Problems to practice - Module-1

1. Evaluate $\int_1^2 \int_0^1 (x^2+y^2)\,dx\,dy$.

2. Evaluate $\int_0^{2\pi} \int_0^{\pi} \int_0^{a} r^4 \sin\phi \,dr\,d\phi\,d\theta$

3. Evaluate $\int_0^1 \int_x^{\sqrt{x}} xy(x+y)\,dx\,dy$

4. Evaluate $\int_0^{\log 2} \int_0^{x} \int_0^{x+y} e^{x+y+z}\,dx\,dy\,dz$

Hint (1) $\int e^{x+y+z} = \int e^x e^y e^z$

(2) $\iiint e^{ax} e^{by} e^{cz}\,dx\,dy\,dz = \iint e^{by} e^{cz} \left[\frac{e^{ax}}{a}\right] dy\,dz$

(3) $a \log p = \log(p^a)$

5. Evaluate $\iint_R \frac{e^{-y}}{y}\,dx\,dy$, given that R is the region bounded by the lines $x=0$, $x=y$ & $y=\infty$.

Ans = 1

6. Evaluate $\iiint_V (x+y+z)\,dx\,dy\,dz$, where V is the volume of the rectangular parallelopiped bounded by $x=0$, $x=a$, $y=0$, $y=b$, $z=0$ & $z=c$.

Ans = $\frac{abc}{2}(a+b+c)$

7. Change the order of integration in $\int_0^a \int_{a-y}^{\sqrt{a^2-y^2}} y\,dx\,dy$ and then evaluate it.

Ans = $\frac{a^3}{6}$

8. Change the order of integration of

$$\int_0^4 \int_{x^2/4}^{2\sqrt{x}} dy\, dx$$ and then evaluate it.

Ans $= 16/3$

9. change the order of integration of

$$\int_0^a \int_{a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2-y^2}} xy\, dx\, dy$$ and then evaluate it.

Ans $= \frac{2}{3} a^4$.

10. Change the order of integration in

$$\int_0^1 \int_y^{2-y} xy\, dx\, dy.$$

11. change the order of integration of

$$\int_0^a \int_{x^2/a}^{2a-x} xy\, dy\, dx.$$

12. Change the order of integration of

$$\int_0^b \int_0^{\frac{a}{b}(b-y)} xy\, dx\, dy$$ and then evaluate it.

Ans $= \frac{a^2 b^2}{24}$.

13. change the order of integration of

$$\int_0^a \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} x^2\, dy\, dx$$ and evaluate it.

Ans $= \frac{\pi}{16} a^3 b$

14. Change the order of integration and evaluate

$$\int_0^1 \int_x^1 \frac{x\, dx\, dy}{x^2+y^2}.$$

Ans $= \frac{1}{2} \log 2$

15. Find $\int_0^\infty x^4 e^{-x^2} dx$

16. Evaluate $\int_0^\infty \sqrt{x}\, e^{-x^2} dx$

17. Evaluate $\int_0^\infty \frac{e^{-x^2}}{\sqrt{x}} dx$

18. Evaluate $\int_0^\infty e^{-x} x^4 dx$

19 Evaluate $\int_0^1 x^3 (1-x)^5 dx$

20. Evaluate $\int_0^{\pi/2} \sin^3\theta \cos^{3/2}\theta \, d\theta$

21. Evaluate $\iint_R x^2 y^2 \, dx dy$, where R is bounded by the region $x=0$, $y=0$ and $x+y \leq 1$.

Problems to Practice - Module-2

1. If $\phi = x^3 + y^3 + z^3 - 3xyz$, then find $\nabla\phi$, $\nabla \cdot \nabla\phi$ and $\nabla \times \nabla\phi$ at $(1,2,3)$.

2. If $\vec{F} = (6xy + z^3)\vec{i} + (3x^2 - cz)\vec{j} + (3xz^2 - y)\vec{k}$ is irrotational, then find c.

3. Find the value of λ, when $\vec{F} = \lambda y^4 z^2 \vec{i} + 4x^3 z^2 \vec{j} + 5x^2 y^2 \vec{k}$ is solenoidal.

4. If $\vec{F} = 3xyz^2 \vec{i} + 2xy^3 \vec{j} - x^2 yz \vec{k}$, then find $\nabla \cdot \vec{F}$ and $\nabla \times \vec{F}$ at the point $(1,-1,1)$.

5. show that $\vec{F} = (z^2 + 2x + 3y)\vec{i} + (3x + 2y + z)\vec{j} + (y + 2zx)\vec{k}$ is irrotational and find its Scalar potential.

6. show that $\vec{F} = (x + 2y)\vec{i} + (y + 3z)\vec{j} + (x - 2z)\vec{k}$ is solenoidal.

7. Find the directional derivative of $2xy + z^2$ at the point $(1,-1,3)$ in the direction of vector $\vec{i} + 2\vec{j} + 2\vec{k}$.

8. Find the directional derivative of xyz at the point $(1,1,1)$ in the direction $\vec{i} + \vec{j} + \vec{k}$.

9. Find the divergence of vector and curl of vector for $\vec{F} = (y^2 + z^2 - x^2)\vec{i} + (x^2 + z^2 - y^2)\vec{j} + (x^2 + y^2 - z^2)\vec{k}$.

enclosed by

10. Find ∇r^3, $\nabla(\log r)$.

11. Verify Green's theorem in a plane to evaluate $\int_C x^2(1+y)\,dx + (x^3+y^3)\,dy$ where C is the square formed by $x = \pm 1$ and $y = \pm 1$.

12. Verify Gauss divergence theorem for $\vec{F} = (x^2 - yz)\vec{i} + (y^2 - zx)\vec{j} + (z^2 - xy)\vec{k}$ and the closed surface of the rectangular parallelopiped formed by $x=0$, $x=1$, $y=0$, $y=2$, $z=0$ & $z=3$.

13. Verify stoke's theorem for $\vec{F} = x^2\vec{i} + xy\vec{j}$, where S is the surface formed by xy plane bounded by $x=0$, $y=0$, $x=2$, $y=2$.

14. Use Green's theorem in a plane to evaluate $\int_C (x^2 - y^2)\,dx + 2xy\,dy$, where C is bounded by $x=0$, $x=a$, $y=0$, $y=b$.

[Hint: Find RHS alone]

15. Evaluate $\int_C \vec{F} \cdot d\vec{r}$ along the curve $x = t^2$, $y = 2t$, $z = t^3$ from $t = 0$ to $t = 1$ given that $\vec{F} = xy\vec{i} - z\vec{j} + x^2\vec{k}$.

16. Find the workdone by the force $\vec{F} = 3xy\vec{i} - y^2\vec{j}$, when it moves a particle along the curve $y = 2x^2$ in the XY plane from $(0,0)$ to $(1,2)$.

~~17. Find the~~

17. Evaluate $\iint_S \vec{F} \cdot \hat{n}\, ds$ where $\vec{F} = y\vec{i} - x\vec{j} + 4\vec{k}$ and S is the part of the sphere $x^2 + y^2 + z^2 = a^2$ that lies in first octant.

18. Evaluate $\iint_S \vec{F} \cdot \hat{n}\, ds$ where $\vec{F} = z\vec{i} + x\vec{j} - y^2 z\vec{k}$ and S is the cylinder $x^2 + y^2 = 1$ included in the first octant between the planes $z = 0$ and $z = 2$.

[Hint: $\phi = x^2 + y^2 - 1$
$ds = \dfrac{dy\,dz}{|\hat{n} \cdot \vec{i}|}$]

19. Evaluate $\iiint_V \nabla \cdot \vec{F}\, dv$, where $\vec{F} = (2x^2 - 8z)\vec{i} - 2xy\vec{j} - 4x\vec{k}$ and V is bounded by $x = 0$, $y = 0$, $z = 0$ and $2x + 2y + z - 4 = 0$.

20. Evaluate $\iiint_V \nabla \cdot \vec{F}\, dv$ where $\vec{F} = 2x^2y\vec{i} - y^2\vec{j} + 4xz^2\vec{k}$ and V is the region in the first octant bounded by the cylinder $y^2 + z^2 = 9$ and $x = 2$.

21. Find the angle between $\phi = 3x^2y - y^3z^2$ at $(1,-2,-1)$ and $(1,-1,2)$.

22. Find the angle between $\phi = x^2 - y - z - 1$ and $\phi = x^2 + y^2 + z^2$ at the point $(1,2,1)$.

23. Find the unit normal vector of $\phi = x^2y - 2y^2z^2$ at $(1,-1,-1)$

24. Find the unit normal vector of $\phi = \frac{1}{2}\log(x^2+y^2)$ at $(1,1,1)$