Module 4

Functions

Function Let X and Y be any two sets.
A relation f from X to Y is called a function if for every $x \in X$, there is a unique $y \in Y$ such that $(x, y) \in f$.

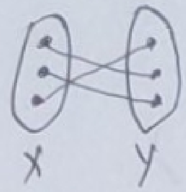
If y is the unique element of Y assigned by the function f to the element x of X, then we write $f(x) = y$.

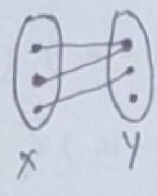
If f is a function from X to Y, then $f : X \longrightarrow Y$.

If $f(x) = y$, then x is called preimage and y is called image of x under f.

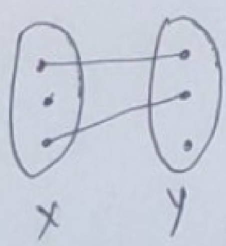
If f is a function from X to Y, then X is domain of f and Y is codomain of f.

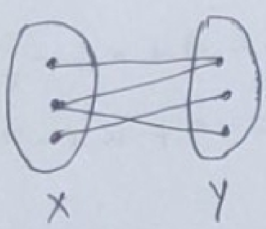
The following are functions.





The following are not functions





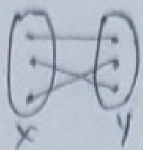
Rule: No element should be free in the domain X.
$\therefore$ not a function

Rule:
More than one mapping from each element of X should not be assigned
$\therefore$ not a function

Types of functions

1. <u>One-to-one</u> <u>(injective)</u>

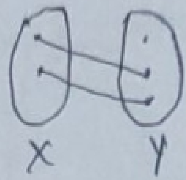
If distinct elements of X is mapped to distinct elements of Y, then the mapping $f: X \to Y$ is called one-to-one



$f(x_1) = f(x_2)$

2. <u>Into (~~surjective~~)</u>

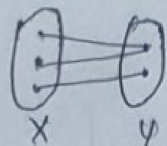
~~If~~ $f: X \to Y$ such that there is atleast one element $b \in Y$ which has nopreimage under f, then f is an into function.



$f(x) = y$

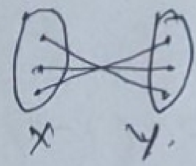
3. <u>Onto (surjective)</u>

If $f: X \to Y$ such that every element in Y has preimage under f is calledan onto function.



4. <u>One-to-one onto (bijective)</u>

A mapping $f: X \to Y$ is called one-~~to~~-one onto if f is both one to one and onto function.



1. Determine whether the function is one to one, onto
$f : Z^+ \to Z^+$ defined by $f(x) = x^2+2$

Z^+ = Set of all positive integers.

<u>To prove</u> f is one to one

$f(x_1) = f(x_2)$

$x_1^2 + 2 = x_2^2 + 2$

$x_1^2 - x_2^2 = 0$

$(x_1 - x_2)(x_1 + x_2) = 0$

$x_1 - x_2 = 0$ [since Z^+ is positive integers, $x_1 + x_2 \neq 0$]

$\Rightarrow x_1 = x_2$

$\therefore f$ is one-to-one

<u>To prove</u> f is onto

$y = f(x) \Rightarrow y = x^2 + 2$

$\Rightarrow x^2 = y - 2$

If $y = 1$, then $x^2 = -2 \Rightarrow x = \pm i\sqrt{2} \notin Z^+$

If $y = 4$, then $x^2 = 2 \Rightarrow x = \pm\sqrt{2} \notin Z^+$.

$\therefore f$ is not onto.

Composition of functions and inverse of function.

1. If $f, g : R \to R$ are defined by $f(x) = x^3 - 4x$ and $g(x) = \frac{1}{x^2+1}$, then find $(f \circ g)(x)$ and $(g \circ f)(x)$.

Soln

$$(f \circ g)(x) = f(g(x)) = f\left(\frac{1}{x^2+1}\right) = \left(\frac{1}{x^2+1}\right)^3 - 4\left(\frac{1}{x^2+1}\right)$$

$$(g \circ f)(x) = g(f(x)) = g(x^3 - 4x) = \left(\frac{1}{x^2+1}\right)^3 - 4\left(\frac{1}{x^2+1}\right)$$

2. If $f : A \to R$ is defined by $f(x) = \frac{4x}{2x-1}$, then find f^{-1}.

Soln

$$f(x) = \frac{4x}{2x-1}$$

$$y = \frac{4x}{2x-1}$$

$$(2x-1)y = 4x$$

$$2xy - y - 4x = 0$$

$$2x(y-2) - y = 0$$

$$2x(y-2) = y$$

$$2x = \frac{y}{y-2} \Rightarrow x = \frac{y}{2(y-2)} \quad \therefore f^{-1}(x) = \frac{y}{2(y-2)}$$

1. If f and g are functions from set of integers to set of integers, defined by $f(x) = 2x+3$ and $g(x) = 3x+2$, find $f \circ g$, $g \circ f$, f^{-1} and g^{-1}.

soln:

$f \circ g = (f \circ g)(x) = f(g(x)) = f(3x+2) = 3(2x+3)+2 \neq 6x+11)$

$g \circ f = (g \circ f)(x) = g(f(x)) = g(2x+3) = 2(3x+2)+3 = 6x+7$

$\boxed{f(x) = 2x+3}$

$y = 2x+3$

$y-3 = 2x$

$\frac{y-3}{2} = x$

$\therefore f^{-1}(x) = \frac{y-3}{2} \Rightarrow f^{-1}$ is $\frac{y-3}{2}$

find g^{-1}

2. Let $f: R \to R$ be given by $f(x) = x^2$ and $g: R - \{2\} \to R$ be given by $g(x) = \frac{x}{x-2}$. find $f \circ g$ and $g \circ f$.

Permutation

A bijection from a set A to itself is called a permutation of A.

1. Let $f : R \to R$ be defined as $f(a) = 2a+1$. Since f is one-to-one and onto, f is a permutation.

2. Let $A = \{1, 2, 3\}$. Then the permutations of A are

$P_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$ $P_2 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$ $P_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$

$P_4 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ $P_5 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$ $P_6 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$.

Compute $P_4 \circ P_3$ and P_5^{-1}.

Soln

$P_4 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ $P_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$

$P_4 \circ P_3 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$

$P_5^{-1} = \begin{pmatrix} 3 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix}$

Also find $P_5 \circ P_6$, P_2^{-1}.

Cyclic Permutation

Let $A = \{1, 2, 3, 4, 5\}$. The cycle $(1, 3, 5)$ denotes the permutation.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 5 & 4 & 1 \end{pmatrix}$$

Disjoint cycles

Two cycles of set A are said to be disjoint if no element of A appears in both cycles.

Let $A = \{1, 2, 3, 4, 5\}$, Then

$(1\ 2\ 5)$ & $(3\ 4)$ are disjoint cycles.

$(1\ 2\ 5)$ & $(2, 3, 4)$ are not disjoint cycles.

1. Let $A = \{1, 2, 3, 4, 5, 6\}$. Then compute $(1, 2, 5) \circ (2, 4, 6)$ and $(2, 4, 6) \circ (1, 2)$

Soln.

$$(1, 2, 5) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 5 & 3 & 4 & 1 & 6 \end{pmatrix}$$

$$(2, 4, 6) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 4 & 3 & 6 & 5 & 2 \end{pmatrix}$$

$$(1, 2) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 3 & 4 & 5 & 6 \end{pmatrix}$$

$$(1, 2, 5) \circ (2, 4, 6) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 3 & 6 & 1 & 5 \end{pmatrix}$$

$$(2, 4, 6) \circ (1, 2) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 1 & 3 & 6 & 5 & 2 \end{pmatrix}$$

Even and Odd permutations

A permutation of a finite set is called even if it can be written as a product of an even number of transpositions.
It is called odd if it can be written as a product of an odd number of transpositions.

1. Is the permutation $P = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 4 & 5 & 7 & 6 & 3 & 1 \end{pmatrix}$ even or odd.

<u>soln</u>

P can be written as product of disjoint cycles

$$P = (1, 2, 4, 7) \circ (3, 5, 6)$$

Each cycle can be written as product of transpositions.

$$(1, 2, 4, 7) = (1, 7) \circ (1, 4) \circ (1, 2)$$
$$(3, 5, 6) = (3, 6) \circ (3, 5)$$

$$P = (1, 7) \circ (1, 4) \circ (1, 2) \circ (3, 6) \circ (3, 5)$$

Since P is a product of odd number of transpositions, P is an odd permutation.

2. Determine whether the permutation is even or odd.

(a) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 2 & 1 & 6 & 5 & 8 & 7 & 3 \end{pmatrix}$ (b) $(6, 4, 2, 1, 5)$

(c) $(4, 8) \circ (3, 5, 2, 1) \circ (2, 4, 7, 1)$

3. Let $A = (1, 2, 3, 4, 5)$. ~~and~~ Let $f = (5, 2, 3)$ and $g = (3, 4, 1)$ be permutation of A. Compute $f \circ g$ and $f^{-1} \circ g^{-1}$.