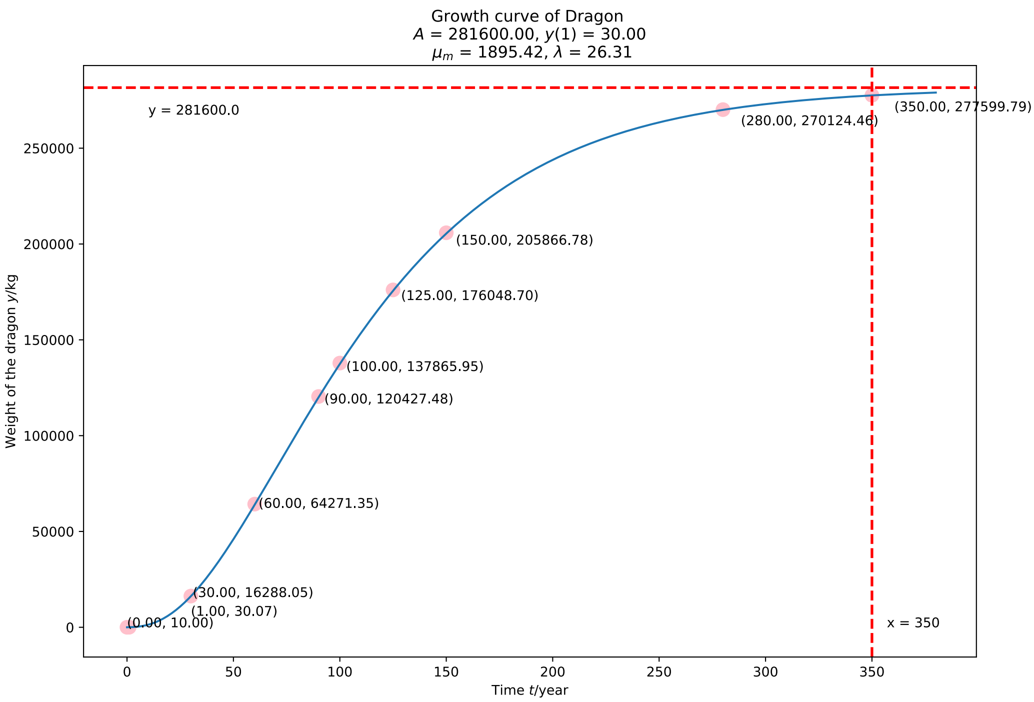

Figure 7.1: Groth curve of a dragon when $y(1) = 30$

Table 7.1: Growth curve function

$y(1)$	$y(t)$	$y'(t)$
30	$2181600 \times (1 - 0.967e^{-0.0151t})^3$	$12373.57 \times (1 - 0.967e^{-0.0151t})^2 \, e^{-0.015t}$
35	$2181600 \times (1 - 0.967e^{-0.0178t})^3$	$14520.21 \times (1 - 0.967e^{-0.0178t})^2 \, e^{-0.0178t}$
40	$2181600 \times (1 - 0.967e^{-0.0202t})^3$	$16475.90 \times (1 - 0.967e^{-0.0202t})^2 \, e^{-0.0202t}$

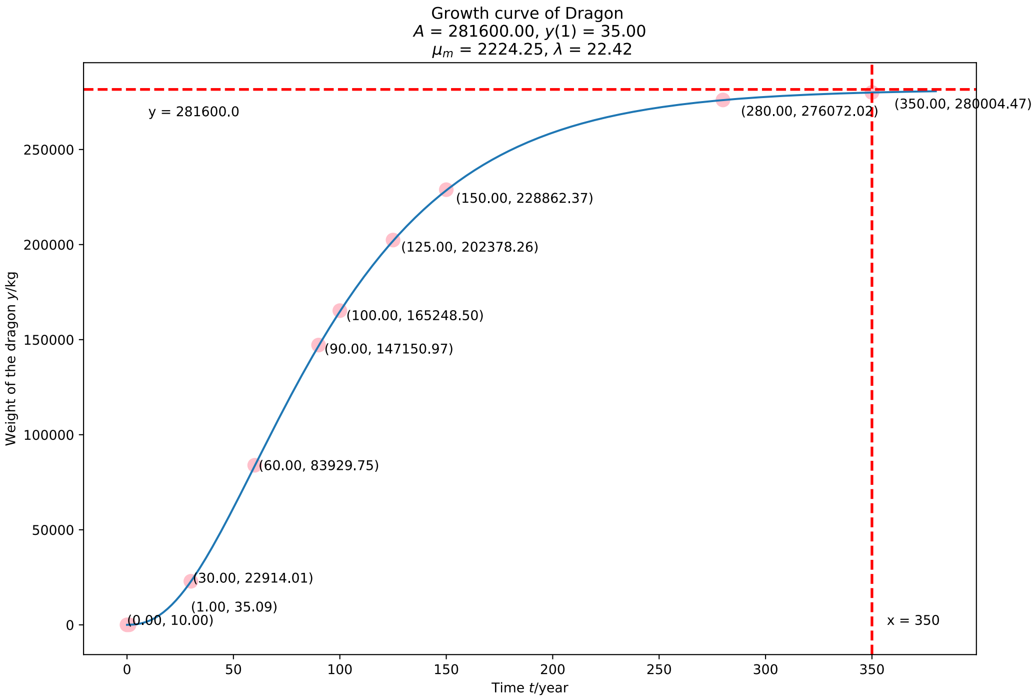

Figure 7.2: Groth curve of a dragon when $y(1) = 40$

Appendix B