UNIT IV

ROBOT KINEMATICS AND ROBOT PROGRAMMING

Forward Kinematics, Inverse Kinematics and Difference; Forward Kinematics and Reverse Kinematics of manipulators with Two, Three Degrees of Freedom (in 2 Dimension), Four Degrees of freedom (in 3 Dimension) Jacobians, Velocity and Forces-Manipulator Dynamics, Trajectory Generator, Manipulator Mechanism Design-Derivations and problems. Lead through Programming, Robot programming Languages-VAL Programming-Motion Commands, Sensor Commands, End Effector commands and simple Programs.

Robot Kinematics: Forward and Inverse Kinematics

- Kinematics studies the motion of bodies without consideration of the forces or moments that cause the motion.
- Robot kinematics refers the analytical study of the motion of a robot manipulator. Formulating the suitable kinematics models for a robot mechanism is very crucial for analyzing the behaviour of industrial manipulators.

6.1.1. Kinematics

- ✓ Kinematics means the analytical study of the geometry of motion of a mechanism.
- ✓ With respect to a fixed reference coordinate system.
- ✓ Without regard to the forces (or) moments that cause the motion.
- ✓ It refers to the study of geometric and time based quantities like position, velocity and acceleration of every part of the robot.

6.1.1.1. Kinematics Vs. Differential Kinematics

- ✓ Kinematics describes the analytical relationship between joint positions and the end effector position and orientations.
- ✓ Differential kinematics describe the analytical relationship between the joint motion and end effector motion in terms of velocities.

6.1.2. Object Manipulation

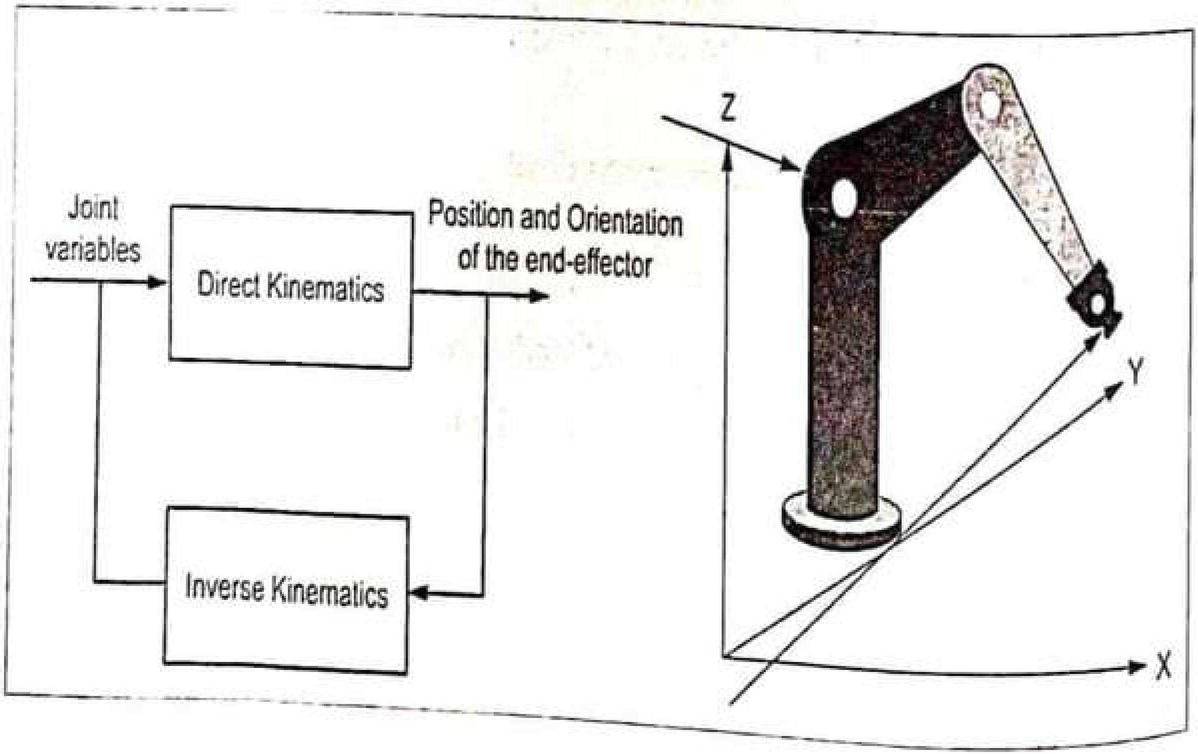
Manipulation is the skillful handling and treating of objects picking them up, moving them, fixing them one to another, and working on them with tools. Before we can program a robot to perform such operations, we require a method of specifying where the object is relative to the robot gripper, and a way of controlling the motion of the gripper.

6.1.3. Kinematic Model

Before a robot can move its hand to an object, the object must be located relative to it. There is currently no simple method for measuring the location of a robot hand. Most robots calculate the position of their hand using a kinematic model of their arm.

.1.4. Position and Orientation

- ✓ The state of the gripper of a robot is described by its position orientation in space with respect to a fixed frame called the base frame.
- ✓ In Fig.6.1, a fixed frame (x_0, y_0, z_0) is chosen to be the base frame. In the study of robotics. we always attach a coordinate system or a rigid frame to every member of the robot and to every object that is of interest.
- ✓ We often think of attribute like position, orientation, velocities, forces and torques being described with respect to some frame of reference.
- ✓ These activities can be redefined with respect to other frame via the transformation matrices.


Joint
variables
Direct Kinematics
Position and Orientation
of the end-effector
Inverse Kinematics
Z
Y
X

.5. Forward Kinematics/Direct Kinematics

- ✓ If we given the joint angles, we have to determine the position and orientation of the end-effector.

$$P_{world} = (x, y, z)$$

- ✓ For example for a revolute robot having, three degrees of freedom, if the joint angles θ_1, θ_2, θ_3 are specified. We can calculate the position and orientation of the end-effector.

$$P_{world} = (x, y, z)$$

- ✓ The outcome of the forward kinematics problem is always unique There are no multiple solutions.

1.6. Inverse Kinematics/Reverse Kinematics

- ✓ If we given the position and the orientation of the end-effector, we have to determine the numerical values for the joint angles.

$$P_{joint} = (\theta_1, \theta_2, \theta_3)$$

- ✓ This problem is not quite straight forward like the forward kinematics problem.
- ✓ In general it is not possible to obtain closed form solutions due to the non-linear simulations equations.
- ✓ Further, the non-linear nature of the problem leads to multiple solutions in certain cases.

6.1.6.1. Difference Between Forward Kinematics and Inverse Kinematics

(a) Forward Kinematics:

- ✓ If we given joint angles, compute the transformation between world and gripper coordinates.

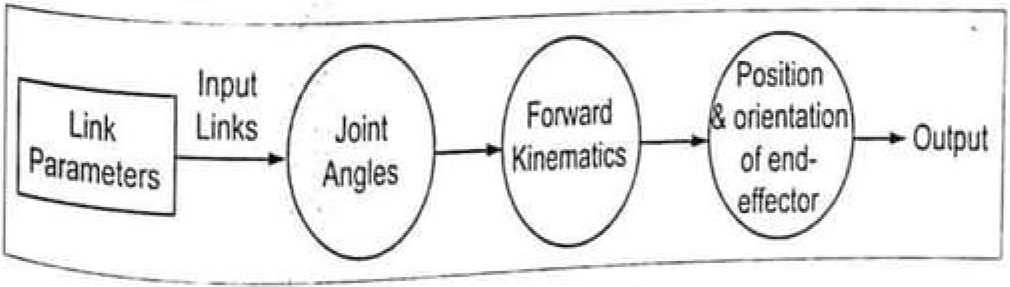


Fig. 6.2. Forward kinematics

- ✓ Relatively straight forward.

(b) I...

...very straight forward.

(b) Inverse (Reverse) Kinematics:

- ✓ If we given the transformation between world coordinates and an arbitrary frame, compute the joint angles that would line gripper coordinates up with that frame.

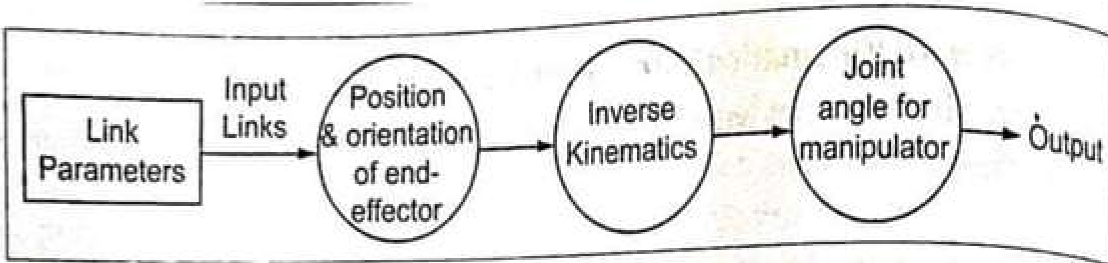


Fig. 6.3. Reverse kinematics

- ✓ For a kinematic mechanism, the inverse kinematic problems difficult to solve.
- ✓ The robot controller must solve a set of non-linear simultaneous algebraic.

Two Frames Kinematic Relationship:

There is a **kinematic relationship** between **two frames**, basically a translation and a rotation.

In manipulator **robotics**, there are **two kinematic** tasks:
Direct (also forward) **kinematics** – Given are
joint **relations** (rotations, translations) for the **robot** arm.

Transformation:

In **robotics** applications, many different coordinate systems can be used to **define** where **robots**, sensors, and other objects are located. You can **transform** between coordinate systems when you apply these representations to 3-D points.

6.1.8. Homogeneous Transformation

1. Homogeneous Transformation Matrix

$$
{}^{A}T_{B} = \begin{bmatrix} \overset{\text{Rotation matrix}}{\boxed{R}_{3\times3}} & \overset{\text{Position matrix}}{\boxed{P}_{3\times1}} \\ 0 & \boxed{1} \end{bmatrix} \leftarrow \text{Scaling}
$$

2. Composite Homogeneous Transformation Matrix

Rules:

- ✓ Transformation (rotation/translation) with respect to (X,Y,Z) (OLD FRAME), using ***pre-multiplication.***
- ✓ Transformation (rotation/translation) with respect to (U,V,W) (NEW FRAME), using ***post-multiplication.***

1.9. Composite Rotation Matrix

A sequence of finite rotations matrix multiplications do not commute rules:

- ✓ If rotating coordinate O-U-V-W is rotating about principal axis of OXYZ frame, then Pre-multiply the previous (resultant) rotation matrix with an appropriate basic rotation matrix.
- ✓ If rotating coordinate O-U-V-W is rotating about its own principal axes, then post-multiply the previous (resultant) rotation matrix with an appropriate basic rotation matrix.

.1.10. Homogeneous Representation

A frame in space (Geometric Interpretation)

$$F = \begin{bmatrix} R_{3\times 3} & P_{3\times 1} \\ 0 & 1 \end{bmatrix}$$

$$F = \begin{bmatrix} n_x & s_x & a_x & p_x \\ n_y & s_y & a_y & p_y \\ n_z & s_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Principal axis n with respect to the reference coordinate system

6.1.11. Manipulator Kinematics

$$\xrightleftharpoons[\text{Inverse}]{\text{Forward}} \ \text{Kinematic} \quad \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \\ \theta_5 \\ \theta_6 \end{bmatrix} \qquad \begin{bmatrix} x \\ y \\ z \\ \phi \\ \theta \\ \varphi \end{bmatrix} \qquad \xLeftrightarrow{\text{Jacobian Matix}} \qquad \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \\ \dot{\theta}_4 \\ \dot{\theta}_5 \\ \dot{\theta}_6 \end{bmatrix} \qquad \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \dot{\phi} \\ \dot{\theta} \\ \dot{\varphi} \end{bmatrix}$$

Joint Space ***Task Space***

Jacobian Matrix: Relationship between joint space velocity with task space velocity

6.1.12. Jacobian

Let the linear velocity and the angular velocity of the end-effector be represented in the vectorial form by

Let the joint angular velocities of a revolute robot be represented by

$$\frac{d\theta}{dt} = \begin{pmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_6 \end{pmatrix}$$

The vector V and $d\theta/dt$ can be connected by a matrix known as the Jacobian , i.e.,

$$V = J\frac{d\theta}{dt}$$

Where, $J = J(\theta)$ is the Jacobian.

Further $$\boxed{V = J\frac{d\theta}{dt}}$$

1.13. Dynamics

- ✓ This field is devoted to the study of motion caused by forces and torques.
- ✓ One method of controlling a robot while following a specified part requires the computation of the reflected torques generated by the joint motion.
- ✓ The computation involves the solution of the dynamic equations of the mainpulator.
- ✓ These equation are non-linear in nature.

1.14. Trajectory Generation

- ✓ Let the end-effector of the robot the required to move from a point A to another point B through some specified intermediate points.
- ✓ A straight line fit for the path from A to B through the intermediate point may not be preferable to many situations.
- ✓ This is due to the discontinues experienced in joint velocities and accelerations. To overcome this problem, cubic spines may be flitted to the path.

5.1.15. Position Control

- ✓ The individual joints are provided with separate systems, if the desired is known.
- ✓ It could serve as the reference input to the control system. The actual displacement θ_1 is measured by the sensors mounted at the with joint.

1.16. Force Control

- ✓ The joints of an industrial robot can be driven in any one of the two following modes. (*a*) Position control mode, or (*b*) Force control mode.
- ✓ A robot in the force control mode has the ability to exert the desired force on the work piece after the contact with the job has been made.
- ✓ This ability is of vital importance when a robot is used for tightening bolts or nuts or when it is employed for spot welding. Force control is complementary to position control

.17. Singularity

- ✓ Singularity problems surface when trying to control robots in Cartesian space.
- ✓ A robot singularity occurs when robot axes are redundant (more axes than necessary to cause the same motion) or when the robot is in certain configurations that require extremely high joint rates to move at some nominal speed in cartesian space.

I.18. Redundancy

- ✓ Most industrial robots have six or less joints, thus, redundancy is not inherent to their design.
- ✓ Some robots, though, do have a certain joint arrangement in their final orientation joints that can lead to redundancy for certain orientations.

S.No	Forward kinematics	Reverse kinematics
2.	To compute the position of the end effector from specified values for the joint parameters. $P_{world} = (x, y, z)$	To determine the joint angle parameters that provides desired position for each of the robot's end effector. $P_{joint} = (\theta_1, \theta_2, \theta_3) = (\theta_1, \theta_2)$
3.	Forward kinematics problem is quite straight forward.	Reverse kinematics problem is not quite straight forward like the forward kinematics.
4.	Easy to compute	The inverse kinematics problem is difficult to solve.
5.	Forward kinematics problems can be solved by using matrix multiplication.	Reverse kinematics problem must solve a set of non linear simultaneous algebraic equation.
6.	Linear equation	Non linear equation such as sin, cos in matrix rotation.
7.	The existence of unique solution	The existence of multiple solution
8.	The possible existence of a solution	The possible non existence of a solution.
9.	Non singularities	Singularities
10.	Forward kinematics is simple.	Inverse kinematics is complex if the number of serial joint increases.

6.6. FORWARD (or) DIRECT KINEMATIC TRANSFORMATION OF TWO DEGREES OF FREEDOM

The transformation of coordinates of the end effector point from the joint space to the world space is known as ***forward kinematics transformation.***

6.6.1. LL ROBOT (OR) PP ROBOT

Let us consider a cartesian LL (or) PP robot.

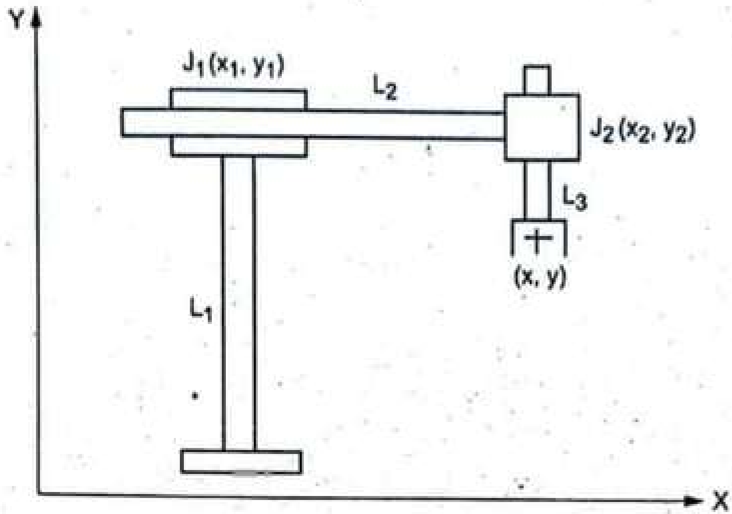


Fig. 6.7. LL (or) PP robot

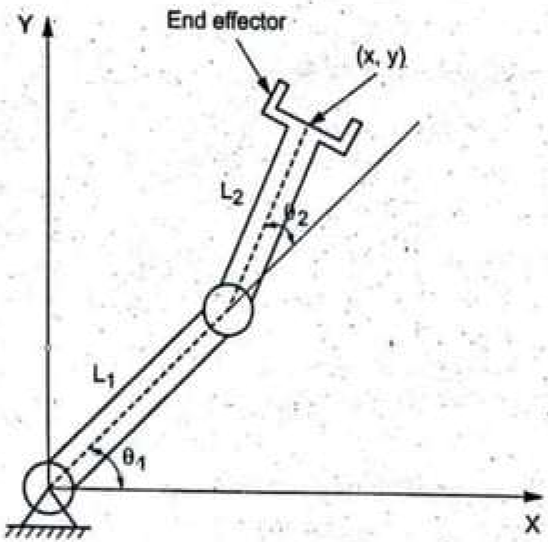


Fig. 6.8. Two manipulator with two degrees of freedom

- Joints J_1 and J_2 are linear joints with links of variable lengths L_1 and L_2. Let joint J_1 be denoted by (x_1, y_1) and joint J_2 (x_2, y_2).
- From geometry, we can easily get the following:

$$x_2 = x_1 + L_2 \quad \dots (6.1)$$

$$y_2 = y_1 \quad \dots (6.2)$$

These relations can be represented in homogeneous matrix form:

$$\begin{bmatrix} x_2 \\ y_2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & L_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix}$$

(or) $$\boxed{X_2 = T_1 X_1}$$

Where, $$X_2 = \begin{bmatrix} x_2 \\ y_2 \\ 1 \end{bmatrix}; \quad T_1 = \begin{bmatrix} 1 & 0 & L_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \quad X_1 = \begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix}.$$

If the end-effector point is denoted by (x, y), then

$$\boxed{x = x_2} \qquad \text{... (6.3)}$$

$$\boxed{y = y_2 - L_3} \qquad \text{... (6.4)}$$

therefore,

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -L_3 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_2 \\ y_2 \\ 1 \end{bmatrix}$$

(or) $$X = T_2 X_2 \qquad \text{... (6.5)}$$

Substitute X_2 value in equation (6.5), we get

$$X = T_2 (T_1 X_1) = T_{LL} X_1 \qquad [\because T_2 T_1 = T_{LL}]$$

Where, $$T_{LL} = T_2 T_1 \qquad \text{... (6.6)}$$

And $$T_{LL} = \begin{bmatrix} 1 & 0 & L_2 \\ 0 & 1 & -L_3 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{... (6.6.1)}$$

6.6.2. RR ROBOT

Let θ and α be the rotations at joints J_1 and J_2 respectively. Let J_1 and J_2 have the coordinates of (x_1, y_1) and (x_2, y_2) respectively.

One can write the following from the geometry:

$$x_2 = x_1 + L_2 \cos(\theta) \qquad \text{... (6.7)}$$

$$y_2 = y_1 + L_2 \sin(\theta) \qquad \text{... (6.8)}$$

Equation (6.7) and (6.8) can be written in matrix form

$$\begin{bmatrix} x_2 \\ y_2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & L_2 \cos(\theta) \\ 0 & 1 & L_2 \sin(\theta) \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix}$$

$[\because\ x = x_1 + y_1 + z_1;\ y = x_1 + y_1 + z_1]$

(or) $$X_2 = T_1 X_1 \qquad \text{... (6.9)}$$

On the other end: $$x = x_2 + L_3 \cos(\alpha - \theta) \quad \text{... (6.10)}$$

$$y = y_2 - L_3 \sin(\alpha - \theta) \quad \text{... (6.11)}$$

Equation (6.9) and (6.10) can be written in matrix form,

$$X = T_2 X_2 \quad \text{... (6.12)}$$

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & L_3 \cos(\alpha - \theta) \\ 0 & 1 & -L_3 \sin(\alpha - \theta) \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_2 \\ y_2 \\ 1 \end{bmatrix}$$

Substitute X_2 value in equation (6.12), we get

Combining the two equation gives,

$$X = T_2 (T_1 X_1) = T_{RR} X_1 \qquad [\because T_2 T_1 = T_{RR}]$$

$$T_{RR} = \begin{bmatrix} 1 & 0 & L_2 \cos(\theta) + L_3 \cos(\alpha - \theta) \\ 0 & 1 & L_2 \sin(\theta) - L_3 \sin(\alpha - \theta) \\ 0 & 0 & 1 \end{bmatrix} \quad \text{... (6.12.1)}$$

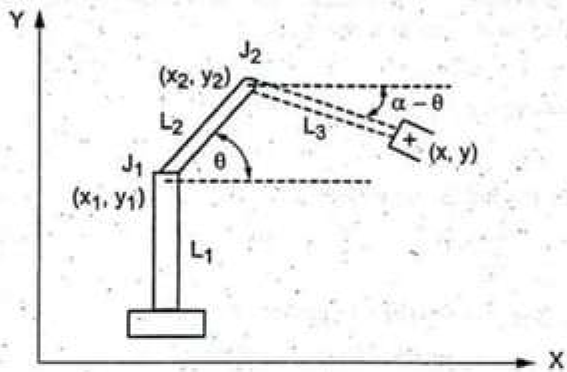


Fig. 6.9. RR robot

6.6.3. TL (OR) TP ROBOT

Let α be the rotation at twisting joint J_1 and L_2 be the variable link length at linear joint J_2.

We can write as

$$x = x_2 + L_2 \cos(\alpha) \quad \text{... (6.13)}$$

$$y = y_2 + L_2 \sin(\alpha) \quad \text{... (6.14)}$$

Equation (6.13) and (6.14) can be written in matrix form

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & L_2 \cos(\alpha) \\ 0 & 1 & L_2 \sin(\alpha) \\ 0 & 0 & 1 \end{bmatrix} : \begin{bmatrix} x_2 \\ y_2 \\ 1 \end{bmatrix} \quad \text{... (6.14.1)}$$

or $$X = T_{TL} X_2 \qquad [\because x = x_2 + y_2 + z_2; y = x_2 + y_2 + z_2]$$

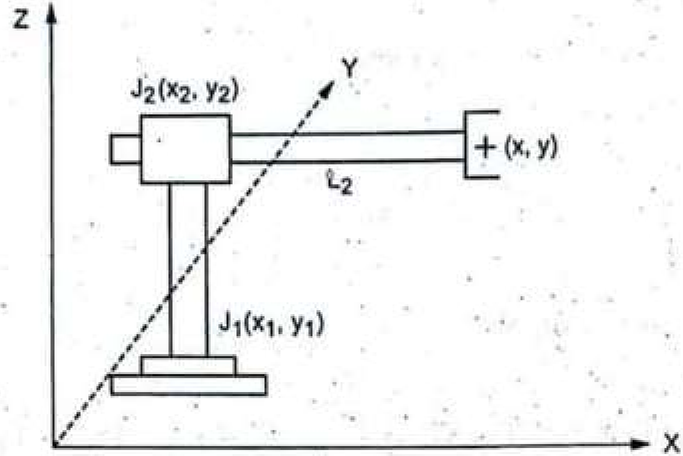


Fig. 6.10. TL (or) TP Robot

6.7. BACKWARD (OR) REVERSE KINEMATIC TRANSFORMATION OF TWO DEGREES OF FREEDOM

6.7.1. LL (OR) PP ROBOT

In backward kinematic transformation, the objective is to drive the variable link lengths from the known position of the end effector in world space.

$$x = x_1 + L_2(\alpha) \quad \text{... (6.15)}$$

$$y = Y_1 - L_3 \quad \text{... (6.16)}$$

$$y_1 = y_2 \quad \text{... (6.17)}$$

By combining equations (6.15) and (6.16), we can get

$$L_2 = x - x_1 \qquad L_3 = -y + y_2$$

6.7.2. RR ROBOT

$$x = x_1 + L_2 \cos(\theta) + L_3 \cos(\alpha - \theta) \quad \text{... (6.18)}$$

$$y = y_1 + L_2 \sin(\theta) - L_3 \sin(\alpha - \theta)(\theta) \quad \text{... (6.19)}$$

Combine the equations (6.18) and (6.19) easily, we can get the angles.

$$\cos(\alpha) = \frac{[(x - x_1)^2 + (y - y_1)^2 - L_2^2 - L_3^2}{2 L_2 L_3} \quad \text{... (6.19.1)}$$

and

$$\tan(\theta) = \frac{(y - y_1)(L_2 + L_3 \cos(\alpha)) + (x - x_1) L_3 \sin(\alpha)}{(x - x_1)(L_2 + L_3 \cos(\alpha)) - (y - y_1) L_3 \sin(\alpha)} \quad \text{... (6.19.2)}$$

6.7.3. TL (OR) TP ROBOT

$$x = x_2 + L \cos(\alpha)(\theta) \quad \text{... (6.20)}$$

$$y = y_2 + L \sin(\alpha)(\theta) \quad \text{... (6.21)}$$

One can easily get the equations for length and angle:

$$L = \sqrt{(x - x_2)^2 + (y - y_2)^2} \quad \text{... (6.22)}$$

Substitute equations (6.20), (6.21) values in equation (6.22), we get

$$\boxed{\sin(\alpha) = \frac{y - y_2}{L}}$$

Example 6.1 *An LL (or) PP robot has two links of variable length. Assuming that the origin of the global coordinate system is defined at joint J_1, determine the following:*

(a) The coordinate of the end effector point if the variable link lengths are 3 m and 5 m.

(b) Variable link lengths if the end effector is located at (3, 5).

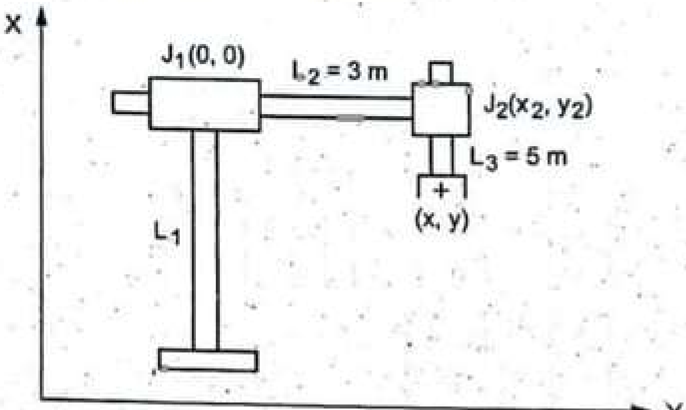


Fig. 6.11. LL (or) PP Robot

☺ ***Solution:***

(a) The coordinate of the end effector point if the variable link lengths are 3 m and 5 m:

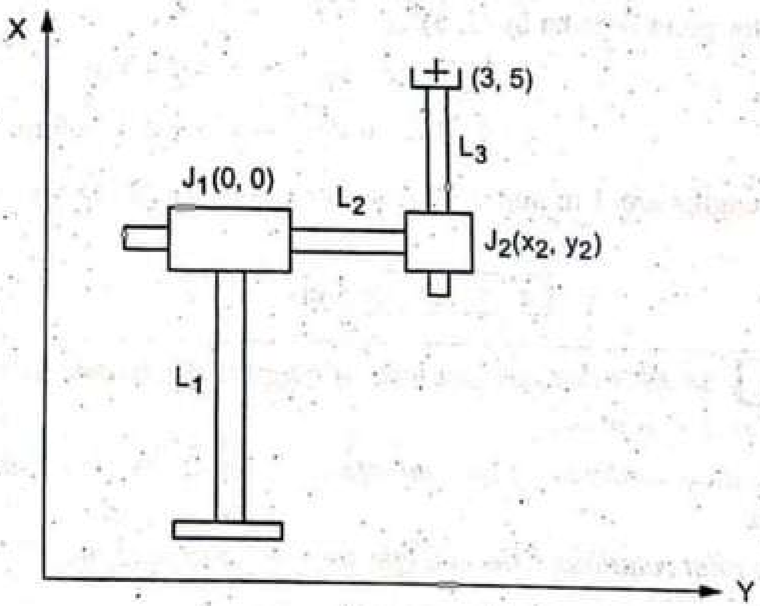


Fig. 6.12. LL (or) PP robot

Given that $(x_1, y_1) = (0, 0),\ x_1 = 0,\ y_1 = 0,\ L_2 = 3,\ L_3 = 5.$

$$T_{LL} = \begin{bmatrix} 1 & 0 & L_2 \\ 0 & 1 & -L_3 \\ 0 & 0 & 1 \end{bmatrix}$$ [From equation (6.6.1)]

Substituting the values

$$T_{LL} = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = T_{LL} \begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \\ 1 \end{bmatrix}$$

Therefore the end-effector point is given by (3, –5).

$$\boxed{x = 3, y = -5}$$

(b) Variable link lengths

The end-effector point is given by (3, 5).

Then:

$$L_2 = x - x_1 = 3 - 0 = 3 \text{ m}$$

$$L_3 = -y + y_1 = -5 + 0 = -5 \text{ m}$$

The variable lengths are 3 m and 5 m. The minus sign is due to the coordinate system used.

$$\boxed{x = 3, \ y = 5 \text{ m}}$$

Example 6.2 . *An RR robot has two links of length 1 m. Assume that the origin of the global coordinate system is at J_1.*

a) Determine the coordinate of the end effector point if the joint rotations are 30° at both joints.

b) Determine joint rotations if the end effector is located at (1, 0).

☺ ***Solution:***

(a) Coordinates of the end effector point if the joint rotations are 30° at both joints:

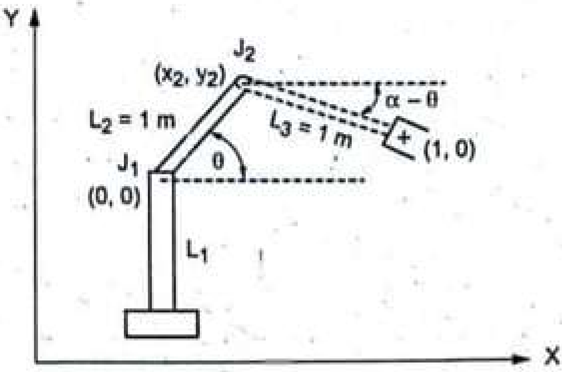


Fig. 6.13. RR robot

Given that $(x_1, y_1) = (0, 0), \quad \theta = 30°; \quad L_2 = 1 \text{ m}; \quad L_3 = 1 \text{ m}$

$$T_{RR} = \begin{bmatrix} 1 & 0 & L_2 \cos(\theta) + L_3 \cos(\alpha - \theta) \\ 0 & 1 & L_2 \sin(\theta) - L_3 \sin(\alpha - \theta) \\ 0 & 0 & 1 \end{bmatrix}$$

... [From equation (6.12.1)]

$$T_{RR} = \begin{bmatrix} 1 & 0 & \frac{\sqrt{3}}{2} + 1 \\ 0 & 1 & \frac{1}{2} + 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\left[\because \begin{aligned} \cos 30° &= \frac{\sqrt{3}}{2} = 1.866 \\ \sin 30° &= \frac{1}{2} = 0.5 \\ \cos(30 - \theta) &= 1 \\ \sin(30 - \theta) &= 0 \end{aligned}\right.$$

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = T_{RR} \begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix}$$

Substituting the values in above equation,

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1.8667 \\ 0 & 1 & 0.5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 1.8667 \\ 0.5 \\ 1 \end{bmatrix}$$

Therefore the end-effector point is given by (1.8667, 0.5).

$$\boxed{x = 1.8667; \quad y = 0.5}$$

(b) Joint rotation of the end effector is located at (1, 0):

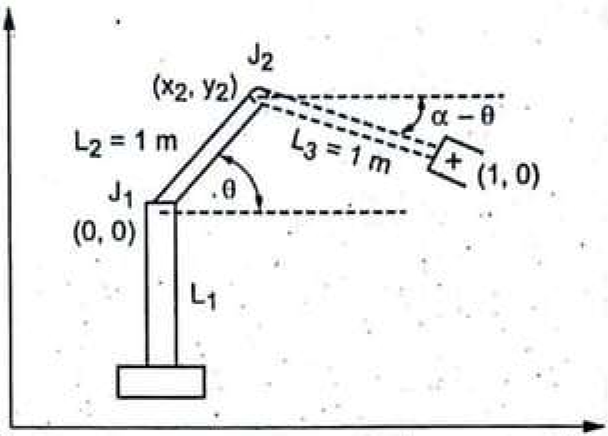


Fig. 6.14. RR robot

Given that $(x, y) = (1, 0), \quad x = 1, \quad y = 0, \quad x_1 = 0, \quad y_1 = 0$

$L_2 = 1$ m; $\quad L_3 = 1$ m.

We know $$\cos(\alpha) = \frac{x^2 + y^2 - L_2^2 - L_3^2}{2\,L_2\,L_3}$$ [From equation (6.19.1)]

$$\cos(\alpha) = \frac{1^2 + 0^2 - 1^2 - 1^2}{2 \times 1 \times 1} = -0.5$$

$$\alpha = \cos^{-1}(-0.5)$$

$$\boxed{\alpha = 120°}$$

$$\tan(\theta) = \frac{(y - y_1)(L_2 + L_3\cos(\alpha)) + (x - x_1) L_3 \sin(\alpha)}{(x - x_1)(L_2 + L_3\cos(\alpha)) - (y - y_1) L_3 \sin(\alpha)}$$

[From equation (6.19.2)]

Substituting the values in equation

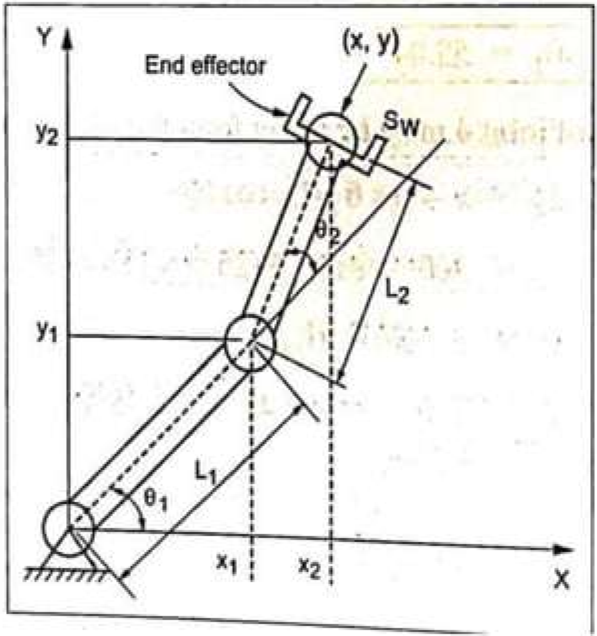
$$\tan(\theta) = \frac{(0 - 0)(1 + 1\cos(120°)) + (1 - 0)\, 1 \sin(120°)}{(1 - 0)(1 + 1\cos(120°)) - (0 - 0)\, 1 \sin(120°)}$$

$$\tan(\theta) = \frac{\frac{\sqrt{3}}{2}}{0.5} = \sqrt{3}$$

$$\theta = \tan^{-1}(\sqrt{3})$$

$$\boxed{\textbf{Angle } (\theta) = 60°}$$

FORWARD AND REVERSE KINEMATIC TRANSFORMATION FOR RR ROBOT WITH TWO DEGREES OF FREEDOM WITH 2D



Forward Kinematics/Direct Kinematics

The position of the end arm in world space by defining vector for link 1 and another for link 2.

From Figure 6.13, $r_1 = L_1 \cos\theta_1,\ L_2 \sin\theta_2$ ... (6.23)

$$r_2 = L_2 \cos(\theta_1 + \theta_2),\ L_2 \sin(\theta_1 + \theta_2) \quad \text{... (6.24)}$$

Let L_1 be the length of the arm 1.

L_2 be the length of the arm 2.

Link L_1 makes an angle θ_1 with horizontal.

Link L_2 makes an angle θ_2 with link L_1.

— The end point of the robot is at $S_W = (x, y)$

— For the forward kinematics, we can compute x and y coordinates.

$$\boxed{\begin{aligned} x &= L_1 \cos\theta_1 + L_2 \cos(\theta_1 + \theta_2) \\ y &= L_1 \sin\theta_1 + L_2 \sin(\theta_1 + \theta_2) \end{aligned}}$$

Reverse / Backward / Inverse kinematics:

✓ In reverse kinematics, we have given world coordinates (x and y) and we want to calculate the joint values θ_1 and θ_2.

$$x = L_1 \cos\theta_1 + L_2 \cos(\theta_1 + \theta_2) \quad \ldots (6.25)$$

$$y = L_1 \sin\theta_1 + L_2 \sin(\theta_1 + \theta_2) \quad \ldots (6.26)$$

Squaring and adding equations (6.25) and (6.26), we get

$$x^2 + y^2 = L_1^2 \cos^2\theta_1 + L_2^2 \cos^2(\theta_1 + \theta_2) + 2 L_1 L_2 \cos\theta_1 \cdot \cos(\theta_1 + \theta_2)$$
$$+ L_1^2 \sin^2\theta_1 + L_2^2 \sin^2(\theta_1 + \theta_2) + 2 L_1 L_2 \sin\theta_1 \cdot \sin(\theta_1 + \theta_2)$$

$$[\because (A + B)^2 = A^2 + B^2 + 2\,AB]$$

$$x^2 + y^2 = L_1^2 \{(\cos^2\theta + \sin^2\theta)\} + L_2^2 \{\cos^2(\theta_1 + \theta_2) + \sin^2(\theta_1 + \theta_2)\}$$
$$+ 2 L_1 L_2 [\cos\theta_1 \cdot \cos(\theta_1 + \theta_2) + \sin\theta_1 \cdot \sin(\theta_1 + \theta_2)]$$

$$x^2 + y^2 = L_1^2\ [1] + L_2^2\ [1] + 2\, L_1\, L_2\, [\cos\theta_1 \cdot \cos(\theta_1 + \theta_2) +$$

$$\sin\theta_1 \cdot \sin(\theta_1 + \theta_2)]$$

$[\because \sin^2\theta + \cos^2\theta = 1 \Rightarrow \sin^2(\theta_1 + \theta_2) + \cos^2(\theta_1 + \theta_2) = 1]$

$$x^2 + y^2 = L_1^2 + L_2^2 + 2\, L_1\, L_2\, [\cos\theta_1 \cos(\theta_1 + \theta_2) + \sin\theta_1 \cdot \sin(\theta_1 + \theta_2)]$$

$$x^2 + y^2 = L_1^2 + L_2^2 + 2\, L_1\, L_2\, [\cos(\theta_1 - (\theta_1 + \theta_2))]$$

$[\because \cos A \cos B + \sin A \cdot \sin B = \cos(A - B)]$

$$x^2 + y^2 = L_1^2 + L_2^2 + 2\, L_1\, L_2\, [\cos\theta_1 - \theta_1 + \theta_2]$$

$$= L_1^2 + L_2^2 + 2\, L_1\, L_2\, [\cos\theta_2]$$

$$\cos\theta_2 = \frac{x^2 + y^2 - L_1^2 - L_2^2}{2\, L_1\, L_2}$$

$$\boxed{\theta_2 = \cos^{-1}\left[\frac{x^2 + y^2 - L_1^2 - L_2^2}{2\, L_1\, L_2}\right]}$$

DETERMINE THE FORWARD AND REVERSE KINEMATICS SOLUTION OF A SPHERICAL ROBOT CONFIGURATION (RRL) (THREE DOF 2D MANIPULATOR)

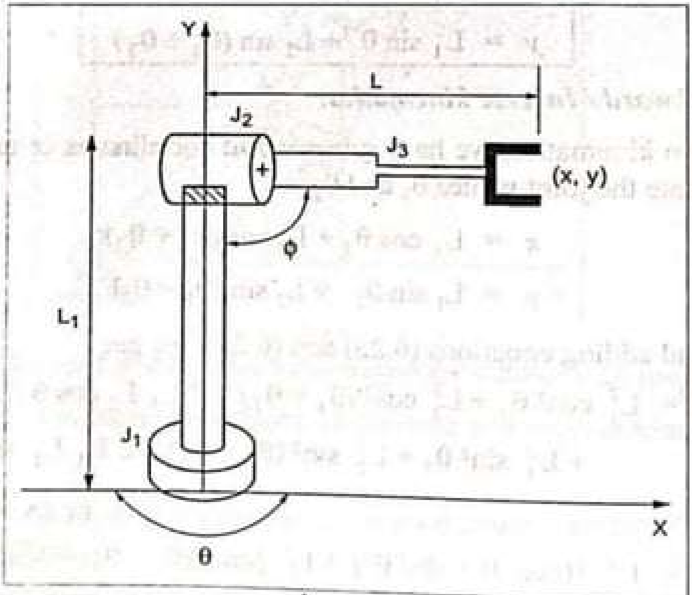


Fig. 6.14. Forward and reverse kinematics of spherical Robot RRL Configuration with three DOF with 2D manipulator

Forward Kinematics:

✓ For the forward kinematics, we can compute the x, y and z coordinates.

$$x = \cos\theta\,(\mathrm{L}\cos\phi) + \mathrm{L}_3\cos\phi$$

$$y = \sin\theta\,(\mathrm{L}\cos\phi) + \mathrm{L}_1 + \mathrm{L}_3 + \mathrm{L}_3\cos\phi$$

Backward (or) Reverse Kinematics:

✓ In the backward kinematics, we have given world coordinates x, y, ϕ and we want to calculate the joint angles θ_1, θ_2 and θ_3.

✓ This is accomplished by first determining the coordinates of joint 3. *i.e.*, x_3, y_3.

$$x_3 = x - \mathrm{L}_3\cos\phi \quad \text{... (6.27)}$$

$$y_3 = y - \mathrm{L}_3\cos\phi \quad \text{... (6.28)}$$

Substituting the 'x' values and 'y' values in the equation (6.27) and (6.28) respectively.

$$x_3 = \cos\theta\,(L\cos\phi) + L_3\cos\phi - L_3\cos\phi$$

$$\boxed{x_3 = \cos\theta\,(L\cos\phi)} \qquad [\because L_3\cos\phi - L_3\cos\phi = 0] \qquad \dots (6.28.1)$$

$$y_3 = y - L_3\cos\phi$$

$$y_3 = \sin\theta\,(L\cos\phi) + L_1 + L_3 + L_3\cos\phi - L_3\cos\phi \quad [\because L_3\cos\phi - L_3\cos\phi = 0]$$

$$\boxed{y_3 = \sin\theta\,(L\cos\phi) + L_1 + L_3}$$

It can be written as

$$\boxed{y_3 - L_1 - L_3 = \sin\theta\,(L\cos\phi)} \qquad \dots (6.28.2)$$

Adding the equations (6.28.1) and (6.28.2) on both sides, we get

$$x_3 + [y_3 - L_1 - L_3] = \cos\theta\,(L\cos\phi) + \sin\theta\,(L\cos\phi) \qquad \dots (6.29)$$

Squaring the equation (6.29) on both sides, we get

$$(x_3)^2 + [y_3 - (L_1 + L_3)]^2 = \cos^2\theta\,(L^2\cos^2\phi) + \sin^2\theta\,(L^2\cos^2\phi)$$

$$x_3^2 + [y_3 - (L_1 + L_3)]^2 = L^2\cos^2\phi\,(\cos^2\theta + \sin^2\theta) \qquad [\because \sin^2\theta + \cos^2\theta = 1]$$

$$x_3^2 + [y_3 - (L_1 + L_3)]^2 = L^2\cos^2\phi \qquad \dots (6.30)$$

$$L^2 = \frac{x_3^2 + [y_3 - (L_1 + L_3)]^2}{\cos^2\phi}$$

$$\boxed{\begin{aligned} L &= \frac{x_3^2 + [\{y_3 - (L_1 + L_3)\}^2]^{1/2}}{\cos\phi} \\ \tan\theta &= \frac{y_3 - [L_1 + L_3]}{x_3} \\ \sin\phi &= \frac{y_3 - [L_1 + L_3]}{L} \end{aligned}}$$

FORWARD AND BACKWARD TRANSMISSION FOR A ROBOT WITH THREE JOINTS

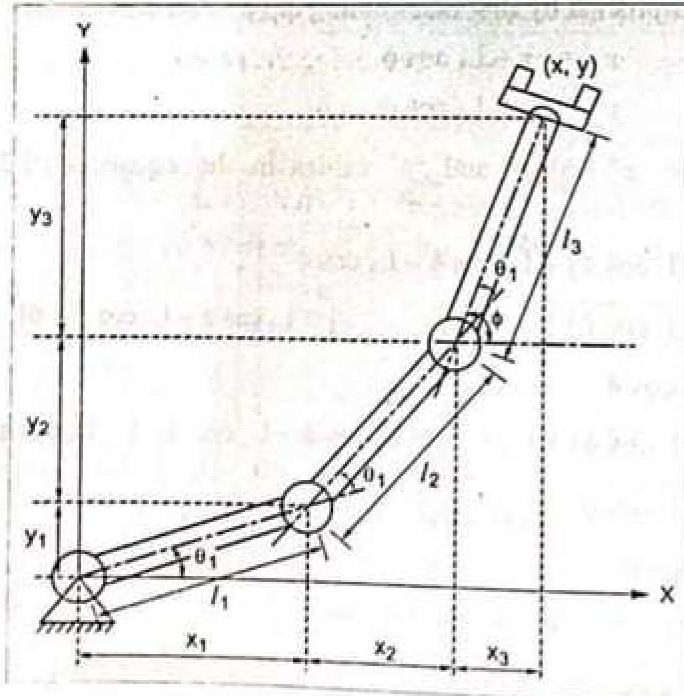


Fig. 6.15. Forward and reverse transformation for a RRR Robot configuration with three DOF with 2D

Forward Kinematics

- ✓ For the forward kinematics, we can compute the x, y and z coordinates.

$$x = L_1 \cos\theta_1 + L_2 \cos(\theta_1 + \theta_2) + L_3 \cos(\theta_1 + \theta_2 + \theta_3)$$

$$z = L_1 \sin\theta_1 + L_2 \sin(\theta_1 + \theta_2) + L_3 \sin(\theta_1 + \theta_2 + \theta_3)$$

- ✓ 'ϕ' as the orientation angle. It is the angle made by the wrist with horizontal. It equals to the algebraic sum of the angles.

$$\phi = \theta_1 + \theta_2 + \theta_3$$

Backward (or) Reverse Kinematics

- ✓ In the backward kinematics, we have given world coordinates x, z and ϕ and we want to calculate the joint values θ_1, θ_2 and θ_3.
- ✓ This is accomplished by first determining the coordinates of joint 3. *i.e.*, x_3, z_3.

$$x_3 = x - L_3 \cos\phi$$

$$z_3 = z - L_3 \sin\phi$$

We can determine the θ_1 and θ_2 values from knowing the coordinates of joint 3.

$$x_3 = L_1 \cos\theta_1 + L_2 \cos(\theta_1+\theta_2) + L_3 \cos(\theta_1+\theta_2+\theta_3) - L_3 \cos\phi$$

$$[\because \theta_1+\theta_2+\theta_3 = \phi]$$

$$x_3 = L_1 \cos\theta_1 + L_2 \cos(\theta_1+\theta_2) + L_3 \cos\phi - L_3 \cos\phi$$

$$\boxed{x_3 = L_1 \cos\theta_1 + L_2 \cos(\theta_1+\theta_2)} \qquad [\because L_3 \cos\phi - L_3 \cos\phi = 0]$$

$$z_3 = z - L_3 \sin\phi$$

$$z_3 = L_1 \sin\theta_1 + L_2 \sin(\theta_1+\theta_2) + L_3 \sin(\theta_1+\theta_2+\theta_3) - L_3 \sin\phi$$

$$z_3 = L_1 \sin\theta_1 + L_2 \sin(\theta_1+\theta_2) + L_3 \sin\phi - L_3 \sin\phi$$

$$\boxed{z_3 = L_1 \sin\theta_1 + L_2 \sin(\theta_1+\theta_2)} \qquad [\because \theta_1+\theta_2+\theta_3 = \phi$$

$$L_3 \sin\phi - L_3 \sin\phi = 0]$$

Squaring and adding x_3 value and z_3 value on both sides, we get

$$x_3^2 + z_3^2 = L_1^2 \cos^2\theta_1 + L_2^2 \cos^2(\theta_1+\theta_2) + 2 L_1 L_2 \cos\theta_1 \cdot \cos(\theta_1+\theta_2) +$$

$$L_1^2 \sin^2\theta_1 + L_2^2 \sin^2(\theta_1+\theta_2) + 2 L_1 L_2 \sin\theta_1 \cdot \sin(\theta_1+\theta_2)$$

$$x_3^2 + z_3^2 = L_1^2\,[\cos^2\theta_1 + \sin^2\theta_1] + L_2^2\,[\cos^2(\theta_1+\theta_2) + \sin^2(\theta_1+\theta_2)] +$$
$$2\,L_1\,L_2\,[\cos\theta_1 \cdot \cos(\theta_1+\theta_2) + \sin\theta_1 \cdot \sin(\theta_1+\theta_2)$$

$[\because (A+B)^2 = A^2 + B^2 + 2\,AB]$

$$x_3^2 + z_3^2 = L_1^2\,[1] + L_2^2\,[1] + 2\,L_1\,L_2\,[\cos\theta_1 \cdot \cos(\theta_1+\theta_2) + \sin\theta_1 \cdot \sin(\theta_1+\theta_2)]$$

$[\because \cos A \cos B + \sin A \sin B = \cos(A-B)]$

$$x_3^2 + z_3^2 = L_1^2 + L_2^2 + 2\,L_1\,L_2\,[\cos(\theta_1 - \theta_1 + \theta_2)]$$
$$x_3^2 + z_3^2 = L_1^2 + L_2^2 + 2\,L_1\,L_2\,[\cos\theta_2] \qquad [\because \theta_1 - \theta_1 = 0]$$

$$\cos\theta_2 = \frac{x_3^2 + z_3^2 - L_1^2 - L_2^2}{2\,L_1\,L_2}$$

$$\theta_2 = \cos^{-1}\left[\frac{x_3^2 + z_3^2 - L_1^2 - L_2^2}{2\,L_1\,L_2}\right]$$

Substituting the value of θ_2 in equation, we get the value of θ_1.

Finally the value of θ_3 can be obtained by using the following relation

$$\boxed{\theta_3 = \phi - (\theta_1 + \theta_2)}$$

. FORWARD AND REVERSE KINEMATIC TRANSFORMATION FOR TRLR ROBOT CONFIGURATION WITH FOUR DOF WITH 3D

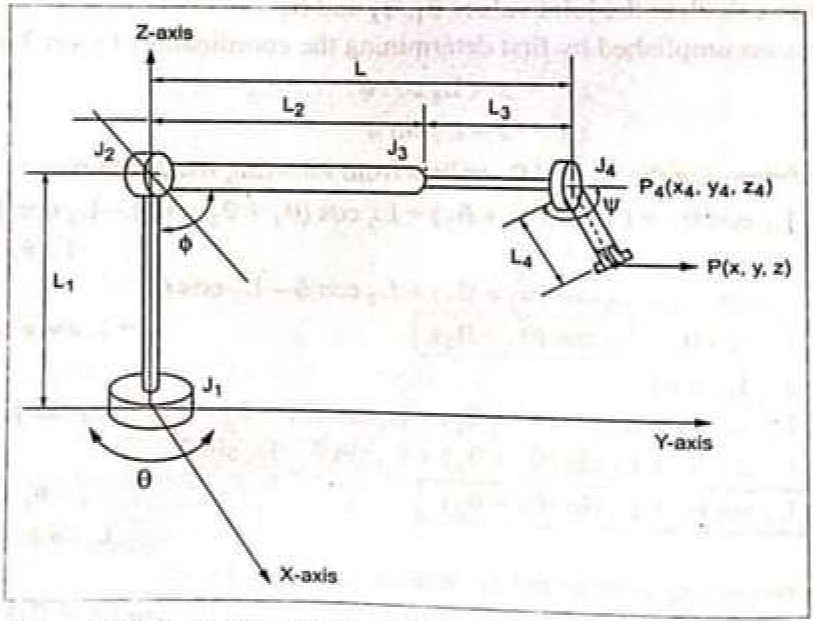


Fig. 6.16. Forward and Reverse Kinematic transformation for TRLR robot configuration with four DOF with 3D

Let Joint 1 (J_1) = Twisted type joined which can be used to move the base (T-type)

Joint 2 (J_2) = Rotation about an axis perpendicular to the z axis. (R-type)

Joint 3 (J_3) = Linear joint which is used for to and fro motion (L-type)

Joint 4 (J_4) = Gripper joint which have a rotational motion (R-type)

θ = Angle of rotation of joint1 (Base rotation)

ϕ = Angle of rotation of joint 2 (Angle made by arm) (Elevation)

ψ = Angle of rotation of joint 4 (Angle made by wrist)

L_1 = Length between base and arm

L_4 = Length between wrist and gripper

L = Length of the linear joint 3 [combination of L_2 and L_3]

The base to end effectors is given by the coordinates

$$= p(x, y, z)$$

The base to wrist is given by the coordinates

$$= p_4(x_4, y_4, z_4)$$

6.8.1. Forward Kinematics

The position of end effector P in world space is given by,

$$x = \cos\theta\,(L\cos\phi + L_4\cos\psi) \quad \ldots (6.31)$$

$$y = \sin\theta\,(L\cos\phi + L_4\cos\psi) \quad \ldots (6.32)$$

$$z = L_1 + L\sin\phi + L_4\sin\psi \quad \ldots (6.33)$$

6.8.2. Reverse Kinematics

In the backward transformation, we are given the world coordinates x, y, z, ψ and ϕ mention orientation. To find out joint values, we define the coordinate joint 4.

The position from the base to wrist x_4, y_4, z_4 is given by,

$$x_4 = x - \cos\theta\,(L_4\cos\psi) \quad \ldots (6.34)$$

$$y_4 = y - \sin\theta\,(L_4\cos\psi) \quad \ldots (6.35)$$

$$z_4 = z - L_4\sin\psi \quad \ldots (6.36)$$

Substituting the value of equations (6.31), (6.32) & (6.33) in (6.34), (6.35), and (6.36) respectively, we get

$$x_4 = \cos\theta\,(L\cos\phi + L_4\cos\psi) - \cos\theta\,(L_4\cos\psi)$$

$$= L\cos\theta\cos\phi + L_4\cos\theta\cos\psi - L_4\cos\theta\cos\psi$$

$$\boxed{x_4 = L\cos\theta\,.\cos\phi} \quad \dots (6.37)$$

$$y_4 = \sin\theta\,(L\cos\phi + L_4\cos\psi) - \sin\theta\,(L_4\cos\psi)$$

$$= L\sin\theta\cos\phi + L_4\sin\theta\cos\psi - L_4\sin\theta\cos\psi$$

$$\boxed{y_4 = L\sin\theta\cos\phi} \quad \dots (6.38)$$

$$z_4 = L_1 + L\sin\phi + L_4\sin\psi - L_4\sin\psi$$

$$z_4 = L_1 + L\sin\phi \qquad [\because L_4\sin\psi - L_4\sin\psi = 0]$$

It can be written as

$$\boxed{z_4 - L_1 = L\sin\phi} \quad \dots (6.39)$$

Squaring and adding equations (6.37), (6.38) and (6.39) on both sides, we get

$$x_4^2 + y_4^2 + (z_4 - L_1)^2 = L^2 \cos^2\theta \cos^2\phi + L^2 \sin^2\theta \cos^2\phi + L^2 \sin^2\phi$$

$$= L^2 \cos^2\phi [\cos^2\theta + \sin^2\theta] + L^2 \sin^2\phi$$

$$= L^2 \cos^2\phi + L^2 \sin^2\phi \qquad [\because \sin^2\theta + \cos^2\theta = 1]$$

$$= L^2 (\cos^2\phi + \sin^2\phi)$$

$$= L^2 \qquad [\because \cos^2\phi + \sin^2\phi = 1]$$

$$x_4^2 + y_4^2 + (z_4 - L_1)^2 = L^2$$

$$L^2 = x_4^2 + y_4^2 + (z_4 - L_1)^2$$

The length is given by $\boxed{L = [x_4^2 + y_4^2 + (z_4 - L_1)^2]^{1/2}}$

$$\boxed{\sin\theta_2 = \frac{z_4 - L_1}{L}}$$

$$\theta_2 = \sin^{-1}\left[\frac{z_4 - L_1}{L}\right]$$

where $\phi = \theta_2 + \theta_4$

$$\boxed{\theta_4 = \phi - \theta_2}$$

9. MANIPULATOR POSITION KINEMATICS – FORWARD POSITION KINEMATICS (3-DEGREES OF FREEDOM 2-D MANIPULATOR) BY USING MATRIX FORM

- ✓ In the forward kinematics, if we given the joint angles, we have to find out the position of the end effector.
- ✓ In this manner, we are construct the different transformation matrices and combine them in the right way. The result being ${}^{ee}_{bs}T$, where {bs} is the pose frame of the robot manipulator.
- ✓ This would be done by the use of the Denavit-Hartenberg convention.
- ✓ The transformation that relates the last and first frames in a serial manipulator arm, and thus, the solution to the forward kinematics problem is then represented by the compound transformation matrix.
- ✓ The axes are moving, thus the compound homogeneous transformation matrix is found by premultiplying the individual transformation matrices.

$$ {}^{ee}_{bs}T = {}^{N}_{0}T = {}^{1}_{0}T \; {}^{2}_{1}T \; {}^{3}_{2}T \ldots\ldots\ldots $$

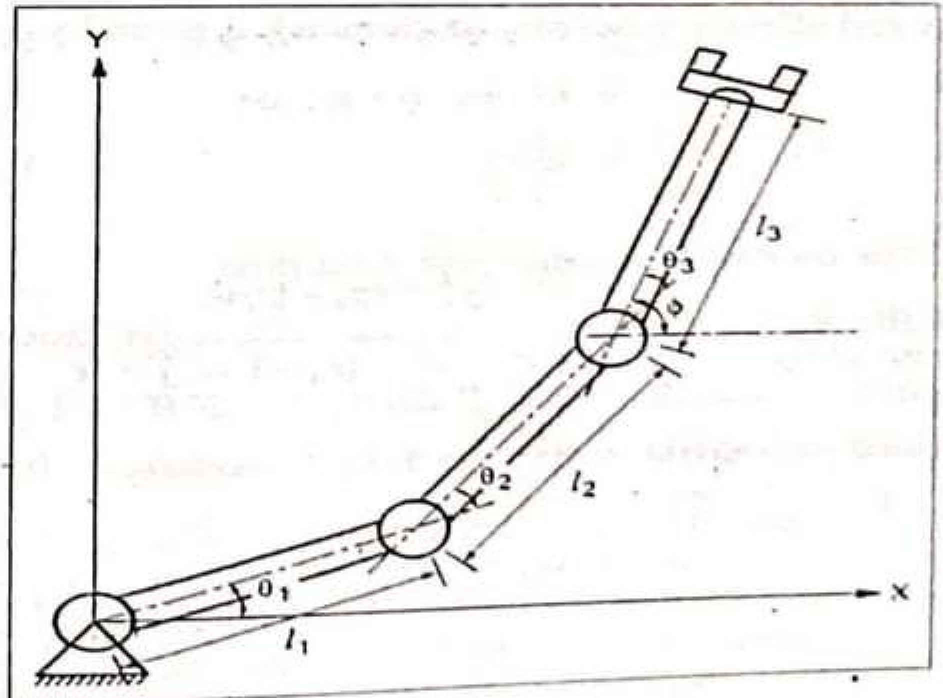


Fig. 6.17. Three DOF-2D Manipulator

Planar Three Manipulator

$$
{}^{3}_{0}T = \begin{bmatrix} \cos(\theta_1+\theta_2+\theta_3) & -\sin(\theta_1+\theta_2+\theta_3) & 0 & L_1\cos\theta_1 + L_2\cos(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2+\theta_3) & \cos(\theta_1+\theta_2+\theta_3) & 0 & L_1\sin\theta_1 + L_2\sin(\theta_1+\theta_2+\theta_3) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$[\because \theta_1+\theta_2+\theta_3 = \phi]$

A given end effector orientation is in the following form:

$$
{}^{ee}_{bs}T = \begin{bmatrix} \cos\phi & -\sin\phi & 0 & x \\ \sin\phi & \cos\phi & 0 & y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \ldots (6.40)
$$

$$\cos\phi = \cos(\phi_1+\phi_2+\phi_3)$$

$$\sin\phi = \sin(\phi_1+\phi_2+\phi_3)$$

$$x = L_1\cos\phi_1 + L_2\cos(\phi_1+\phi_2) \quad \ldots (6.41)$$

$$y = L_1\sin\phi_1 + L_2\sin(\phi_1+\phi_2) \quad \ldots (6.42)$$

Squaring and adding equations (6.41) and (6.42), we get

$$x^2 + y^2 = L_1^2 \cos^2 \phi_1 + L_2^2 \cos^2 (\phi_1 + \phi_2) + 2 L_1 L_2 \cos \phi_1 \cdot \cos (\phi_1 + \phi_2) +$$

$$L_1^2 \sin^2 \phi_1 + L_2^2 \sin^2 (\phi_1 + \phi_2) + 2 L_1 L_2 \sin \phi_1 \cdot \sin (\phi_1 + \phi_2)$$

$$[\because (A + B)^2 = A^2 + B^2 + 2 AB]$$

$$x^2 + y^2 = L_1^2 \{(\cos^2 \phi + \sin^2 \phi)\} + L_2^2 \{\cos^2 (\phi_1 + \phi_2) + \sin^2 (\phi_1 + \phi_2)\}$$

$$+ 2 L_1 L_2 [\cos \phi_1 \cdot \cos (\phi_1 + \phi_2) + \sin \phi_1 \cdot \sin (\phi_1 + \phi_2)]$$

$$x^2 + y^2 = L_1^2 [1] + L_2^2 [1] + 2 L_1 L_2 [\cos \phi_1 \cdot \cos (\phi_1 + \phi_2) + \sin \phi_1 \cdot \sin (\phi_1 + \phi_2)]$$

$$[\because \sin^2 \phi + \cos^2 \phi_1 = 1$$

$$\sin^2 (\phi_1 + \phi_2) + \cos^2 (\phi_1 + \phi_2) = 1]$$

$$= L_1^2 + L_2^2 + 2 L_1 L_2 [\cos \phi_1 \cdot \cos (\phi_1 + \phi_2) + \sin \phi_1 \cdot \sin (\phi_1 + \phi_2)]$$

$$x^2 + y^2 = L_1^2 + L_2^2 + 2 L_1 L_2 [\cos \phi_1 - (\phi_1 + \phi_2)]$$

$$[\because \cos A \cos B + \sin A \sin B = \cos (A - B)]$$

$$x^2 + y^2 = L_1^2 + L_2^2 + 2 L_1 L_2 [\cos \phi_1 - \phi_1 + \phi_2]$$

$$= L_1^2 + L_2^2 + 2 L_1 L_2 [\cos \phi_2]$$

$$\boxed{\cos \phi_2 = \frac{x^2 + y^2 - L_1^2 - L_2^2}{2 L_1 L_2}}$$

We know that, $\sin^2 \phi + \cos^2 \phi = 1$

$$\sin^2 \phi = 1 - \cos^2 \phi$$

$$\sin \phi = \sqrt{1 - \cos^2 \phi}$$

$$\sin \phi_2 = \pm\sqrt{1 - \cos \phi_2}$$

$$\boxed{\theta_2 = A \tan 2 (\sin \phi_2, \cos \phi_2)}$$

$$x = k_1 \cos \phi_1 - k_2 \sin \phi_1$$

$$y = k_1 \sin \phi_1 + k_2 \cos \phi_1$$

$$k_1 = L_1 + L_2 \cos \phi_2$$

$$k_2 = L_2 \sin \phi_2$$

$$r = \sqrt{k_1^2 + k_2^2}$$

$$\gamma = A \tan 2\,(k_2, k_1)$$

Then, $k_1 = r \cos \gamma$

$$k_2 = r \sin \gamma$$

Applying these in the equation for x and y.

$$x/r = \cos \gamma \cos \phi_1 + \sin \gamma \sin \phi_1$$

$$y/r = \cos \gamma \sin \phi_1 + \sin \gamma \cos \phi_1$$

It can be written as, $\cos (\gamma + \theta_1) = \dfrac{x}{r}$

$$\sin (\gamma + \theta_1) = \frac{y}{r}$$

$$r+\theta_1 = A\tan 2\,(y, x)$$

Hence, $$\boxed{\theta_1 = A\tan 2\,(y, x) - A\tan 2\,(k_2, k_1)}$$

$$\text{If } x = y = 0$$

θ_1 becomes arbitrary.

θ_3 can be solved from the two equations for $\sin\phi$ and $\cos\phi$.

$$\theta_3 = \psi - \theta_1 - \theta_2$$

$$\boxed{\theta_3 = A\tan 2\,(\sin\psi, \cos\psi) - \theta_1 - \theta_2}$$

.11. TRANSFORMATION

.11.1. Representation of Transformation

- ✓ Transformation is defined as making a movement in space.
- ✓ When a frame moves in space relative to fixed reference frame, we can represent this motion in a form similar to a frame representation.
- ✓ This is because a transformation itself is a change in the state of a frame (representing the change in its location and orientation) and thus it can be represented as a frame.
- ✓ A transformation may be in one of the following forms:

1. Pure translation
2. Pure rotation about an axis.
3. A combination of translation and rotation.

Pure Translation

- ✓ If a frame moves in space without any change in its orientation, the transformation is a pure translation.
- ✓ The directional unit vectors remain in the same direction and thus do not change.
- ✓ The new location of the frame relative to the fixed reference frame can be found by adding the vector representing the translation to the vector representing the original location of the origin of the frame.
- ✓ It matrix form, new frame representation may be found by premultiplying the frame with a matrix representing the transformation.

- ✓ Since the directional vectors do not change in a pure translation, the transformation T will simply be,

$$T = \begin{bmatrix} 1 & 0 & 0 & dx \\ 0 & 1 & 0 & dy \\ 0 & 0 & 1 & dz \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where dx, dy and dz are the three components of a pure translation vectors T relative to the x, y and z axes of the reference frame.

- ✓ First three columns represents no rotational movement (Equivalent to unity), while the last column represents the translation. The new location of the frame will be

$$F_{new} = \begin{bmatrix} 1 & 0 & 0 & dx \\ 0 & 1 & 0 & dy \\ 0 & 0 & 1 & dz \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} n_x & o_x & a_x & p_x + dx \\ n_y & o_y & a_y & p_y + dy \\ n_z & o_z & a_z & p_z + dz \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This equation is written as,

$$F_{new} = \text{Trans}(dx, dy, dz) \times F_{old}$$

COORDINATE SYSTEM

1. Robotics Control

Coordinate Transformation

1. Reference coordinate frame OXYZ
2. Body-attached frame O'uvw:

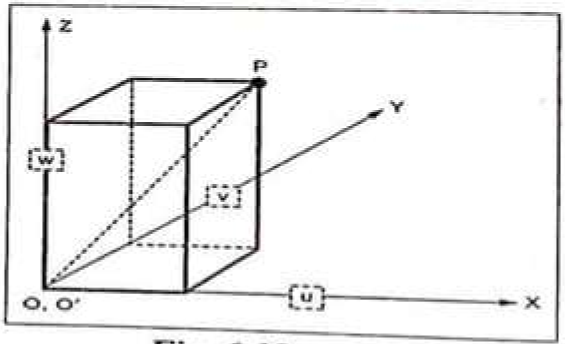


Fig. 6.19.

Point represented in OXYZ:

$$\mathbf{P}_{xyz} = [p_x, p_y, p_z]^T$$

$$\overline{\mathbf{P}}_{xyz} = p_x\, i_x + p_y\, j_y + p_z\, k_z$$

Point represented in O'uvw:

$$\overline{P}_{uvw} = p_u\, i_u + p_v\, j_v + p_w\, k_w$$

Two frames coincide $\Rightarrow$ $p_u = p_x; \quad p_v = p_y; \quad p_w = p_z$

Properties of Dot Product:

Let x and y be arbitrary vectors in R^3 and θ be the angle from x to y, then

$$x - y = |x|\ |y| \cos\theta$$

Properties of ortho normal coordinate frame:

Mutually perpendicular	Unit vectors
$\overline{i} \cdot \overline{j} = 0$	$\lvert\overline{i}\rvert = 1$
$\overline{i} \cdot \overline{k} = 0$	$\lvert\overline{j}\rvert = 1$
$\overline{k} \cdot \overline{j} = 0$	$\lvert\overline{k}\rvert = 1$

Coordinate Transformation

Rotation only: $\overline{P}_{xyz} = p_x\, i_x + p_y\, j_y + p_z\, k_z$

$$\overline{P}_{uvw} = p_u\, i_u + p_v\, j_v + p_w\, k_w$$

$$P_{xyz} = R\, P_{uvw}$$

The coordinates rotation in these two frames are:

p_x, p_y and p_z represent the projections of P onto OX, OY, OZ axes, respectively.

Since
$$\mathrm{P} = p_u\, i_u + p_v\, i_v + p_w\, k_w$$

$$p_x = i_x \cdot \mathrm{P} = i_x \cdot i_u p_u + i_x \cdot j_v p_v + i_x \cdot k_w p_w$$

$$p_y = j_y \cdot \mathrm{P} = j_y \cdot i_u p_u + j_y \cdot j_v p_v + j_y \cdot k_w p_w$$

$$p_z = k_z \cdot \mathrm{P} = k_z \cdot i_u p_u + k_z \cdot j_v p_v + k_z \cdot k_w p_w$$

Expressed in matrix form

$$\begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = \begin{bmatrix} i_x \cdot i_u & i_x \cdot j_v & i_x \cdot k_w \\ j_y \cdot i_u & j_y \cdot j_v & j_y \cdot k_w \\ k_z \cdot i_u & k_z \cdot j_v & k_z \cdot k_w \end{bmatrix} \begin{bmatrix} p_u \\ p_v \\ p_w \end{bmatrix}$$

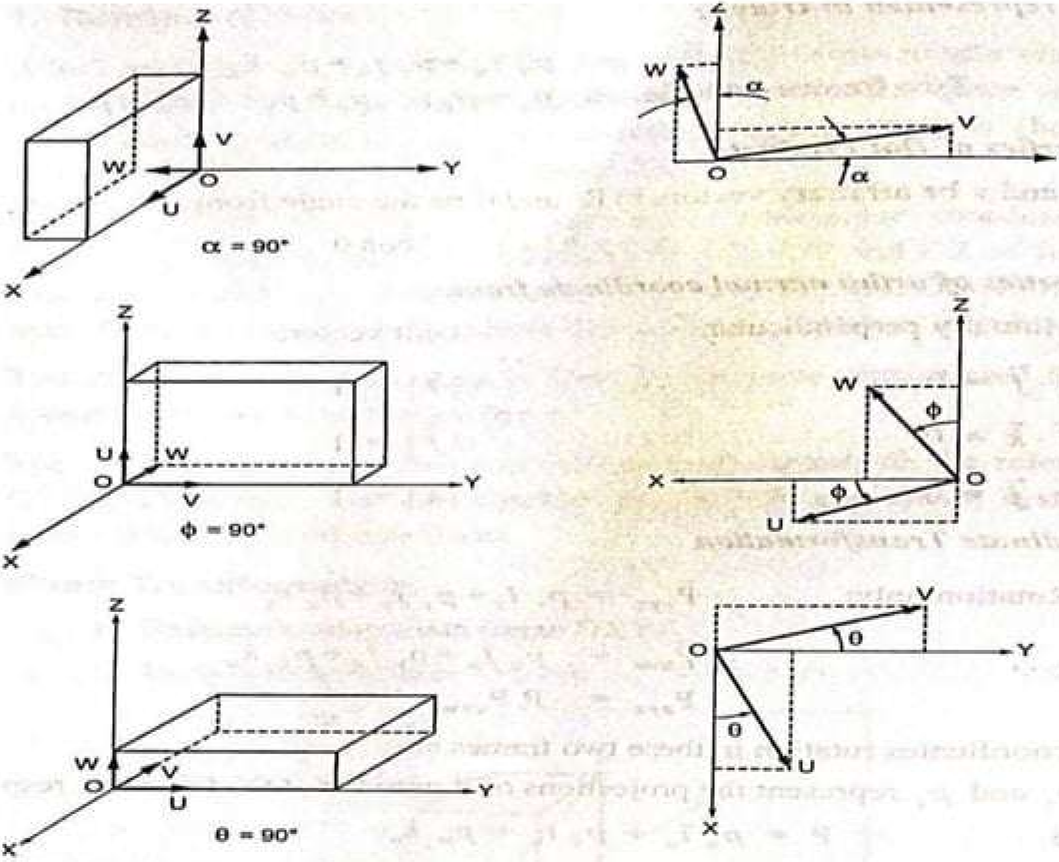


Fig. 6.20 Rotating coordinat

Rotation about x axis with θ:

$$\begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} p_u \\ p_v \\ p_w \end{bmatrix}$$

$$p_x = p_u$$

$$p_y = p_v \cos\theta - p_w \sin\theta$$

$$p_z = p_v \sin\theta + p_w \cos\theta$$

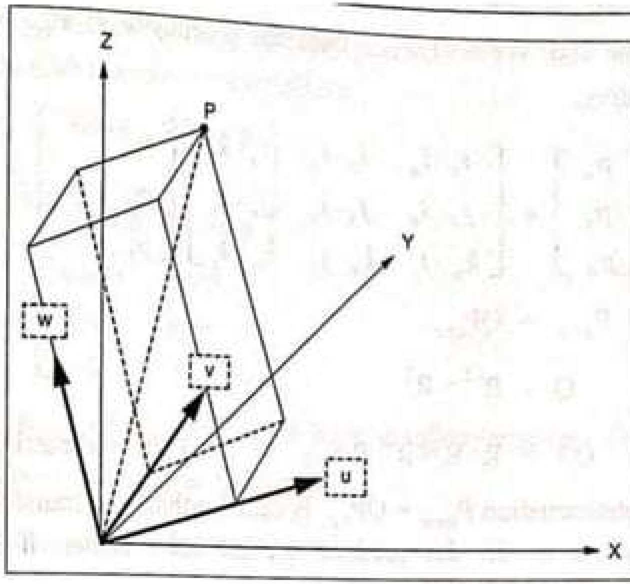


Fig. 6.21.

Rotation about x-axis with θ:

$$\mathrm{Rot}(x, \theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

Rotation about y-axis with θ:

$$\mathrm{Rot}(y, \theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

Rotation about z-axis with θ:

$$P_{xyz} = RP_{uvw}$$

$$\mathrm{Rot}(z, \theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Basic Rotation Matrix:

$$R = \begin{bmatrix} i_x \cdot i_u & i_x \cdot j_v & i_x \cdot k_w \\ j_y \cdot i_u & j_y \cdot j_v & j_y \cdot k_w \\ k_z \cdot i_u & k_z \cdot j_v & k_z \cdot k_w \end{bmatrix}; \quad P_{xyz} = RP_{uvw}$$

$$\begin{bmatrix} p_u \\ p_v \\ p_w \end{bmatrix} = \begin{bmatrix} i_x \cdot i_u & i_x \cdot j_y & i_u \cdot k_z \\ j_v \cdot i_x & j_v \cdot j_y & j_v \cdot k_z \\ k_w \cdot i_x & k_w \cdot j_y & k_w \cdot k_z \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix};$$

$$P_{uvw} = QP_{xyz}$$

$$Q = R^{-1} = R^T$$

$$QR = R^T R = R^{-1} R = I_3 \quad \Leftarrow 3 \times 3 \text{ identify matrix}$$

$$P_{xyz} = R_{x\alpha} \; P_{uvw}$$

$$i_x = i_u$$

$$R_{x\alpha} = \begin{bmatrix} i_x \cdot i_u & i_x \cdot i_v & i_x \cdot k_w \cdot \\ j_y \cdot i_u & j_y \cdot j_v & j_y \cdot k_w \\ k_z \cdot i_u & k_z \cdot j_v & k_z \cdot k_w \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 0 & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix}$$

Similarly, the 3×3 rotation matrices for rotation about the OY axis with ϕ angle and about the OZ axis with θ angle are respectively.

$$R_{y\phi} = \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix}$$

$$R_{z\theta} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hints: Inverse transformation (Rotation matrix)

$$X = R \cdot A$$

$$A = R^{-1} \cdot X$$

Rotation	R Matrix	$R^{-1} = R^T$
$R(x, \theta)$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}$
$R(y, \theta)$	$\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$	$\begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$
$R(z, \theta)$	$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Example 6.6 *A point $P_{uvw} = (4, 3, 2)$ is attached to a rotating frame, the frame rotates 60 degrees about the OZ axis of the reference frame. Find the coordinates of the point relative to the reference frame after the rotation.*

Given that $u = 4,\ v = 3,\ w = 2,\ \theta = 60°$

☺ ***Solution:***

$$P_{xyz} = R_{z,60} \cdot P_{uvw}$$

We know that,

$$R_{z,\theta} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P_{xyz} = \begin{bmatrix} \cos 60° & -\sin 60° & 0 \\ \sin 60° & \cos 60° & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

$$P_{xyz} = \begin{bmatrix} 0.5 & -0.866 & 0 \\ 0.866 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}$$

$$P_{xyz} = \begin{bmatrix} 0.5\,(4) & -0.866\,(3) & 0\,(2) \\ 0.866\,(4) & 0.5\,(3) & 0\,(3) \\ 0\,(4) & 0\,(3) & 1\,(2) \end{bmatrix}$$

$$= \begin{bmatrix} 2 & - & 2.598 & + & 0 \\ 3.464 & + & 1.5 & + & 0 \\ 0 & + & 0 & + & 2 \end{bmatrix}$$

$$P_{xyz} = \begin{bmatrix} -0.598 \\ 4.964 \\ 2 \end{bmatrix}$$

New position points $= [-0.598,\ 4.964,\ 2]$

6.12.2. Homogonoous Transformations

- ✓ The use of homogeneous transformations is a general method for solving the kinematics equations of a robot manipulator with many joints.
- ✓ A generalized transformation is now described by a single matrix that combines the effects of translation and rotation.

Rotation matrices (4×4) are defined as:

$$\mathrm{Rot}(x, 0) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathrm{Rot}(y, 0) = \begin{bmatrix} \cos\theta & 0 & -\sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathrm{Rot}(z, 0) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The translation matrix (4×4) is also defined as:

$$\text{Trans}\left(a,b,c\right)=\begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

6.13. 3D TRANSFORMATION

Sequences of transformations can be combined into a single transformation using the concatenation (combined) process. For example, consider the rotation of a line about an arbitrary point. Line AB is to be rotated through 45° in anticlockwise direction about point A. An inverse translation of AB to A_1B_1. A_1B_1 is then rotated through 45° to A_2B_2. The line A_2B_2 is then translated to A_3B_3.

The respective transformation matrices are:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -T_x & -T_y & 1 \end{bmatrix}$$

$$\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ T_x & T_y & 1 \end{bmatrix}$$

The same effect can be achieved using the concatenated (combined) matrix given below:

$$[X_1\ Y_1\ 1] = [X\ \ Y\ \ 1] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -T_x & -T_y & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ T_x & T_y & 1 \end{bmatrix}$$

6.14. TRANSFORMATION MATRIX FROM 2D TO 3D

It is often necessary to display objects in 3-D on the graphics screen. The transformation matrices developed for 2-dimensions can be extended to 3-D.

1. Scaling: The scaling matrix in 3-D is

$$\begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2. *Translation:* The translation matrix is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ T_x & T_y & T_z & 1 \end{bmatrix}$$

(i) Translation along the X axis is dx.

$$\text{Trans}(x, dx) = \begin{bmatrix} 1 & 0 & 0 & dx \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(ii) Translation along the Y axis is dy.

$$\text{Trans}(y, dy) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & dy \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(*iii*) Translation along the Z axis is dz.

$$\text{Trans}(z, dz) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & dz \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T = \begin{bmatrix} n_x & s_x & a_x & p_x \\ n_y & s_y & a_y & p_y \\ n_z & s_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T = \begin{bmatrix} n & s & a & p \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where $p = (p_x, p_y, p_z)$ is position vectors.

3. *Rotation:* Rotation in 3-D can be about X, Y, or Z axis.

(i) Rotation about X axis:

$$R_x = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(ii) Rotation about Y axis:

$$R_y = \begin{bmatrix} \cos\theta & 0 & -\sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(iii) Rotation about Z axis:

$$R_z = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

15. COMPOSITE ROTATION MATRIX

- ✓ Basic rotation matrices can be multiplied together to represent a sequence of finite rotation about the principle axes of OXYZ coordinate system.
- ✓ To develop a rotation matrix representing a rotation of α angle about the OX axis followed by a rotation of θ angle about the OZ axis followed by a rotation of ϕ angle about the OY axis.

✓ The resultant rotation matrix representing the rotation is,

$$\mathrm{Rot}(X, \alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix}$$

$$\mathrm{Rot}(Y, \phi) = \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix}$$

$$\mathrm{Rot}(Z, \theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R = R_{y\phi}\ R_{z\theta}\ R_{x\alpha}$$

$$R = \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix}$$

$$R_{y\phi} \times R_{z\theta}$$

$$= \begin{bmatrix} \cos\phi\cos\theta + 0 + 0 & -\cos\phi\sin\theta + 0 + 0 & 0 + 0 + \sin\phi \\ 0 + \sin\theta + 0 & 0 + \cos\theta + 0 & 0 + 0 + 0 \\ -\sin\phi\cos\theta + 0 + 0 & \sin\phi\sin\theta + 0 + 0 & 0 + 0 + \cos\theta \end{bmatrix}$$

$$R = (R_{y\phi} \cdot R_{z\theta})\, R_{x\alpha}$$

$$= \begin{bmatrix} \cos\phi\cos\theta & -\cos\theta\sin\phi & \sin\phi \\ \sin\theta & \cos\theta & 0 \\ -\sin\theta\cos\theta & \sin\phi\sin\theta & \cos\phi \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix}$$

... will also rotate about its principle axes.

Example 6.8.1 *P_{uvw} $(5, 4, 3)^T$ and $q_{uvw} = (6, 2, 4)^T$ with respect to rotated OUVW coordinate system. Determine the corresponding point P_{xyz}, q_{xyz} with respect to reference coordinate system, if it has been rotated about OZ axis.*

☺ ***Solution:*** Given that $u = 5$; $v = 4$; $w = 3$.

$$R_{y\phi} = \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix}$$

Case (i):

$$p_{xyz} = R_{z,60} \cdot p_{uvw}$$

$$R_{z,0} = \begin{bmatrix} \cos 0 & \cos(90° + 0) & \cos 90° \\ \cos(90° - 0) & \cos 0 & \cos 90° \\ \cos 90° & \cos 90° & \cos 0° \end{bmatrix}$$

$$R_{z,0} = \begin{bmatrix} \cos 0 & -\sin 0 & 0 \\ \sin 0 & \cos 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\left[\because \begin{aligned} \cos(90° + 0) &= -\sin 0 \\ \cos 90° &= 0 \\ \cos(90° - 0) &= \sin 0 \\ \cos 0 &= 1 \end{aligned}\right]$$

$$p_{xyz} = R_{z,0} \cdot p_{uvw}$$

Substituting the values of $R_{z,60}$ and p_{uvw}.

$$p_{xyz} = \begin{bmatrix} \cos 60° & -\sin 60° & 0 \\ \sin 60° & \cos 60° & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 4 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0.5 & -0.866 & 0 \\ 0.866 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 4 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0.5\,(5) & -0.866\,(4) & 0 \times 3 \\ 0.866\,(5) & 0.5\,(4) & 0 \times 3 \\ 0\,(5) & 0\,(4) & 1 \times 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2.5 & -3.464 & +0 \\ 4.33 & + \quad 2 & +0 \\ 0 & + \quad 0 & +3 \end{bmatrix}$$

$$p_{xyz} = \begin{bmatrix} -0.964 \\ 6.330 \\ 3.000 \end{bmatrix}$$

$$\boxed{\text{New position points} = [-0.964,\ 6.330,\ 3.000]}$$

Case (ii): Coordinates of point q_{xyz}.

Given that $u = 6$, $v = 2$, $w = 4$, $\theta = 60°$

We know that, $q_{xyz} = \text{Rot}(z, 60°) \cdot q_{uvw}$

$$= \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

$$= \begin{bmatrix} \cos 60° & -\sin 60° & 0 \\ \sin 60° & \cos 60° & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 2 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0.5 & -0.866 & 0 \\ 0.866 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 2 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0.5\,(6) & - & 0.866\,(2) & + & 0 \times 4 \\ 0.866\,(6) & + & 0.5\,(2) & + & 0 \times 4 \\ 0\,(6) & + & 0\,(2) & + & 1 \times 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1.5 & - & 1.732 & + & 0 \\ 5.196 & + & 1 & + & 0 \\ 0 & + & 0 & + & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -0.232 \\ 6.196 \\ 4 \end{bmatrix}$$

New position points $q_{xyz} = [-0.232, \ 6.196, \ 4]$

Example 6.12 *q(u, v, w) are given by $(4, 3, 2)^T$ which are rotated about X-axis of the reference frame by angle of 45°. Determine the point q_{xyz}.*

☺ ***Solution:*** Given: $q(u, v, w) = (4, 3, 2)^T$; $\alpha = 45°$, $u = 4, v = 3, w = 3$

$$q(x, y, z) = \mathrm{Rot}(x, 45) \cdot q_{uvw}$$

$$q(x, y, z) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 45° & -\sin 45° \\ 0 & \sin 45° & \cos 45° \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.707 & -0.707 \\ 0 & 0.707 & 0.707 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}$$

$$= [4 \times 1 + 0 \times 3 + 0 \times 2] \; [4 \times 0 + 3 \times 0.707 + 2 \times (-0.707)]$$

$$[4 \times 0 + 3 \times 0.707 + 2 \times 0.707]$$

$$q(x, y, z) = [4, 0.707, 3.535]$$

Hence the new position point =

$$\boxed{q_{xyz} = \begin{bmatrix} 4 \\ 0.707 \\ 3.535 \end{bmatrix}}$$ **Ans.**

COMPOSITE HOMOGENEOUS TRANSFORMATION MATRIX

The homogeneous rotation and translation matrices can be multiplied together to obtain a composite homogeneous transformation matrix (T matrix).

Since matrix multiplication is not commutative, careful attention must be paid to the order in which these matrices are multiplied.

The following rules are useful for finding a composite homogeneous transformation matrix.

Rules:

- ✓ Transformation (rotation/translation) with respect to (X, Y, Z) (OLD FRAME) using premultiplication.
- ✓ Transformation (rotation/translation) with respect to (U, V, W) (NEW FRAME) using postmultiplication.

.17.3. Rotation Matrix with Euler Angles Representations

- ✓ The matrix representation for rotation of a rigid body simplifies many operations, but it needs nine elements to completely describe the orientation of a rotating rigid body.

- ✓ It does not lead directly to a complete set of generalized coordinates. Such a set of generalized coordinates can describe the orientation of a rotating rigid body with respect to a reference coordinate frame.

- ✓ They can be provided by three angles called Euler angles, ϕ, θ and ψ. Although Euler angles describe the orientation of a rigid body with respect to a fixed reference frame, there are many different types of Euler angle representations.

Orientation Representation

$$F = \begin{bmatrix} R_{3\times3} & P_{3\times1} \\ 0 & 1 \end{bmatrix}$$

- ✓ Rotation matrix representation needs 9 elements to completely describe the orientation of a rotating rigid body.
- ✓ Euler angles representation (ϕ, θ, ψ)
- ✓ Many different types
- ✓ Description of Euler angle representations

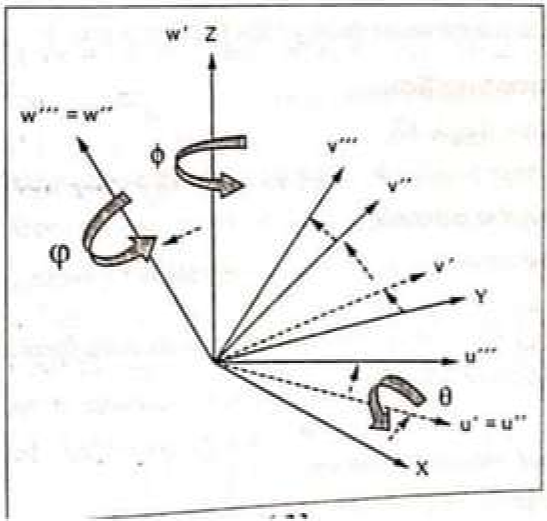
	Euler angle I	Euler angle II	Roll-Pitch Yaw
Sequence of rotation	ϕ about OZ axis	ϕ about OZ axis	ψ about OX axis
	θ about OU axis	θ About OV axis	θ about OY axis
	ψ about OW axis	ψ about OW axis	ϕ about OZ axis

1. **Euler Angle I, Animated**

✓ ***Euler Angle I:***

$$R_{z\phi} = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad R_{u',\theta} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

$$R_{w',\phi} = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



Resultant Eulerian rotation matrix:

$$R = R_{z\phi}\, R_{u',\theta}\, R_{w',\phi}$$

$$\begin{pmatrix} \cos\phi\cos\varphi - \sin\phi\sin\varphi\cos\theta & -\cos\phi\sin\varphi - \sin\phi\cos\varphi\cos\theta & \sin\varphi\sin\theta \\ \sin\phi\cos\varphi + \cos\phi\sin\varphi\cos\theta & -\sin\phi\sin\varphi + \cos\phi\cos\varphi\cos\theta & -\cos\phi\sin\theta \\ \sin\varphi\sin\theta & \cos\varphi\sin\theta & \cos\theta \end{pmatrix}$$

2. Euler Angle II, Animated

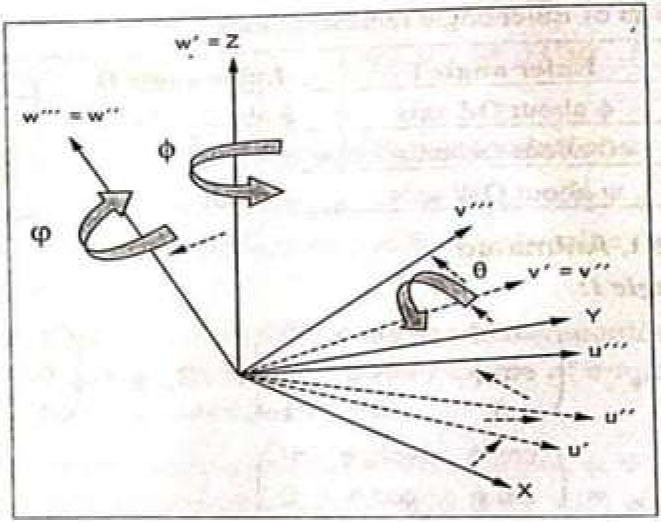


Fig. 6.24.

Note the opposite (clockwise) sense of the third rotation, ϕ.

Orientation Representation

Matrix with Euler Angle II:

$$\begin{pmatrix} \sin\phi\sin\varphi + \cos\phi\sin\varphi\cos\theta & -\sin\phi\cos\varphi - \sin\phi\cos\varphi\cos\theta & \cos\phi\sin\theta \\ \cos\phi\sin\varphi + \sin\phi\cos\varphi\cos\theta & \cos\phi\cos\varphi - \sin\phi\cos\varphi\cos\theta & \sin\varphi\sin\theta \\ -\cos\varphi\sin\theta & \sin\varphi\sin\theta & \cos\theta \end{pmatrix}$$

Example 6.16 Calculate the homogeneous transformation matrix for the following transform sequence that describes frame {1} with respect to the reference frame.

Trans (4, – 3, 7) Rot (x, – 60) Rot (y, 45) Rot (z, 90)

(a) What is the origin of this frame? (with respect to the reference frame)

(b) What are the coordinates of a point at (1, 1, 1) in frame {1} with respect to the reference frame?

Solution:

$$(a)\ {}^{0}_{1}T = \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(-60°) & -\sin(-60°) & 0 \\ 0 & \sin(-60°) & \cos(-60°) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \cos(45°) & 0 & \sin(45°) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(45°) & 0 & \cos(45°) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(90°) & -\sin(90°) & 0 & 0 \\ \sin(90°) & \cos(90°) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$ {}^{0}_{1}T = \begin{bmatrix} 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 4 \\ \frac{1}{2} & \frac{\sqrt{6}}{4} & \frac{\sqrt{6}}{4} & -3 \\ -\frac{\sqrt{3}}{2} & \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(i) The origin of this frame (with respect to the reference frame) is the last column, *i.e.*,

$$= [4, -3, 7]^T$$

(ii) The coordinate of a point at (1, 1, 1) in frame {1} with respect to the reference frame are:

$$
{}^{0}P = {}^{0}_{1}T\ {}^{1}P = \begin{bmatrix} 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 4 \\ \frac{1}{2} & \frac{\sqrt{6}}{4} & \frac{\sqrt{6}}{4} & -3 \\ -\frac{\sqrt{3}}{2} & \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}
$$

$$
= \begin{bmatrix} 4 \\ -\frac{5}{2} + \frac{\sqrt{2}}{2} \\ 7 + \frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2} \\ 1 \end{bmatrix}
$$

$$
= \begin{bmatrix} 4 \\ -1.27526 \\ 6.84108 \\ 1 \end{bmatrix}
$$

Ans.

Example 6.19 *Frame (2) is rotated with respect to frame (1) about the X-axis by an angle of 60°. The position of the origin of frame (2) as seen from frame (1) is 1D2 = [9.0, 7.0, 5.0]^T. Obtain the transformation $^1D_2 = [9.0 \quad 7.0 \quad 5.0]^T$. Obtain the transformation matrix $^1T_2$ which describes ($^1T_2$) frame (2) relative to frame (1). Also find the description of point P in frame (1), if $^2P = [4.0 \quad 6.0 \quad 8.0]^T$.*

☺ ***Solution:*** The homogeneous transform matrix describes frame {2} with respect to frame {1}.

$$^1T_2 = \left[\begin{array}{c:c} ^1T_2 & ^1D_2 \\ \hdashline 0\,0\,0 & 1 \end{array}\right] \qquad \ldots(6.19.1)$$

Frame (2) is rotated relative to frame (1) about X-axis by 60°.

We know that, rotated about X-axis,

$$ {}^{1}R_{2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 60^\circ & -\sin 60^\circ \\ 0 & \sin 60^\circ & \cos 60^\circ \end{bmatrix} $$

$$ {}^{1}R_{2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.500 & -0.866 \\ 0 & 0.866 & 0.500 \end{bmatrix} $$

Substituting ${}^{1}R_{2}$ and ${}^{1}D_{2}$ in the above equation,

$$ {}^{1}T_{2} = \left[\begin{array}{ccc:c} 1 & 0 & 0 & 9.000 \\ 0 & 0.500 & -0.866 & 7.000 \\ 0 & 0.866 & 0.500 & 5.000 \\ \hdashline 0 & 0 & 0 & 1 \end{array}\right] $$

$$ \text{Given } {}^{2}P = [4.0 \quad 6.0 \quad 8.0]^{T} $$

Point P in frame (1) is given by

$$^{1}P = {}^{1}T_2 \cdot {}^{2}P$$

Substituting the values,

$$^{1}P = \begin{bmatrix} 1 & 0 & 0 & 9.000 \\ 0 & 0.500 & -0.866 & 7.000 \\ 0 & 0.866 & 0.500 & 5.000 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4.0 \\ 6.0 \\ 8.0 \\ 1 \end{bmatrix}$$

$$^{1}P = \begin{bmatrix} 1\,(4.0) + 0\,(6.0) + 0\,(8.0) + 9.000\,(1) \\ 0\,(4.0) + 0.500\,(6.0) + -0.866\,(8.0) + 7.000\,(1) \\ 0\,(4.0) + 0.866\,(6.0) + 0.500\,(8.0) + 5.000\,(1) \\ 0\,(4.0) + 0\,(6) + 0\,(8.0) + 1\,(1) \end{bmatrix}$$

$$^{1}P = \begin{bmatrix} 4 + 0 + 9 \\ 3 - 6.928 + 7 \\ 5.196 + 4 + 5 \\ 1 + 0 + 0 \end{bmatrix}$$

$$= \begin{bmatrix} 13.000 \\ 3.072 \\ 14.196 \\ 1.000 \end{bmatrix}$$

$${}^{1}P = [13.000 \quad 3.072 \quad 14.196 \quad 1.000]^T \qquad \text{...(6.19.5)}$$

The 3 × 1 position of vector of point P in frame (1) in physical coordinate is then

$$\boxed{{}^{1}P = [13.000 \quad 3.072 \quad 14.196]^T}$$

6.18. KINEMATIC MODEL (Forward (Direct) Kinematics)

Before a robot can move its hand to an object, the object must be located relative to it. There is currently no simple method for measuring the location of a robot hand. Most robots calculate the position of their hand using a kinematic model of their arm.

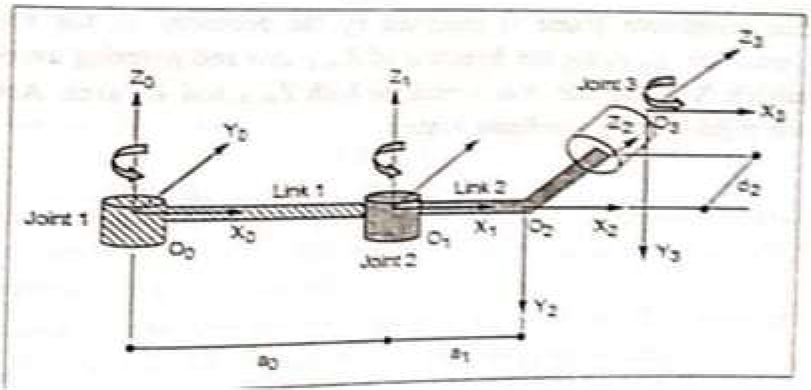
Steps to derive kinematics model

1) Assign D-H coordinates frames
2) Find link parameters
3) Transformation matrices of adjacent joints
4) Calculate kinematics matrix
5) When necessary, Euler angle representation

LINK COORDINATE FRAMES

(1) Assign Link Coordinate Frames

- ✓ To describe the geometry of robot motion, we assign a Cartesian coordinate frame (O_i, X_i, Y_i, Z_i) to each link, as follows:
- ✓ Establish a right-handed orthonormal coordinate frame O_0 at the supporting base with Z_0 lying along joint 1 motion axis.
- ✓ The Z_i axis is directed along the axis of motion of joint ($i+1$), that is, ($i+1$) rotates about or translates along Z_i.



2) Link Coordinate Frames

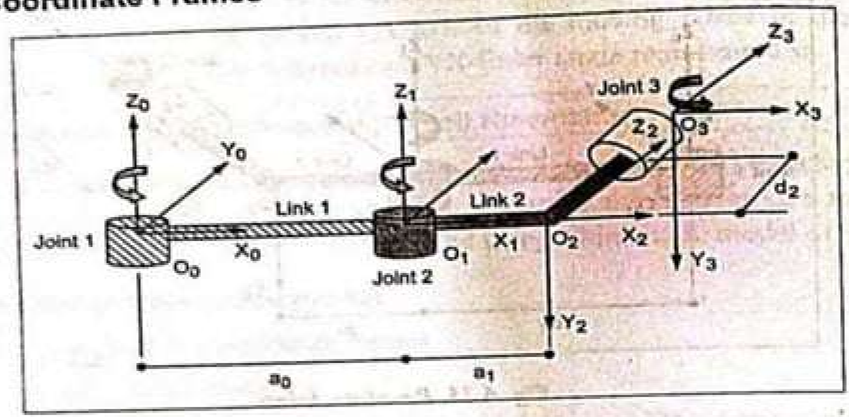


Fig. 6.27.

- Locate the origin of the i^{th} coordinate at the intersection of the Z_i and Z_{i-1} or at the intersection of common normal between the Z_i and Z_{i-1} axes and the Z_i axis.
- The X_i axis lies along the common normal from the Z_{i-1} axis to the Z_i axis $X_i = \pm (Z_{i-1} \times Z_i) / \| Z_{i-1} \times Z_i \|$, (if Z_{i-1} is parallel to Z_i, then X_i is specified arbitrarily, subject only to X_i being perpendicular to Z_i).
- Assign $Y_i = + (Z_i \times X_i) / \| Z_i \times X_i \|$ to complete the right-handed coordinate system.

3) Link and Joint Parameters

- ✓ Joint angle θ_i: the angle of rotation from the X_{i-1} axis to the X_i axis about the Z_{i-1} axis. It is the joint variable if joint i is rotary.

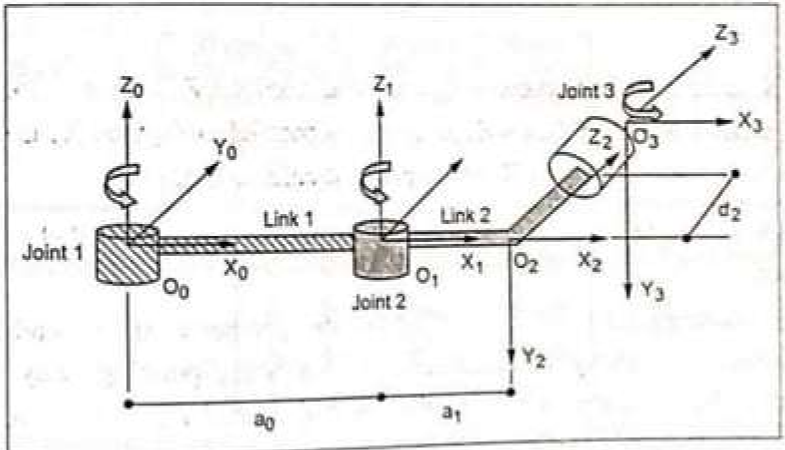


Fig. 6.29.

- ✓ Joint distances d_i: The distance from the origin of the (i-1) coordinate system to the intersection of the Z_{i-1} axis and the X_i axis along the Z_{i-1} axis. It is the joint variable if joint I is prismatic.

✓ ... the intersection of the Z_{i-1} axis and the X_i

Joint i	α_i	a_i	d_i	θ_i
1	0	a_0	0	θ_0
2	-90	a_1	0	θ_1
3	0	0	d_2	θ_2

Joint i	α_i	a_i	d_i	θ_i
1	0	a_0	0	θ_0
2	-90	a_1	0	θ_1
3	0	0	d_2	θ_2

$$T_{i-1}^{i} = \begin{bmatrix} C\theta_i & -C\alpha_i S\theta_i & S\alpha_i S\theta_i & \alpha_i C\theta_i \\ S\theta_i & C\alpha_i C\theta_i & -S\alpha_i C\theta_i & a_i S\theta_i \\ 0 & S\alpha_i & C\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_0^1 = \begin{bmatrix} \cos\vartheta_0 & -\sin\theta_0 & 0 & a_0\cos\theta_0 \\ \sin\theta_0 & \cos\theta_0 & 0 & a_0\sin\theta_0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_2^3 = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 & 0 \\ \sin\theta_2 & \cos\theta_2 & 0 & 0 \\ 0 & 0 & 1 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_0^3 = (T_0^1)\,(T_1^2)\,(T_2^3)$$

3.18.2. Forward Kinematics of PUMA Industrial Robot (PUMA 562)

- ✓ Usually, any tools or handling devices of a manipulator can be described with the help a constant homogeneous transformation, if related to the last, outmost link of the robot.
- ✓ In this tool frame, environmental objects can be easily located relative to the tool tip or gripper. So here it is quiet simple to find relative motions necessary for the tool tip in order to handle objects.
- ✓ However, to really achieve the related joint motion to move the tool tip appropriately, the necessary motion trajectory of the tip with respect to the base frame has to be calculated. The resulting base frame description can then be used to solve the inverse kinematics problem, that is to find the related joint coordinates.

✓ Let us assume, we have to treat an industrial manipulator with 6 revolute joints:

θ_i	Joint variables (actual arm posture)
a_i, α_i, d_i	Constant Denavit-Hartenberg parameters
^6T_E	Tool frame, described in the coordinate frame system of the outmost link

The homogeneous transformation relating the tool frame to the fixed base frame is given by:

$$^0T_E = {}^0T_1 \; {}^1T_2 \; {}^2T_3 \; {}^3T_4 \; {}^4T_5 \; {}^5T_6 \; {}^6T_e \quad \dots (6.45)$$

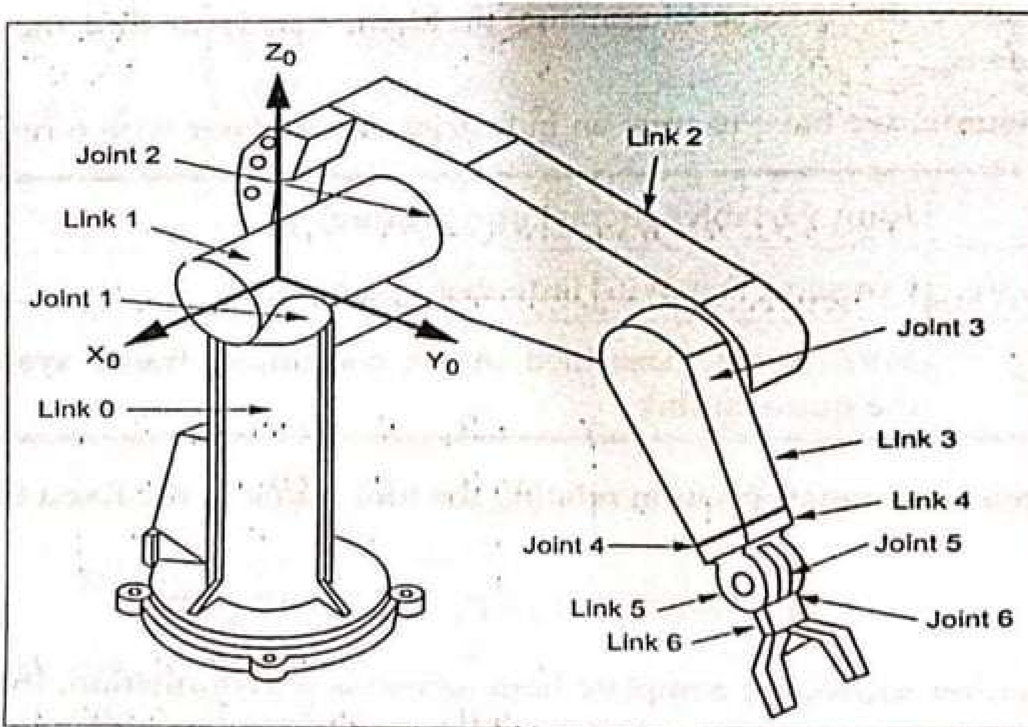


Fig. 6.30. PUMA industrial robot with 6 revolute joints

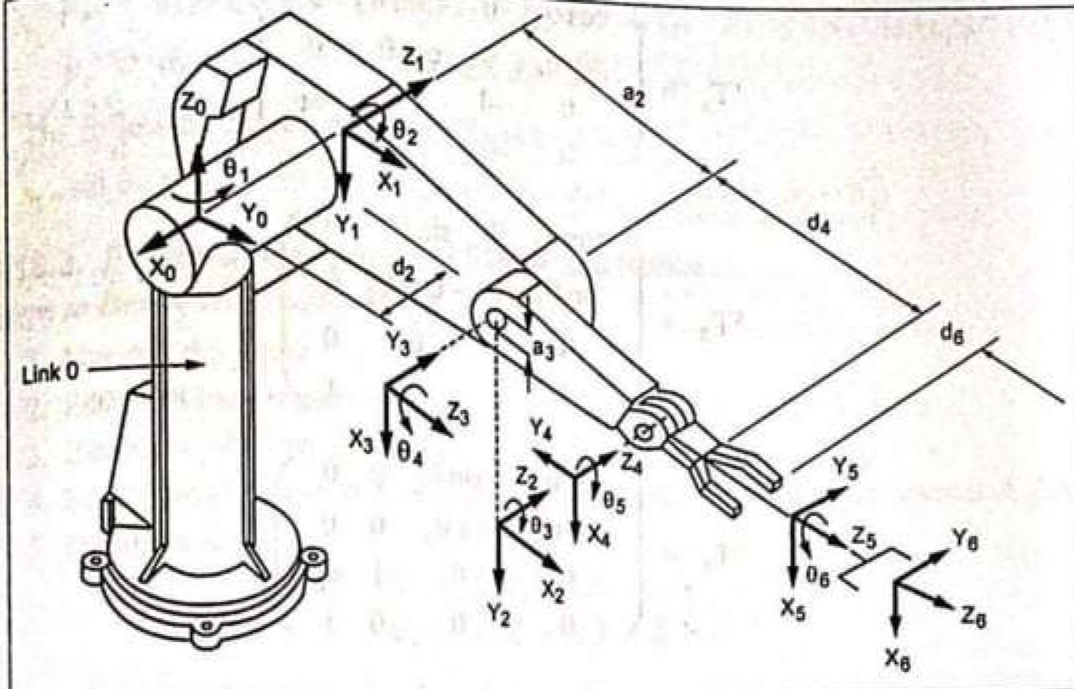


Fig. 6.31. DH frame assignment for the PUMA robot

Transformations between all neighboring frames are given by the following DH-transformations:

$$
{}^{0}T_{1} = \begin{pmatrix} \cos\theta_1 & 0 & -\sin\theta_1 & 0 \\ \sin\theta_1 & 0 & \cos\theta_1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

$$
{}^{1}T_{2} = \begin{pmatrix} \cos\theta_2 & -\sin\theta_2 & 0 & a_2\cos\theta_2 \\ \sin\theta_2 & \cos\theta_2 & 0 & a_2\sin\theta_2 \\ 0 & 0 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

$$
{}^{2}T_{3} = \begin{pmatrix} \cos\theta_3 & 0 & \sin\theta_3 & a_3\cos\theta_3 \\ \sin\theta_3 & 0 & -\cos\theta_3 & a_3\sin\theta_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

$$
{}^{3}T_{4} = \begin{pmatrix} \cos\theta_4 & 0 & -\sin\theta_4 & 0 \\ \sin\theta_4 & 0 & \cos\theta_4 & 0 \\ 0 & -1 & 0 & d_4 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

$$
{}^{4}T_{5} = \begin{pmatrix} \cos\theta_5 & 0 & \sin\theta_5 & 0 \\ \sin\theta_5 & 0 & -\cos\theta_5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

$$
{}^{5}T_{6} = \begin{pmatrix} \cos\theta_6 & -\sin\theta_6 & 0 & 0 \\ \sin\theta_6 & \cos\theta_6 & 0 & 0 \\ 0 & 0 & 1 & d_6 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

$$
{}^{0}T_{6} = {}^{0}T_{1}\,{}^{1}T_{2}\,{}^{2}T_{3}\,{}^{3}T_{4}\,{}^{4}T_{5}\,{}^{5}T_{6} = \begin{bmatrix} n_x & s_x & a_x & p_x \\ n_y & s_y & a_y & p_y \\ n_z & s_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
\begin{aligned}
n_x &= c_1\,(c_{23}\,(c_4\,c_5\,c_6 - s_4\,s_6) - s_{23}\,s_5\,c_6) - s_1\,(s_4\,c_5\,c_6 + c_4\,s_6) \\
n_y &= s_1\,(c_{23}\,(c_4\,c_5\,c_6 - s_4\,s_6) - s_{23}\,s_5\,c_6) + c_1\,(s_4\,c_5\,c_6 + c_4\,s_6) \\
n_z &= -s_{23}\,(c_4\,c_5\,c_6 - s_4\,s_6) - c_{23}\,s_5\,c_6 \\
s_x &= c_1\,(-c_{23}\,(c_4\,c_5\,s_6 + s_4\,c_6) + s_{23}\,s_5\,s_6) - s_4\,(-s_4\,c_5\,s_6 + c_4\,c_6) \\
s_y &= s_1\,(-c_{23}\,(c_4\,c_5\,s_6 + s_4\,c_6) + s_{23}\,s_5\,s_6) + c_1\,(-s_4\,c_5\,s_6 + c_4\,c_6) \\
s_z &= s_{23}\,(c_4\,c_5\,c_6 + s_4\,c_6) - c_{23}\,s_5\,s_6 \\
a_x &= c_1\,(c_{23}\,c_4\,s_5 + s_{23}\,c_5) - s_4\,s_4\,s_5 \\
a_y &= s_1\,(c_{23}\,c_4\,s_5 + s_{23}\,c_5) + c_4\,s_4\,s_5 \\
a_z &= -s_{23}\,(c_4\,s_5 + c_{23}\,c_5) \\
p_x &= c_1\,(d_6\,(c_{23}\,c_4\,s_5 + s_{23}\,c_5) + s_{23}\,d_4 + a_3\,c_{23} + a_2\,c_2) - s_1\,(d_6\,s_4\,s_5 + d_2)
\end{aligned}
$$

$$p_y = s_1 \left(d_6 \left(c_{23}\, c_4\, s_5 + s_{23}\, c_5\right) + s_{23}\, d_4 + a_3\, c_{23} + a_2\, c_2\right) + c_1 \left(d_6\, s_4\, s_5 + d_2\right)$$

$$p_z = d_6 \left(c_{23}\, c_5 - s_{23}\, c_4\, s_5\right) + c_{23}\, d_4 - a_3\, s_{23} - a_2\, s_2$$

The following abbreviations have been used here:

$s_2 = \sin\theta_2$; $c_2 = \cos\theta_2$; $s_{23} = \sin(\theta_2 + \theta_3)$; $c_{23} = \cos(\theta_2 + \theta_3)$

What is meant by jacobian Matrix?

- The jacobian matrix is defined as that linear relationship between incremental joint movements and unincremental tip movements.
- Let the linear velocity and the angular velocity of the end effector be represented in the vectorial form
- Let the joint angular velocities of a revolute robot be represented by

$$\frac{d\theta}{dt} = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_6 \end{bmatrix}$$

- The vector V and $\frac{d\theta}{dt}$ can be connected by a matrix known as the Jacobian (*i.e.,*)

$$V = J\frac{d\theta}{dt},$$

Where $J = J(\theta)$ is the jacobian

Further $\boxed{V = J\frac{d\theta}{dt}}$

6.17.5. DIFFERENT MODEL OF THE JACOBIAN OF A ROBOT

Example 1

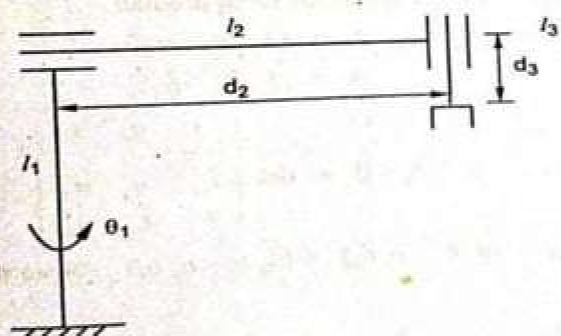
Calculate the jacobian of a IR2P robot

- Consider IR2P robot
- Let us link length l_1, l_2 and l_3.
- Depth d_2, d_3 .
- Makes an angle θ_1 .
- Generation of jacobian matrix.

$$f\begin{pmatrix}\theta_1\\ d_2\\ d_3\end{pmatrix} = \begin{pmatrix}d_2\cos\theta_1\\ d_2\sin\theta_1\\ l_1 - d_3\end{pmatrix} = \begin{pmatrix}x\\ y\\ z\end{pmatrix}$$

Thus

$$\begin{pmatrix}\dot{X}\\ \dot{Y}\\ \dot{Z}\end{pmatrix} = \begin{pmatrix}-d_2\sin\theta_1 & \cos\theta_1 & 0\\ d_1\cos\theta_1 & \sin\theta_1 & 0\\ 0 & 0 & -1\end{pmatrix}\begin{pmatrix}\dot{\theta}_1\\ \dot{d}_2\\ \dot{d}_3\end{pmatrix}$$



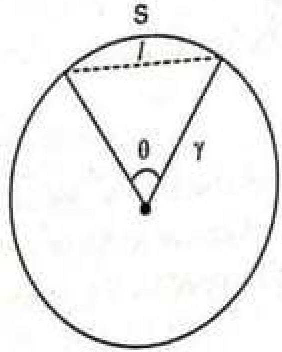


Fig. 6.52. Length of circular segment

Before we developed it, we need to recall two simple facts.

1. Given a circle of radius r, the length of a circular segment 'S' is given by

$$S = r\,\theta$$

Where θ = measured in radius, when θ is small in this segments tends to the arc 'l' (tends)

2. Given any two independent vectors $\underline{A}$ and $\underline{B}$, we can construct a third vector that is orthogonal to both $\underline{A}$ and $\underline{B}$ using the vector cross product.

$$\underline{C} = \underline{A} \times \underline{B}$$

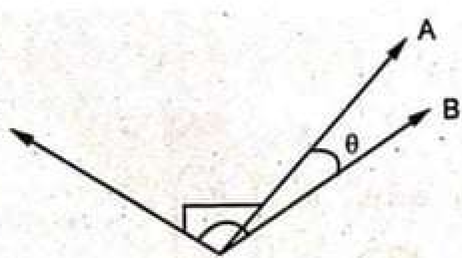


Fig. 6.53. Length of circular segment

$$|\underline{C}| = |\underline{A}|\,|\underline{B}|\,\sin\theta$$

We can use the determinant to calculate the cross product

If

$$\underline{A} = (a_x, a_y, a_z)$$

$$\underline{B} = (b_x, b_y, b_z)$$

$$\underline{C} = \underline{A} \times \underline{B} = \det \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

$\underline{C} = (a_z b_z - a_z b_y)_i - (a_x b_x - a_z b_x)_j + (a_x b_y - a_y b_x)_k$ (Note that order counts)

Example 2

6.18. TRAJECTORY GENERATION

6.18.1. DEFINE TRAJECTORY

Trajectory means path (or) track. It generates sequence of time based control set points for the control of manipulator from the initial configuration to its destination. Interpolate (or) approximate the desired path by a class of polynomial functions. Trajectory generation means construct trajectory (path + time scaling) so that the robot reaches a sequence of points in a given time. Trajectory should be sufficiently smooth and respect limits on joint variables, velocities, accelerations (or) torques.

6.18.1.1. Aim of the Trajectory Generation

Aim of the trajectory generation is to generate inputs to the motion control system which ensures that the planned trajectory is executed.

6.18.1.2. Description of Trajectory Generation

The user (or) upper level planner describes the desired trajectory by some parameters usually.

1. Initial and final point (point to point control).
2. Finite sequence of point along the path (Motion through sequence of points).

- Trajectory planning (or) generation can be performed either in the joint space (or) operational space.

1. Joint Space

- The vector product of the translational and angular displacement of each joint of a robotic link. All the places that a robotic arm can access.
- Joint space in robotics is defined by a vector whose components are the translational and angular displacement of each joint of a robotic link.

2. Joint Space : Description

The description of the motion to be made by the robot by its joint value. The motion between the two points is unpredictable.

1. Trajectory in the Joint Space

- Calculate the inverse kinematics solution from initial point to the final point.
- Assign total time T_{path} using maximum velocities in joints.
- Discretize the individual joint trajectories in time.
- Blend a continuous function between these point.

Advantages

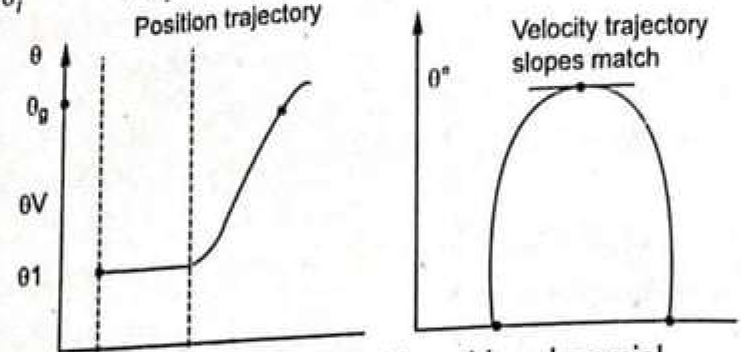
Trajectory in the Operational Space

- Calculate path from the initial point to the final point.
- Assign a total time T_{path} to traverse the path.
- Discretization the points in time and space.
- Blend a continuous time function between these points.
- Solve inverse kinematics at each step.

Advantages

- We want a smooth function $\theta(t)$ that start at θ_1 and stops at θ_f in time t_f. This implies the following constraints for each joint value θ.

$$\theta(0) = \theta_i \qquad \theta(t_f) = \theta_f \qquad \dot{\theta}(0) = 0 \qquad \dot{\theta}(t_f) = 0$$



- Four such constraints can be satisfied by cubic polynomial

Put $\quad \theta(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^2$

$\Rightarrow \quad \dot{\theta}(t) = a_1 + 2a\,2t + 3a\,3t^2$

- Solving for the unknowns a_0, a_3 we get

$a_0 = \theta_0,\ a_1 = 0,\ a_2 = \frac{3}{tf^2}(\theta_f - f_0)$

and $a_3 = \frac{-2}{t_f^3}(\theta_f - \theta_0)$

- This is just the standard hemite interpolation polynomial. Note that we specify such a cubic polynomial for each of the n joints.
- The above scheme can be extended by adding a number of via points.
- In this case, we specify not rely the start and goal points. But intermediary p_1, p_1 ..... $p_i n$ = goal.
- Between p_{i-1} and p_i we fit a cubic polynomial with velocity specifications. $V_0 = V_n = 0$, and intermediate velocity V_i chosen according to some heuristic.

- Computationally expensive due to inverse kinematics

xample 1

- Suppose we are given the initial and final position of the end effector and a time (t_f) to accomplish a trajectory from start to stop.
- We use inverse kinematics to map the initial and final workspace coordinates to the corresponding joint angle.

$$\theta_i = \begin{pmatrix} \theta_{1i} \\ \theta_{2i} \\ \vdots \\ \theta_{ni} \end{pmatrix} \qquad \theta_f = \begin{pmatrix} \theta_{1f} \\ \theta_{2f} \\ \vdots \\ \theta_{nf} \end{pmatrix}$$