Problems to Practice

Module - 3

Complex Differentiation

1. Test whether the following are analytic.

a. $f(z) = e^{2z}$

(Hint: $e^{z} = e^{x+iy} = e^{x}e^{iy} = e^{x}[\cos y + i\sin y]$)

b. $f(z) = \cos z$

c. $f(z) = \bar{z}$

d. $f(z) = z^2$

e. $f(z) = \sin z$

f. $f(z) = z^3 + z$

g. $f(z) = (x^2 - y^2 + 2xy) + i(x^2 - y^2 - 2xy)$

h. $f(z) = \cosh z$

i. $f(z) = y$

2. Test whether the following are harmonic.

a. $u = 3x^2y + 2x^3 - y^3 - 2y^2$

b. $v = e^{x}[\cos y - \sin y]$

c. $u = \frac{1}{2}\log(x^2 + y^2)$

d. $u = x^4 - 6x^2y^2 + y^4$.

3. Find the analytic function whose imaginary part is $V = x^3 - 3xy^2 + y + 1$.

4. Construct an analytic function where $u = e^{2x}[x\cos 2y - y\sin 2y]$. Also find its harmonic conjugate.

5. If $u = e^x \cos y$, find $f(z) = u + iv$.

6. Find the analytic function $f(z)$ whose real part is $u = x^2 - y^2 + 2xy - 3x - 2y$ and also find its harmonic conjugate.

7. Find the bilinear transformation which maps $1, i, -1$ onto the points $0, 1, \infty$.

8. Find the bilinear transformation which maps $z = \infty, i, 0$ into the points $w = 0, i, \infty$.

9. Find the bilinear transformation which maps $z_1 = 2, z_2 = i, z_3 = -2$ onto $w_1 = 1, w_2 = i, w_3 = -1$.

10. Find the image of the circle $|z| = 2$ under the transformation $w = z + 2 + 3i$

11. Find the image of the circle $|z| = 3$ under the transformation $w = 5z$.

12. Find the fixed points or invariant points of

(a) $W = \frac{2zi+5}{z-4i}$

(b) $W = \frac{z-1}{z+1}$

(c) $W = 2 - \frac{2}{z}$

13. Find the critical points of

(a) $W = \sqrt{(z-a)(z-b)}$

(b) $w^2 = z^4$

(c) $W = e^{z^2}$

14. ~~Repeat~~ Prove the following

(a) $\nabla^2 \log|f(z)| = 0$

(b) $\nabla^2 |f(z)| = 4\,|f'(z)|^2$

Problems to Practice

Module - 4

Complex Integration.

1. Evaluate the following by Cauchy Integral Formula.

a. $\int_C \frac{z^2+3}{z-3} dz$ where C is the circle $|z|=4$.

b. $\int_C \frac{1}{2z-3} dz$ where C is the circle $|z|=1$. "0"

c. $\int_C \frac{3z^2+7z+1}{z+1} dz$ where C is the circle $|z|=\frac{1}{2}$. "0"

d. $\int_C \frac{z}{(z-2)(z-4)} dz$ where C is $|z|=3$

e. $\int_C \frac{\sin\pi z^2+\cos\pi z^2}{(z-1)(z-2)}$ where C : $|z|=3$. "$4\pi i$"

f. $\int_C \frac{\cos\pi z^2}{(z-1)(z-2)}$ where C : $|z|=3$. "$4\pi i$"

g. $\int_C \frac{e^{2z}}{(z+1)^4}$ where C : $|z|=2$ "$\frac{8}{3}\pi i e^{-2}$"

h. $\int_C \frac{z^3-z}{(z-2)^3} dz$ where C : $|z|=3$ "Ans $=12\pi i$"

i. $\int_C \frac{z}{(z-1)(z-2)^3} dz$ where C is $|z|=5/2$

2. Find residue of $f(z) = \dfrac{1-e^{2z}}{z^3}$. "$-2$"

3. Calculate residue of $f(z) = \dfrac{z}{(z-1)^2}$ at its pole.

"1"

4. ~~Find~~ Find residue of $f(z) = \dfrac{z^3}{(z-1)^4(z-2)(z-3)}$ at its simple poles.

"-8 & $\frac{27}{16}$"

5. Evaluate the following using Cauchy Residue Theorem.

a. $\displaystyle\int_C \frac{1-2z}{z(z-1)(z-2)}\,dz$ where C is $|z|=1.5$ "$3\pi i$"

b. $\displaystyle\int_C \frac{e^{-z}}{z^2}\,dz$ where C is $|z|=1$ "$-2\pi i$"

6. Evaluate the following. (use Cauchy Residue Theorem or Cauchy Integral formula)

a. $\displaystyle\int_C \frac{(z^2+2)\,dz}{z+4}$ where C is $|z|=2$.

b. $\displaystyle\int_C \frac{4z^2-4z+1}{(z-2)(4+z^2)}\,dz$ where C is $|z|=1$ "0"

7. Evaluate the following by Taylor's Series.

a. $f(z) = \sin z$ about $z = \pi/4$.

b. $f(z) = \log(1+z)$ about $z = 0$.

c. $f(z) = \dfrac{z}{(z+4)(z+5)}$ at $|z| < 4$

8. Evaluate by Contour integration.

(a) $\displaystyle\int_0^{2\pi} \frac{d\theta}{5 - 3\cos\theta}$

b) $\displaystyle\int_0^{2\pi} \frac{1}{a + b\sin\theta}\, d\theta$

c). $\displaystyle\int_0^{2\pi} \frac{d\theta}{5 - 4\sin\theta}$ "$\frac{2\pi}{3}$"

d) $\displaystyle\int_0^{2\pi} \frac{\cos 2\theta}{5 - 4\cos\theta}\, d\theta$ "$\frac{\pi}{6}$"

[Hint: Replace $\cos 2\theta$ = Real part of $e^{i2\theta} = z^2$

$$\int_0^{2\pi} \frac{\cos 2\theta}{5 - 4\cos\theta}\, d\theta = R.P \int_C \frac{z^2}{5 - 4\left(\frac{z^2+1}{2z}\right)} dz$$]

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