# Methodologies for Origin-Destination Travel Demand Estimation within a Strategic Traffic Assignment Model

by

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**Day-to-day demand volatility is an inherent property of transport systems, which can substantially impact network performance and must therefore be incorporated into the transport planning process. The thesis proposes methodologies to explicitly account for such demand volatility in two aspects of the transportation planning process: the traffic assignment and the estimation of Origin-Destination demand.**

**The traffic assignment modelling contributions are based on a novel traffic assignment model - strategic user equilibrium (STRUE). Under STRUE travellers choose routes to minimize their expected travel cost, where their decision is based on knowledge of a demand distribution, rather than a deterministic demand. STRUE is defined such that at equilibrium, all used paths have equal and minimal expected travel costs. Two extensions of STRUE which relax previous simplifying assumptions and enhance the model's applicability are proposed. For each extension a convex mathematical program is formulated, and the model's performance is evaluated. The numerical analysis demonstrates that the STRUE can account for demand volatility while maintaining computation efficiency.**

**In addition to traffic assignment modelling, Origin-Destination demand estimation is addressed. A novel bi-level programming framework is proposed to estimate the total demand, which calls on the aforementioned STRUE model, thus incorporating day-to-day demand volatility into the estimation process. A mathematical proof demonstrates the convexity of the proposed framework. The numerical analysis illustrates the efficiency and sensitivity of the proposed framework.**

**An additional challenge for estimating O-D demand matrices results from the large number of O-D pair demands to be estimated, which is often much greater than the number of monitored links. The assumption of sparse O-D matrix can be used to address the under-determination problem. Thus, sparsity regularization is combined with link flow correlation to provide additional inputs for the O-D estimation process to improve the solution quality. The model is formulated as a convex generalized least squares problem with regularization, the usefulness of sparsity assumption and link flow correlation is presented in the numerical analysis.**

**To summarize, the thesis generalizes the novel STRUE model to improve its applicability, and proposes two robust demand estimation models while explicitly accounting for both demand and link flow variation.**



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“There are only two ways to live your life. One is as though nothing is a miracle. The other is as though everything is a miracle.”

--Albert Einstein

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Ph.D. is just the beginning of my career. Many opportunities still lie ahead, and I am ready to grasp them.

Tao Wen, July 2016

# Methodologies for Origin-Destination Travel Demand Estimation within a Strategic Traffic Assignment Model

Tao Wen, Ph.D.

The University of New South Wales, July 2016

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# ABSTRACT

Day-to-day demand volatility is an inherent property of transport systems, which can substantially impact network performance and must therefore be incorporated into the transport planning process. The thesis proposes methodologies to explicitly account for such demand volatility in two aspects of the transportation planning process: the traffic assignment and the estimation of Origin-Destination demand.

The traffic assignment modelling contributions are based on a novel traffic assignment model - strategic user equilibrium (STRUE). Under STRUE travellers choose routes to minimize their expected travel cost, where their decision is based on knowledge of a demand distribution, rather than a deterministic demand. STRUE is defined such that at equilibrium, all used paths have equal and minimal expected travel costs. Two extensions of STRUE which relax previous simplifying assumptions and enhance the model's applicability are proposed. For each extension, a convex mathematical program is formulated, and the model's performance is evaluated. The numerical analysis demonstrates that the STRUE can account for demand volatility while maintaining computation efficiency.

In addition to traffic assignment modelling, Origin-Destination demand estimation is addressed. A novel bi-level programming framework is proposed to estimate the total demand, which calls on the aforementioned STRUE model, thus incorporating day-to-day demand volatility into the estimation process. A

mathematical proof demonstrates the convexity of the proposed framework. The numerical analysis illustrates the efficiency and sensitivity of the proposed framework.

An additional challenge for estimating O-D demand matrices results from the large number of O-D pair demands to be estimated, which is often much greater than the number of monitored links. The assumption of the sparse O-D matrix can be used to address the under-determination problem. Thus, sparsity regularization is combined with link flow correlation to provide additional inputs for the O-D estimation process to improve the solution quality. The model is formulated as a convex generalized least squares problem with regularization, the usefulness of sparsity assumption and link flow correlation is presented in the numerical analysis.

To summarize, the thesis generalizes the novel STRUE model to improve its applicability and proposes two robust demand estimation models while explicitly accounting for both demand and link flow variation.

## LIST OF RELEVANT PUBLICATIONS

(1) Wen T; Gardner L; Dixit V; Duell M; Waller ST, 2016, 'A Strategic User Equilibrium for Independently Distributed Origin-Destination Demands', in TRB 2016 Compendium of Papers DVD, presented at Transportation Research Board 95th Annual Meeting, Washington, DC, 10 - 14 January 2016

(2) Wen T; Wijayaratna K; Gardner L; Dixit VV; Waller ST, 2015, 'A Learning Model for Traffic Assignment: Incorporating Bayesian Inference within the Strategic User Equilibrium Model', in 37th Australasian Transport Research Forum (ATRF), presented at 37th Australasian Transport Research Forum (ATRF), Sydney, Australia, 30 September - 2 October 2015

(3) Wen T; Cai C; Gardner L; Dixit VV; Waller ST; Chen F, 2015, 'A Maximum Likelihood Estimation of Trip Tables for The Strategic User Equilibrium Model', in TRB 94th Annual Meeting Compendium of Papers, presented at The 94th Annual Meeting of the Transportation Research Board, Washington D.C, 11 - 15 January 2015

(4) Wen T; Cai C; Gardner L; Dixit VV; Waller ST, 2014, 'A Least Squares Method for Origin-Destination Estimation Incorporating Variability of Day-To-Day Travel Demand', presented at 19th Hong Kong Society for Transportation Studies, Hong Kong, 13 - 15 December 2014

(5) Wen T; Gardner L; Dixit V; Duell M; Waller ST, 2014, 'A Strategic User Equilibrium Model Incorporating Both Demand and Capacity Uncertainty', in TRB 2014 Compendium of Papers DVD, presented at Transportation Research Board 93rd Annual Meeting, Washington, DC, 12 - 16 January 2014

(6) Duell M; Wen T; Waller ST, 2013, 'A Linear Programming Network Design Model Incorporating System Optimal Strategic Dynamic Traffic Assignment Behaviour', in Proceedings of the 18th International Conference of Hong Kong Society for Transportation Studies, HKSTS 2013 - Travel Behaviour and Society, pp. 437 – 444

(7) Menon A, Cai C, et al, 2014, 'An Approach to Sparse, Fine-Grained OD Estimation', in TRB 94th Annual Meeting Compendium of Papers, presented at The 94th Annual Meeting of the Transportation Research Board, Washington D.C, 11 - 15 January 2015

(8) Menon A, Cai C, et al, 2015, 'Fine-Grained OD Estimation with Automated Zoning and Sparsity Regularisation', Transportation Research Part B: Methodological, Volume 80, October 2015, Pages 150-172, ISSN 0191-2615.

**Papers submitted and under review:**

(1) Wen T et al, 2016, 'Two Methods to Calibrate the Total Travel Demand and Variability for a Regional Transport Network', under review of the Computer-Aided Civil and Infrastructure Engineering.

(2) Wen T et al, 2016, ''A Strategic User Equilibrium for Independently Distributed Origin-Destination Demands', under review of the Computer-Aided Civil and Infrastructure Engineering.

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# Chapter 1

# Thesis overview and contribution

## 1.1 Introduction

Among all the developing and developed countries, transportation is always one of the paramount things that both government and people care about. If the world is a human body, then transportation is the cardiovascular system- it carries nutrition to the human body and maintains the body in a perfect condition. Nowadays, the recovery from the global economy crisis has imposed burdens on the majority of cities, and congestion is very likely to increase. As a response, the Australian government has announced an AU$5 billion commitment to improve Australia's road, rail and airport infrastructure. In the United States, a budget of over US$84 Billion will be invested in transportation infrastructure and planning. In developed countries, it is reported that over 70 percent of domestic passenger movements occur on roads. Driving remains by far the preferred means of transport within cities and for trips up to 400 kilometres. Thus, efficient and robust macro transportation planning tools are in a tremendous need, these models urgently need improving.

George Box said, 'All models are wrong, but some are useful'. In transportation research, this sentence could be interpreted as: various transportation planning models have their own applicability; therefore, a variety of models should be proposed and examined meticulously so that researchers and practitioners can decide on the most suitable models for different situations. In the thesis, methodologies are proposed to capture the inherent uncertainties in a network, which is a phenomenon commonly observed in daily life. For example, the number of trips between two places may vary day-to-day; however, traditional models often treat the traffic demand as a deterministic value, which neglects the volatility of traffic demand. Also, the capacity of a road system may fluctuate due to adverse weather conditions, traffic rules or traffic incidents, but the fluctuation is normally not considered in traditional models. These uncertainties may have significant impact on the prediction of transportation planning models; hence, traditional transport models may not be applicable. To improve the models' applicability, the thesis accounts for these uncertainties and their corresponding impacts, aiming at providing more insights for policy makers and investors.

In the thesis, contributions are made in two aspects of the transportation planning process: traffic assignment and Origin-Destination travel demand estimation. They are closely related to each other, due to that a traffic assignment model provides users' route choice information and allocates traffic trips to different road segments of a network, while most statistical O-D matrix estimation techniques require users' route choice information to infer the O-D matrix. Novel methodologies are proposed in both aspects. The thesis builds a bridge to connect these two aspects by explicitly accounting for demand volatility, so that consistent decisions can be made effectively and reliably.

To account for demand volatility in the traffic assignment model, a novel strategic user equilibrium model is proposed. The strategic user equilibrium (STRUE) assumes that travellers choose route to minimize their expected travel cost, where their decision is based on knowledge of a demand distribution, rather than a deterministic demand value. They then stick to the route choice strategy regardless of the day-to-day realized demand. STRUE is defined such that at equilibrium, all used paths have equal and minimal expected travel costs, and no user can reduce his expected travel cost by unilaterally switching his routes. The background of the strategic user equilibrium is demonstrated in Chapter 2. The notion of STRUE was originally proposed by (Dixit et al., 2013), which assumes that each O-D pair is perfectly-correlated with each other, also, capacity is assumed to be a deterministic value. These assumptions limit the applicability of the model, which inspired the following generalization of the STRUE in Chapter 3 and Chapter 4.

Firstly, the impact of both demand and capacity uncertainties is explored in Chapter 3. As mentioned earlier in this section, capacity may vary day-to-day due to various factors such as adverse weather conditions, transport policies or driver behaviours, and thus the fluctuation in capacity should be accounted for in users' route choice behaviour. A novel mathematical program is formulated for the extension of the STRUE, which incorporates both demand and capacity uncertainties (referred to as C-STRUE in the thesis). In addition, it should be noted that O-D pair demands are not always perfectly-correlated. The other extension of the original STRUE is proposed to address the issue when O-D demand is independent of each other (referred to as I-STRUE in the thesis) in Chapter 4. These two extensions of the STRUE has relaxed previous simplifying assumptions and enhanced the applicability of STRUE. Further, both expected link flow and users' route choice are proved to be unique, which is critical to guarantee these models' applicability in transportation planning, because uniqueness implies that the prediction will not change based on the same inputs, which is vital for the subsequent analysis in the transportation planning process. Due to that the formulation of STRUE is based on the framework of (Beckmann et al., 1956), the computation is simple and efficient. From practitioners' perspective, the calibration burden of both I-STRUE and C-STRUE mainly comes from the distributional assumptions on demand. The calibration of volume-delay function may also be necessary.

In addition to the traffic assignment modelling, Origin-Destination demand estimation methodologies have also been proposed. Previous literature has paid little attention to demand volatility and its impact on the O-D estimation results. This thesis explicitly considers link flow variation as a result of demand uncertainty, which can be collated by day-to-day loop detector data. The utilization of such link flow fluctuation improves the estimation quality. On the other hand, as users' route choice information is normally required in statistical O-D estimation models, the thesis implements STRUE and its extensions to provide the route choice information while accounting for demand volatility. The consistent incorporation of demand volatility in both route choice assignment and O-D demand estimation helps provide a robust estimation of O-D matrix.

Two methodologies are proposed in this thesis, which focus on different aspects of the O-D estimation problem- total demand calibration and O-D matrix estimation. In Chapter 6, a novel bi-level programming framework is proposed to estimate the total demand and its variability. This framework calls on the aforementioned STRUE model, and thus incorporates day-to-day demand volatility into the estimation process. A mathematical proof is provided to demonstrate the convexity of the proposed framework. Numerical analysis quantifies the efficiency and sensitivity of the proposed framework.

Unlike the total demand calibration, there is an additional challenge in the traditional O-D estimation problem: the large number of demands to be estimated is often much greater than the number of monitored links. The assumption that O-D matrix tends to be sparse can be used to address such an under-determination issue. The assumption of a sparse matrix originates from the commonly-observed phenomenon that some centroids tend to be more popular than others, and only few trips are made for intro-zonal travel. Consequently, a large portion of trips will be made for a small portion of O-D pairs, that is, there are a lot of O-D pairs with only a few or even zero trips. Mathematically, this implies that the O-D matrix is a sparse matrix. Thus, sparsity regularization is combined with link flow correlation to provide additional inputs for the O-D estimation process to mitigate the issue of under-determination. Note that the sparsity assumption would not limit the model's applicability in general cases as the sparsity level can be tuned as a parameter in the optimization problem. The usefulness of sparsity assumption and link flow correlation is presented in the numerical analysis, and the uniqueness of the model is proved.

It is worth noting that the aforementioned models all serve for the purpose of long-term transportation planning. To assess the impact of new developments, the long-term post-re-equilibration state is commonly employed. However, critical factors that are often unaccounted for, yet essential to the success of the planning process, are the time taken for users to learn about and adjust to a given change within the system and the corresponding impact. Chapter 5 is an application of the STRUE; it models the day-to-day learning process of road users, and the corresponding system performance over time with a focus on the impact of specific new developments.

Modern technology has improved data availability significantly; we are now in an era of 'big data'. From the perspective of a transport planner, it brings us a host of data which comprises people's day-to-day travel record, trip variation and many more. As a consequence, transportation planning models should account for the stochasticity in a network. The thesis aims to provide methodologies by utilizing these elaborate data effectively, so that a robust estimation can be made for policy makers and investors. Also, the computation simplicity and efficiency of the proposed models in the thesis have made it possible to apply them in transportation software. Thesis organization and contributions are presented in the next section.

## 1.2 Thesis organization and contributions

This section outlines research contributions of this thesis.

This thesis introduces methodologies for estimating the Origin-Destination demand matrix with the strategic user equilibrium. The main contributions are the generalization of the STRUE model to improve its applicability and the proposal of two reliable demand estimation models which explicitly account for both demand and link flow variations. Thus, contributions of this thesis lie in two aspects of the transportation planning process: traffic assignment and O-D demand estimation.

On the traffic assignment aspect, two extensions of the STRUE are proposed to improve the model's applicability; both extensions can provide network performance measures analytically, which reduces computation complexity significantly. In addition, a unique solution is proved for each extension, which guarantees the models' applicability in transportation planning process. The main contributions in chapter order are:

1) In Chapter 3, the capacity uncertainty is incorporated in C-STRUE, so users will make their route decisions based on not only the demand distribution but also

the capacity variability. That is, the capacity is now treated as a random variable, which generalizes the assumption made in the original STRUE.

2) In Chapter 4, STRUE is generalized to model independently distributed O-D demands in I-STRUE, which allows O-D demands to inflate or deflate independently. I-STRUE can also provide each user's unique O-D specific route choice.

3) In Chapter 5, P-STRUE is applied to account for the impact of users' learning behaviour caused by infrastructure development.

On the O-D estimation side, two methodologies are proposed which explicitly account for demand and link flow volatility. Both methods can provide unique estimation of O-D demand. The main contributions on this problem are:

4) In Chapter 6, a Bi-level programming model is proposed to calibrate the total demand and its variability from day-to-day observed link flows. The upper level is either maximum likelihood estimation or least squares method, and the lower level is the P-STRUE which accounts for demand volatility.

5) In Chapter 7, a generalized least squares method is proposed to estimate independent O-D pairs; the proposed method incorporates link flow correlation and O-D matrix sparsity, I-STRUE is implemented to provide users' route choice while accounting for demand volatility. The sparsity assumption on O-D matrix has the potential to improve estimation quality.

Due to that the thesis' contribution is on both traffic assignment and O-D demand estimation, a literature review of the traffic assignment models and the strategic user equilibrium are presented in Chapter 2, and a literature review of travel demand estimation is provided in Chapter 6. A background review, detailed analysis on research gap and research contributions are also demonstrated in each chapter. Figure 1-1 presents the structure of the thesis. After the figure a detailed summary of each chapter will be demonstrated.

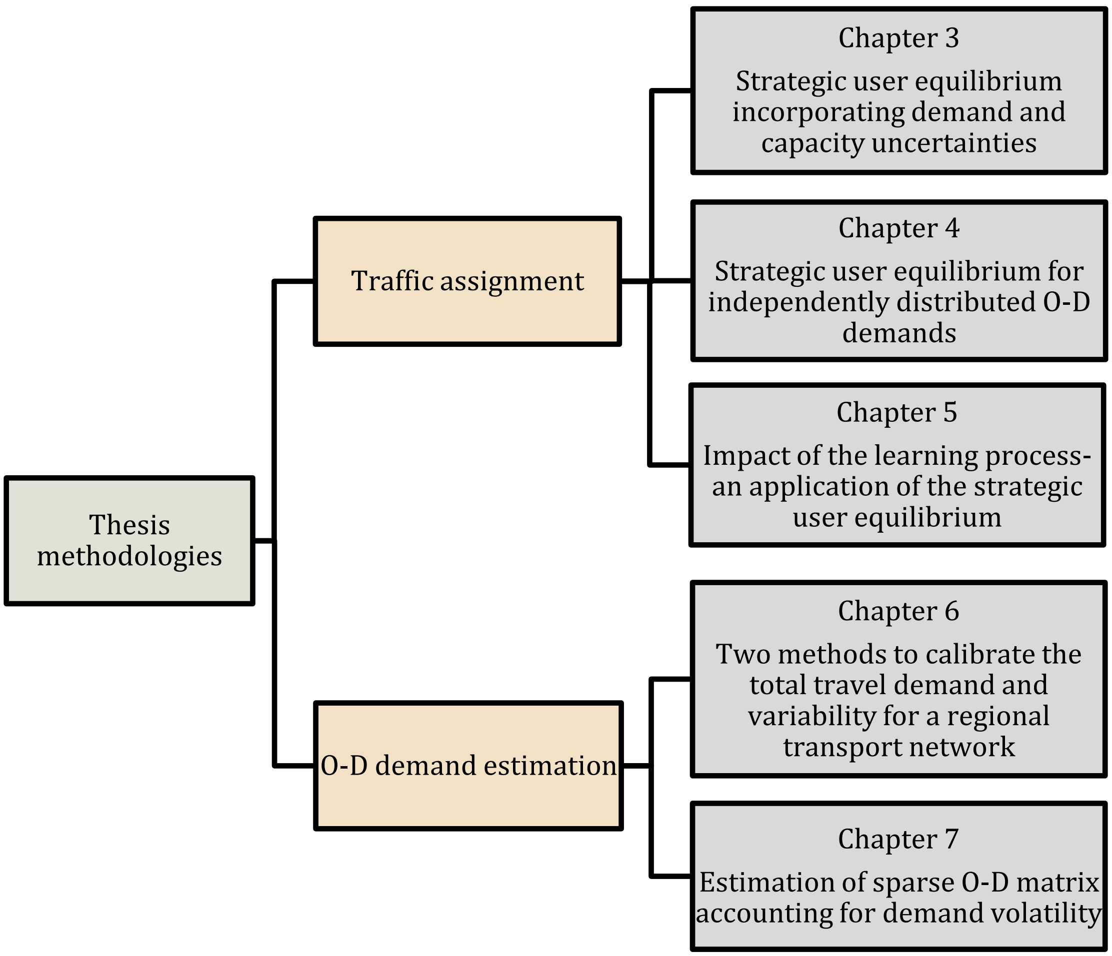



Figure 1-1 Thesis structure.

**Chapter 1: Thesis overview**

This chapter gives a general overview of the thesis, concisely introduces the strength and contribution of the research, and demonstrates the structure of the thesis.

**Chapter 2: Introduction and literature review of the strategic user equilibrium**

Firstly, background knowledge of traffic assignment models and the four step transportation planning process are demonstrated. Then this chapter introduces the network stochasticity including demand and capacity uncertainties, and the corresponding impact on system reliability and route choice behaviour. Also, the original notion of STRUE (as explained in in introduction) is illustrated. The last section of this chapter compares P-STRUE and its two extensions, and explains the contributions that this thesis has made on the strategic user equilibrium.

**Chapter 3: The strategic user equilibrium incorporating demand and capacity uncertainties**

In this chapter, an extension of P-STRUE is proposed which accounts for both demand and capacity uncertainties. Two different types of statistical distributions are used to model these uncertainties independently and a mathematical program is constructed to solve the problem analytically. The uniqueness of the model is proved, which guarantees the model's applicability in transportation planning applications. Network performance measures are provided to evaluate the impact of demand and capacity uncertainties in a virtual network; numerical analysis also demonstrates the efficiency of the model.

**Chapter 4: The strategic user equilibrium for independently distributed O-D demands**

This chapter introduces a model that relaxes the assumption of proportional O-D demand in the P-STRUE, as it accounts for users' strategic link choice under independently distributed O-D demands. The convexity of the mathematical formulation is proved when each O-D demand is assumed to follow a Poisson distribution independently; link flow distributions and users' strategic link choice are also proved to be unique. Network performance measures are given analytically. A numerical analysis is conducted on the Sioux Falls network. A Monte Carlo method is used to simulate network performance measures, which are then compared to the results computed from the analytical expression. It is illustrated that the model is capable of accounting for demand volatility while maintaining computation efficiency.

**Chapter 5: Impact of the learning process- an application of the strategic user equilibrium**

This chapter is an application of the STRUE, which focuses on the short-term impact of users' learning process when new infrastructure development occurs. Travellers assume an initial demand distribution, and incrementally update it based on their day-to-day travel experiences. Bayesian Inference is used to update the travel demand distribution, and the strategic user equilibrium model is used to compute the underlying traffic assignment pattern. Numerical analysis is conducted on a test network to demonstrate the learning process in terms of the perceived travel demand, path choice, and perceived path travel times.

**Chapter 6: Two methods to calibrate the total travel demand and variability for a regional transport network**

The state-of-the-art O-D traffic demand estimation is illustrated in conjunction with traditional O-D estimation approaches in this chapter. A bi-level programming method is proposed to calibrate the total demand and its variability, where the upper-level is either a new maximum likelihood estimation method or a least squares method, and the lower-level is the strategic user equilibrium assignment model (P-STRUE) which accounts for the day-to-day demand volatility. Model sensitivity to biased link flow data is analytically derived. The numerical analysis demonstrates the robustness of the proposed method.

**Chapter 7: Estimation of sparse O-D matrix accounting for demand volatility**

A common issue in traffic count based O-D estimation method is under-determination: the number of O-D pairs to be estimated is often much greater than the number of monitored links. It is observed that some centroids tend to be more popular than others, in addition, only a few trips are made for intro-zonal travel. Consequently, a large portion of trips will be made for a small portion of O-D pairs, that is, there are a lot of O-D pairs with only a few or even zero trips. Mathematically, this implied that the O-D matrix is a sparse matrix. Thus, sparsity regularization is combined with link flow correlation to provide additional inputs for the O-D estimation process to mitigate the issue of under-determination and improve estimation quality thereby. The model is formulated as a convex generalized least squares problem with regularization, the usefulness of sparsity assumption and link flow correlation is presented in the numerical analysis.

**Chapter 8: Conclusion and future research**

This chapter contains a concise conclusion for each chapter, points out the limitations of the methodologies proposed in the thesis, and presents an outlook of possible future research.

# Chapter 2

# Introduction and background of the strategic user equilibrium

## 2.1 Traffic assignment and user equilibrium models

The widely-used four-step transportation planning process has enabled transport engineers to scrutinize transport problems in a consistent and efficient way. Various transportation models have been applied in each step focusing on different factors of a transport system. Due to the fact that the output of each step may be the input of next step, the depicted model for each step may have a significant impact on the entire prediction results. It is therefore vital to ensure appropriate models are applicable to account for a variety of factors such as demand volatility, capacity uncertainty, travel time reliability, travel behaviour and etc.

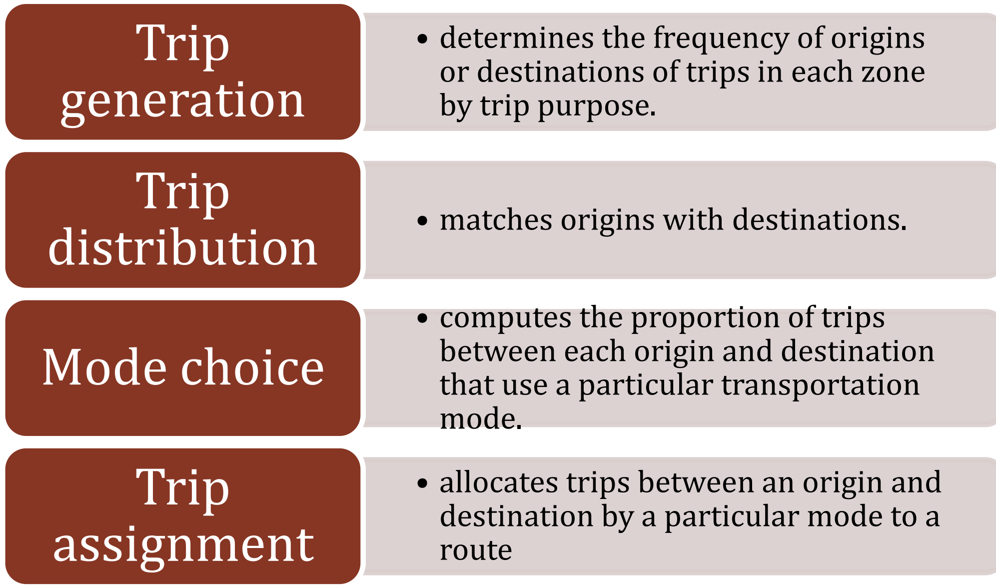



Figure 2-1 The four-step transportation planning process.

Specifically, this chapter focuses on the last step - trip assignment, which allocates trips from an origin to a destination to a designated route. In this thesis commuter trips are mainly considered due to their dominant impact on a road system. Among all the trip assignment models, the user equilibrium principle, which was firstly introduced by Wardrop (Wardrop, 1952), is a widely-accepted constraint. In essence, the Wardrop's user equilibrium states that under the equilibrium condition, all used routes have equal and minimum travel cost, and this travel cost is less than that of any unused routes. It assumes users will attempt to minimize their travel cost until no one can unilaterally change his travel cost (equilibrium condition). Wardrop's user equilibrium focuses on the macro and long-term side of a network and provides a way to predict traffic pattern in a transportation network.

The user equilibrium was formulated as a mathematical optimization problem (Beckmann et al., 1956) as shown in the equations below:

$$\min z([l]) = \sum_{n \in N} \int_0^{x_n} t_n(l_n)\, dl_n \tag{2-1}$$

Subject to:

$$\sum_k p_k^m = q_m \qquad \forall m \in M \tag{2-2}$$

$$p_k^m \geq 0 \qquad \forall m \in M \tag{2-3}$$

$$l_n = \sum_m \sum_k p_k^m \delta_{n,k}^m \qquad \forall m \in M \tag{2-4}$$

Where, $l_n$ represents the flow on link *n*, and each OD demand $q_m$ is deterministic. Flow on path $k$ for O-D pair $m$ is represented by $p_k^m$, and $\delta_{n,k}^m$ is an indicator equal to 1 if link *n* belongs to path $k$ between O-D pair $m$, and 0 otherwise. In this formulation, the objective function is the sum of the integrals of the link cost functions. Equation 2-2 represents a set of flow conservation constraints, i.e. the sum of path flows for every OD pair $m$ should be equal to the O-D pair demand, which preserves the trips out of and in each O-D centroid. Equation 2-3 indicates that the path flow must be non-negative. Equation 2-4 indicates the link flow in terms of path flows and OD demand. The gradient of the objective function is a vector of the travel time on a link. The above mathematical formulation has been proved to be equivalent to Wardrop's user equilibrium conditions and has a unique solution with respect to link flow. The mathematical program in Equation 2-1 to Equation 2-4 can be solved by many convex optimization techniques such as the Newton method, the gradient descent method and so forth, among which the Frank-Wolfe method is the commonly-used one.

However, the traditional Wardrop's user equilibrium may suffer from two limitations:

1. Broadly, each individual's route choice behaviour may not be perfectly represented by the Wardrop's user equilibrium; individuals may evaluate other factors such as travel time reliability, past travel experience and so forth when making a route choice. Hence, this traditional user equilibrium may be biased when these factors have a significant impact on the prediction.
2. The stochasticity in a transportation network is not considered. Due to day-to-day variation in O-D demand and road capacity, the network equilibrium established in a real-world road network will not exactly follow the user equilibrium. Therefore, it is vital to account for these stochastic elements in a transportation network.

Due to these limitations, some extensions of the Wardrop's user equilibrium, or several alternative user equilibrium concepts were introduced in previous literature, such as stochastic user equilibrium (SUE), strategic user equilibrium (STRUE), dynamic user equilibrium, boundedly rational user equilibrium (BRUE), perceived value equilibrium (PV-UE), late arrival penalized user equilibrium

(LAPUE), probabilistic user equilibrium (PUE) and user equilibrium with recourse (UER), these models are summarized in the table below. It is worth mentioning that these equilibrium models may provide different or even conflicted prediction, due to their assumptions being distinct from each other. Hence the implementation of these models is contingent on network properties and local policies. The applicability of these equilibrium models still remains an open question and is subject to different limitations.

Table 2-1 Summary of a few user equilibrium models.

| **Model name** | **Model description** (1-Key literature,2-Equilibrium condition. 3-Important assumptions, 4-Model limitations) | |
|---|---|---|
| **Deterministic user equilibrium (DUE)** | 1 | (Wardrop, 1952, Beckmann et al., 1956) |
| | 2 | All used paths of each O-D pair have minimal and equal travel cost for each departure time. |
| | 3 | Deterministic demand and capacity, and users have perfect information on the demand and capacity. |
| | 4 | Rarely observed in reality, but easy to implement. |
| | 1 | (Daganzo and Sheffi, 1977) |

| | | |
|---|---|---|
| **Stochastic user equilibrium (SUE)** | 2 | No individual can improve his perceived travel time by unilaterally changing routes. |
| | 3 | Individuals do not have perfect information due to random perception errors; route choice follows the logit assignment. |
| | 4 | The model requires path enumeration, which increases computation burden; parameter is difficult to be justified. |
| **Strategic user equilibrium (STRUE)** | 1 | (Waller et al., 2013, Dixit et al., 2013) |
| | 2 | Users make their route choice to minimize their expected travel time and stick to this decision regardless of day-to-day realized demand. |
| | 3 | Demand uncertainty, and users know the demand distribution |
| | 4 | O-D demands are assumed to be perfectly correlated, which limits its applicability. |
| **Dynamic user equilibrium** | 1 | (Friesz et al., 1993) |
| | 2 | All used paths of each O-D pair have minimal and equal travel cost for each departure time. |
| | 3 | Demand is time dependent. |

| | | |
|---|---|---|
| | 4 | Formulation may be complicated and computation may be time-consuming. |
| **Boundedly rational user equilibrium (BRUE)** | 1 | (Zhang, 2011, Mahmassani and Chang, 1987) |
| | 2 | State of a transportation system in which all users are satisfied with their current choices and thus do not intend to switch. |
| | 3 | Fixed capacity, and day-to-day demand volatility. |
| | 4 | An individual can take any route whose travel costs are within this indifference band, and therefore the equilibrium solution is not unique (Zhang, 2011). |
| **Perceived value equilibrium (PV-UE)** | 1 | (Connors and Sumalee, 2009) |
| | 2 | At equilibrium, all used routes have equal (maximum) perceived value which is a summary of the overall attractiveness of the alternative under risk. |
| | 3 | Demand is fixed, and travel time is a variable. |
| | 4 | The weight parameter is difficult to calibrate in reality. |
| **Late arrival penalized user** | 1 | (Watling, 2006) |
| | 2 | Each user equilibrates such that their valuation of the path's disutility is equal and minimum. Late arrival penalties are |

| | | |
|---|---|---|
| **equilibrium (LAPUE)** | | incorporated in the path disutility function under fixed departure times. |
| | 3 | Travel time is assumed to follow certain statistical distribution, demand is fixed and drivers are risk-neutral. |
| | 4 | The assumption of risk-neutral drivers may limit its applicability. |
| **Probabilistic user equilibrium (PUE)** | 1 | (Lo and Tung, 2003, Lo et al., 2006) |
| | 2 | The travel time distributions of any used routes have the same mean, which is equal to the minimum mean origin–destination (OD) travel time. Moreover, the travel time distributions of any used routes have variabilities that are within specified bounds. |
| | 3 | Link capacity degradation causes travel time variability. |
| | 4 | Users' perception of travel time reliability might be different. |
| **User equilibrium with recourse (UER)** | 1 | (Unnikrishnan and Waller, 2009) |
| | 2 | A traffic network is in UER if each user chooses a minimum expected cost routing policy and no user can unilaterally change their routing policies to improve the experienced expected cost. |

| | | |
|---|---|---|
| | 3 | The arcs in the network are uncertain to users but follow discrete states defined by a known probability mass function. |
| | 4 | Employed the online shortest path as the sub-problem. |

These studies show that based on the assumptions about individual choice behaviour, different equilibria can be determined. This to a greater or lesser extent may approach the actual network state within a certain road network. It is worth noting that although unique network equilibria in terms of traffic flow exist, these do not contain unique route patterns. That is, an equilibrium state may be associated with a number of possible route flow patterns.

## 2.2 Demand uncertainty, capacity uncertainty and their impact

Realism and computational complexity are both critical factors in transportation planning models and must be equally considered to determine a model's practical applicability. A major complication in transportation modelling is the ability to properly account for the inherent uncertainties such as demand (Kim et al., 2009, Bellei et al., 2006, Axhausen et al., 2002, Richardson, 2003, Stopher et al., 2008), capacity (Brilon et al., 2005, Wu et al., 2010) and connectivity (Bell and Iida, 1997, Iida and Wakabayashi, 1989). Furthermore, as has been noted, uncertainty surrounding these variables directly impacts route choice behaviour (Uchida and

Iida, 1993, Abdel-Aty et al., 1995, Brownstone et al., 2003, Lam and Small, 2001, de Palma and Picard, 2005). To address the limitations of the Wardrop's user equilibrium, two aspects of uncertainty of a network are considered in the thesis: demand and capacity uncertainties. Researchers have focused on a variety of user equilibrium models such as the stochastic user equilibrium (SUE) to account for the uncertainties in a network (Nakayama and Takayama, 2003, Watling, 2002, Meng and Wang, 2008, Sumalee et al., 2011, Bekhor et al., 2009), which requires path enumeration for all O-D pairs. However, little attention has been paid to the user equilibrium that can account for demand volatility while maintaining the computation efficiency. It is therefore, necessary to incorporate these stochastic elements into models to ensure robust planning capabilities but to do so in a manner that maintains computational tractability.

Improper consideration of demand variability in planning models can result in gross underestimation of travel time (Waller et al., 2001). Numerous research efforts have focused on the impact of day-to-day stochasticity regarding demand through specific model variations. For example, Clark and Watling proposed an assignment model with stochastic demand to determine the impact on variance in total system travel time (Clark and Watling, 2005). The demand volatility can be modelled as various statistical distributions such as Poisson distribution (Watling, 2002, Bell, 1991, Hazelton, 2003), log-normal distribution (Kuang and Huang, 2013, Zhou and Chen, 2008), normal distribution (Shao et al., 2006b) or Binomial distribution (Nakayama and Takayama, 2003). Also, there has been a significant amount of research to account for demand uncertainty in the network design problem (NDP) perspective (Chow and Regan, 2011, Sharma et al., 2009, Ukkusuri and Patil, 2009, Gardner et al., 2008, Yin et al., 2009, Sumalee et al., 2006, Ukkusuri et al., 2007).

Capacity has been found to be a random variable with stochastic variations affected by driver behaviour and adverse weather conditions (Zhang et al., 2009, Lam et al., 2008). For this reason, it is critical to incorporate stochastic capacity in our planning models. Stochastic variations in capacity from the mean were found to be normally distributed under different traffic flow conditions (Wu et al., 2010). In fact, there is also extensive evidence that stochastic variations in capacity are found to follow a gamma distribution fairly well (Brilon et al., 2005). Lo and Tung (2003) introduced the stochastic variation in capacity in the probabilistic user equilibrium, but assumed a simplistic uniform distribution (Lo and Tung, 2003).

It is due to these uncertainties in the network that has led to concerns about ensuring reliability. In part, this has come about due to the finding that road users tend to value reliability at about the same magnitude as delays (Asensio and Matas, 2008, Bates et al., 2001, Dixit et al., 2013, Abdel-Aty et al., 1995, Brownstone et al., 2003, Uchida, 2015), hence the impact of travel time variation should not be neglected. The reliability of travel time may be affected by capacity degradation, for example, Lo extended the user equilibrium to a reliability-based user equilibrium (RUE) to account for travel time reliability by adding a safety margin in the travel time budget (Lo et al., 2006). The RUE was further expanded to incorporate multi-modal transport (Fu et al., 2014, Shao et al., 2006b), travellers' perception errors (Clark and Watling, 2005, Shao et al., 2006b) and network uncertainties (Zhou and Chen, 2008, Shao et al., 2006a). Further, a number of previous publications considered travel time reliability as a result of path flow correlation (Shao et al., 2013, Lam et al., 2008, Clark and Watling, 2005), where a priori information on path flows and path enumeration are required, which may be computationally complicated. Under the framework, the correlation between links is considered hence path travel time is non-additive (Fan et al., 2005, Dong and Mahmassani, 2009). Path travel time can be derived from a link travel time covariance matrix (Xing and Zhou, 2011, Shahabi et al., 2013, Sen et al., 2001), temporal and spatial correlations (Gao and Chabini, 2006, Miller-Hooks and Mahmassani, 2003), or simulation-based approaches (Huang and Gao, 2012, Ji et al., 2011, Zockaie et al., 2014, Zockaie et al., 2013). Some research also focuses on the penalties due to late arrival and the corresponding route choice (Chen and Zhou, 2010, Watling, 2006). The stochastic variations in transportation systems can be attributed to variations in demand, capacity or behaviour (Siu and Lo, 2008, Shao et al., 2006b, Van Lint et al., 2008). The risk of variation of travel demand and its effects on route choice are explained by Uchida and Iida (Uchida and Iida, 1993), who developed a new assignment model to consider the impact of variation in travel time. It should also be noted that the impact of not considering these uncertainties leads to significant biases and errors (Duthie et al., 2011). Hence, it is important to incorporate the critical realities of demand uncertainties in our transportation models for travel behaviour, so that consistent decisions can be made based on cost-benefit analysis associated with improving reliability by affecting variations in demand. In the thesis, we account for the link travel time reliability by formulating the strategic user equilibrium as a convex optimization problem, which can be efficiently solved by some numerical methods such as the widely-used Frank-Wolfe algorithm.

## 2.3 The concept of strategic user equilibrium

The concept of considering the strategic choice in a user equilibrium context was proposed by Nguyen and Pallottino, specifically in the context of transit assignment (Nguyen and Pallottino, 1989). This notion was later expanded in the dynamic traffic assignment formulation (Hamdouch et al., 2004) and static traffic assignment (Marcotte et al., 2004). Sumalee et al. proposed a new model for demand and capacity uncertainty where users' strategic choice are obtained via a transformation of cumulative prospect theory (Sumalee et al., 2009). Dixit et al. proposed a strategic user equilibrium under trip variability (Dixit et al., 2013), which was further expanded the linear programming formulation for Dynamic Traffic Assignment, by dividing it into strategic stage and realized demand stage (Waller et al., 2013).

Generally, the strategic user equilibrium is based on the premise revealed by previous research that users tend to be "sticky" with regards to routing decisions – once they have selected an optimal route for their regular commute they tend to follow that route day-to-day. The model attempts to capture this behaviour in the resultant assignment pattern by assuming commuters and other road users are informed of the traffic conditions and that they are rational decision makers. The model relies on users minimizing their expected travel time based on the previous trip experiences in which they have gathered knowledge on demand (daily trips) or capacity variability (availability due to incidents, weather, etc.). The user's knowledge of each can be represented by a given distribution, while a known expected value and variance; a separate distribution is generated for demand and capacity respectively. Based on these known distributions, each user selects a travel route, to minimize their expected travel time subject to Wardrop's UE conditions – for each O-D pair the expected travel time on any used path is equal and less than the expected travel time on any unused path (Wardrop, 1952). Users then follow their chosen strategy (i.e., route) independent of the day-to-day realized demand or capacity. We define the resultant strategy differently in the models proposed in the thesis; this strategy represents users' route choice and remains fixed and independent of daily traffic conditions. However, the actual link flows on any given day, which are determined by variable O-D demand and users' strategy, will vary day-to-day, dependent on the realized demand. Therefore, the travel patterns produced by the strategic user equilibrium on any given day need not conform to a state of equilibrium. In fact, the most common observed link flows for all days will likely be in a state of dis-equilibrium (Watling and Hazelton, 2003). This is consistent with the lack of observed equilibrium in real networks.

### 2.3.1 Introduction to the strategic user equilibrium for perfectly-correlated demands

In the thesis, two extensions of the strategic user equilibrium originally proposed by Dixit et al. (2013) are presented. An important assumption in the original formulation is that the demand for each OD pair $m$ is fixed promotional to the total demand, and therefore each O-D demand varies according to the aggregated total demand (the sum of all the O-D demands). This implies that each O-D demand is perfectly-correlated to each other. This model will be referred to as P-STRUE in the thesis. The mathematical formulation of P-STRUE is presented below as a background. The symbols used in this subsection are summarised in the table below.

Table 2-2 A summary of notations in this section.

| | |
|---|---|
| $N$ | Link (index) set. |
| $K_m$ | Path set for O-D pair *m.* |
| $M$ | O-D pair set. |
| $f_n$ | Proportion of total demand on link $n$; $f= (\ldots,f_n )$. |
| $t_n$ | Travel time on link $n$; $t= (\ldots,t_n,\ldots)$. |
| $p_k^m$ | The proportion of flow on path $k$, connecting OD pair $m$, must be non-negative. |

| | |
|---|---|
| $q_m$ | The proportion of total demand that are between OD pair $m$; $1 = \sum_{\forall M} q_m$. |
| $T$ | Random variable for total demand with probability distribution $g(T)$. |
| $g(T)$ | Probability distribution for total demand of the network. |
| $\delta_{n,k}^m$ | Link-Path indicator variable. $\delta_{n,k}^m = \begin{cases} 1 & if\ link\ n\ is\ on\ path\ k\ between\ OD\ pair\ m \\ 0 & otherwise \end{cases}$ |
| $t_{nf}$ | Free flow travel time on link *n.* |
| $C_n$ | The capacity on link *n.* |
| $E()$ | The expectation of a variable. |
| $Var()$ | The variance of a variable. |
| $Std()$ | The standard deviation of a variable. |
| $CV$ | The coefficient of variation, which is defined as the standard deviation of a variable divided by the mean of that variable. |

In this model, the mathematical problem is to find the proportion of link flows that satisfy the strategic user equilibrium criterion. This link-flow pattern can be obtained by solving the following mathematical program:

$$\min z([f]) = \int_{-\infty}^{\infty} \sum_{n \in N} \int_{0}^{x_n} t_n(y_n T)\, G(T) dy_n dT \qquad 2\text{-}5$$

Subject to:

$$\sum_{k \epsilon K_m} p_k^m = q_m \qquad \forall m \epsilon M \qquad 2\text{-}6$$

$$p_k^m \geq 0 \qquad \forall m \epsilon M, k \epsilon K_m \qquad 2\text{-}7$$

$$f_n = \sum_{m \epsilon M} \sum_{k \epsilon K_m} p_k^m \delta_{n,k}^m \qquad \forall\, n \epsilon N \qquad 2\text{-}8$$

In this formulation, users' strategic choice is denoted as the link proportions $f_n$, i.e. the link flow divided by the total demand $T$. Symbol $[f]$ represents a vector of all the link proportions. Symbol $y_n$ is a dummy variable for integration. Each O-D demand is the fixed demand proportion $q_m$ multiplied by the total demand $T$, that is, each O-D demand is proportional to the total demand, and the proportions are fixed constants. This implies that all O-D demands are perfectly correlated. The total demand is assumed to follow a certain distribution. The objective function is the sum of the integrals of the expected value of the link cost functions; the link proportions are proved to be unique under the assumption of fixed demand proportions. Note that the link proportion does not change in line with the realized total demand day-to-day, while link flow does vary every day corresponding to the realized total demand.

It is important to recognize that for the P-STRUE model the path (and link) proportions are invariant. However, the actual link flow volumes vary as a function of the realized demand (since the link flows would be the product of the realized demand and the link proportions); meaning equilibrium conditions are unlikely to be satisfied for many demand realizations. This outcome is consistent with real world traffic networks where equilibrium conditions are not observed on a day-to-day basis. One of the strengths of this approach is that the uncertainty in travel times and flows can be analytically tied back to the demand uncertainty.

The P-STRUE is not explained in detail here due to it not being part of the contribution of the thesis; focus will be put on its two extensions. Further explanation and details can be found in (Dixit et al., 2013).

## 2.4 Two extensions of the strategic user equilibrium

The impact of demand and capacity variability, in conjunction with the merits of the strategic approach in traffic assignment models are introduced in the above sections. However, the assumption of perfect correlated O-D demands in P-STRUE implies that all O-D demands should vary correspondingly to the variation in total aggregated demand, that is, each O-D demand is a fixed demand proportion multiplied by the variable total demand. In addition, as stated in Section 2.2, the impact of capacity uncertainty should not be neglected; however, this is not accounted for in the P-STRUE model either. Therefore, the rest of this thesis focuses on generalizing the original strategic user equilibrium model, so that the aforementioned limitations can be relaxed. Chapter 3 presents a mathematical framework to incorporate demand and capacity uncertainty into the strategic user equilibrium paradigm, in which the capacity variation also has an impact on the expected travel time (C-STRUE). In Chapter 4, each O-D demand is assumed to be independently distributed (I-STRUE), so each O-D demand can inflate or deflate asymmetrically. The key assumptions and core contributions of the proposed models in this thesis are summarized in the table below. The next chapter demonstrates the C-STRUE in detail.

Table 2-3 Summary of the strategic user equilibrium models in this chapter.

| **Model name** | **Key assumptions** | **Contributions** |
|---|---|---|
| The strategic user equilibrium model for perfectly correlated Origin-Destination demands (P-STRUE) | Perfectly correlated O-D demands, and fixed capacity.<br>Assumption level: Tight | Account for demand uncertainty and users' strategic route choice. |
| The strategic user equilibrium model incorporating both demand and capacity uncertainty (C-STRUE) | Perfectly correlated O-D demands, and capacity uncertainty. | Account for both demand and capacity uncertainty, and users' strategic route choice. |

| | | |
|---|---|---|
| | Assumption level: Medium | |
| The strategic user equilibrium for independently distributed Origin-Destination demands (I-STRUE) | O-D demand is independent of each other, and fixed capacity.<br>Assumption level: loose | Account for demand volatility when each O-D demand is independent distributed. Provide a unique O-D specific strategic link choice. |

All three STRUE models have implemented the Dijkstra's algorithm to solve the shortest path problem, in the worst case scenario the computation complexity of Dijkstra's algorithm will be $n^2$, where $n$ is the number of nodes in a network. In the F-W algorithm, the Dijkstra's algorithm will be run for several iterations until the solution to the optimization problem converges, and the number of iterations is contingent on network size, connectivity and other factors. Note that the path enumeration process is not required which greatly reduces the computation complexity. In the real world, the number of nodes in a network is normally less than 10000, which makes the STRUE models a viable option for transportation planning.

# Chapter 3

# The strategic user equilibrium incorporating demand and capacity uncertainties

## 3.1 Chapter introduction

In this chapter, we present an extension of the P-STRUE originally proposed by Dixit et al (2013). This extension addresses day-to-day volatility in travel conditions resulting from demand uncertainty, by incorporating the impact of capacity uncertainty. Specifically, the demand and capacity variability are represented independently using assumed known distributions. The proposed model is based on the premise that users gain knowledge of the demand and capacity distributions through their past travel experience. Using this knowledge, users seek to minimize their *expected* travel time and choose a strategy (i.e. travel route) accordingly. They then follow this strategy invariantly, independent of the realized traffic demand and capacity. The network may therefore result in non-equilibrium assignment patterns, which is consistent with the lack of observed equilibrium in field networks.

A mathematical framework is formulated to incorporate demand and capacity uncertainty into the strategic user equilibrium paradigm, expressions for the analytical link travel times and the corresponding variability are derived and the uniqueness of the assignment solution regarding link flows is proved. Numerical analysis is included to demonstrate the performance and reliability of the model by comparing the analytical results with simulated assignment. Section 3.4 provides a summary of this chapter.

## 3.2 Problem formulation

This section defines the mathematical concept of the C-STRUE, Table 3-1 explains the notations used in the section. Some of the symbols have already been explained in subsection 2.3.1, Table 3-1 summarizes the notations specifically used in this chapter.

Table 3-1 Summary of notations for C-STRUE.

| | |
|---|---|
| $l_n$ | Variable of flow on link *n*. |
| $G(\,)$ | The probability density function of the capacity variable. |
| $C_{inv,n}$ | The inverse of the capacity variable on link *n*. |
| $\rho()$ | Inverse gamma probability distribution for the inverse of the capacity $\frac{1}{C_n}$ |
| $u_m$ | The Lagrangian multiplier. |
| $c_k^m$ | the expected travel time on a path *k* between OD pair *m*. |
| $\mu_T$ | The parameter of the total demand distribution. |

| | |
|---|---|
| $\sigma_T$ | The parameter of the total demand distribution. |
| $\varphi_n$ | The parameter of the gamma distribution for capacity on link *n.* |
| $\omega_n$ | The parameter of the gamma distribution for capacity on link *n.* |

The specific variation of strategic user equilibrium which is employed here was first introduced by Dixit et al (*2013*) and aims to find a set of path proportions to minimize the cost on a path, these costs are the same on all used paths and smaller than costs on any other unused paths. The proportion of total demand for O-D pair *m* is fixed, while the total demand is a variable. The strategy of users is related to the path proportions and users will not change the strategy once they select it. The total demand is assumed to follow the lognormal distribution while the capacity on a link is a random variable and is assumed to follow the gamma distribution (Brilon et al., 2005). Traffic data shows that capacity uncertainty is highly dependent on the level of traffic or the volume/capacity ratio, and almost no probability distribution can perfectly model its performance. However, the gamma distribution can fit the empirical capacity distribution well due to its capability to approximate various types of distribution curve. An important aspect to recognize is that if the capacity has a gamma distribution then the inverse of the capacity has an inverse gamma distribution. The link travel time function is assumed to be evaluated by the Bureau of Public Roads function (U.S, 1964):

$$t_n(l_n) = t_{nf}\left[1 + \alpha\left(\frac{l_n}{C_n}\right)^{\beta}\right] \tag{3-1}$$

The BPR travel time function is computed by the $\beta_{th}$ power of the inverse of capacity, and therefore we use the inverse variable to represent the inverse of capacity ($C_{inv,n} = 1/C_n$).

The proportion of total trips on a link is only a function of the path proportions $p_k^m$. The equilibrium is reached when the following objective function with respect to $f$ is satisfied. Both capacity and demand distributions are included when users making their routing decision:

$$min\, z([f]) = \sum_{n \epsilon N} \int_0^{f_n} \int_0^{\infty} \int_0^{\infty} t_n(fT)\ \ g(T)\ \ \rho\left(C_{inv,n}\right) dT dC_{inv,n}\ \ df \tag{3-2}$$

Subject to:

$$\sum_{k \epsilon K_m} p_k^m = q_m \qquad \forall m \epsilon M \tag{3-3}$$

$$p_k^m \geq 0 \qquad\qquad \forall m \epsilon M, k \epsilon K_m \tag{3-4}$$

$$f_n = \sum_{m \epsilon M} \sum_{k \epsilon K_m} p_k^m \delta_{n,k}^m \qquad\qquad \forall\, n \epsilon N \tag{3-5}$$

Equation 3-2 is the objective function which is later shown to be equivalent to find an assignment solution for the C-STRUE. Equation 3-3 represents the conservation of OD flows, i.e. the sum of the proportion of total demand on all paths for O-D pair $m$ should be equal to the proportion of total demand for OD pair $m$.

Equation 3-4 ensures the non-negativity of path flows. Equation 3-5 ensures that the proportion of total trips on a link is the sum of all path proportions using the link, where $\delta_{n,k}^{m}$ is an indicator variable that indicates whether path $k$ between OD pair $m$ uses link $n$ (1 if link $n$ is included in path $k$ and 0 otherwise), which is determined by the network itself.

The optimization model (Equations 3-2 to 3-5) can be re-written as a Lagrangian problem:

$$L(p,u) = z[f(p)] + \sum_{m} u_m \left( q_m - \sum_{k} p_k^m \right) \qquad 3\text{-}6$$

The first order optimality conditions of the Lagrangian are:

$$p_k^m \frac{\partial L(p,u)}{\partial p_k^m} = 0 \quad and \quad \frac{\partial L(p,u)}{\partial p_k^m} \geq 0 \qquad \forall k, m \qquad 3\text{-}7$$

$$\frac{\partial L(p,u)}{\partial u_m} = 0 \qquad 3\text{-}8$$

For Equation 3-7,

$$\frac{\partial L(p,u)}{\partial p_k^m} = \underbrace{\frac{\partial z[f(p)]}{\partial p_k^m}}_{\text{First part}} + \underbrace{\frac{\partial}{\partial p_k^m} \sum_m u_m \left(q_m - \sum_k p_k^m\right)}_{\text{Second part}} \qquad 3\text{-}9$$

We use chain rule to decompose the first part of the equation:

$$\frac{\partial z[f(p)]}{\partial p_k^m} = \sum_{n \epsilon N} \frac{\partial z(f)}{\partial f_n} \frac{\partial f_n}{\partial p_k^m} \tag{3-10}$$

Since $z(f)$ is the integral of $f$ from 0 to $f_n$, the partial derivative of z$(f)$ with respect to $f_n$ is:

$$\frac{\partial z(x)}{\partial f_n} = \frac{\partial}{\partial f_n} \sum_{n \epsilon N} \int_0^{f_n} \int_0^{\infty} \int_0^{\infty} t_n(fT)\, g(T)\, \rho\big(C_{inv,n}\big)\, dT\, dC_{inv,n}\, df$$

$$\therefore \frac{\partial z(x)}{\partial f_n} = \int_0^{\infty} \int_0^{\infty} t_n(f_n T) g(T) \rho\big(C_{inv,n}\big) dT dC_{inv,n} \tag{3-11}$$

Equation 3-11 implies that the partial derivative of the objective function with respect to link flows is equal to the expected travel time on the corresponding link.

From Equation 3-5 we know:

$$\frac{\partial f_n}{\partial p_k^m} = \delta_{a,l}^m \tag{3-12}$$

Using Equations 3-10, 3-11 and 3-12, the first part in equation 3-9 can be rewritten as:

$$\frac{\partial z[f(p)]}{\partial p_k^m} = c_k^m \tag{3-13}$$

Where $c_k^m$ is the expected travel time on a path *k* between OD pair *m*.

The second part of equation 3-9 is:

$$\frac{\partial}{\partial p_k^m}\sum_m u_m\left(q_m - \sum_k p_k^m\right) = -u_m \qquad 3\text{-}14$$

Therefore, the optimality conditions can be written as:

$$p_k^m(c_k^m - u_m) = 0 \qquad and \qquad (c_k^m - u_m) \geq 0 \quad \forall k, m \qquad 3\text{-}15$$

$$\sum_k p_k^m = q_m \qquad \forall k, m \qquad 3\text{-}16$$

$$p_k^m \geq 0 \qquad \forall k, m \qquad 3\text{-}17$$

Equation 3-16 and 3-17 are the flow conservation and non-negativity constraints for the proportions, respectively. Equation 3-16 holds for all paths connecting each OD pair in the network. When these conditions are met the expected cost of any used path between OD pair $m$ will be equal and minimal, where a used path is defined as any path that has a non-zero proportion. If the proportion on a path between $m$ is zero, then the expected travel time on the path is greater than the expected travel time at equilibrium. This shows the equivalence between the optimization model (Equations 3-2 to 3-5) and the equilibrium conditions of the C-STRUE.

An important property for any traffic assignment problem is the uniqueness in link flows so that project evaluation can be conducted in a consistent manner. This is possible if the Hessian matrix of the Lagrangian function is positive definite. The elements of the Hessian matrix can be represented as below:

$$\frac{\partial^2 z(f)}{\partial f_a \partial f_b} = \int_0^\infty \int_0^\infty \frac{\partial t_a(f_a T)}{\partial f_b} g(T)\rho\left(C_{inv,n}\right) dT dC_{inv,n}$$

$$= \begin{cases} \int_0^\infty \int_0^\infty \frac{\partial t_a(fT)}{\partial f_b} g(T)\rho C_{inv,n} dT\, dC_{inv,n} & if a = b \\ 0 & otherwise \end{cases} \quad 3\text{-}18$$

As is observed in Equation 3-18, the diagonal elements are positive, this implies that the Hessian matrix is positive definite, and therefore the objective function is strictly convex, and hence there exists a unique solution to the optimization problem.

As mentioned earlier, the total demand distribution is considered to follow the lognormal distribution defined by two parameters $\mu_T$ and $\sigma_T$

$$g(T) = Lognormal\ distribution(\mu_T, \sigma_T) \quad 3\text{-}19$$

The capacity of a link is assumed to follow the gamma distribution(Brilon et al., 2005) with scale parameters $\varphi, \omega$ which describe the shape and location of the distribution.

$$G(C_n) = Gamma(\varphi_n, \omega_n) \quad 3\text{-}20$$

Therefore the mean and variance of the capacity are defined as:

$$\begin{cases} E(C_n) = \varphi_n * \omega_n \\ Var(C_n) = \varphi_n * \omega_n^2 \end{cases}, for\ all\ n \in N \quad 3\text{-}21$$

If the capacity has a gamma distribution, the inverse of capacity ($C_{inv,n}$) has an inverse gamma distribution, with the following parameters:

$$\rho(C_{inv,n}) = Inv\_Gamma\left(\varphi_n, \frac{1}{\omega_n}\right) \qquad 3\text{-}22$$

Therefore, the objective function in Equation 3-2 can be solved and re-written as:

$$z(f) = \sum_{n \epsilon A} t_{nf}[f_n + \frac{\alpha}{\beta + 1} L_\beta(n) M_\beta f_n^{\beta+1}] \qquad 3\text{-}23$$

Where, $L_\beta(n)$ is the $\beta th$ moment of the inverse gamma distribution for capacity on link $n$, $M_\beta$ is the $\beta th$ moment of lognormal distribution for the total demand.

$$L_\beta(n) = \frac{(\frac{1}{\omega_n})^\beta}{(\varphi_n - 1) \dots (\varphi_n - \beta)} \qquad 3\text{-}24$$

$$M_\beta = e^{\beta \mu_T + \frac{1}{2}\sigma_T \beta^2} \qquad 3\text{-}25$$

Using the aforementioned moment generating function, the expected travel time and variance of link travel times can be evaluated based on the probability distributions of demand and capacity.

The expected travel time on a link is:

$$E(t_n) = t_{nf} + \alpha t_{nf} f_n^\beta M_\beta L_\beta(n) \qquad 3\text{-}26$$

The variance of travel time on a link is:

$$var(t_n) = E(t_n^2) - [E(t_n)]^2$$

$$= \alpha^2 t_{nf}^2 f_n^{2\beta} [M_{2\beta} L_{2\beta}(n) - M_\beta^2 L_\beta(n)^2] \quad 3\text{-}27$$

The expected total system travel time is:

$$E(TSTT) = E\left\{\sum_n [f_n T * t_n(f_n T)]\right\}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sum_n f_n T[t_{nf} + \alpha t_{nf} T^\beta \left(\frac{f_n}{C_n}\right)^\beta ] g(T)\rho\left(\frac{1}{C_n}\right)\, dTd\left(\frac{1}{C_n}\right)$$

$$= \sum_n f_n t_{nf} M_1 + \alpha t_{nf} f_n^{\beta+1}\, M_{\beta+1} L_\beta(n) \quad 3\text{-}28$$

The variance of total system travel time is:

$$var(TSTT) = E\left[\left(\sum_n [f_n T * t_n(f_n T)]\right)^2\right] - \left(E\left[\sum_n f_n T * t_n(f_n T)\right]\right)^2 \quad 3\text{-}29$$

One of the strengths of this model is that we have tied back the variability in travel time to the variability in demand and capacity through a simple analytical expression.

## 3.3 Numerical results and analysis

The C-STRUE model is implemented to evaluate network performance under a range of demand and capacity uncertainty levels. The analysis is conducted in the Sioux Falls network (Bar-Gera, 2012b) with 24 nodes and 76 links (see Figure 3-1). The total demand *T* is the summation of all O-D demands, and *E(T)* is the mean total demand for all O-D pairs, and *Std(T)* is the standard deviation of total demand. The

O-D demand is specified as proportions of the total network demand; therefore, the demand for a given O-D is the O-D proportion multiplied by the total demand ( $T * q_m$). The mean capacity and standard deviation of a link $n$ is $E(C_n)$ and $Std(C_n)$, respectively. Free flow travel time on a link is given by $t_{nf}$. The BPR parameters α and β are equal to 0.15 and 4, respectively. The network properties can be found in the Sioux Falls network properties data.

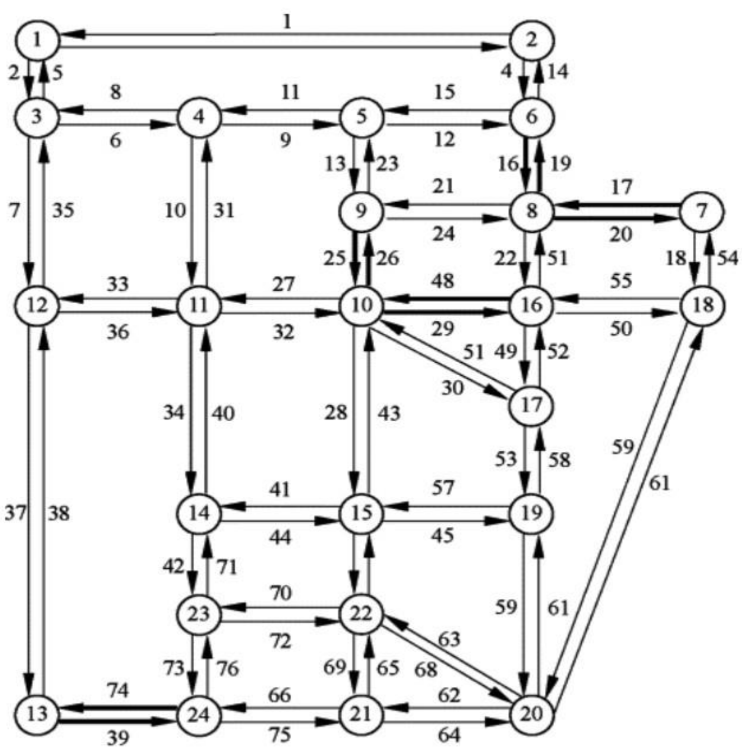



Figure 3-1 Sioux Falls network

The level of uncertainty is quantified for the total aggregate demand and capacity of each link, and is described by the coefficient of variation (CV), where $CV(Total\ demand) = \frac{E[T]}{Std[T]}$, and $CV(Capacity) = \frac{E[C_n]}{Std[C_n]}$. The same capacity uncertainty level applies to all links in the network, representative of a reduction is total capacity on a network, for instance due to weather; however, each link capacity is sampled independently. The CV for both total demand and link capacity may vary from 0% to 20% with an increment of 2.5%; and 0% represents the deterministic case where total demand or capacity is a constant instead of a variable. The two forms of uncertainty are treated independently.

The mean and standard deviation of link travel times can be computed for the model in two ways: i) estimated from the analytical model using Equations 3-26 and 3-27, and ii) empirically through simulation. The analytical solution can estimate the travel time and its variation without sampling the demand volatility, which significantly reduces computation burden. For the simulation analysis, Monte Carlo sampling is implemented to generate demands from a set of lognormal demand distributions (i.e. each distribution was defined by a mean and variance of the total trip distribution), and link capacities are sampled from a Gamma distribution. The total demand and capacity are sampled independently and simultaneously. For each realized demand and capacity combination, the C-STRUE link flows are calculated based on the set of link proportions identified by the Frank-Wolfe algorithm, from which the corresponding link travel times are computed. The mean and standard deviation of link travel times are estimated based on the entire demand and capacity sample set, equal to 1000 demand-capacity scenarios. For the remainder of the analysis the estimated link travel time mean and standard deviation are calculated using equations 3-26 and 3-27 and are referred to as Estimated $E[t_n]$ and Estimated

*Std[$t_n$]*. The simulated mean and standard deviation are referred to as Simulated *E[$t_n$]* and Simulated *Std[$t_n$]*. Similarly, for the Total System Travel Time (*TSTT*) the *E[TSTT]* and *Std[TSTT]* are estimated using equations 3-28 and 3-29 from the analytical model, hereby referred to as Estimated *E[TSTT]* and Estimated *Std[TSTT]*. The mean and standard deviation computed via simulation based on the sampled set are referred to as Simulated *E[TSTT]* and Simulated *Std[TSTT].*

Figure 3-2 provides the *E[TSTT]* for the Sioux Falls network from both the analytically estimated (shown in x axis) and simulated (shown in y axis) C-STRUE at different levels of demand and capacity uncertainty. The *CV* ranges from 2.5% to 20% for both demand and capacity variables. In total, 64 different combinations of demand-capacity distribution scenarios are evaluated using each approach. In Figure 3-2, the *R*-squared value is 0.98, which is close to 1, indicating that the simulated *E[TSTT]* closely approximates the estimated *E[TSTT].*

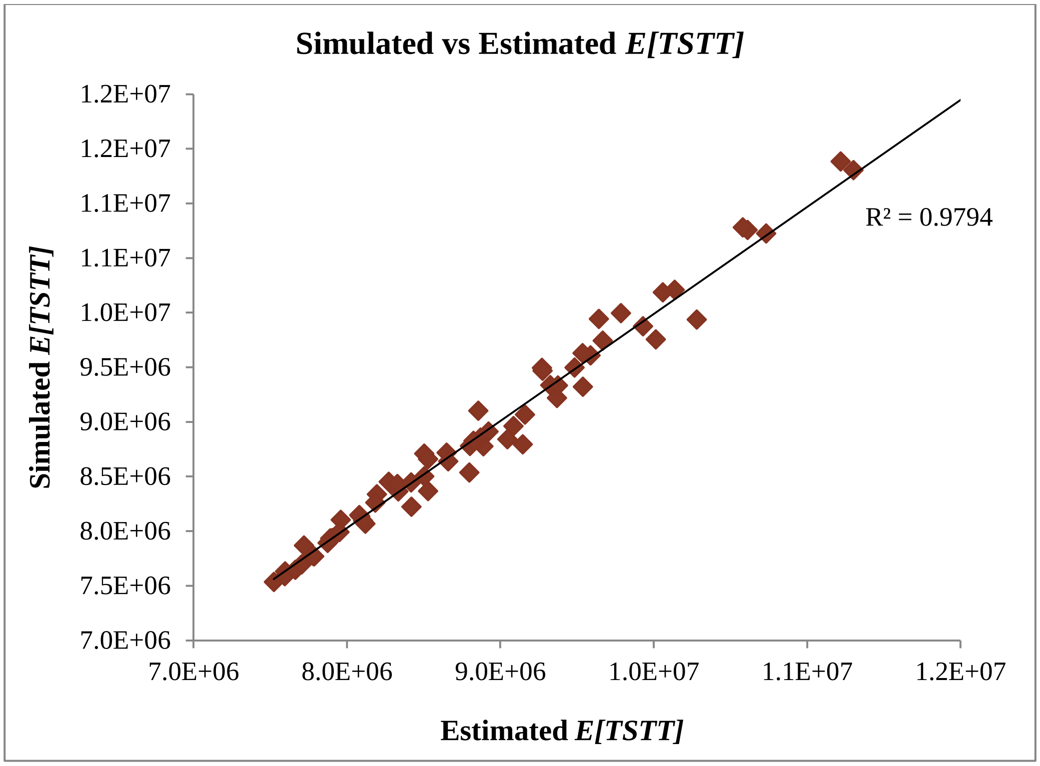



Figure 3-2 Simulated vs estimated *E[TSTT].*

In Figure 3-3 the *E[TSTT]* under different uncertainty levels are presented. The *CV* of capacity is represented on the x-axis and each series corresponds to a different *CV* of total demand, as indicated by the legend. The *CV* ranges from 2.5% to 20% with an increment of 2.5%.

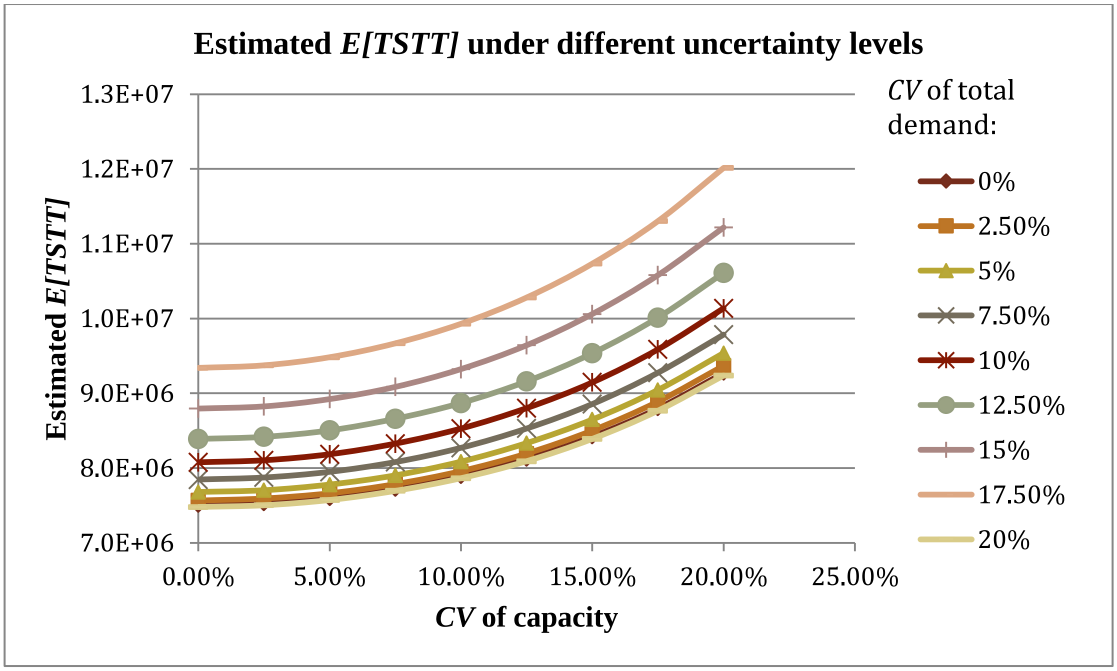



Figure 3-3 Estimated *E[TSTT]* under varying uncertainty.

The *E[TSTT]* rises more than 20% when the *CV* of capacity increases from 0% to 20%. Similarly, an increase in the CV of total demand also results in an increased *E[TSTT]*. The results illustrate that both demand and capacity uncertainty corresponds to a decrease in system performance, which increases with the level of uncertainty. The effect is further exaggerated when both types of uncertainty are accounted for. This is shown by that when both capacity and total demand are deterministic, the *TSTT* is significantly smaller than the case when the variations of both total demand and capacity are high. Therefore, incorporating both demand and capacity uncertainty into the C-STRUE model can help avoid underestimation of future system performance.

Figure 3-4 provides the *Std[TSTT]* for the Sioux Falls network from both the analytically estimated and simulated C-STRUE evaluation approaches for different combinations of demand and capacity uncertainty. Similarly to Figure 2, the *CV* ranges from 2.5% to 20% for both demand and capacity variables, and 64 different combinations of demand-capacity distribution scenarios are evaluated using each approach.  In Figure 3-4, the R squared value is 0.9867, which shows that the simulated *Std[TSTT]* closely approximates the estimated *Std[TSTT].*

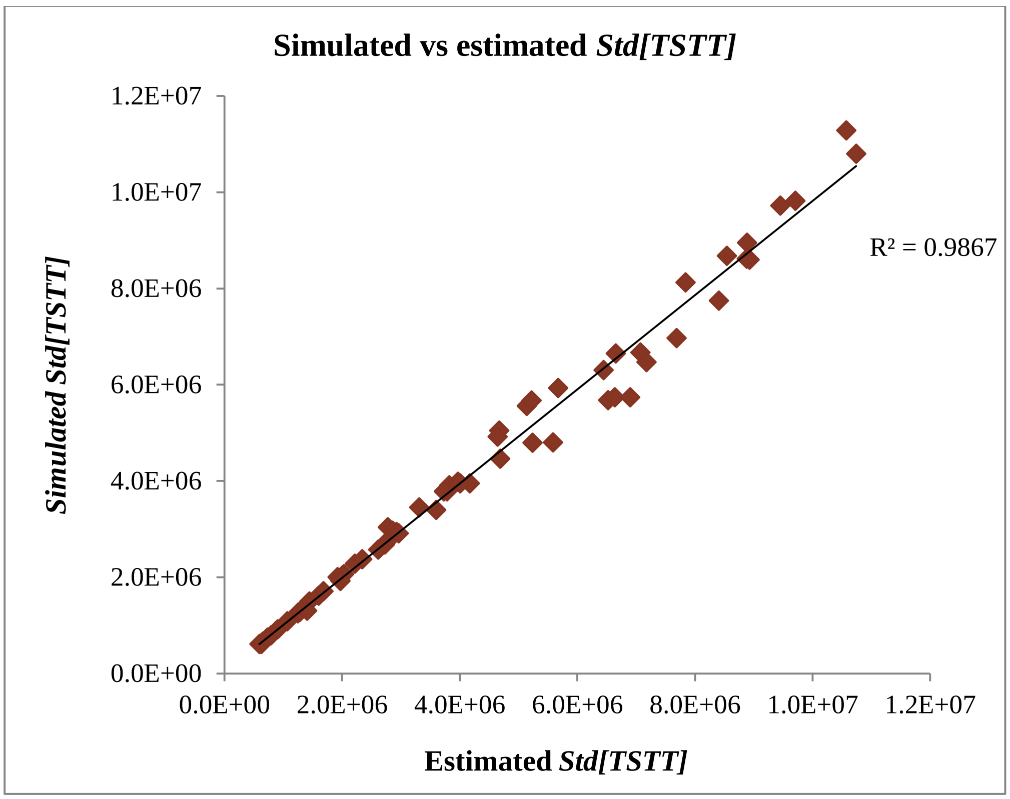



Figure 3-4 Simulated *Std[TSTT]* compared to Estimated *Std[TSTT].*

In Figure 3-5 the impact of demand and capacity uncertainty on system variability is illustrated. The *CV* of capacity is represented on the x-axis and each series corresponds to a different *CV* of total demand, as indicated by the legend. The *CV* ranges from 0% to 20%.

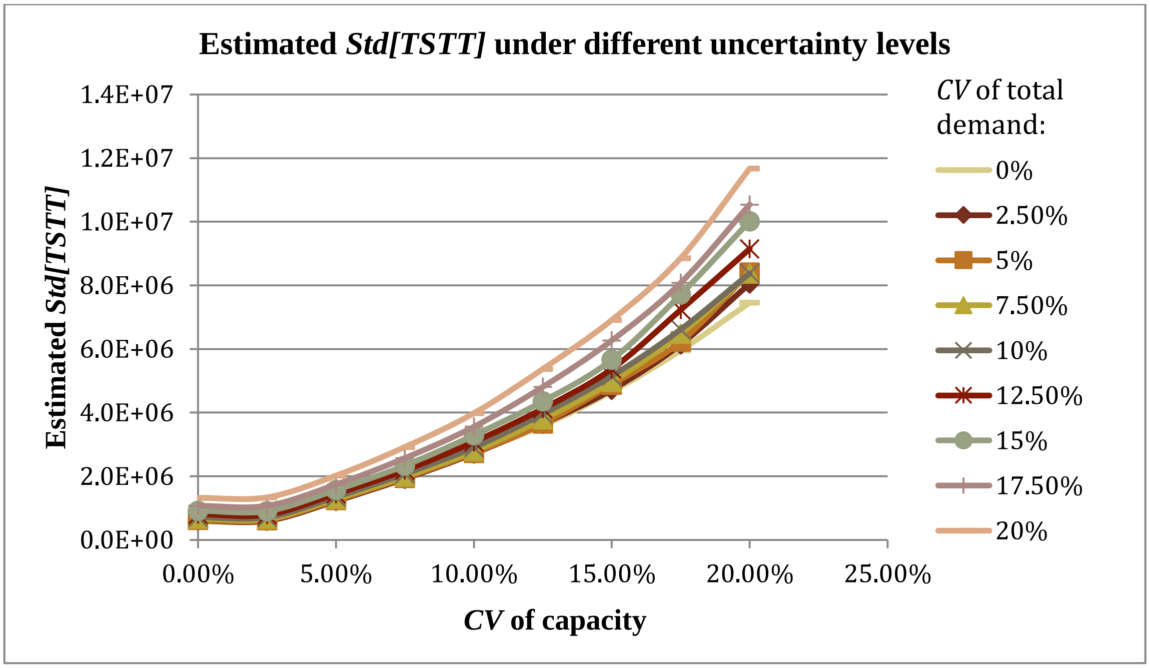



Figure 3-5 Estimated *Std[TSTT].*

Figure 3-5 illustrates the increased variability in system performance that results from an increase in demand and capacity uncertainty. For each *CV* of total demand, as represented by each line in the figure, the *Std[TSTT]* increases at least 700%. However, when *CV* of capacity equals 20%, the estimated *Std[TSTT]* only increases 50%, from 8 million to 12 million. So in contrast to the expected system performance, the capacity uncertainty has a more exaggerated impact on *Std[TSTT]* than demand uncertainty. This can be explained by the model assumptions whereby the O-D proportions remain fixed, and the aggregate demand is treated as the random variable, whereas the link capacities are sampled independently of one another. However, the *Std[TSTT]* increases with both types of uncertainty; therefore, neglecting either demand or capacity uncertainty will lead to a significant deviation in the estimation of system performance.

Figure 3-6 (a) further supports the C-STRUE model predictions by comparing the estimated and simulated $E[t_n]$ and $Std[t_n]$ for 2 links, link 10 and link 15 of the Sioux Falls network. The results are representative of the other links in the network. The same 64 combinations of demand and capacity uncertainty levels, as in Figure 3-2 and Figure 3-4, are evaluated using both the estimated and simulated evaluation methods. In Figure 3-6 (a) the x-axis represents the simulated E[TT] and the y-axis represents the estimated $E[t_n]$. In Figure 3-6 (b) the x-axis represents the simulated $Std[t_n]$ and the y-axis represents the estimated $Std[t_n]$.

The R-squared values are close to 1, which suggests the estimated link travel time and the corresponding variability closely approximates the simulated results. This supports the reliability of the analytical expressions for $E[t_n]$ and $Std[t_n]$ from the C-STRUE model. These results also illustrate the ability of the C-STRUE model to capture both link level and system level performance. It is notable that a higher variation in demand/capacity results in a greater variance between estimated and simulated results because of the random generation process of the Monte-Carlo simulation.

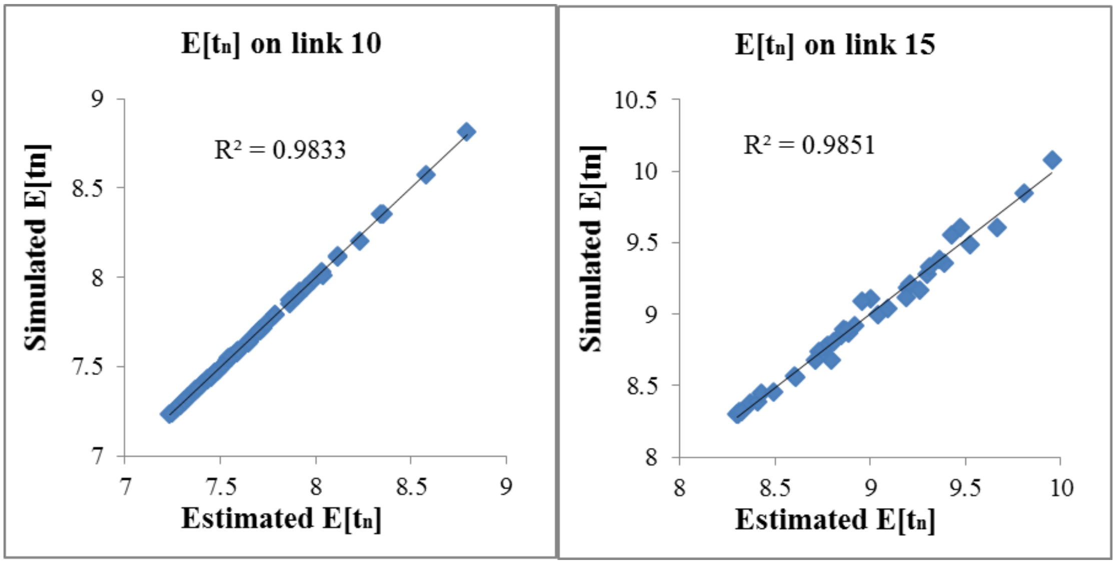



Figure 3-6 (a) Simulated vs estimated travel time on link 10. (b) Simulated vs estimated travel time on link 15.

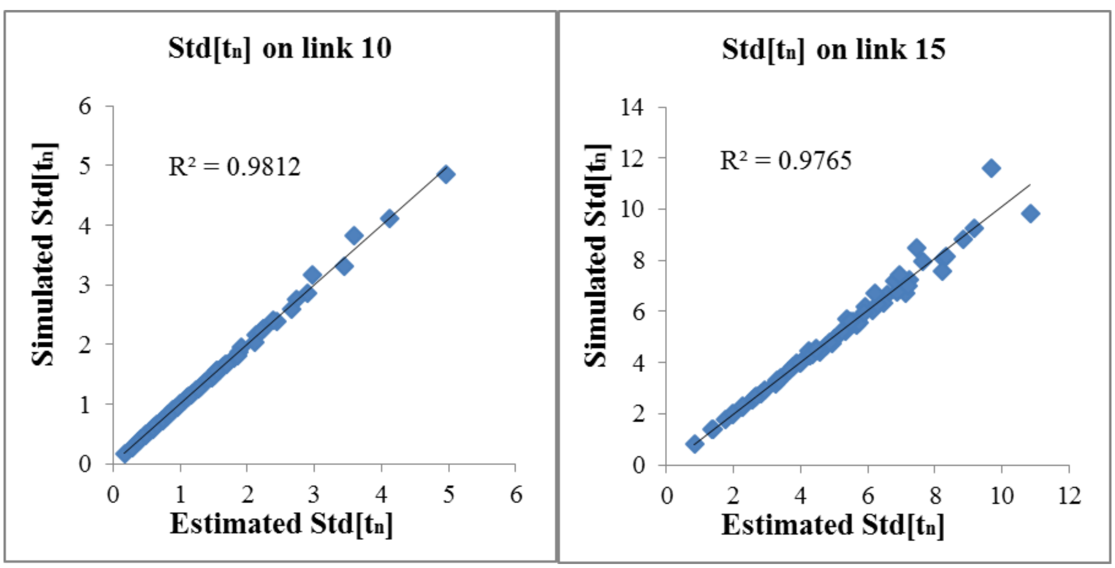



Figure 3-7 (a) Simulated vs estimated standard deviation of travel time on link 10. (b) Simulated vs estimated standard deviation of travel time on link 15.

## 3.4 Chapter summary

In this chapter, we introduced the C-STRUE model which deals with day-to-day route assignment in the context of demand and capacity uncertainty. The analytical solution to the C-STRUE is derived mathematically and the uniqueness of the optimal solution is proven. This ensures the model's applicability to transportation planning applications that require unique project rankings. Further, numerical analyses are conducted employing this model to quantify the impact of considering demand and capacity uncertainties on transport network analysis. The results demonstrate that the analytical solution is well approximated by the simulated results, and thus supports the use of the analytical method as a computationally efficient method for considering traffic flow and network reliability in a planning context. Additionally, it is presented that both demand and capacity uncertainty levels can significantly increase *E[TSTT]* and *Std[TSTT],* and thus incorporating them reduced the bias of in estimation.

However, a key consideration is that due to the mathematical restrictions of the gamma distribution, the coefficient of variance of capacity must be within 34 percent; otherwise the solution will not be a natural number. In real life, capacity variation can hardly reach this upper bound, and even if it does, some simulation techniques can be used to solve the problem. A number of future research directions would be valuable to further facilitate the usefulness of this work. For instance, alternative probability distributions should be explored with consideration for empirical fit to both the demand and capacity uncertainties. Further, dynamic variations of this problem are critical to capture both the day-to-day volatility of conditions, use behaviour and the realism of congestion dynamics.

# Chapter 4

# The strategic user equilibrium for independently distributed O-D demands

## 4.1 Motivation and introduction

In both P-STRUE and C-STRUE, it is assumed that the O-D demands are perfectly-correlated. However, such an assumption may limit the applicability of the model, because each O-D demand may vary differently. Therefore, a generalization of the strategic user equilibrium is proposed here to mitigate the issue, which assumes that each OD demand is independently distributed. The equilibrium assignment problem is to find the link flow distributions that satisfy the user-equilibrium (Wardrop, 1952) criterion. The equilibrium in the I-STRUE is a situation where users equilibrate to minimize their expected travel cost based on the demand distribution, and the expected travel cost is less than the cost on any unused paths. An important aspect of this model is that users make their strategic link choices based on the demand distribution, and this set of strategic link choice remains invariant, and thus the prediction of link flow will likely vary with the realized demand. That is, dis-equilibrium may be observed every day; however, the expected link flow will arrive at the state-of-equilibrium. Watling and Hazleton provided an in-depth discussion on the definition of equilibrium, and also identify the existence of dis-equilibrium due to the underlying variations such as demand (Watling and Hazelton, 2003). This model therefore captures the demand uncertainty and its impact on the travel time reliability, while maintaining the computation tractability and simplicity within the static user equilibrium framework. The difference between the C-STRUE and the I-STRUE is that users' strategic choice is a set of link proportions in C-STRUE, while it is users' O-D specific link choice in the I-STRUE, because each O-D pair demand is no longer a fixed demand proportion multiplied by the total demand.

The other limitation to be concerned about is that the link flows are uniquely defined under the static user equilibrium framework which was originally formulated by Beckmann's transformation (Beckmann et al., 1956); however, multiple path flow solutions are possible depending on the model assumptions and methodologies (Larsson et al., 2001). Analyses based on an arbitrary choice among the infinite number of possible route flow solutions could cause inconsistencies in transportation planning applications such as O-D matrix estimation, emission analysis and many more. Rossi firstly suggested that the entropy-maximizing pattern is the most likely route flow pattern, and thus the implication is to split flows evenly across all UE paths (Rossi et al., 1989). Later the stability and continuity of this approach are theoretically proved (Lu and Nie, 2010). However, the scalability of this approach remains an issue due to that the set of UE paths needs to be enumerated. Janson provided a link-based problem which is equivalent to the maximum entropy approach, and proposed a modified Frank-Wolfe algorithm for the implementation of the model (Janson, 1993). However, the mathematical proof of its equivalency to the original maximum entropy approach is pointed out to be inconsistent in (Akamatsu, 1997). The efficiency of the maximum entropy approach was later improved in various ways: Bar-Gera applied the condition of proportionality (Bar-Gera and Boyce, 1999, Bar-Gera, 2010) and found an approximate set of UE paths (Bar-Gera, 2006). Some dual methods were exploited to find the most likely path flows (Bell and Iida, 1997, Larsson et al., 2001); in addition, (Kumar and Peeta, 2015) proposed an entropy weighted average method to minimize the expected Euclidean distance from all other path flow solution vectors of the static user equilibrium. However, few attentions have been paid on these two aspects: firstly, most of the methods rely on Stirling's approximation to convert the entropy into a continuous function, and is subjected to the limitations of this approximation (Schrödinger, 1957); secondly, the path flow is treated as a deterministic variable, however it is possible to define the entropy of a probability distribution in information theory and to interpret this as a measure of uncertainty associated with that distribution. The maximum entropy of probability distribution method in this model addresses both issues by considering the entropy of path flow distributions, and can eventually provide users' OD specific link choices, which will be referred to as the strategic link choice in this chapter.

The next section presents the mathematical formulation and proof of the I-STRUE, Section 4.3 is a numerical analysis of the model, and Section 4.4 gives a summary of the chapter.

## 4.2 Problem formulation

This section defines the mathematical notations of the I-STRUE mode, due to the substantial difference in model formulation, all notations used in this section are specifically summarized in Table 4.1, including those notations appeared before.

Table 4-1 Summary of notations.

| | |
|---|---|
| $N$ | Link (index) set. |
| $M$ | O-D pair (index) set. |
| $K$ | Path set. |
| $l_n$ | Flow variable for link *n*. |
| $t_n(\,)$ | The function of travel time on link *n.* |
| $\overline{t_n}$ | The expected travel time on link *n.* |
| $t_{nf}$ | The free flow travel time on link *n*. |
| $C_n$ | The capacity on link *n*. |
| $p_k^m$ | Users' path choice on path *k*, connecting O-D pair *m.* |
| $d_n^m$ | Users' strategic link choice, which represents the proportion of O-D |
| $h_k^m$ | The flow on path *k*, connecting O-D pair *m.* |
| $M_n^\theta$ | The $\theta^{th}$raw moment of the link flow distribution on link *n.* |
| $E()$ | The expectation of a variable. |
| $var()$ | The variance of a variable. |
| $G(\,)$ | The probability density function of a variable. |

| | |
|---|---|
| $q_m$ | Proportion of total trips that are between O-D pair $m$; $1 = \sum_{\forall m} q_m$. |
| $T^m$ | Demand variable for O-D pair *m.* |
| $\lambda_n$ | The parameter of the Poisson distribution for flow on link *n.* $[\lambda] = [\lambda_1, \dots, \lambda_n]^T$ |
| $s_m$ | Parameter of the Poisson distribution for O-D pair demand $T^m$. |
| $g(l_n; \lambda_n)$ | Probability mass function of a Poisson distributed variable $l_n$, defined by the parameter $\lambda_n$. |
| $\delta_{n,k}^m$ | Link-Path indicator variable. $\delta_{n,k}^m = \begin{cases} 1 & if\ link\ n\ is\ on\ path\ k\ between\ OD\ pair\ m \\ 0 & \quad otherwise \end{cases}$ |
| $K_m$ | The path set for O-D pair $m$. |
| $K_m^{UE}$ | Shortest path set for O-D pair $m$. |
| $S()$ | The entropy of a variable. |
| $\alpha$ | The parameter of the BPR function. |
| $\nabla z()$ | The gradient of an objective function (). |

### 4.2.1 Equilibrium formulation

This model is an extension of the strategic user equilibrium proposed by (Dixit et al., 2013, Waller et al., 2013), which was formulated as a mathematical program in Section 2.3.

However, the assumption of fixed demand proportion limits the applicability of the model; therefore, in this chapter we propose a generalization of the strategic user equilibrium. It is assumed here that each O-D demand is independently distributed. The equilibrium assignment problem is to find the link flow distributions that satisfy the user-equilibrium (Wardrop, 1952) criterion. In I-STRUE, it is defined as:

**Definition 1**: The strategic user equilibrium is defined such that the expected travel costs are equal on all used paths, and this commonly expected travel time is less than the actual expected travel time on any unused path. In other words, given user equilibrium expected path cost, any deviation from the existing expected path flows cannot reduce the expected path cost.

Based on **Definition 1**, the equilibrium condition can be formulated by the following link-based mathematical program:

$$\min z([l]) = \sum_{n \in N} \int_0^{l_n} \int_0^{+\infty} t_n(w) G(l_n) dl_n dw \tag{4-1}$$

Subject to:

$$\sum_{k \in K_m} p_k^m = 1 \qquad \forall m \in M \tag{4-2}$$

$$p_k^m \geq 0 \qquad \forall m \in M, k \in K_m \tag{4-3}$$

$$l_n = \sum_{m \in M} \sum_{k \in K_m} p_k^m \delta_{n,k}^m T^m \qquad \forall n \in N \tag{4-4}$$

Where, $w$ is a dummy integration variable which represents link flow. In this formulation, the objective function is the sum of the integrals of the expected value of the link cost functions from zero to expected link flow. Its equivalence to the variational inequality for user equilibrium is proved later. The behavioural implication and its equivalence of this formulation to user equilibrium is shown in this section. Equation 4-2 represents a set of flow conservation constraints, i.e. the sum of path flow proportions for every O-D pair *m* should be equal to 1, which preserves the trips out of and in each O-D centroid. Equation 4-3 indicates that the path flow must be non-negative. Equation 4-4 indicates the link flow in terms of path flows and O-D demand. The topology of the network is represented by Equation 4-4.

Since $\delta_{n,k}^{m}$ is dependent only on the network topology and is a constant, $\sum_{k\in K_m} p_k^m \delta_{n,k}^m$ can be integrated into one term $d_n^m$ which is defined here as the users' strategic link choice. It represents the proportion of O-D pair demand $T_m$ traversing link *n*, and the strategic link choice indicates users' link choice disaggregated by O-D pairs, which is extremely important in many transportation applications such as O-D matrix estimation, emission analysis and network design problem.

$$d_n^m = \sum_{k\in K_m} p_k^m \delta_{n,k}^m \quad \forall m \in M, \forall n \in N \tag{4-5}$$

Under the proposed framework, two further statistical assumptions are made here:

**A1.** The actual O-D demand varies day-to-day and follows a Poisson distribution independently of each other (Clark and Watling, 2005, Bell, 1991, Hazelton, 2003, Bera and Rao, 2011, Appiah, 2009) defined by the probability mass function: $T^m \sim g(T^m; s_m)$, where $s_m > 0$. This assumption is discussed later in this Section.

**A2.** Conditional on the realized demand on any given day, each user (driver) is assumed to choose independently between the alternative paths with a fixed strategic path choice $p_k^m$.

Before proceeding to the mathematical proof, some clarifications are made below.

The uniqueness and equivalence conditions are guaranteed by the assumption of Poisson distributed demands, and the non-negativity of Poisson distribution is consistent with real world positive demands; however, the actual demand distribution may vary depending on network type, time frame and many other factors, such as which distribution best fit the actual demand is an open question and demand may not always follow Poisson distribution. In this case, other distributions may be applicable as long as they are non-negative and preserves the uniqueness and equivalence conditions. If the depicted distribution is not supported on the positive semi-infinite interval, the truncated distribution techniques may be applied, such as the truncated normal distribution, which is defined on the domain of positive real numbers.

Each O-D demand follows a Poisson distribution; however, each realised demand $T_m^*$ is a constant. i.e. on that day, there are $T_m^*$ travellers between O-D pair *m.* Each of the $T_m^*$ travellers will choose between $k$ alternative paths, each with a probability $p_k^m$. Also, note that the strategic path choice should be treated as a probability instead of a constant. One possible misunderstanding is to treat the strategic path choice as a constant, in this case, the path flow variable would become: $h_k^m = p_k^m T^m$, namely, a constant multiplied by a Poisson variable. Since $0 \leq p_k^m \leq 1$, the path flow would not follow a Poisson distribution, which is inconsistent with our assumption.

**Proposition 1**: Given assumptions 1 and 2, the unconditional path flow $h_k^m$ follows a Poisson distribution $g(h_k^m; p_k^m s_m)$ independently. The unconditional link flow *ln* also follows a Poisson distribution.

**Proof**:

Assumptions 1 and 2 together imply that for each $m \in M$, conditional on a realized demand $T_m^*$, each user has a probability of choosing path *k*. By definition, the probability of observing a set of path flow $[h_1^{m*}, h_2^{m*}, \dots, h_k^{m*} \,\middle|\, m \in M]$ has a multinomial distribution:

$$P\left(h_1^m = h_1^{m*}, h_2^m = h_2^{m*} \dots h_k^m = h_k^{m*} \,\middle|\, T_m^*\right)$$
$$= \frac{T_m!}{h_1^{m*}!\, h_2^{m*}! \dots h_k^{m*}!} (p_1^m)^{h_1^{m*}} (p_2^m)^{h_2^{m*}} (p_k^m)^{h_k^{m*}} \quad \forall m \in M, k \in K_m \tag{4-6}$$

Where, $\sum_1^k h_k^{m*} = T_m^*$. In the I-STRUE, we are more interested in the unconditional flows; here the path flow conditional on $T_m = T_m^*$ has a multinomial distribution, and $T_m$ follows a Poisson distribution. Then from the relationship of unconditional and conditional probability we obtain:

$$P\{\left(h_1^m = h_1^{m*}, h_2^m = h_2^{m*} \dots h_k^m = h_k^{m*}\right) \cap (\sum_1^k h_k^{m*} = T_m^*)$$

$$= P\left( h_1^m = h_1^{m*}, h_2^m = h_2^{m*} \dots h_k^m = h_k^{m*} \middle| T_m^* = \sum_1^k h_k^{m*} \right)$$

$$* P\left( T_m^* = \sum_1^k h_k^{m*} \right) \qquad 4\text{-}7$$

$$= \frac{\left(\sum_1^k h_k^{m*}\right)!}{h_1^{m*}!\, h_2^{m*}! \dots h_k^{m*}!} (p_1^m)^{h_1^{m*}} (p_2^m)^{h_2^{m*}} (p_k^m)^{h_k^{m*}} * \frac{e^{-s_m} s_m^{\sum_1^k h_k^{m*}}}{\left(\sum_1^k h_k^{m*}\right)!}$$

Because $\sum_{k \in K_m} p_k^m = 1$, the above equation can be rearranged as:

$$\frac{\left(\sum_1^k h_k^{m*}\right)!}{h_1^{m*}!\, h_2^{m*}! \dots h_k^{m*}!} (p_1^m)^{h_1^{m*}} (p_2^m)^{h_2^{m*}} (p_k^m)^{h_k^{m*}} * \frac{e^{-s_m} s_m^{\sum_1^k h_k^{m*}}}{\left(\sum_1^k h_k^{m*}\right)!}$$

$$= \frac{e^{-p_1^m s_m} (p_1^m s_m)^{h_1^{m*}}}{h_1^{m*}!} * \frac{e^{-p_2^m s_m} (p_2^m s_m)^{h_2^{m*}}}{h_2^{m*}!} \dots \frac{e^{-p_k^m s_m} (p_k^m s_m)^{h_k^{m*}}}{h_k^{m*}!} \qquad 4\text{-}8$$

$$= \prod_1^k g(h_k^m; p_k^m s_m) \quad \forall m \in M, k \in K_m$$

**Q.E.D.**

Note that the proof of Proposition 1 is also known as ‘thinning of a Poisson process’, a general and more detailed proof can be found in Corollary 9.17 in

(Boucherie, 2001) or in (Clark and Watling, 2005, Castillo et al., 2014a). The right hand side of Equation 4-8 is a product of a series of Poisson variables $h_k^m$, whose parameters are $p_k^m s_m$. So we have proved that each unconditional path flow follows an independent Poisson distribution, and note that the conclusion of independent path flow does not violate the flow conservation constraints since the path flow conditional on a realized demand still must sum up to each realized demand. From Equation 4-4 we have:

$$l_n = \sum_{m \in M} \sum_{k \in K_m} p_k^m \delta_{n,k}^m T^m = \sum_{m \in M} \sum_{k \in K_m} h_k^m \delta_{n,k}^m \ , \forall n \in N \qquad 4\text{-}9$$

The link-path indicator variable can only be either zero or one; therefore, link flow is the sum of several independent Poisson distributions, which also follows a Poisson distribution (Lehmann and Romano, 2006). The parameter of link flow distribution is then defined by the equations below:

$$l_n \sim g(\lambda_n) \ \ \forall n \in N \qquad 4\text{-}10$$

$$\lambda_n = \sum_{m \in M} \sum_{k \in K_m} p_k^m \delta_{n,k}^m s_m = \sum_{m \in M} \sum_{k \in K_m} E(h_k^m) \delta_{n,k}^m \ \ \forall n \in N \qquad 4\text{-}11$$

**Corollary 1**: The flow conservation constraint can be expressed in terms of expected path flow and expected demand.

**Proof:**

Each O-D demand follows a Poisson distribution with parameter $s_m > 0$, which is the expected O-D demand. Multiply both sides of the flow conservation constraints by the expected demand for O-D pair $m$, and since the summation is taken over $k$ instead of $m$, equation 4-2 can be rewritten as:

$$\sum_{k \in K_m} p_k^m = 1 \rightarrow \sum_{k \in K_m} p_k^m * s_m = 1 * s_m = \sum_{k \in K_m} E(h_k^m) \quad , \forall m \in M \qquad 4\text{-}12$$

**Proposition 2**: The parameters of the link flow distributions are unique under A1 and A2. That is, there exists a unique solution to the mathematical program defined in Equations 4-1 to 4-4.

Use BPR function as the travel cost function:

$$t_n(l_n) = t_{nf}\left(1 + \alpha\left(\frac{l_n}{C_n}\right)^4\right) \qquad 4\text{-}13$$

Note that the travel cost is positive, and due to the equilibrium conditions that all used paths have minimum travel costs, a path with cycles will not be considered as the user equilibrium paths. Because the link flow variable $l_n$ follows the discrete Poisson distribution with a probability mass function $g(l_n; \lambda_n)$ which is defined on positive integers, based on the aforementioned proposition and corollary, the objective function of I-STRUE can then be rewritten as:

$$\min z([\lambda]) = \sum_{n\in N}\int_0^{\lambda_n}\sum_{l_n=0}^{\infty}\left[t_{nf}\left(1+\alpha\left(\frac{l_n}{C_n}\right)^4\right)\right]g(l_n;\lambda_n)d\lambda_n$$

$$= \sum_{n\in N}\int_0^{\lambda_n}\left(t_{nf}+\alpha t_{nf}\left(\frac{1}{C_n}\right)^4 M_n^4\right)d\lambda_n \qquad 4\text{-}14$$

Subject to:

$$\sum_{k\in K_m} E(h_k^m) = s_m \qquad \forall m\in M \qquad 4\text{-}15$$

$$h_k^m \geq 0 \qquad \forall m\in M, k\in K_m \qquad 4\text{-}16$$

$$\lambda_n = \sum_{m\in M}\sum_{k\in K_m} E(h_k^m)\delta_{n,k}^m \qquad \forall n\in N \qquad 4\text{-}17$$

Where, $M_n^4$ is the 4th raw moment of the corresponding Poisson distribution for link *n*:

$$M_n^4 = \lambda_n(1+7\lambda_n+6\lambda_n^2+\lambda_n^3)\ ,\forall n\in N \qquad 4\text{-}18$$

The gradient of the objective function is:

$$\nabla z([\lambda]) = \begin{bmatrix} \dfrac{\partial z([\lambda])}{\partial \lambda_1} \\ \vdots \\ \dfrac{\partial z([\lambda])}{\partial \lambda_n} \end{bmatrix} = \begin{bmatrix} \sum_{l_1=0}^{\infty} \left[ t_{1f} \left( 1 + \alpha \left( \frac{l_1}{C_1} \right)^4 \right) \right] g(l_1; \lambda_1) d\lambda_1 \\ \vdots \\ \sum_{l_n=0}^{\infty} \left[ t_{nf} \left( 1 + \alpha \left( \frac{l_n}{C_n} \right)^4 \right) \right] g(l_n; \lambda_n) d\lambda_n \end{bmatrix} \quad \text{4-19}$$

$$= \begin{bmatrix} \left( t_{1f} + \alpha t_{1f} \left( \frac{1}{C_1} \right)^4 M_1^4 \right) \\ \vdots \\ \left( t_{nf} + \alpha t_{nf} \left( \frac{1}{C_n} \right)^4 M_n^4 \right) \end{bmatrix} = \begin{bmatrix} \overline{t_1} \\ \vdots \\ \overline{t_n} \end{bmatrix}$$

Where, $\overline{t_n}$ is the expected travel time on link *n.* Since the expected travel time depends only on the link flow distribution:

$$\frac{\partial z([\lambda])}{\partial \lambda_a \lambda_b} = \begin{cases} \dfrac{\partial \left( t_{nf} + \alpha t_{nf} \left( \frac{1}{C_n} \right)^4 M_b^4 \right)}{\partial \lambda_a} & if\ a = b\ \ \forall a, b \in N \\ 0 & if\ a \neq b \end{cases} \quad \text{4-20}$$

The Hessian matrix of the objective function can be expressed as:

$$Hessian = \begin{bmatrix} 1 + 14\lambda_1 + 18\lambda_1^2 + 4\lambda_1^3 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 1 + 14\lambda_a + 18\lambda_a^2 + 4\lambda_a^3 \end{bmatrix}, \quad a \in N \quad \text{4-21}$$

Since $\lambda_a$ is greater than 0, the Hessian matrix is positive definite, in addition, all constraints are linear. Therefore, the objective function is convex and has a unique minimum with respect to link flow distribution. The uniqueness is extremely important to ensure the stability of the project rankings in transportation planning models. Note that since the $\theta^{th}$ raw moment of a Poisson distribution can always be expressed in a polynomial form, other travel cost functions may also be used as long as they are monotonically increasing with respect to link flow. For example, changing the exponent parameter for BPR function to two or three clearly does not change the positive-definiteness of the Hessian matrix of the objective function.

**Q.E.D.**

### 4.2.2 Proof of equivalency

In this part we demonstrate that the mathematical formulation is equivalent to the notion of strategic user equilibrium as proposed by **Definition 1**.

**Theorem 1**: The mathematical program defined in Equations 4-14 to Equation 4-17 is equivalent to the strategic user equilibrium condition defined in **Definition 1**.

**Proof:**

For convenience, here we number the paths distinctly and consecutively by dropping the subscripts $k$ and $m$ and replacing them with $e$, that is, there are $e = \sum_{m \in M} \sum_{k \in K_m} k$ (i.e. the sum of all possible paths across all O-D pairs) distinct paths for the network. Let $H$ denote the path flow vector and $\overline{H}$ represent the corresponding expected flow vector. Equation 4-17 can then be written in the following matrix form:

$$H = \begin{bmatrix} h_1 \\ \vdots \\ h_e \end{bmatrix}, \bar{H} = \begin{bmatrix} E(h_1) \\ \vdots \\ E(h_e) \end{bmatrix}, A = \begin{bmatrix} a_{11} & \cdots & a_{1e} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{ne} \end{bmatrix} \rightarrow [\lambda] = A * \bar{H} \qquad 4\text{-}22$$

Where, $[\lambda]$ is a vector of $\lambda_n (n \in N)$, let $[\lambda^*]$ be the vector of a local minimum, $A$ is the link-path incidence matrix which represents whether link $n$ is included in path $e$ or not, the entry of $A$ is either 0 or 1. Equation 4-22 indicates the constraints with respect to $\lambda_n$; clearly the constraints are convex with respect to $\lambda_n$. The convexity of the constraints implies that another vector $[\tilde{\lambda}] = q[\lambda] + (1-q)[\lambda^*]$ is also a feasible set for $0 \le q \le 1$. Thus:

$$[\tilde{\lambda}] - [\lambda^*] = q[\lambda] + (1-q)[\lambda^*] - [\lambda^*] = q([\lambda] - [\lambda^*]) \qquad 4\text{-}23$$

This indicates a step move in a feasible direction. For a $q$ that is small enough, the convexity of the objective function tells us that:

$$q([\lambda] - [\lambda^*])\nabla z(\lambda_n) \ge 0 \qquad 4\text{-}24$$

Where, $\nabla z(\lambda_n)$ is the gradient of the objective function. Dividing both sides of Equation 4-24 by $q$ yields:

$$([\lambda] - [\lambda^*])\nabla z(\lambda_n) \ge 0 \qquad 4\text{-}25$$

As shown above in the calculation in equation 4-19, the gradient of the objective function is a vector of the expected travel cost on link *n,* and the expected cost of any path $e$ consists of the sum of the expected costs of the constituent links:

$$O^* = \begin{bmatrix} o_1 \\ \vdots \\ o_e \end{bmatrix} = A^T * \begin{bmatrix} \overline{t_1} \\ \vdots \\ \overline{t_n} \end{bmatrix} \quad 4\text{-}26$$

Where, $O^*$ represents the path cost vector. Therefore from equation 4-22, we have shown that:

$$(\overline{H} - \overline{H}^*)^T O^* = (\overline{H} - \overline{H}^*)^T * A^T * \begin{bmatrix} \overline{t_1} \\ \vdots \\ \overline{t_n} \end{bmatrix} = ([\lambda] - [\lambda^*])^T * \begin{bmatrix} \overline{t_1} \\ \vdots \\ \overline{t_n} \end{bmatrix} \geq 0 \quad 4\text{-}27$$

Therefore, the formulation is equivalent to, given user equilibrium expected path cost, and any deviation from the existing expected path flows cannot reduce the expected path cost. In another word, the strategic equilibrium is reached when the expected travel times are equal on all used paths, and this common expected travel time is less than the actual expected travel time on any unused path.

The uniqueness of link flow is extremely important to ensure the model's applicability in the transportation planning process. In the proposed framework, other distributions of O-D demand may also be considered, provided that the gradient of the equivalent optimization objective function represents the expected travel cost on each path, which is necessary to guarantee the variational inequality of equilibrium. In addition, the Jacobian of the link cost functions with respect to link flow must be positive definite to assure the uniqueness of expected link flow. That is, the cost function should be monotonically increasing in terms of link flow, and the dominant effect on the cost of a link should be the flow. The convexity of constraints in the mathematical program is sufficient and necessary for the existence of equilibrium.

### 4.2.3 Analytical expression

One of the strengths of this model is that we have tied back the demand uncertainty to the mathematical expression, which can substantially decrease computation steps needed. Note that the link flow parameter $\lambda_n$ and users' strategic link choice are derived from numerical method and are treated invariant. They are substituted into those mathematical expressions to calculate those performance measures.

The expected travel time on a link is given by:

$$\bar{t_n} = E[t_n(\lambda_n)]$$

$$= \sum_{l_n=0}^{\infty} \left[ t_{nf} \left( 1 + \alpha \left( \frac{l_n}{C_n} \right)^4 \right) \right] g(l_n; \lambda_n) d\lambda_n = t_{nf} + \alpha t_{nf} \left( \frac{1}{C_n} \right)^4 M_n^4 \qquad \text{4-28}$$

The variance of travel time on a link is given by:

$$var[t_n(\lambda_n)] = E[t_n^2(\lambda_n)] - E[t_n(\lambda_n)]^2 = \alpha^2 t_{nf}^2 (\frac{1}{C})^8 [M_n^8 - (M_n^4)^2] \qquad \text{4-29}$$

$$M_n^\theta = \sum_{i=1}^{\theta} \lambda_n^i \left\{ \begin{matrix} \theta \\ i \end{matrix} \right\} \quad , \forall n \in N \qquad \text{4-30}$$

Where the braces in Equation 4-30 denotes the Stirling's number of the second kind. Therefore, the raw moment of link flow is monotonically increasing with respect to $\lambda_n$, and therefore it can be characterized that the variance of travel time increases as the expected travel time increase. This is a commonly observed phenomenon on transportation networks around the world (Van Lint et al., 2008, Systematics, 2013).

Once we have $\lambda_n$, the I-STRUE model can provide estimations on the total system travel time ($TSTT$) analytically, which are shown in the equations below. Some characteristics of these expressions will be discussed in the numerical analysis.

$$E(TSTT) = \sum_{n\in N} \sum_{l_n=0}^{\infty} \left[ l_n t_{nf} \left( 1 + \alpha \left( \frac{l_n}{C_n} \right)^4 \right) \right] g(l_n; \lambda_n) d\lambda_n$$

4-31

$$= \sum_{n\in N} t_{nf} M_n^1 + \alpha t_{nf} \left( \frac{1}{C_n} \right)^4 M_n^5$$

$$var(TSTT) = E[(TSTT - E[TSTT])^2] = E(TSTT^2) - E^2(TSTT)$$

$$= \sum_{n\in N} t_{nf}^2 [M_n^2 - (M_n^1)^2] + \alpha^2 t_{nf}^2 \left( \frac{1}{C_n} \right)^8 [M_n^{10} - (M_n^5)^2]$$

4-32

$$+2\alpha t_{nf}^2 \left( \frac{1}{C_n} \right)^4 [M_n^6 - M_n^1 M_n^5]$$

### 4.2.4 The most likely strategic link choice

Under the strategic user equilibrium, the link flows are uniquely defined; however, the strategic link choice matrix (sometimes referred to as the assignment map or O-D specific link choice in literature) is not unique. That is, there might be an infinite number of sets of strategic link choices that produces the same link flows. Although if we run the F-W algorithm and store all the temporary shortest paths for each iteration, the F-W algorithm may provide a different link choice each time, while producing the same link flows. The strategic link choice is extremely useful especially in transportation planning models such as O-D estimation problem, emission analysis and so forth. In these cases only having the aggregated link flows is not sufficient, and hence a uniquely-determined strategic link choice is required to ensure the stability and applicability of this model. To address this issue, some researchers have proposed the following maximum entropy optimization problem to determine the most likely path flows (Larsson et al., 2001, Rossi et al., 1989, Janson, 1993) (which can provide strategic link choice subsequently), where entropy is defined as the number of possible route choice decisions made by individual travellers, and path flow and O-D demand are treated as deterministic variables, where normally only the expected path flow and O-D demand are considered:

$$S_{system} = \prod_{m \in M} \frac{E(T_m)!}{\prod_{k \in K_m} E(h_k^m)!} \qquad 4\text{-}33$$

In principle, entropy gives the number of possible route choice decisions made by individual travellers within a specific route flow solution. The objective is to maximize the entropy of the system (the sum of the entropy of all shortest paths), which can be formulated as the following equivalent mathematical program:

$$Max: -\sum_{m\in M}\sum_{k\in K_m^{UE}} E(h_k^m)\ln[E(h_k^m)] \quad \text{4-34}$$

Subject to:

$$\sum_{k\in K_m^{UE}} E(h_k^m) = E(T_m) \qquad \forall m \in M \quad \text{4-35}$$

$$h_k^m \geq 0 \qquad \forall m \in M, k \in K_m^{UE} \quad \text{4-36}$$

$$l_n = \sum_{m\in M}\sum_{k\in K_m^{UE}} p_k^m E(h_k^m) \quad \forall\, n \in N \quad \text{4-37}$$

Note that only the shortest paths for the strategic user equilibrium are considered, the problem is based on the conditions that the user equilibrium is solved and the equilibrium link flow is obtained. Solving the Lagrangian program above shows that the most likely path flow follows the logit assignment, where the 'link cost' is the Lagrangian multipliers corresponding to Constraint 4-37 (Akamatsu, 1996, Akamatsu, 1997). However, the above formulation may suffer from two limitations. Firstly, The transformation to the equivalent mathematical program relies on Stirling's approximation to convert the entropy into a continuous function, and is subjected to the limitations of this approximation (Schrödinger, 1957). Secondly, the path flow is treated as a deterministic variable hence it neglects the volatility in O-D demand and path flows. It is possible to define the entropy of a probability distribution in information theory and to interpret this as a measure of uncertainty associated with that distribution. In the formulation below, we will show that if path flow and demand volatility are considered, the path flow will *not* follow the logit assignment, and therefore Dial's algorithm may not be applicable under this consideration.

**Definition 2:** The entropy of any path flow state (hereby referred to as path flow distribution) is the measure of randomness or uncertainty of locating an individual random network user. All possible states of this distribution are considered. The higher the number of possible states, the higher the randomness or uncertainty of locating an individual network user in that path flow state. Based on the definition, the following assumption is made:

**A.3.** Under the strategic user equilibrium conditions, users make their strategic path choice (and the corresponding strategic link choice), to maximize the probability distribution entropy of the system.

In the aforementioned formulation, the strategic path choice is not uniquely defined, i.e. there may be several sets of path flows that can provide the same link flow distributions. However, the capability of estimating path flow may be important in various transportation models. Hence, we extend the notion of entropy maximization method used in many previous research (Akamatsu, 1997, Rossi et al., 1989, Kumar and Peeta, 2015), here the entropy of a random variable (instead of a deterministic variable) which follows a certain statistical probability distribution is defined as (Ochs, 1976):

$$S(h_k^m) = -\sum_{h_k^m=0}^{\infty} g(h_k^m; p_k^m s_m)\, ln\, g(h_k^m; p_k^m s_m) \qquad \forall m \in M, k \in K_m^{UE} \tag{4-38}$$

Where, $h_k^m$ is the path flow variable that follows a Poisson distribution with parameter $p_k^m s_m$. The base of the logarithm is not important as long as the same one is consistently used: change of base merely results in a rescaling of the entropy, and thus here the natural logarithm is used. Given that the unconditional path flow follows a Poisson distribution independently of each other, the entropy corresponding to a strategic path choice $p_k^m$ is:

$$S(h_k^m) = p_k^m s_m[1 - ln(p_k^m s_m)] + e^{-p_k^m s_m} \sum_{h_k^m=0}^{\infty} \frac{(p_k^m s_m)^{h_k^m}\, ln\, h_k^m!}{h_k^m!} \tag{4-39}$$

When $p_k^m s_m$ is large, we have the following approximation(Evans et al., 1988):

$$S(h_k^m) \approx \frac{1}{2} ln(2\pi e p_k^m s_m) \quad \forall m \in M, k \in K_m^{UE} \tag{4-40}$$

Considering all the origin-destination pairs, the objective is to find a set of strategic path choice to maximize the entropy of the system:

$$Max \rightarrow z_{entropy}([p_k^m]) = \sum_{m \in M} \sum_{k \in K_m^{UE}} \frac{1}{2} ln(2\pi e p_k^m s_m) \tag{4-41}$$

Subject to:

$$\sum_{k\in K_m^{UE}} p_k^m = 1, \qquad \forall m \in M \tag{4-42}$$

$$p_k^m \geq 0, \qquad \forall k \in K_m^{UE}, m \in M \tag{4-43}$$

$$\lambda_n = \sum_{m\in M} \sum_{k\in K_m^{UE}} p_k^m \delta_{n,k}^m\, s_m \qquad \forall n \in N \tag{4-44}$$

**Proposition 3:** The strategic path choice is unique under Assumption A.3.

Equivalently, the objective function can be transformed into the following minimization problem with the same constraints:

$$min \to -z_{entropy}([p_k^m]) = -\sum_{m\in M} \sum_{k\in K_m^{UE}} \frac{1}{2} ln(2\pi e p_k^m s_m) \tag{4-45}$$

The objective function is the sum of several convex functions, and all three constraints are also convex; therefore a unique optimal solution exists. Since the strategic link choice is the sum of several corresponding strategic path choices, it is also uniquely determined. The optimal solution can be obtained by various numerical methods such as Newton's method and the gradient descent methods.

### 4.2.5 Implementation algorithms of the model

Note that the identification of the equilibrium path set is required when solving the maximum entropy problem above. Although this equilibrium path set is unique, it is difficult to obtain in practice due to computational precision limits. Hence, an approximation method similar to (Larsson et al., 2001) is proposed here.

In this method, all paths used in the all or nothing assignment procedure during the Frank-Wolfe algorithm are stored, until the strategic user equilibrium is reached. Then the expected costs of all these paths are computed, and those paths whose expected costs are within a tolerance threshold are saved as the approximated shortest paths, which will be used to solve the maximum entropy problem here. The tolerance threshold is defined as a proportion of the shortest path cost:

$$\frac{t(h_k^m) - t\left(h_k^{m*}\right)}{t(h_k^{m*})} \leq Tol \qquad \text{4-46}$$

Where, $t(h_k^m)$ represents the expected cost of path $k$ for O-D pair $m$, $t\left(h_k^{m*}\right)$ and represents the shortest path cost for O-D pair $m$. It must be mentioned that by doing so, only a subset of the equilibrium paths may be included. This error may be mitigated by setting a high relative gap (the difference between two consecutive iterations) for the Frank-Wolfe algorithm. The choice of tolerance threshold is also critical: low tolerance value may cause the optimization problem to be infeasible, while high tolerance value may lead to inclusion of non-equilibrium paths. So the choice of the tolerance threshold should be carefully made. The solution procedure of the model is demonstrated below:

**Solution Algorithm:**

Step 1: Initialisation: load free flow $[\lambda]_0$ to the network, and then find the free flow travel cost on each link.

Step 2: All or Nothing Assignment: Based on the travel cost, find a set of expected link flows $[\lambda]_1 \to \min z([\lambda])$, subject to Equations 4-15 to 4-17, and store the shortest path for each all or nothing assignment.

Step 3: Line Search: Find the step size $\beta \to \min z(\beta[\lambda]_{n+1} + (1-\beta)[\lambda]_n)$, subject to $0 \leq \beta \leq 1$.

Step 4: Repeat Step 2 and 3, until $\frac{[\lambda]_{n+1}-[\lambda]_n}{[\lambda]_{n+1}} \leq \varepsilon$, where $\varepsilon$ is the critical value which can be artificially set. In this step, the strategic user equilibrium is reached.

Step 5: Compute the expected cost of each stored path based on the equilibrium link cost.

Step 6: If $\frac{t(h_k^m)-t\left(h_k^{m^*}\right)}{t(h_k^{m^*})} \leq Tol$, save the path as a user equilibrium path.

Step 7: Solve the maximum entropy optimization program in Equations 4-42 to 4-44, a set of strategic path choice $[\dots, p_k^m \dots]$ is obtained. Various numerical methods, such as Newton's method, may be applicable here.

Step 8: Calculate the strategic link choice from the strategic path choice.

Step 1 provides an initial feasible solution to start the algorithm, and Step 2 builds the least cost paths set and load corresponding traffic on these paths. In Step 3, a step size parameter $\beta$ is sought to minimize the objective function, and numerous methods are applicable here such as the Golden section method or Bisection method. Step 4 assesses the degree of convergence by computing the relative change in the expected link flow vector between iterations. Step 5 and Step 6 provide the equilibrium path set which will be used to determine their corresponding strategic path choice. Step 7 solves the optimization maximum entropy problem. Step 8 computes the strategic link choice from the strategic path choice.

## 4.3 Numerical demonstration

The link flow volumes (and travel times) may differ from the deterministic set of flows (and travel times) when demand variability is introduced. The strength of the proposed I-STRUE model is the incorporation of demand uncertainty into the user's decision-making process; therefore, the remainder of the analysis considers only stochastic demand conditions. Specifically, the demand follows a Poisson distribution with a prescribed parameter lambda. In the I-STRUE model, path proportions (and the corresponding link choice probabilities) are dependent on the demand distribution, and not any particular demand realization.

### 4.3.1 Monte-Carlo simulation:

To evaluate the I-STRUE model under demand variability, 100 randomly selected demand scenarios for a given demand distribution curve (with a prescribed Poisson parameter lambda) were generated through Monte Carlo sampling, then for each realized demand sample, 100 sets of multinomially distributed path flows are sampled based on the strategic path choices. As a result, we have 10000 sets of sampled path flows and 10000 sets of corresponding sampled link flows, which represents the unconditional distributions of link flow. These results, derived from the Monte-Carlo method, are indicated as simulated results, and those ones computed from the analytical equations are indicated as estimated results. The simulation aims to show that the derived analytical solution in I-STRUE can capture the demand volatility and the corresponding travel time variation, which avoids the computation burden due to demand sampling. Based on the simulation the following performance measures are computed:

(1) Expected total system travel time ETSTT and Standard deviation of total system travel time, StdTSTT.

(2) Expected link travel times, ETT and Standard deviation of link travel time, StdTT.

(3) Expected link flow and standard deviation of link flow.

The mathematical expression of expected travel time on a link is given by the BPR function introduced in Equation 4-28. The Frank-Wolfe algorithm is implemented with the modified expected link travel time function to determine the strategic choice, and the corresponding link flow distributions, and the relative gap for the F-W method is set to $1^*e^{-5}$. The following results are based on the network depicted in Figure 3-1. The network has 24 nodes and 76 links. The capacity, the free flow speed, the length of each link, as well as other network attributes can be found on (Bar-Gera, 2012b). The BPR parameters $\alpha$ and $\beta$ are taken to be 0.15 and 4, respectively.

Figure 4-1 illustrates the probability distribution of flow on several randomly chosen links, while the x axis denotes the link flow and the y axis represents the probability of observing the corresponding link flow. In total, there are 10000 sets of sampled link flows which are generated from the 100 demand scenarios, and these simulated link flows are plotted as a bar figure; the black curve represents the standard Poisson distribution with the parameter lambda derived from the strategic user equilibrium. It is demonstrated that the standard Poisson distribution curve is well approximated by the simulated results. This is further validated by a Chi-square test on the selected 4 links: the corresponding p values are all equal to 1, which indicates that the simulated link flow is not significantly different from the corresponding standard Poisson distribution. Similar performance can be observed on other links too. Therefore, if each OD demand follows a Poisson distribution independently, the corresponding unconditional link flow estimated by the model also follows a Poisson distribution, whose parameter is determined by users' strategic choice and demand distribution.

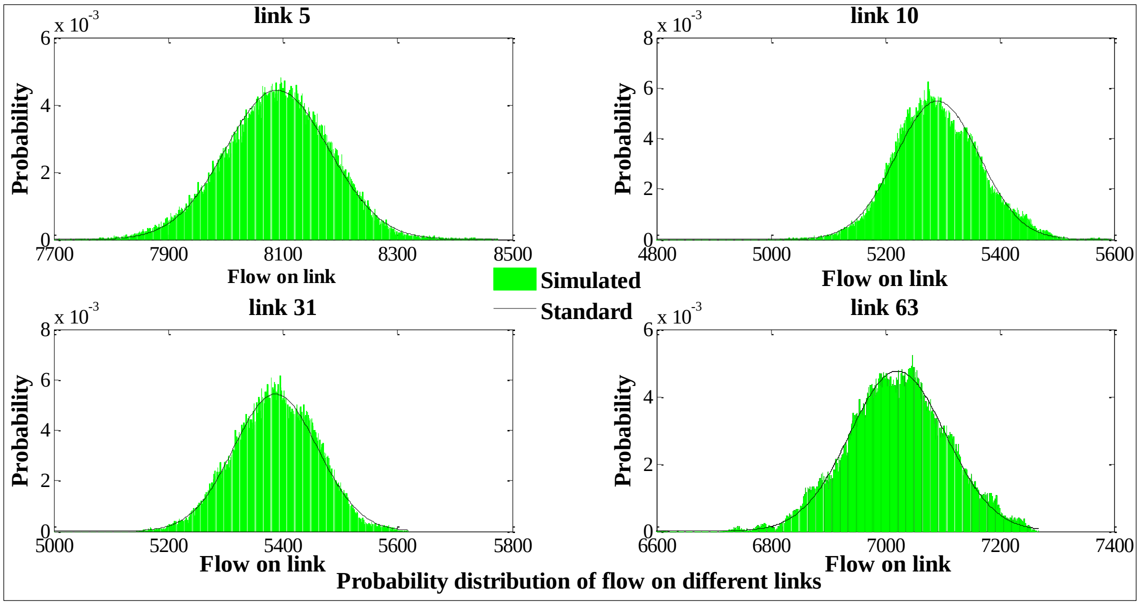



Figure 4-1 The probability distribution of flow on several links.

In Figure 4-2 and Figure 4-3, the 10000 sets of link flows are substituted in the BPR function to evaluate link travel times, which provide the expected link travel times (and respective standard deviation) for all 76 links. In Figure 4-2, the x-axis represents the analytical travel time estimated from Equation 4-28, and the y-axis represents the expected link travel time computed from the simulated results. It is shown in Figure 4-2 that the analytically estimated expected link travel time closely approximates the simulated expected link travel times (as evident by the $R^2$=0.99).

One of the strengths of the I-STRUE model is its capability of estimating the link flow variation analytically. In Figure 4-3 the simulated standard deviation of link travel times are compared with the analytically derived values from Equation 4-29. Similar to the expected link travel time, a linear regression analysis is done on the estimated and simulated standard deviation of link travel time on all 76 links. The R-squared value is very close to 1 despite it being smaller than that of the expected link travel time. It is notable that the simulated standard deviation of link flow may deviate from the estimated ones when the expected travel time is large due to the sample size. Figure 4-3 demonstrates the model's ability to capture variability in travel time as a consequence of demand volatility.

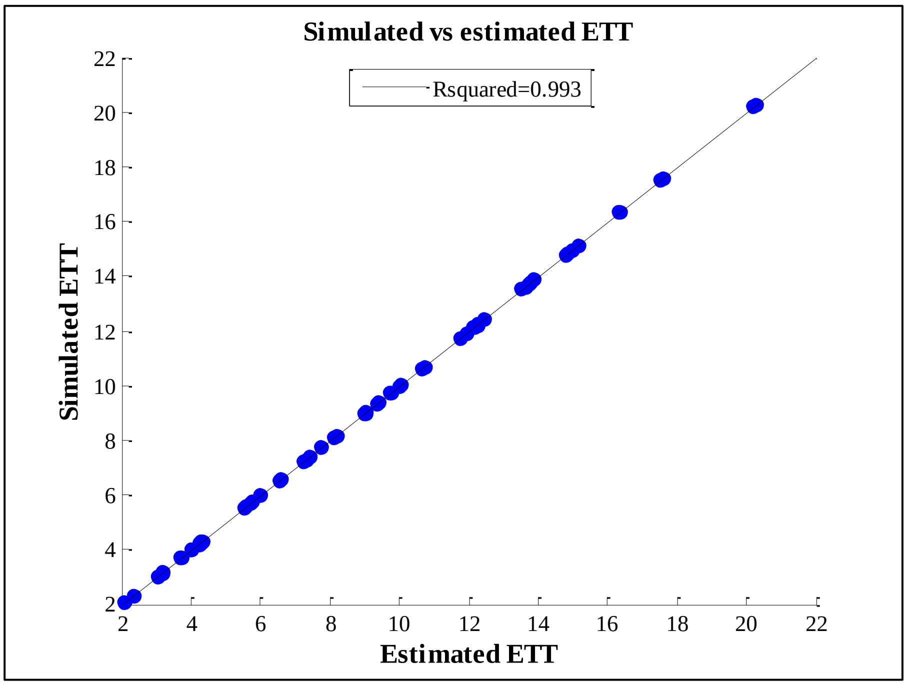



Figure 4-2 Linear regression analysis of expected link travel time.

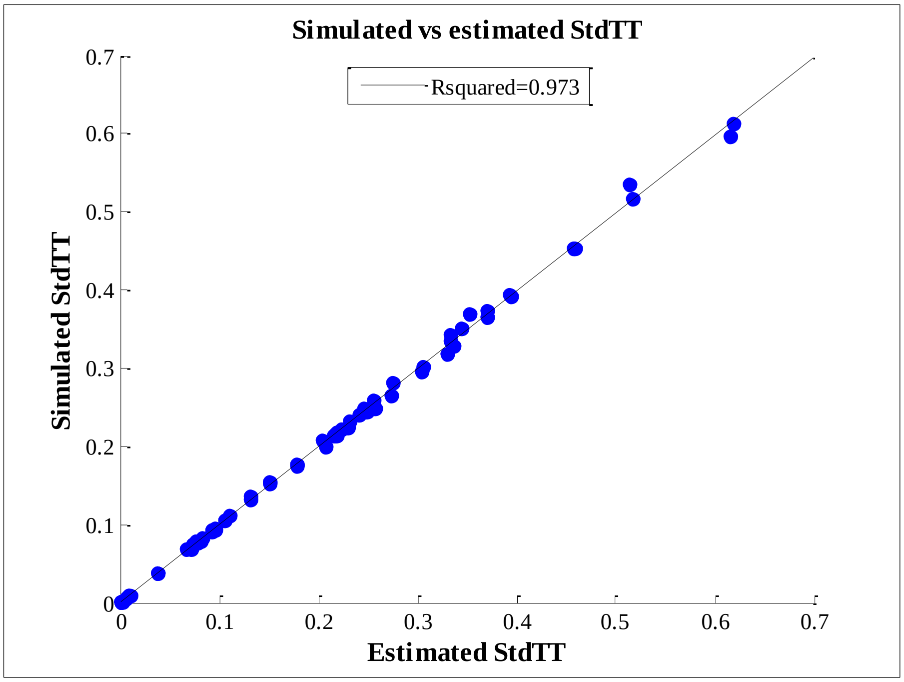



Figure 4-3 Linear regression analysis of standard deviation of link travel time.

Table 4-2 compares the system performance of ETSTT and StdTSTT of the network; the estimated results are computed from Equations 4-30 and 4-31. As demonstrated by the table, the simulated ETSTT is very close to the estimated ETSTT, as indicated by the relative error of just 1.3 percent. However, the StdTSTT is moderately different, due to the Monte-Carlo sampling process as firstly, 100 realized demand are sampled, then 100 multinomially distributed path flows are sampled based on each realized demand, so the path flows are actually correlated in each realized total system travel time; however, the analytical expressions of StdTSTT are derived based on the unconditional path flows which are independent of each other. This inconsistency (correlation and independence) leads to the difference in estimated and simulated StdTSTT, and therefore the analytical expression of StdTSTT should be applied cautiously. Despite this, the ETSTT still presents a reliable approximation; in addition, the estimated expected link flows and the corresponding standard deviations satisfactorily match the simulated results, as shown in Figure 4-2 and Figure 4-3.

Table 4-2 Network Performance Measures

| *Performance measures* | *Estimated results* | *Simulated results* | *Difference* |
|---|---|---|---|
| ETSTT | 7481223.1 | 7582740.9 | 1.3% |
| StdTSTT | 32090.97 | 46547.93 | 31.0% |

### 4.3.2 The most likely strategic link choice

Sometimes the aggregated link flows would not suffice when O-D specific information is required, such as O-D matrix estimation, emission analysis and many other transportation applications. The maximum entropy provides a way to find the 'most likely' strategic link choice. In the analysis, the relative gap (termination criterion) for the F-W method is set to $1*e^{-5}$, and the tolerance threshold is set to ten percent. In total, the F-W method provides 1434 paths, among which 907 paths are identified as the equilibrium paths. To demonstrate the difference between the traditional maximum entropy method (the deterministic case, hereby referred to as MED) and the maximum entropy of probability distribution (where the entropy of path flow distribution is considered, hereby referred to as MEP), the results of these two methods are compared in Table 4-3. In MED, we firstly solve the Wardrop's user equilibrium, where all OD demands are treated as deterministic variables. Then the optimization program in Equations 4-34 to 4-37 is solved to provide the path choice probabilities. In MEP, we solve the optimization program in Equations 4-41 to 4-44 and the strategic path choice is obtained. In Table 4-3, O-D pair (3-16) is chosen to demonstrate the difference between MED and MEP, where the demand for this O-D pair is 200. The differences in path choice can be seen in paths 1, 2, 3 and 5, and such difference may be more significant if only a few paths are considered as user equilibrium paths. No ground truth is found yet to prove which method is better, but the results clearly indicate the necessity and importance of accounting for uncertainties in entropy.

Table 4-3 Path choices of MED and MEP for O-D pair (3-16).

| *Path (Represented by a sequence of nodes)* | *Path choice probability (MED)* | *Strategic path choice for (MEP)* |
|---|---|---|
| [3,4,5,6,8,16] | 0.235 | 0.234 |
| [3,4,5,9,8,7,18,16] | 0.093 | 0.089 |
| [3,1,2,6,8,16] | 0.153 | 0.157 |
| [3,4,5,9,10,16] | 0.166 | 0.166 |
| [3,4,5,6,8,7,18,16] | 0.098 | 0.099 |

| | | |
|---|---|---|
| [3,4,5,9,8,16] | 0.17 | 0.17 |
| [3,1,2,6,8,7,18,16] | 0.085 | 0.085 |

Table 4-4 demonstrates the strategic link choice on link 24, and the O-D pairs with no trips assigned to link 24 are not presented in the table. There are 25 O-D pairs choosing link 24, out of 576 O-D pairs, which is a comparatively small proportion. The figure makes intuitive sense because link 24 is an east-bound link, only travellers choosing nodes 7, 8, 16, 18 as their destinations are very likely to choose it, which is exactly what is shown in Table 4-4. If the strategic link choice is one, it means all the users for that O-D pair will choose this link. The strategic link choice should not be greater than one. The importance of the strategic link choice lies in that it provides the link flow information disaggregated by O-D pairs. In the real world, obtaining aggregate link flow is viable and efficient, and the strategic link choice can clearly present the 'from-and-to' information on links.

Table 4-4 The unique strategic link choice for link 24.

| *From Origin* | *To Destination* | *Strategic link choice* |
|---|---|---|
| 3 | 7 | 0.309 |
| 3 | 8 | 0.270 |
| 3 | 16 | 0.259 |

| | | |
|---|---|---|
| 4 | 7 | 0.207 |
| 4 | 8 | 0.125 |
| 4 | 16 | 0.146 |
| 4 | 18 | 0.398 |
| 4 | 20 | 0.250 |
| 5 | 7 | 0.318 |
| 5 | 8 | 0.172 |
| 5 | 16 | 0.213 |
| 5 | 20 | 0.398 |
| 9 | 7 | 1.000 |
| 9 | 8 | 1.000 |
| 9 | 16 | 0.678 |
| 9 | 18 | 1.000 |
| 9 | 20 | 1.000 |
| 10 | 8 | 1.000 |
| 11 | 8 | 1.000 |
| 12 | 7 | 0.134 |
| 12 | 8 | 0.148 |

| | | |
|---|---|---|
| 12 | 16 | 0.152 |
| 12 | 18 | 0.268 |
| 13 | 7 | 0.051 |
| 13 | 8 | 0.148 |

## 4.4 Chapter summary

In this chapter, we introduced the novel I-STRUE model which relaxes the assumption of proportional demand and hence improves the model fidelity. In this model, users equilibrate based on an expected condition as opposed to a deterministic cost. To capture this behaviour, the model assumes that users rationally make their strategic link choice by considering all possible demand scenarios (all the OD demands are independent of each other) in a known distribution. This strategic link choice is then followed regardless of the realized travel demand on any given scenario. Therefore, the state-of-equilibrium may not be observed on a given day. As such, the proposed model is illustrated to replicate the behaviour of observed link travel time variability. In the proposed model, link flow distributions and users' strategic link choice are proved to be unique mathematically, and network performance measures are given in analytical expressions, which reduce the computation burden of network performance prediction. The efficiency and accuracy of these analytical expressions are demonstrated with a numerical example, and the importance of accounting for probability distribution in entropy function is also presented. Therefore, this model accounts for the demand uncertainty and users' strategic choice while maintaining computation simplicity and tractability.

However, every model has its limitations. In this I-STRUE model, it is assumed that OD demand follows a Poisson distribution to ensure uniqueness. This forces the expected demand to be equal to the variance of demand, and this assumption may limit the applicability of the model.

Many possibilities are still yet to be explored under the model's framework. OD demands may be assumed to follow some bivariate probability distributions if the uniqueness is guaranteed, which allows mean and variance to be different. In addition, we may account for capacity uncertainty by assuming the capacity also follows a certain kind of distribution. Finally, integrating this model into the OD estimation problem would be a straight-forward contribution.

In short, the strategic user equilibrium presents another insight in the fourth step- traffic assignment by accounting for network uncertainty. Its significance and connection to the O-D demand estimation problem are demonstrated in Chapter 7.

# Chapter 5

# Impact of the learning process- an application of the strategic user equilibrium

## 5.1 Introduction

In previous chapters, we have focused on the long-term transportation prediction. In this chapter, we will address adjusted travel route choice in the context of new transport developments and incremental traveller learning. New infrastructure development has the potential to fundamentally change the traffic demand throughout a transport network, which can impact traveller's perceptions and adjustments in multiple ways. For instance, if travellers expect a project to significantly increase or decrease overall travel demand they may change their daily route choice based on those new expectations. Further, over time, travellers will learn actual network demand, and adapt their route choice accordingly. To assess the impact of new developments, the post-re-equilibration state is commonly employed. However, a critical factor that is often unaccounted for, yet essential to the success of the planning process, is the time taken for users to learn about and adjust to a given change within the system. This study addresses this gap, and proposes a methodological framework which can be used to model the day-to-day learning process of road users, and the corresponding system performance over time. The aim is to help identify an appropriate modelling “horizon”, or time period after a project has been completed, for which the transport system impact can be accurately assessed.

Currently, common practice to determine horizon time periods for future traffic impact assessments are based on the scheduled completion of works and the addition of a fixed time period to account for the users learning and transforming their route choice. The fixed time period of “learning” of users’ is based on standard practice and engineering judgement with no definitive method or approach to its calculation. This study investigates the duration of the learning period of users’ in adjusting to the presence of a new development or infrastructure change within the urban environment. The impact of a new development is inferred as a change to the total travel demand distribution generated between origin and destination pairs throughout the network. The time taken by users to transform their initial perceived travel demand distribution to the actual demand distribution is defined as the learning period. The model uses the P-STRUE model (Dixit et al., 2013) as the foundation of the analysis and incorporates learning through Bayesian inference.

In the P-STRUE model, for each user presented in a given demand scenario, the route choice is based on the demand distribution and the route is followed regardless of the realized travel demand on a given day. In this model, users will update their perceived distribution curve based on day-to-day travel experience. Studies completed in the recent past make the assumption that the demand fits a lognormal distribution (Duell et al., 2014, Wen et al., 2014, Zhao and Kockelman, 2002, Kamath and Pakkala, 2002).

In particular, this research employs a methodological framework to model the day-to-day learning process of road users, and the corresponding system performance over time with a focus on the impact of specific new developments. Travellers assume an initial demand distribution, and incrementally update it based on their day-to-day travel experiences. Bayesian Inference is used to update the travel demand distribution, and the strategic user equilibrium model is used to compute the underlying traffic assignment pattern.

In this chapter, further details of the background to the problem are presented in Section 5.2. The next section provides an explanation of the modelling framework as well as the assumptions made to devise the model. The model is then applied to a sample network and the results of the application are then analysed and discussed in Section 5.4. Finally, future extensions and applications of the model are discussed in the chapter summary.

## 5.2 Background

Traffic impact assessment and traffic modelling guidelines provide practitioners advice on how to assess the future traffic impacts which will result from the establishment of a new urban development. The guidelines provide detailed methods to forecast the level of travel demand for the future year assessments by either *(i)* using calibrated and validated regional travel demand models or *(ii)* by using population and development data (Florida Department of Transport (FDOT), 2014, Roads and Maritime Services (RMS), 2013, The California Department of Transportation (Caltrans), 2002). For example, the Roads and Maritime Services Traffic Modelling Guidelines (2013) states the following; “Planners need to analyse historic data and develop a forecasting methodology appropriate to model future time horizons”, providing no clear distinction of how these time horizons are determined. To the authors’ best knowledge there is little to any discussion regarding how the actual horizon time period for assessment is determined in practice. To address this gap in the literature this study specifically addresses the impact of new developments on changes to users’ route choice over time based on their daily travel experience. The contribution of this study is a methodological framework to determine the horizon time period which should be chosen for project assessment.

Within-day traffic assignment has shown its capability of taking implicitly into account the variability of the flow state along the arc accordingly to any concave fundamental diagram, and modelling real-time traffic (Bellei et al., 2005, Gentile et al., 2005, Helbing et al., 2006). However, most commuters tend to update their commute experiences on a day-to-day basis. Day-to-day travel experiences within a transport network affect future travel decisions, extending from mode choice to route choice along a road transport network (Ben-Elia et al., 2013, Ben-Elia and Shiftan, 2010, Mahmassani and Liu, 1999). Day-to-day dynamics of traffic assignment, which investigates the evolution of travel choices and traffic congestion over time, has been addressed in a number of previous studies (Smith et al., 2014, He et al., 2010, Watling and Hazelton, 2003, Daganzo, 1983, Cascetta and Cantarella, 1991, He and Liu, 2012, Watling and Cantarella, 2013, Wang et al., 2013, Zhao and Orosz, 2014, Hazelton, 2002, Han and Du, 2012, Hazelton and Watling, 2004, Zhang and Nagurney, 1996). Previous research has addressed both deterministic process models (He et al., 2010, Han and Du, 2012, Zhang and Nagurney, 1996) and stochastic process models (Cascetta and Cantarella, 1991, Hazelton, 2002, Hazelton and Watling, 2004). A detailed discussion of the literature on this topic is presented by (Watling and Cantarella, 2013). A majority of these studies focused on long-term traffic equilibration as a result of day-to-day traffic variations and seasonal changes which are expected by the user. Previous studies have also investigated the impact of disruptions (He and Liu, 2012, Wang et al., 2013), providing insight into how long people take to learn about the impacts of a major disruption and how they adjust their routing decisions in the long term. All of these studies provide great insights into long-term user route choice. The work presented in this chapter instead focuses on changes in route choice during the adjustment period immediately following the significant change of demand, rather than normal day-to-day conditions.

When considering route choice of road network users, travellers learn about their available routes from their experiences of performing the same trip over an extended period of time. Within the context of this study explicit learning could also potentially arise from the marketing and media of new residential land releases or the opening of new urban infrastructure, as this information has the potential to affect how people perceive the travel conditions. There have been a number of approaches to modelling and understanding about learning in a route choice context and how this affects network performance. (Bogers et al., 2008) suggested that two types of learning, derived from theories within psychology, play a critical role in day-to-day route choice; implicit or reinforcement based learning and explicit or belief based learning. Implicit learning arises for users as a consequence of travel; a higher relative travel time from a trip would be a negative reinforcement whilst a lower relative travel time would be a positive reinforcement for future decision making (Erev and Barron, 2005). In general, people are habitual decision makers and once an efficient method to complete an activity is devised it is used repeatedly, and this holds true for travel decisions (Jager, 2003). However, when characteristics of the network change, such as the establishment of a new residential development, habitual route choice may not be the most efficient method resulting in implicit learning. Additionally, in a transport context, explicit learning will also occur when users gain knowledge from information sources and their beliefs of these information sources (Arentze and Timmermans, 2003).

The learning behaviour has also been investigated in the game theory (Crawford, 2013, Yang, 2005). The strategic learning in the Nash equilibria context was also discussed in (Hoffmann, 2014). Controlled laboratory experiments using repeated route choice games have been conducted to understand users' learning behaviour and results have been adapted to discrete choice models (Ben-Elia and Shiftan, 2010, Cominetti et al., 2010, Bogers et al., 2005). In particular, Ben-Elia and

Shiftan (2010) presented that initial risk seeking behaviour in route choice transforms into risk averse behaviour as learning progresses which is consistent with the findings of (Arentze and Timmermans, 2005). Experimental approaches provide the ability to investigate dynamic system evolution and the behavioural implications of users' day-to-day choices. However, a shortcoming with this method is that there is difficulty in resolving the biases that may occur within the simulated environment as compared to the real environment (Chen and Mahmassani, 2004b). In terms of econometric modelling, (Horowitz, 1984) developed an updating version of EUT to analyse repeated travel choice situations using a weighted average approach in calculating the perceived travel cost of a route. Further studies have also used this concept where the route choice is determined by a process of adaptive learning whereby the information affects the utility of the route and the knowledge of the road network for future decisions (Mahmassani and Liu, 1999, Srinivasan and Mahmassani, 2003, Mahmassani et al., 1986). (De Palma and Marchal, 2002) investigated day-to-day learning using an exponential Markov process representing learning; however, this model was not validated with empirical data. A drawback of all the perception updating methodologies described above is that they do not capture drivers' uncertainty in their estimation of travel time which can be accounted for using a Bayesian updating approach (Jha et al., 1998, Chen and Mahmassani, 2004b). Jha et al. (1998) used a Bayesian updating model to capture the mechanism by which travellers update their day-to-day travel time perceptions based on previous experiences and information from ATIS systems. (Chen and

Mahmassani, 2004a) extended the use of Bayesian Inference by also considering heuristics to trigger and terminate the learning process to depict a users' salience to new information. The social impact under uncertainty on traffic was also demonstrated (Sunitiyoso et al., 2011), the parameterization of modelling the learning, or evolution of urban network was discussed as an extension of physical rules (Helbing and Nagel, 2004). All these studies provided a background in developing the methodology of this study. Specifically, this study utilizes Bayesian Inference to model the learning process within the P-STRUE model, which explicitly incorporates uncertainty into the traffic assignment problem. The analysis of travel behaviour in uncertain conditions has historically focussed on three economic theories, Expected Utility Theory (EUT), Prospect Theory (PT) and Regret Theory (Ben-Elia et al., 2013). Research and models developed using these theoretical frameworks provide great insight into one-shot decision making where the outcome of one decision has no relationship to the next (Arentze and Timmermans, 2005). In contrast, the focus of this study is to incorporate experiential information into the users' decision process.

Travel demand is a critical factor that affects travel time on a network. Though demand data is difficult to obtain, the expected demand and an estimate for the distribution of travel demand can be obtained through loop detector data, household survey data and through many other approaches. However, there is a certain degree of uncertainty that exists with these estimations. This study incorporates this uncertainty by providing partial information regarding the demand distribution to the user as well as including a perception component which interprets the user's confidence level of their estimation of the travel demand. A Bayesian Inference Model was implemented to update users' perceived travel demand distributions based on previous travel experiences, which contrasts previous studies which investigated the update of travel time perceptions. Furthermore, the study utilizes the strategic user equilibrium (Dixit et al., 2013) to determine the traffic assignment pattern corresponding to each step in the learning process, and quantify various system performance metrics. The static version of the STRUE model, which has been developed in a static and dynamic context (Waller et al., 2013), was specifically selected because it offers a way to incorporate day-to-day demand variability. The importance of accounting for demand volatility has also been discussed in many previous literature (Clark and Watling, 2005, Duthie et al., 2011, Uchida and Iida, 1993); the model ensures that users recognize the uncertainty or variability in travel time to their destination and rationally choose routes while considering all possible demand scenarios from a known (or perceived) distribution. In addition, the model provides the link flow variability as a result of the demand volatility.

## 5.3 Problem formulation

In this section we describe the Bayesian inference process which is used to model the learning behaviour of users. The underlying traffic assignment model implemented in this model framework is the P-STRUE assignment model. A brief description of this assignment model is also included for completeness in this section. Table 5-1 lists the notation used in this section in addition to Table 2-2.

Table 5-1 Summary of notations.

| | |
|---|---|
| $T_a$ | Actual total trip demand. |
| $T_p$ | Users' perceived total trip demand. |
| $x_i$ | The realized total trip demand observed by users on day *i*. |
| $P(x \mid y)$ | The conditional probability of $x$ given $y$. |
| $M_\beta$ | $\beta_{th}$ raw moment of the total demand distribution. |
| $\mu$ | The location parameter of a lognormal distribution. |
| $\sigma$ | The scale parameter of a lognormal distribution. |
| $\varphi$ | The shape parameter of a gamma distribution. |
| $\omega$ | The scale parameter of a gamma distribution. |

The details of this optimization problem were explained in Section 2.3.1. The link travel time function, $t_n$, is modelled after the Bureau of Public Roads cost function (U.S, 1964), which is defined in Equation 3-1.

The probability density function for the total trips $g(T)$ is assumed to follow a lognormal distribution. The positiveness of the lognormal distribution and the ease of determining conjugate priors necessary for the Bayesian inference process ease the computational process and as such are assumed for this study as well. However, it must be noted that other distributions could be assumed; however, significant mathematical manipulation would be required to ensure positivity and apply Bayesian inference. With P-STRUE, the expected link travel time and variability of link travel time can be shown to be strictly a manifestation of travel demand uncertainty, and is analytically defined as follows:

$$E(t_n) = t_{nf} + \alpha t_{nf}(\frac{f_n}{C_n})^{\beta} M_{\beta} \qquad 5\text{-}1$$

$$var(t_n) = E(t_n^2) - [E(t_n)]^2 = \alpha^2 t_{nf}^2 (\frac{f_n}{C_n})^{2\beta}[M_{2\beta} - M_{\beta}^2] \qquad 5\text{-}2$$

Equation 5-1 defines the expected link travel time and equation 5-2 defines the variance of link travel time, where $M_{\beta}$ is the $\beta_{th}$ moment of the demand distribution and can be found analytically using the moment generating function. For a given demand distribution, the P-STRUE assignment problem can be solved using any algorithm capable of solving the static traffic assignment problem. In this work the Frank-Wolfe algorithm is implemented to compute the link proportions. The difference is we use equation 5-1 with the moment generating function to calculate and update the expected travel time instead of a deterministic travel time; hence, the shortest path cost may change accordingly.

As stated previously, the main contribution of this work is that the learning process is incorporated into the novel strategic user equilibrium model. To incorporate learning, the user's *perceived* demand distribution is used to compute the link costs in P-STRUE, and thus results in a given system assignment pattern. Furthermore, the perceived demand distribution is assumed to change over time based on knowledge gained by users through their past travel experiences. Every time the perceived demand distribution is updated, the link costs functions will change, resulting in a new set of equilibrium-based path choices. To update the perceived distribution a Bayesian inference process was implemented, which is described below. The learning model with underlying P-STRUE assignment is hereby referred to as L_STRUE.

Firstly, two demand distributions are defined, i) the actual demand distribution and ii) the perceived demand distribution. The actual distribution represents the true state, from which day-to-day demands are sampled. The perceived distribution is what the users assume to be true at the time. It is assumed that the actual distribution does not change during the timeframe of concern. The perceived distribution is assumed to initially underestimate or overestimate the expected trip demand and variance. Both distributions are assumed to follow lognormal distributions with known, but different, parameters. The actual demand distribution is defined as $T_a \sim Lognormal(\mu_a, \rho_a)$, and the perceived demand distribution is defined as $T_p \sim Lognormal(\mu_p, \rho_p)$. Note that $\mu$ and $\rho$ are simply parameters of the lognormal distribution, and have a direct relation to the mean and variance of the lognormal distribution, defined in Equations 5-3 and 5-4, respectively. These equations represent the mean and variance of the total trip demand distribution.

$$E(T) = e^{\mu + \frac{1}{2\sigma}} \tag{5-3}$$

$$var(T) = \left(e^{\frac{1}{\sigma}} - 1\right)[E(T)]^2 \tag{5-4}$$

Because providing the mean and variance of total trip demand is more intuitive than simply assuming the corresponding lognormal parameters, the lognormal parameters $\mu_a$ and $\rho_a$ were back calculated based on the actual demand distribution, $E(T_a)$ and variance of $Var(T_a)$. It was further assumed *i)* that the perceived location parameter $\mu_p$ is identical to the location parameter of the actual demand distribution, $\mu_a$, *ii)* it is known by the users, and *iii)* remains fixed over the course of the learning process. This is based on the assumption that users have some level of prior knowledge which they base their initial perceived distribution on (i.e. they are not unfamiliar drivers). The assumption also allows us to compute the perceived expected demand, $E(T_p)$ and variance of perceived demand $Var(T_p)$, for any precision parameter $\sigma_p$. Note that the assumption of identical location parameters and the assumption below of the gamma distribution are made because they can reduce the computation complexity without compromise in the investigation of the learning process. When these assumptions are not applicable, numerical integration can be used.

The learning process is modelled using Bayesian Inference to update the precision parameter $\sigma_p$ based on users' previous travel experiences. The initial perceived precision parameter $\sigma_p$ is assumed to be a random variable, and follows a gamma distribution $\rho \sim Gamma(\varphi, \omega)$. The Gamma distribution is capable of describing various kinds of probability curves and is always positive. From Bayesian inference, the posterior distribution is a function of both the prior distribution and the likelihood function. The posterior distribution of the precision $\sigma_p$ given that a set of data $t$ is observed can be expressed as:

$$P(\sigma_p \,|\, \boldsymbol{t}) = \frac{P(\boldsymbol{t} \,|\, \sigma_p) P(\sigma_p)}{\int P(\boldsymbol{t} \,|\, \sigma_p) P(\sigma_p) d\sigma_p} \tag{5-5}$$

Where, $\boldsymbol{t} = [t_1 \quad \cdots \quad t_k]'$ 5-6

Note that $t_k$ is the observed total demand on day $k$. The gamma distribution is the conjugate prior of the lognormal likelihood function, i.e. if the actual demand has a lognormal distribution, from Bayesian inference, the closed form probability distribution function of the posterior distribution exists, and is also a gamma distribution. The precision variables for the prior and posterior distributions are thus defined as $\sigma_{prior} \sim Gamma(\varphi_0, \omega_0)$ and $\sigma_{posterior} \sim Gamma(\varphi, \omega)$, respectively.

The parameters $\varphi$ and $\omega$ are initialized as $(\varphi_0, \omega_0)$, and updated each day based on users' travel experience, as defined below:

$$\varphi = \varphi_0 + \frac{k}{2} \tag{5-7}$$

$$\omega = \omega_0 + \frac{\sum_1^k \left(lnt_i - \mu_p\right)^2}{2} \tag{5-8}$$

The expected precision and variance of precision can therefore be defined in terms of $\varphi$ and $\omega$ as follows:

$$E\left(\rho_{posterior}\right) = \frac{\varphi}{\omega} \tag{5-9}$$

$$Var\left(\rho_{posterior}\right) = \frac{\varphi}{\omega^2} \tag{5-10}$$

The variance of the precision can be interpreted as the confidence level of a user group, and reflects their willingness to adapt their route choice (i.e. update the perceived demand distribution) based on past travel experiences. A low precision variance represents a user group who is more confident in their initial perception of the travel conditions, and is therefore going to be less willing to change their perception based on travel experiences. A higher precision variance represents a user group who is less confident in their initial perception of the travel conditions, and is therefore going to be more willing to adapt their route choice based on past travel experiences. The impact of this variable is illustrated in a sensitivity study presented in the numerical analysis section.

In the analysis conducted, a single iteration is equivalent to a day during which users commute to work. Each day the users will select a route based on their perceived demand at the time. On the same day, a demand will be realized, which is sampled from the actual demand distribution, resulting in a set of path flows on the network and the consequent link and path travel times. The users observe these travel times, and update their perceived demand curves accordingly, by updating parameters, $\varphi$ and $\omega$. The updated parameters are then used to compute the updated perceived precision parameter $\rho_p$ using the expected posterior precision from Equation 5-9. At the end of each iteration the updated perceived demand distribution will be:

$$T_p \sim Lognormal(\mu_p, E(\sigma_{posterior})) \quad \text{5-11}$$

At the end of the entire learning process the users' final perceived demand distribution will be defined by $T_p \sim Lognormal(\mu_p, \sigma_{posterior})$. As the users are updating their perceived demand every day according to the Bayesian inference, their perceived demand distribution is expected to have converged from the initial perceived distribution, $T_p \sim Lognormal(\mu_p, \sigma_{prior})$ to the actual distribution, $T_a \sim Lognormal(\mu_a, \sigma_a)$. The convergence rate may be affected by many factors including network size, demand variability and many more. Note that because $\varphi_0$ and $\omega_0$ determine the prior precision variable, assuming the initial perceived distribution is equivalent to assuming a mean and variance of the precision variable. To further explore the impact of the initial perception distribution on the learning process, the next section illustrates the convergence behaviour of L_STRUE for a range of initial perceived distributions modelled on a test network.

## 5.4 Numerical analysis

The developed L_STRUE model has been demonstrated using a test network with 4 nodes and 5 links (Braess Network) as presented in Figure 5-1. The assumed network properties are defined within Figure 5-1, free flow travel times and capacity for each link are shown in parentheses in units of miles per hour and vehicles per hour, respectively. In addition, the BPR parameters α and $\beta$ for all links are equal to 0.15 and 4 respectively, the length of each link is 1 mile.

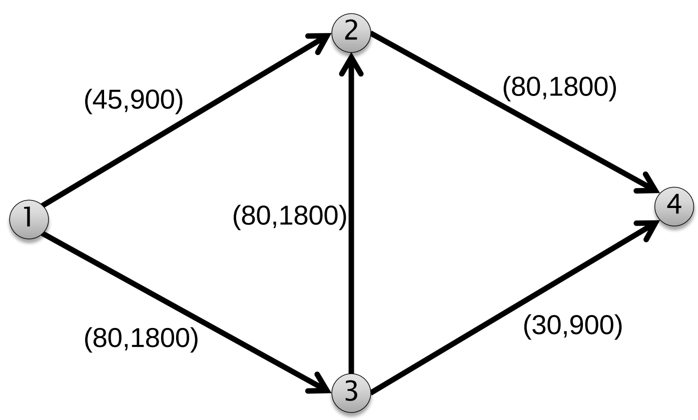



Figure 5-1 The test network.

The analysis investigated the sensitivity of the network performance regarding two main components: *i)* different initial perceived demand distributions (i.e. how accurate drivers' initial perception is relative to the actual travel demand distribution) and *ii)* the impact of an increasing precision variance (i.e. confidence level of the drivers in regards to their initial perception). The purpose of conducting these two sensitivity analyses was; firstly, to compare the learning process when the initial perceived and actual demand distributions varied; and secondly, reveal the role of the precision variance in the learning process. For all scenarios evaluated the actual demand curve was fixed, with a mean demand of 2700 and standard deviation of 270, or 10% of the mean. Twelve scenarios were selected in a systematic fashion representing different combinations of perceived overestimation believed by users, and precision variance (confidence levels). The scenario selection was based on the idea that road network users would have knowledge of a new (recent) development and as a result perceive conditions which are inflated relative to the historical traffic conditions. The precision variance levels were chosen to demonstrate the system impact of users' willingness to adapt. The set of scenarios assessed in this analysis is presented in Table 5-2. The perceived standard deviation of total trips $STD[T_p]$, is presented as a percentage of the perceived expected total trips $E[T_p]$. For each scenario 2000 iterations were run in order to capture the entire learning process.

Table 5-2 Scenarios assessed for numerical analysis.

| Scenario | Variance of precision | $E[T_p]$ | $STD[T_p]$ |
|---|---|---|---|
| 1 | 0.1 | 2835 | 34% |
| 2 | 0.1 | 3240 | 67% |
| 3 | 0.1 | 3510 | 84% |

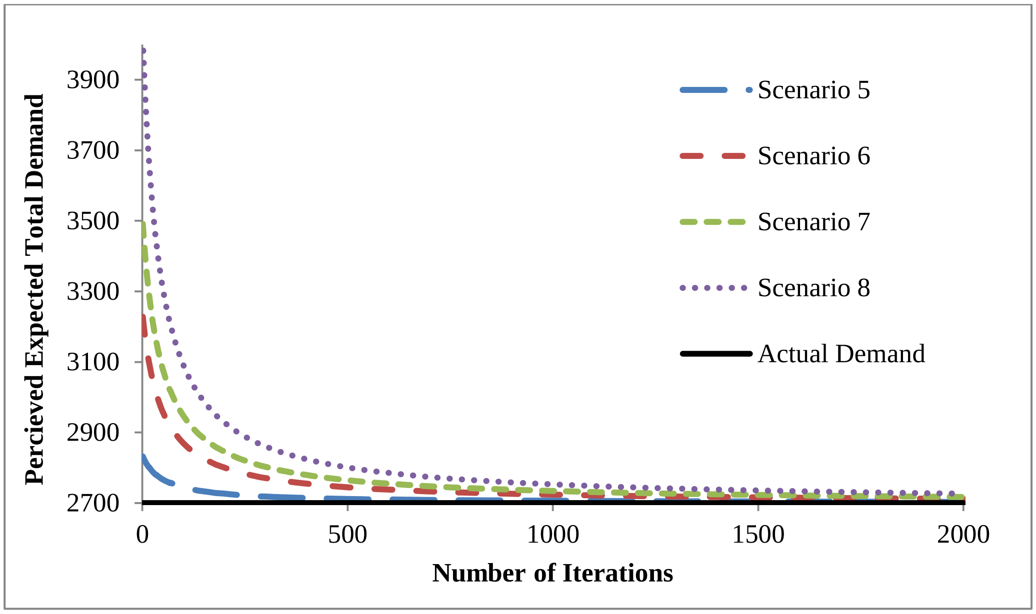
| | | | |
|---|---|---|---|
| 4 | 0.1 | 4050 | 113% |
| 5 | 0.2 | 2835 | 34% |
| 6 | 0.2 | 3240 | 67% |
| 7 | 0.2 | 3510 | 84% |
| 8 | 0.2 | 4050 | 113% |
| 9 | 0.3 | 2835 | 34% |
| 10 | 0.3 | 3240 | 67% |
| 11 | 0.3 | 3510 | 84% |
| 12 | 0.3 | 4050 | 113% |

### 5.4.1 System level performance

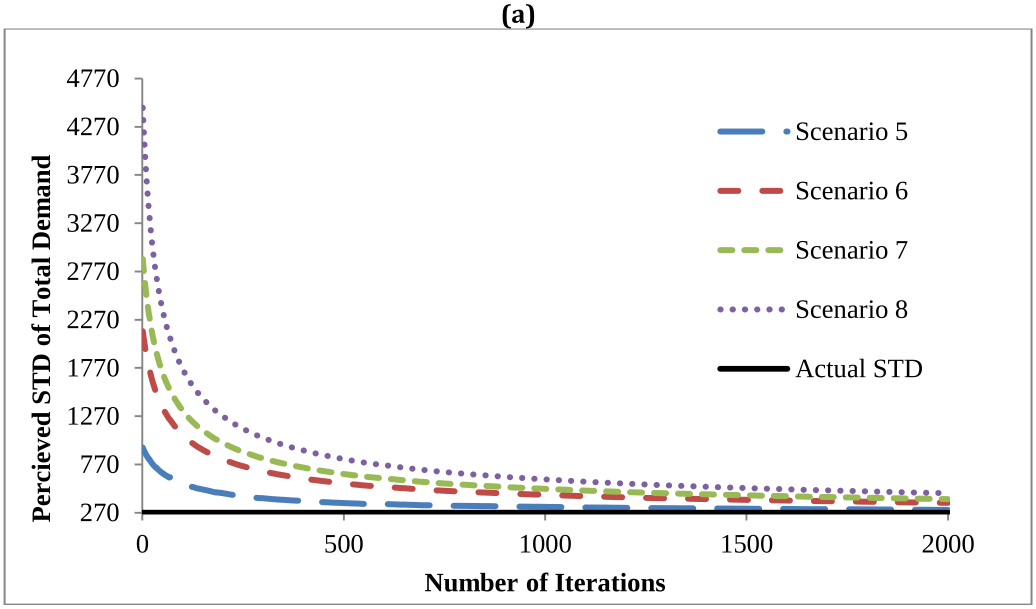
A system level performance assessment was conducted to obtain an understanding of the convergence of the L_STRUE model under the different scenarios tested. The purpose of the analysis was to identify system level performance metrics for different initial perceived demand distributions and ii) variance of the precision variable. In the analysis presented the time to convergence provides a proxy for the time taken for users' to learn the actual travel demand conditions, which is one of the main objectives of this study. It is however important to note, the numerical results of the L_STRUE model are specific to this case study, and at this point cannot be extrapolated to alternative network structures. The main contribution of this work is the proposed framework for modelling the learning process travellers go through due to changes to the network conditions. This study also serves to demonstrate various potential applications of the model. Throughout the following sections, results from a subset of the scenarios are presented, which are representative of the trends observed across all the scenarios tested.

#### 5.4.1.1 Sensitivity to initial perceived demand distribution

The convergence of the perceived expected demand over the learning period is illustrated in Figure 5-2(a). The horizontal axis represents the learning time, or the number of learning iterations, while the vertical axis represents users' perceived expected total demand. The figure shows Scenarios 5 through to 8, which consider the range of initial perceived demand curves presented in Table 5-2 and a fixed precision variance of 0.2. The horizontal line in the figure depicts the actual total demand. The figure illustrates the convergence of all the scenarios; after 365 iterations (which can be interpreted as a year of daily travel) the perceived demand is within 5% of the actual demand. The initial perceived demand distribution is shown to have a significant impact on the learning process. Figure 5-2(a) reveals the most inflated initial perceived distribution has taken almost twice as long to converge than the least inflated scenario. Figure 5-2(b) presents the convergence of the perceived standard deviation of demand over the learning period, illustrating a similar trend to what was observed in Figure 5-2(a). Both these figures suggest that when people's perceived demand distributions more closely reflect the actual demand distribution, the learning time is reduced significantly. This can potentially be a source of information provided to users in order to reduce the level of learning required within a system.




**(a)**



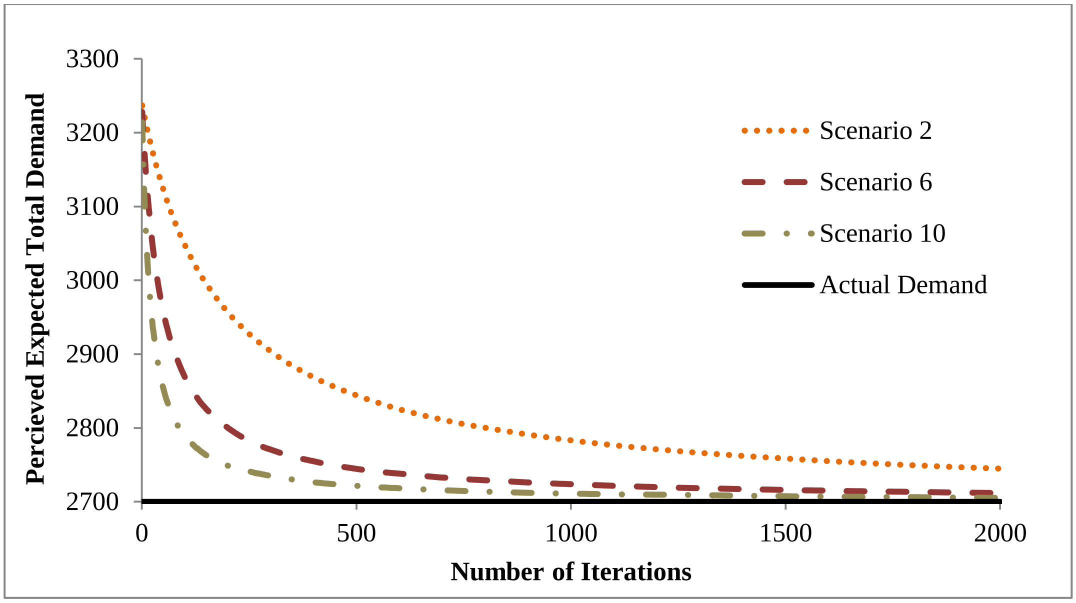

**(b)**

Figure 5-2: Illustration of Convergence of a) Perceived Expected Demand b) Perceived Variation of Demand under different initial perceived demand distributions.

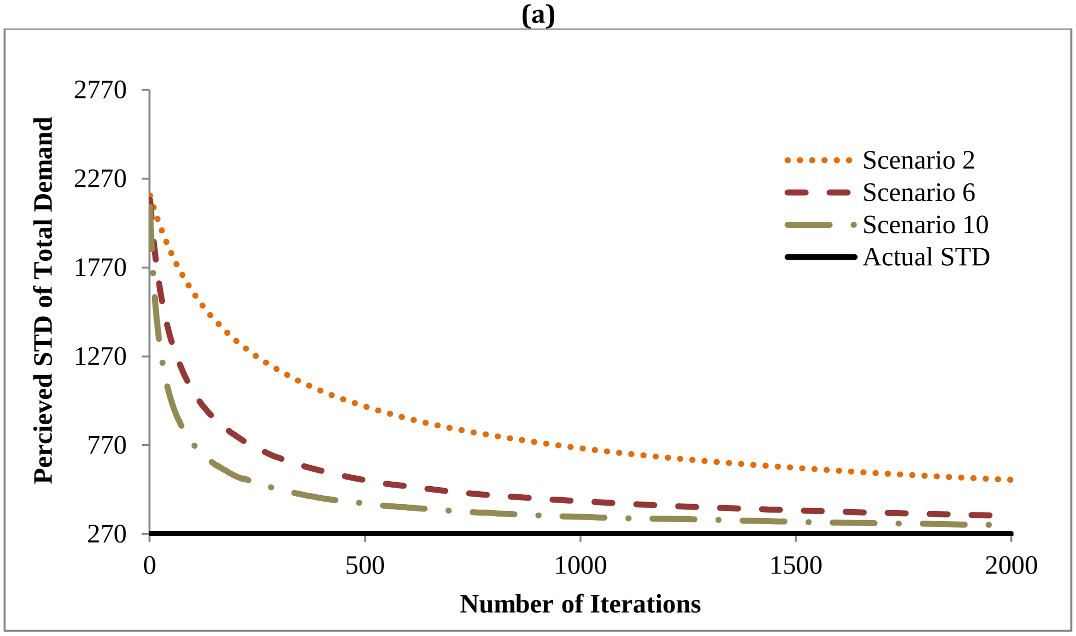
#### 5.4.1.2 Sensitivity to precision variance

The impact of increased precision variance on the convergence of the L_STRUE model is presented in this section. The convergence of perceived expected demand and perceived standard deviation of demand are presented in Figure 5-3(a) and Figure 5-3(b), respectively. The figures illustrate Scenarios 2, 6 and 10 corresponding to a precision variance of 0.1, 0.2 and 0.3, where the initial perceived demand distribution remains fixed.

Figure 5-3(a) and Figure 5-3(b) clearly indicate convergence of the perceived expected value and standard deviation of the total demand to the actual distribution values. However, the rate of convergence is significantly affected by the value of the precision variance. The lowest precision variance results in a considerably slower convergence when compared with the other two scenarios. Again, the lower precision variance represents a user group that is more confident in their initial perception, and therefore less willing to change their route choice. Similarly, a higher precision variance indicates a user group who is less certain about the prevailing traffic conditions and therefore less confident in his/her initial perception of the travel demand. These users can possibly be categorised as “new road users” or an “unfamiliar road users”, as they are more willing to update their route choice based on previous travel experiences, and therefore learn the actual demand faster, as illustrated by increased rate of convergence of the L_STRUE model. An alternative explanation of this behaviour is that the user is a “fast learner” and someone who is aware of the presence of any new developments and has rationalised the potential effect on traffic and is willing to adjust his travel patterns. In contrast, users with a lower precision variance could be classified as “stubborn users” who are determined to believe that their initial perceptions reflect the actual traffic conditions, and refuse to accept the changed resulting from a new development. These users exhibit a slower rate of learning, as illustrated by decreased rate of convergence of the L_STRUE model. The results from the L_STRUE model therefore provide a behavioural intuition regarding the precision variance. Note that in larger-scale networks, convergence rate may also be affected by network characteristics and many other factors, but due to the assumption that users will learn according to the Bayesian inference, the precision variance will still remain a critical factor.




**(a)**




**(b)**

Figure 5-3: Illustration of Convergence of a) Perceived Expected Demand and b) Perceived Variation of Demand under different precision variances.

### 5.4.2 Path level analysis

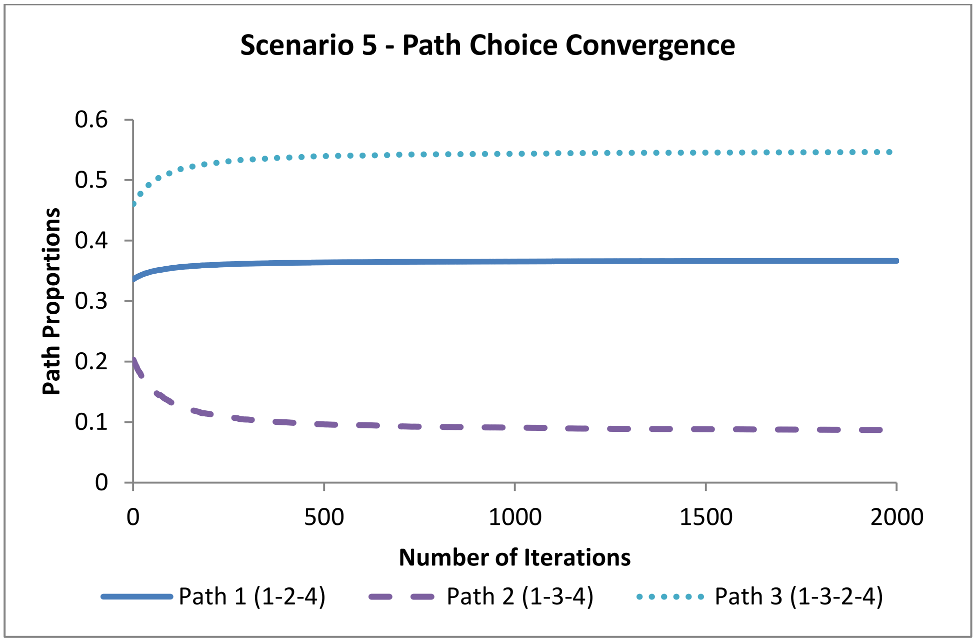
In addition to evaluating the learning process at the system level, a path level assessment was conducted to explore the impact of the learning process on user route choice under the different scenarios tested. Of specific interest was the changes to path flows and path travel times over time as users learned the actual conditions of the network. Similar sensitivity analysis was conducted to explore the changes in path flows over time relative to *i)* the variation of the initial perception distribution and *ii)* variance of the precision variable.

As described in the methodology section, the path assignment is computed using the P-STRUE model. It is important to note that the P-STRUE model provides unique link proportions and ultimately unique link flows, but not unique path flows. However, for the test network used within this study, it was possible to obtain path performance statistics because there were distinct links associated to individual paths. Path-based statistics are presented instead of link level statistics because they provide a more intuitive illustration of the network performance. As with the system level analysis, results from a subset of scenarios evaluated are presented, which are representative of the trends observed across all the scenarios tested. The paths are hereby referred to as Path 1, 2 and 3, where Path 1 connects nodes 1-2-3, Path 2 connects nodes 1-3-4, and Path 3 connects nodes 1-3-2-4.

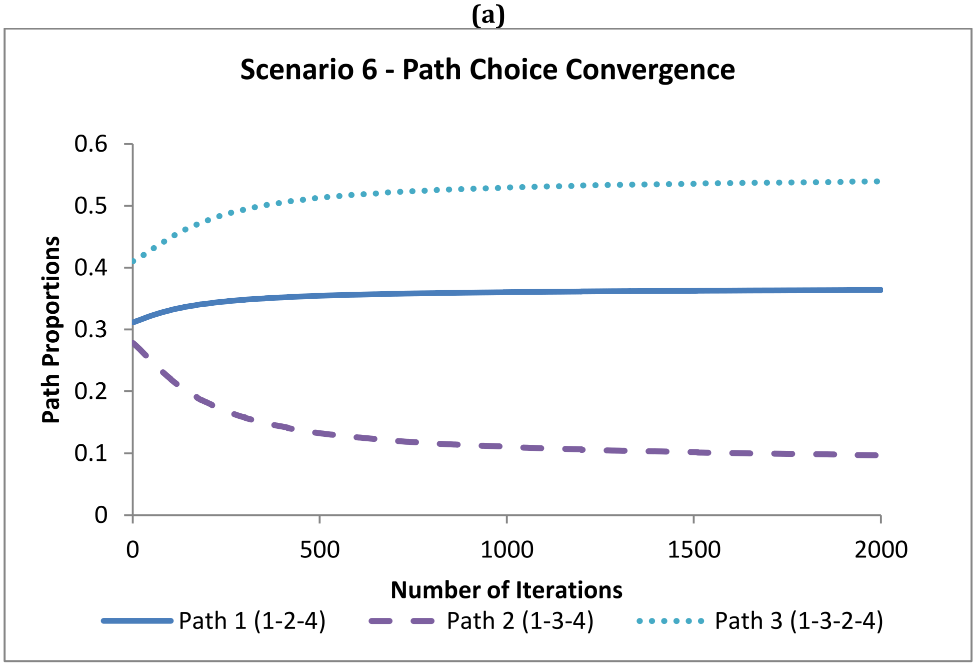
#### 5.4.2.1 Path choice convergence: sensitivity to initial perceived demand distribution

The convergence of the path proportions over the learning period is illustrated in Figure 5-4 for four different initial perceived demand distributions. As with the system level analysis, Scenarios 5 through 8 are presented for consistency. The results illustrate that the path proportions converge to within 5% of the actual expected demand for all the scenarios (5 through to 8) within 2000 iterations. Similar to the system level analysis, the convergence rate of the path proportions is sensitive to the accuracy of the initial perception of the users. The results illustrate a clear increased rate of convergence when the initial perceived demand is closer to the actual demand.

Across all the scenarios the path proportions deviate from their initial state. Initially the path proportions for Path 1 ranges between 0.31 and 0.34, Path 2 ranges between 0.20 and 0.29 and Path 3 ranges between 0.41 and 0.46. The differences in the initial proportions are a result of the differences in the initial perceived distributions. As users learn over time the proportions across all scenarios converge to the same values. The changes in path proportions represent a considerable change in link flow over time. In particular, the flow on Path 2 has halved over the course of the learning process. These results illustrate the importance of accounting for the learning process in conjunction with new developments that may impact demand, which can have major implications in how we forecast and manage traffic throughout the network. In addition, the process can affect infrastructure planning and potentially the ranking of the suitability of infrastructure projects.




**(a)**

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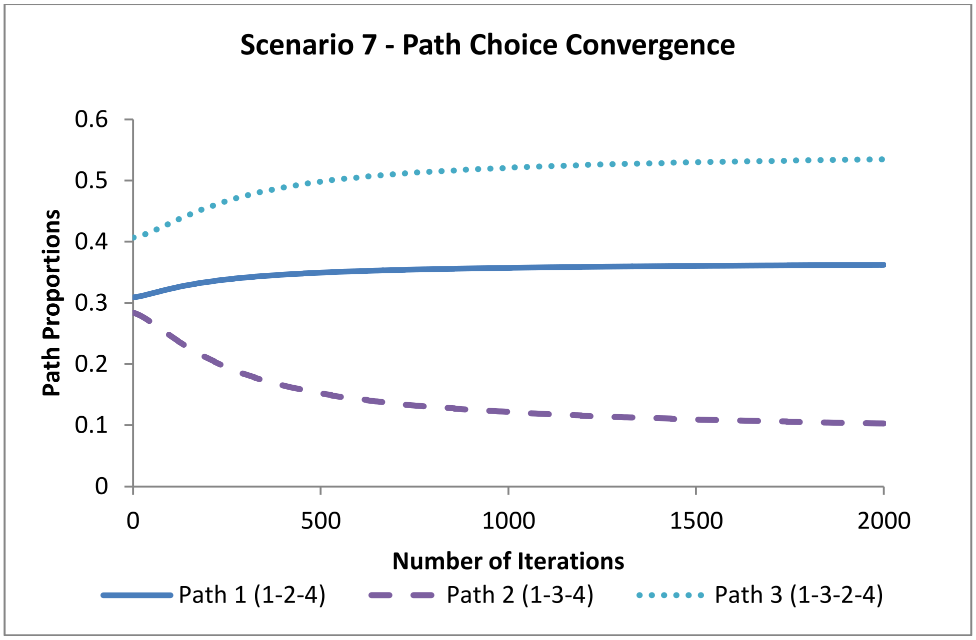


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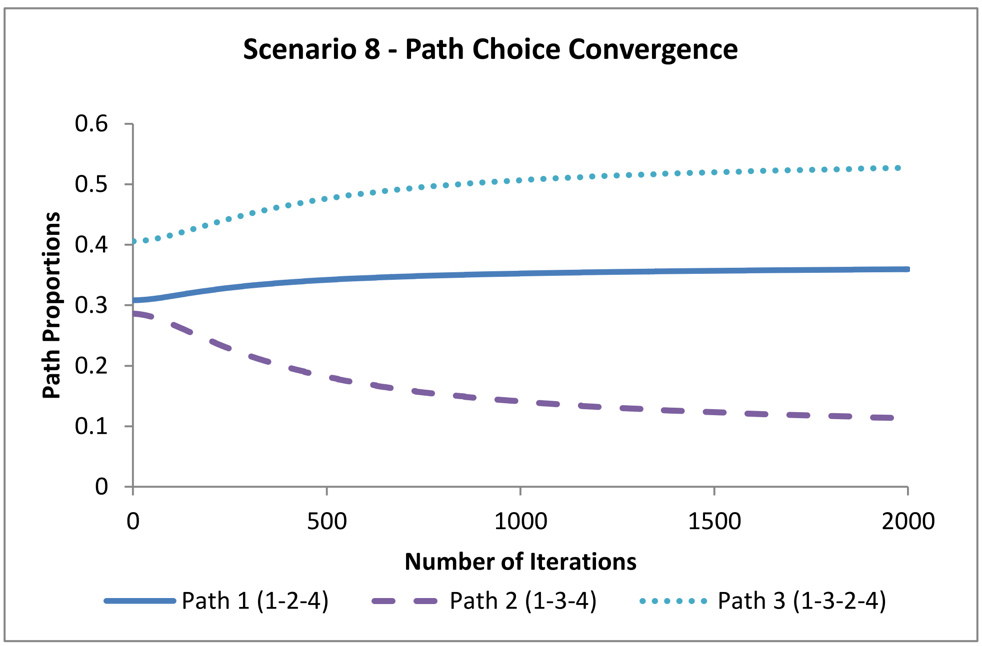
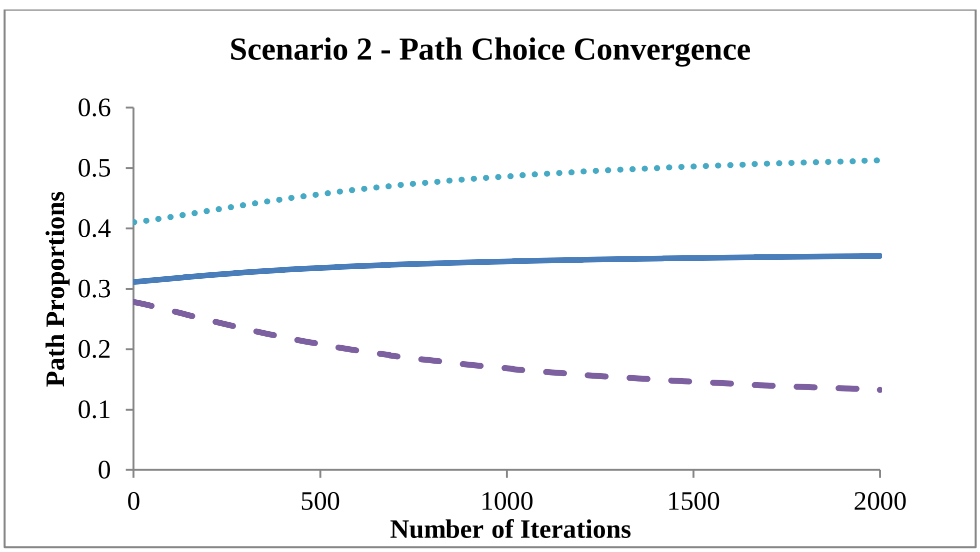



**(d)**

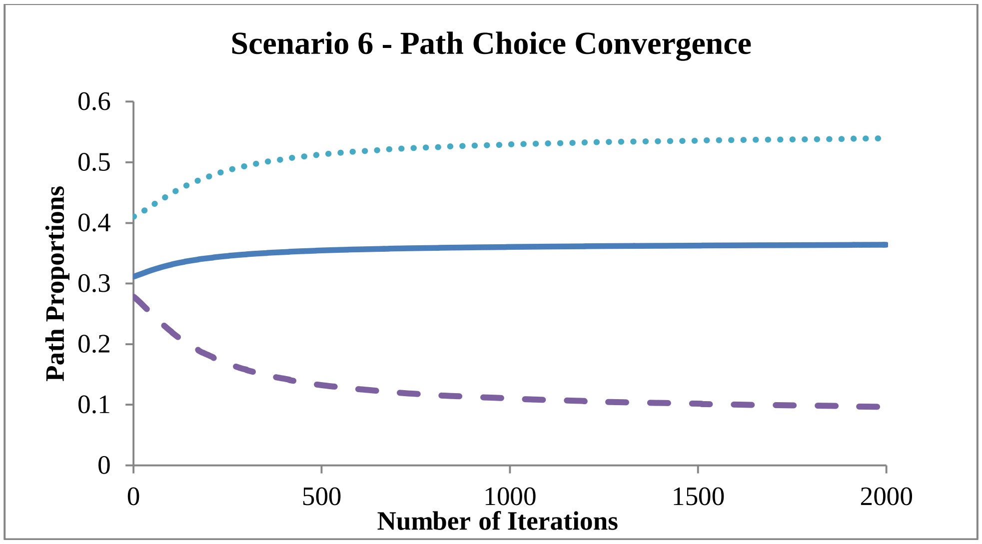
Figure 5-4 Impact of Perceived Demand Distribution on Path Choice for four scenarios a) Scenario 5 b) Scenario 6 c) Scenario 7 d) Scenario 8, corresponding to different initial perceived demand distributions.

#### 5.4.2.2 Path choice convergence: sensitivity to precision variance

The sensitivity of the precision variance on path choice is illustrated in Figure 5-5. As with the previous sensitivity analysis, Scenario 2, 6, and 10 have been presented, which correspond to a precision variance of 0.1, 0.2 and 0.3, respectively, and a fixed initial perceived demand. The results illustrate the same trends as what was observed in Figure 5-3. The rate of path choice convergence is significantly affected by the precision variance, with a lower variance corresponding to a slower rate of familiarity or degree of stubbornness of users of the network. Accordingly, further investigation is required to calibrate the true value of the precision variance for a given user group and network, and will be addressed in future work using controlled behavioural experimental procedures.




**(a)**

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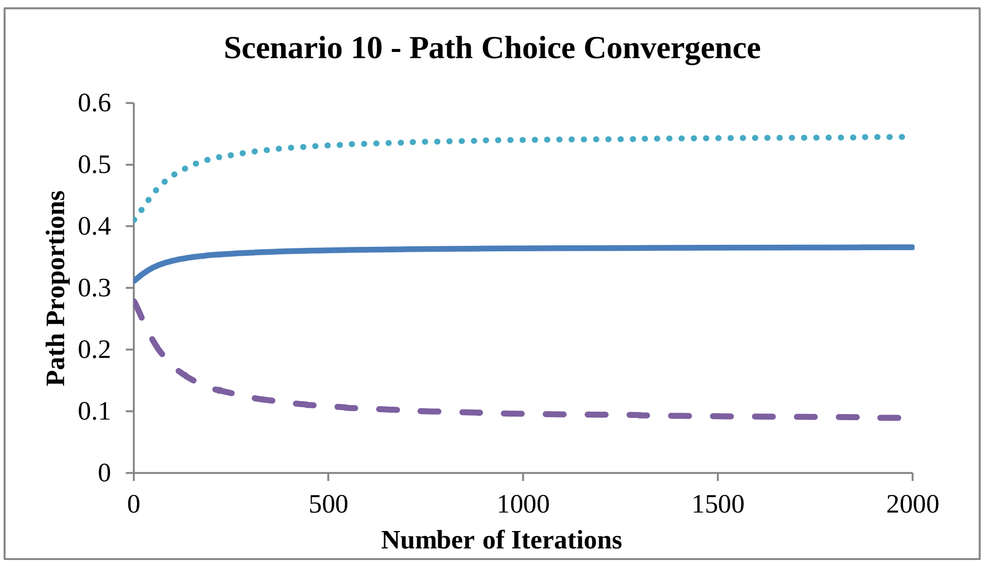



**(c)**

Figure 5-5 Impact of Precision Variance on Path Choice. The three figures graphs correspond to a variance precision of (a) 0.1, (b) 0.2, (c) 0.3.

#### 5.4.2.3 Convergence of perceived expected path travel time

Finally, we explore the changes in the perceived expected path travel times by the users over the course of the learning process. The results are depicted in

Figure 5-6. The perceived travel times provide the basis for the users' route choice decisions. Thus, evaluating how these costs change throughout the learning period can provide insight into users' expected route choice. In Figure 5-6 the expected path travel times for all three paths are shown to overlap. This is consistent with the definition of the P-STRUE model, for which a Wardrop's Equilibrium solution is based on the expected path costs, and in the case of L_STRUE, the perceived expected path costs. The figure also illustrates that perceived expected path travel times are initially much greater than they are under the actual demand distribution. The results also illustrate a quick convergence to the correct distribution. For Scenario 6, the perceived expected travel times converges in around 100 iterations, to 2.84. This time period is consistent with Figure 5-5(b), in which the path proportions stabilize after the same number of iterations. The results from this type of path level analysis can be used to reveal how quickly the impact of a new development is learnt by users.

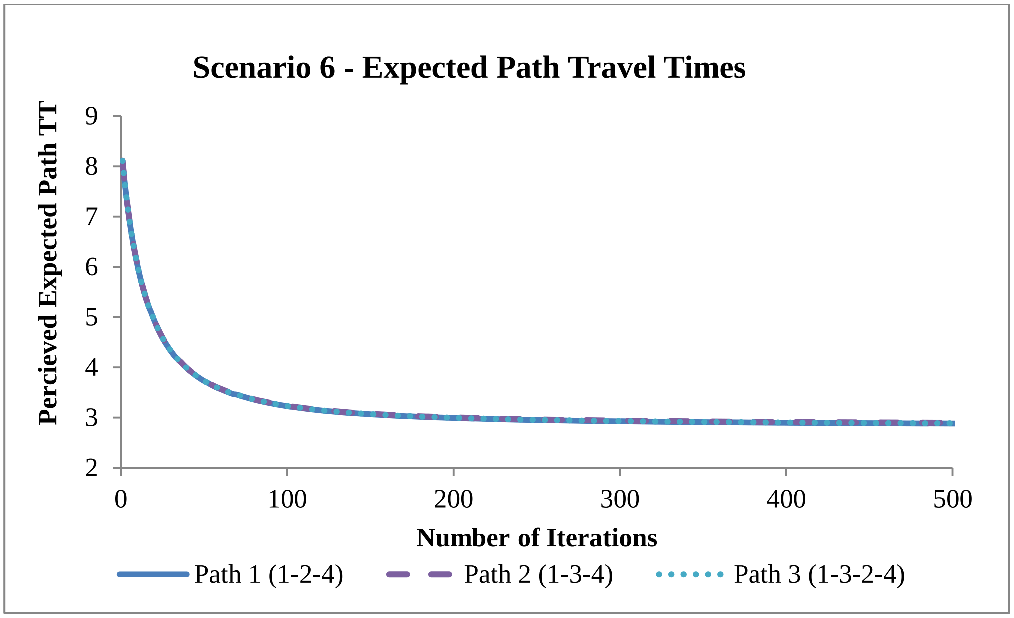



Figure 5-6 Illustration of Convergence of Perceived Expected Path Travel Times

## 5.5 Chapter summary

This study proposes a methodological framework that can be used to model the day-to-day learning process of road users after a new development or infrastructure project is in place. The contribution lies in two main ways: *(i)* the application of focus here is the impact of specific new developments on route choice and the immediate adjustment period, versus normal day-to-day conditions; and *(ii)* the Bayesian Inference model is employed to model the learning process within the P-STRUE assignment model, which is implemented to compute the underlying traffic assignment pattern each day. Numerical analysis is conducted to investigate the sensitivity of the learning process with respect to two main factors: how accurate drivers' initial perception is relative to the actual travel demand distribution; and the impact of the drivers' confidence in their initial perception.

Results illustrate that drivers learned the true demand distribution for all scenarios evaluated. The learning period is shown to be highly dependent on the precision variance, namely the drivers' level of confidence in their initial perception. The lowest precision variance, corresponding to a higher confidence level, resulted in a considerably slower convergence process, and longer learning period. In contrast, higher precision variances, representing "new road users" or an "unfamiliar road users", corresponded to much shorter learning periods. Similar trends are evident at the path level and system level. The results from this type of analysis can be used to reveal how quickly users learn the true impact of a new development, and provide insight into users' expected route choice throughout the assessment period.

Future research will address the development of L-STRUE as well as the application into different transport contexts. There is considerable scope to further develop the L-STRUE model. The assumption regarding the equality of the location parameter of the actual demand and perceived demand can be relaxed and different conjugate priors, such as the normal distribution, can be applied to observe any differences in behaviour. Also, due to that the network analysed is a tiny artificial network, it is worth investigating the performance of L_STRUE on a larger-scale network, to evaluate the impact of learning process, as the number of alternative paths for each O-D pair, the learning behaviour of each user class and the travel time reliability will be far more complicated in the network presented. Furthermore, controlled behavioural experimental procedures need to be conducted to understand the true value of the precision variance for a given user group and then used as a calibration tool for the model. These developments will enhance the modelling and understanding of the cognitive learning processes that a user makes whilst travelling.

A learning model such as the L_STRUE has a number of extensions in addition to the assessment of changes to infrastructure and the urban environment within a transportation context. The L_STRUE can be further developed to understand the impact of major disruptions and disasters to a network. The removal of an area of a network will affect the actual demand distribution and perceived demand distribution of the users, resulting in a learning process. Another key area where an adaptation of the L_STRUE can be applied is within public transit modelling. The impact of learning within a road network can affect the performance and reliability of bus systems, and consequently impact the way we value the implementation of these systems. These and further applications will be considered as future research efforts.

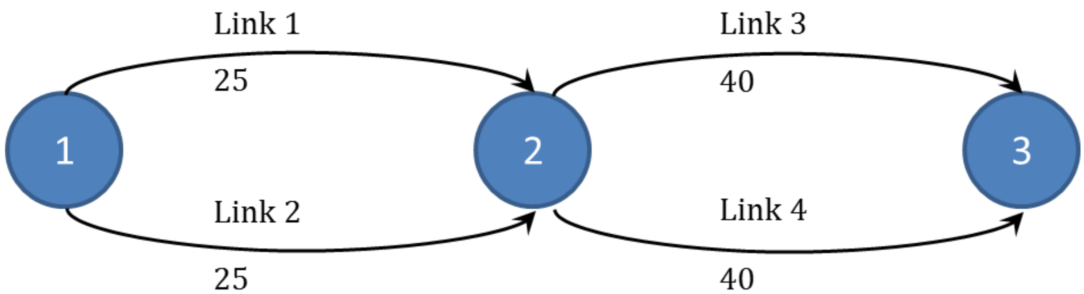
# Chapter 6

# Two methods to calibrate the total travel demand and variability for a regional transport network

## 6.1 Background

### 6.1.1 The research motivation

The development of a country brings changes in the land-use and economic state of affairs, and thus the number of trips could vary accordingly. It is therefore important to identify the frequency of trips between different centroids in a network for policy makers and transport planners. A simple example below illustrates why an O-D matrix is necessary for the transportation planning process.




| O-D pair | Number of trips |
|---|---|
| 1-2 | 20 |
| 1-3 | 30 |
| 2-3 | 50 |

| O-D pair | Number of trips |
|---|---|
| 1-2 | 50 |
| 1-3 | 0 |
| 2-3 | 80 |

| O-D pair | Number of trips |
|---|---|
| 1-2 | 30 |
| 1-3 | 20 |
| 2-3 | 60 |

Figure 6-1 A simple example

In Figure 6-1, assume people will evenly split themselves among all possible routes. The number below each link indicates the number of people on that link. Under the assumption all three O-D matrices can provide the same set of link flows specified in the figure. However, they differ from each other significantly. Now, say an expressway will be added which directly connects centroid 1 and 3, but what if the second matrix in the figure is the true one? In this case, the express way would just be a waste of money. Therefore, predicting/obtaining a reliable O-D matrix is fundamental and vital for transportation planning.

Traditionally, the O-D matrix is obtained from plate surveys, household surveys or roadside surveys (Castillo et al., 2014b, Cremer and Keller, 1987). Such survey activities may suffer from limited response, financial constraint or sampling coverage (Nihan and Davis, 1987). Additionally, by the time the survey data is collected and processed, the O-D data obtained becomes obsolete. Nowadays, the dissemination and application of some modern technologies such as inductive loop detectors, cellular phones and automatic license plate recognition systems have provided efficient ways to collect up-to-date traffic count data, and as an alternative to the traditional approaches, enhanced origin-destination (O-D) matrix calibration (or estimation) methodologies could prove useful for transportation planning.

A statistical approach is used for estimating or calibrating an O-D matrix from observed traffic counts. Normally, some prior knowledge of the O-D demand is needed. In the past, many models have been proposed and widely applied for O-D matrix calibration. The accuracy of these calibrated matrices depends on the calibration model used, the input data errors and availability of link count data. On comparing the traditional approaches, O-D matrix calibration is much more cost-efficient and time-efficient, besides, due to the fact that traffic counts are collected automatically every day, the calibrated results are always up-to-date. However, it is difficult to infer a unique O-D matrix directly using these approaches because the number of O-D pairs is much larger than the number of links, and users' routing mechanism is unknown; therefore, some assumptions or some prior information of O-D matrix is necessary to guarantee a unique solution.

Unlike the traditional O-D estimation, this chapter proposes a complimentary approach to calibrate the aggregated O-D demand (total demand) and corresponding variability for a regional traffic network using day-to-day observed link counts. The proposed method explicitly considers the inherent uncertainty in demand which is observed in traffic networks and applies P-STRUE to obtain users' route choice for calibration. As such, it is assumed that each O-D pair demand is perfectly-correlated and varies in line with the total demand. For the case of independently-distributed O-D demands, another model is proposed in Chapter 7. In the next section the motivation and background of O-D demand estimation are demonstrated in detail.

### 6.1.2 Widely-used O-D matrix estimation methodologies

From the past studies, O-D matrix estimation can be categorised as gravity model (Högberg, 1976, Robillard, 1975, Willumsen, 1981), growth factor model (Appiah, 2009, Ortúzar and Willumsen, 2002, Bierlaire, 1997) and traffic count data-based O-D matrix estimation (Bera and Rao, 2011). Growth factor model calibrates prior O-D matrix using information on the observed or projected growth rate. The gravity model is based on Newton's gravitational law, and the travel cost represents the 'impedance' between two centroids. The traffic count data-based model seeks for an O-D matrix that can reproduce a set of link flows that is as close to the traffic count data as possible. The main strengths and drawbacks of these three models are summarized in the table below.

Table 6-1 A summary of commonly-used O-D estimation models.

| Model name | Strengths | Limitations |
|---|---|---|
| **Growth factor model** | Easy to implement in a spreadsheet; and considers the inflation of demand. | Information on travel cost is not utilised; and any O-D pair with zero demand will remain so in the updated matrix. |
| **Gravity model** | Information on travel cost is included; and implementation and calibration are easy. | Survey data is required for calibrating the parameters; and it cannot handle with accuracy external-external trips (Willumsen, 1981). |
| **Traffic count data-based model** | Prior O-D matrix may be helpful but not necessary; and model input reflects up-to-date traffic pattern; Cost-efficient. | Data availability; and traffic count data coverage. |

### 6.1.3 Traffic count data-based O-D estimation problem

In this chapter, we will focus on the traffic count data-based O-D matrix estimation problem, which mainly relies on statistical approaches using traffic count data. However, the problem is often challenging due to that the number of observable links in a traffic network is often much smaller than the number of O-D pairs to be estimated. Therefore, it may not be possible to obtain a unique solution from a single set of link counts alone. As a consequence, various forms of additional assumptions and a priori knowledge are required to obtain a unique solution.

A wide range of methods have been proposed according to their assumptions, which include generalized least square method (Cascetta, 1984, Bell, 1991), the maximum likelihood method (Spiess, 1987), bi-level programming approach (Yang et al., 1992, Codina et al., 2002, Yang, 1995, Kim et al., 2001), Bayesian approaches (Tebaldi and West, 1998, Maher, 1983, Dey and Fricker, 1994, Li, 2012, Hazelton, 2001) and maximum entropy (Fisk, 1988, van Zuylen and Branston, 1982, Van Zuylen and Willumsen, 1980). Integration of the methods mentioned above was also of recent interest (Castillo et al., 2014b, Aerde et al., 2003).

Basically, the objective of traffic count data-based O-D matrix estimation is to optimize an objective function (which may vary based on model requirements) subject to a set of constraints (typically the assignment of O-D flows, such as Equation 6-3; and positiveness of O-D trips and link flows, such as Equations 6-2 and 6-4). Mathematically, the problem is to find an optimal O-D matrix $\boldsymbol{T}^* = [T_1, \dots, T_m]^`$ such that (the letter T in bold represents a matrix or vector):

$$\boldsymbol{T}^* = argmin: f_1(\boldsymbol{T}, \widetilde{\boldsymbol{T}}) + f_2(\boldsymbol{L}, \widetilde{\boldsymbol{L}}) \qquad 6\text{-}1$$

Subject to:

$$\boldsymbol{T} \geq 0 \tag{6-2}$$

$$\boldsymbol{L} = Assign(\boldsymbol{T}) \tag{6-3}$$

$$\boldsymbol{L} \geq 0 \tag{6-4}$$

Where,

$\boldsymbol{T}$ − The target O-D matrix to be estimated.

$\boldsymbol{T}^*$ − The optimal/estimated O-D matrix.

$\widetilde{\boldsymbol{T}}$ − A prior O-D matrix.

$Assign(\ )$ −The assignment of O-D trips to links.

$f_1(\ ), f_2(\ )$–The measurement function.

$\boldsymbol{L}$ − A vector of link flow produced by the target O-D matrix.

$\widetilde{\boldsymbol{L}}$ − A vector of observed link flow, normally obtained from traffic count data.

As aforementioned, additional assumptions are required to ensure solution uniqueness, which is reflected in the measurement functions $f_1(\ ), f_2(\ )$. In this chapter, a modification of the maximum likelihood method and the generalized least squares method is introduced. We will first demonstrate how the traditional maximum likelihood and generalized least squares method work.

#### 6.1.3.1 The maximum likelihood method

Statistically, the maximum likelihood method estimates a set of parameters for a probability density function to fit the observed data, i.e. it maximizes the joint probability of observing the existing data (prior O-D matrix and observed link flows) conditional on the target O-D matrix. The objective function can be expressed as:

$$T^{ML} = argmax: P(\widetilde{\boldsymbol{T}}|\boldsymbol{T}) + P(\tilde{\boldsymbol{L}}|\boldsymbol{T}) \quad 6\text{-}5$$

Subject to constraints 6-2 to 6-4.

Where,

$P(\widetilde{\boldsymbol{T}}|\boldsymbol{T})$ −The probability of observing the prior O-D matrix conditional on the target O-D matrix.

$P(\tilde{\boldsymbol{L}}|\boldsymbol{T})$ − The probability of observing the average traffic counts conditional on the target O-D matrix.

A pre-assumed distribution of the O-D matrix and traffic counts is normally required. The benefit is that multiple sets of traffic counts can be utilised instead of only using the average link flow, and additionally, a confidence interval can be given by the model output, which is useful for transportation assessment.

#### 6.1.3.2 The generalized least squares method

The generalized least squares method is widely-applied in regression models especially when correlation needs to be considered. In O-D estimation problems, it minimizes the sum of the squared residuals of O-D matrix and link flows:

$$\boldsymbol{T}^{GLS} = argmin: \left(\boldsymbol{T} - \widetilde{\boldsymbol{T}}\right)^{`} Y^{-1}\left(\boldsymbol{T} - \widetilde{\boldsymbol{T}}\right) + \left(\boldsymbol{L} - \tilde{\boldsymbol{L}}\right)^{`} Z^{-1}\left(\boldsymbol{L} - \tilde{\boldsymbol{L}}\right) \quad \text{6-6}$$

Subject to constraints 6-2 to 6-4.

Where,

$Y$ −The covariance matrix for the O-D matrix.

$Z$ −The covariance matrix for the traffic counts.

The advantage is that no distributional assumptions on the data are required, which increases the method's flexibility. Also, the method associates survey data directly with traffic count data, while considering the relative accuracy of these data (Bell, 1991). The method is also proved to be useful in exploiting the O-D matrix structure (Bierlaire and Toint, 1995).

#### 6.1.3.3 Bi-level programming approach

Initially, estimation of the O-D matrix is a once-off procedure, that is, calibration/calculation of O-D matrix is only performed once. This may lead to some issues when a network is congested, the prior information on O-D matrix is inaccurate, or data noise is not insignificant. In this approach, the upper level is an

O-D matrix estimation problem and the lower level is the assignment of O-D trips. (Yang et al., 1992) first introduced the convex bi-level optimization problem, which was later extended to account for link flow correlation (Yang, 1995). The heuristic algorithm, which is a global optimum technique, was applied to solve the upper level in (Stathopoulos and Tsekeris, 2004, Yang, 1995, Kim et al., 2001); however, the solution was not proved to be optimal mathematically. (Codina et al., 2002) proposed two algorithms under the bi-level programming framework: one sought for an approximation of the steepest descent direction for the upper level and one linearized the lower level assignment problem. In addition, an iterative column generation algorithm was demonstrated based on the characteristics of path cost function continuity, and the solution was proved to be a local minimum (Garcia-Rodenas and Verastegui-Rayo, 2008). However, the aforementioned algorithms were only applied on a small network. In this chapter the proposed bi-level programming approach is proved to be applicable to medium-scale networks such as the Anaheim network.

### 6.1.4 The research trend- how will the problem be explored?

The aforementioned statistical approaches are mainly applied in static networks. However, sometimes the impact of time should not be neglected; in this case, a time-dependent O-D matrix is required (Frederix et al., 2011, Bierlaire and Crittin, 2004, Cipriani et al., 2014, Nie and Zhang, 2008, Sherali and Park, 2001).

Basically, the dynamic O-D estimation divides O-D trips and link flows by several time intervals:

$$\boldsymbol{T} = \begin{bmatrix} T_1^{t_1} & \cdots & T_1^{t_i} \\ \vdots & \ddots & \vdots \\ T_m^{t_1} & \cdots & T_m^{t_i} \end{bmatrix} \quad 6\text{-}7$$

$$\boldsymbol{L} = \begin{bmatrix} l_1^{t_1} & \cdots & l_1^{t_i} \\ \vdots & \ddots & \vdots \\ l_n^{t_1} & \cdots & l_n^{t_i} \end{bmatrix} \quad 6\text{-}8$$

Where, $n$ is the number of links, $m$ is the number of O-D pairs and there are $i$ time intervals.

The general idea of dynamic O-D estimation is to find the O-D matrix $\boldsymbol{T}$ that can reproduce the matrix $\boldsymbol{L}$ across all time intervals (Cremer and Keller, 1987). Normally, the traffic count data at time interval $t_i$ is used to calibrate the O-D matrix at a previous time interval, namely $t_{i-1}$, and recursive methods are preferred in such a case (Cao et al., 2000). The incorporation of automatic vehicle identification data in the dynamic O-D estimation problem has also drawn much attention recently (Antoniou et al., 2004, Zhou and Mahmassani, 2006, Dixon and Rilett, 2002, Van Der Zijpp, 1997).

Additionally, the problem has been extended to account for the stochastic nature of observed flows (Lo et al., 1996, Lo et al., 1999), Some computer-aided heuristic algorithms are also applied to this problem (Stathopoulos and Tsekeris, 2004, Yang, 1995). The genetic algorithm, which is a heuristic search method, plays an important role in the O-D estimation problem (Kattan and Abdulhai, 2006, Yun and Park, 2005, Kim et al., 2001, Baek et al., 2004). The genetic algorithm mimics the process of natural selection-evaluation, selection, crossover, and mutation (Mitchell, 1998). An evaluation function is required to assess the quality of each solution (Foy et al., 1992). The main advantage of genetic algorithm is its capability of solving non-convex, complex optimization, while the drawback is that the solution is not guaranteed to be optimal.

Some other methods have also been proposed by researchers to enhance the model applicability, such as multi-class O-D estimation (Wong et al., 2005, Baek et al., 2004), fuzzy-based approach (Xu and Chan, 1993, Foulds et al., 2013, Reddy and Chakroborty, 1998) and neutral network based approach (Gong, 1998). However, issues regarding computation complexity and the application to large-scale networks still remain a challenge. The statistical O-D estimation techniques are still proved to be efficient and easy-to-implement.

On the other hand, higher order information of a network, such as the variance and covariance of observed link flows, can potentially provide more constraints to the traffic count data-based model. This is considered as network tomography problems in statistics and computer science literature (Vardi, 1996, Hazelton, 2015, Cao et al., 2000, Airoldi and Blocker, 2013), but its application in transportation models is yet to be fully explored. Cremer and Keller demonstrated that aggregating or averaging link count data collected over a sequence of time period may result in the loss of important information (Cremer and Keller, 1987). Hazelton (Hazelton, 2003) proposed a weighted least squares method to account for the covariance of links, and assumed a parameter to explain the circumstances when the variance exceeds the mean if a Poisson distribution is used. Bell (Bell, 1983) proposed a maximum likelihood method and found the analytical solution to the covariance of O-D matrix by using a Taylor approximation. However, the assumptions in these studies may limit the model applicability. For example, the O-D demand was assumed to follow the Poisson or multinomial distribution, which stipulates certain relationships between the mean and variance of the O-D demand. In monitored networks, loop detectors can easily provide link counts on a day-to-day basis; therefore, it is important to consider the variation of link flows and the distribution of O-D total demand as effective information to calibrate the O-D trip matrix. In this study, the maximum likelihood method assumes the O-D demand follows a lognormal distribution, which allows the mean and variance of total demand to be independent of each other, and assures the non-negativity of the demand. The proposed model in this chapter therefore estimates the distribution of the total O-D demand and thus significantly reduces computation complexity.

Estimation of the O-D trip matrix also requires a robust assignment model. The Logit based stochastic assignment model was incorporated into a linear programming model. Such models are called path flow estimation based models (Sherali et al., 1994, Chootinan et al., 2005, Chen et al., 2009), and these model either needs path enumeration (Sherali et al., 2003) or information on the set of shortest paths (Nie et al., 2005, Nie and Lee, 2002). However, when applying the assignment model to a large network, realism and computational complexity are both critical in determining a model's practical applicability. Further, a major complication in transportation modelling is the ability to properly account for the inherent uncertainties regarding demand (Kim et al., 2009, Bellei et al., 2006, Szeto et al., 2011) and capacity levels (Brilon et al., 2005, Wu et al., 2010). Additionally, as has been noted, uncertainty regarding these variables directly affects route choice behaviour (Uchida and Iida, 1993) and traffic predictions (Duthie et al., 2011). It is, therefore, necessary to incorporate these stochastic elements into models to ensure robust planning capabilities, but to do so in a manner that maintains computational tractability. The strategic user equilibrium (Dixit et al., 2013) effectively accounts for the impact of demand uncertainty subject to Wardrop's UE conditions, and under the static user equilibrium framework, the computation tractability and simplicity are preserved. The model was extended to the dynamic traffic assignment (Waller et al., 2013) and road pricing scheme (Duell et al., 2014).

In the real world, issues concerning the quality and availability always exist in traffic count data. Failures during transmission of data, malfunction of loop detectors, resource limitations and breakdown of storage device in traffic management centre could all lead to a loss of data (Gajewski et al., 2002). Some issues on data may be diagnosed by outlier identification techniques (Kim, 2006, Vanajakshi and Rilett, 2004). Additionally, to mitigate the issue of low coverage of loop detectors due to budget limitations, researchers have proposed the sensor location problem, which investigates the optimal location for installing loop detectors (Larsson et al., 2010, Gentili and Mirchandani, 2012). Various algorithms have been proposed to enhance the computation efficiency and accuracy (Yang et al., 2003, Bianco et al., 2001, Kim et al., 2003). Ehlert (Ehlert et al., 2006) further extends the problem to find optimal sensor location in addition to existing detectors. The sensor location problem's novelty lies in its focus on data collection side instead of model calibration side, and hence improves the fundamental input of the O-D estimation problem.

## 6.2 Chapter introduction

In this chapter we propose a complementary approach to calibrate the aggregated O-D demands (i.e. the sum of all O-D trips- the total demand) and corresponding variability for a regional traffic network using day-to-day observed link counts. The proposed method explicitly considers the inherent uncertainty in demand which is observed in traffic networks. A bi-level programming formulation is used to calibrate the expected total demand and the corresponding demand variability of traffic networks. In the bi-level formulation the upper-level is either:

1) The modified maximum likelihood method for O-D matrix estimation with P-STRUE which is hereby referred to as the MLOD method. The MLOD method has the ability to utilize information from day-to-day observed link flows.

2) The modified least squares method for O-D matrix estimation with P-STRUE is hereby referred to as LSOD. The LSOD method is capable of capturing link flow variations

The novelty of both methods evolves from the incorporation of the P-STRUE model for traffic assignment (Dixit et al., 2013) in the lower level of the bi-level programming approach. The P-STRUE model was proposed to capture the impact of day to day demand volatility on network performance, and eventually route choice. In terms of the applicability to the proposed O-D estimation model, the P-STRUE model can take the total demand distribution as input (estimated at the upper level), and output a set of link flow distributions which can then be compared to the link level observations.

As demonstrated in Section 2.3, in the P-STRUE model, each O-D pair demand is assumed to be a fixed proportion of the total demand in the system; and hence each O-D pair demand varies according to the change in total demand. Therefore, the objective of this work is to estimate the total demand, and perhaps more importantly, a fundamental parameter which captures the variance in the total trip distribution. Given the estimated demand distribution, the proposed method also provides the variance of link flows (from P-STRUE), which can be used as a measurement of reliability for planning purposes.

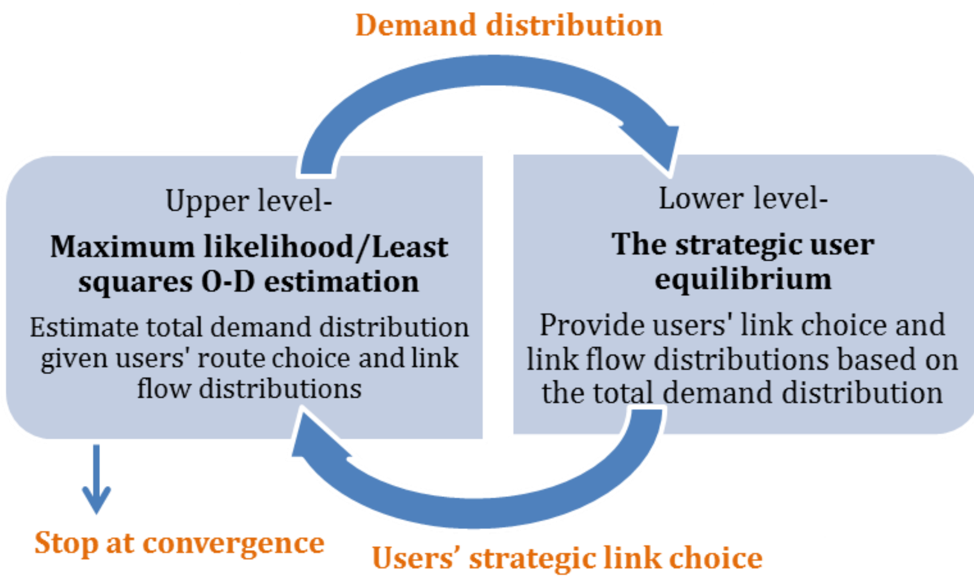



Figure 6-2 The Bi-level programming method

The bi-level programming method is proposed to eliminate the impact of strongly biased prior estimates, as demonstrated in Figure 6-2, while the upper level provides information about the total demand distribution to the lower level P-STRUE model, and the results from the P-STRUE model can provide link flow distributions back to the upper level. A benefit of the proposed model includes the incorporation of observed day-to-day link flows, instead of aggregated or averaged link flows. Additionally, the performance of both MLOD and LSOD methods can be assessed based on the accuracy of their estimations for both expected link flows and link flow distributions, which are a direct output of the P-STRUE model.

The association of link flow variables to the total demand in P-STRUE allows for the use of day-to-day link flows (which in return provide actual link flow distributions) to calibrate the total demand distribution. The calibration is accomplished by implementing the following methods: 1) MLOD method, in which we maximize the joint probability of observing all sets of link flows within a time period; and 2) LSOD method, in which we minimize the sum of the residuals of mean and standard deviation of link flows. The main difference between the two methods is that the MLOD method considers every observation of link flow, and seeks to find a distribution that fits the observed link flow best, while the LSOD method uses only the mean and standard deviation of link flow as the inputs. When data sets are typically small or moderate in size, extensive simulation studies show that in small sample designs where there are only a few failures, the maximum likelihood estimation is better than the least squares method (Genschel and Meeker, 2010, Maus et al., 2001). Meanwhile, the method of least squares is a standard approach to approximate the solution of over-determined systems, i.e. sets of equations in which there are more equations than unknowns, such as linear regression. Also, no distributional assumptions need to be made in least squares method.

An additional contribution of this research addresses the issue of data availability and quality, which can lead to error in observed link flows (Zhou and List, 2010). Sensitivity tests of model estimation, such as the error bound measures, the travel demand scale measure (Bierlaire, 2002) and maximum possible relative error (Yang et al., 1991) have been proved useful due to these unavoidable errors in traffic data. In this chapter, sensitivity analysis is conducted to demonstrate the model's robustness against varying levels of detector error. In addition, the sensitivity to error in observed link flows is analytically derived for the proposed model, and the results are validated using simulation.

In this chapter, we have reviewed the relevant literature in Section 6.1. Section 6.3 outlines the contribution of this chapter. Section 6.4 defines the mathematical model and includes a derivation for the analytical solution to the total demand estimation. The mathematical proof demonstrates the convexity of the model, and the sensitivity to the prediction error is analytically derived. Numerical analysis is demonstrated in Section 6.5, where the estimated results of both methods are compared to illustrate the efficiency and sensitivity of the proposed model; while conclusions, limitations of the model and future research are discussed in Section 6.6.

## 6.3 Chapter contribution

The contribution of this chapter can be summarized as follows:

1) We apply the strategic user equilibrium model to account for the impact of demand volatility on users' route choice. The total demand is considered as a stochastic variable and is assumed to follow a certain distribution. In return the total demand volatility causes link flow variation.

2) Given the day-to-day link flow fluctuations we estimate the total network travel demand distribution using two different methods: i) MLOD method and ii)

LSOD method. The performance of both methods is evaluated and compared on a virtual network and a medium-sized network.

3) A Bi-level formulation is proposed which reduces the impact of initial biased estimates. Both upper level and lower level are proven to be strictly convex.

4) O-D estimation results from the proposed methods are presented and compared for both analytical and simulated analysis.

5) Sensitivity analysis is conducted to test the model's robustness against erroneous input data.

The next section presents the mathematical formulation of the model.

## 6.4 Problem formulation

This section defines the mathematical concept of the model. Table 6-2 defines the notations used in this chapter in addition to Table 2-2.

Table 6-2. Summary of notations.

| *Symbol* | *Definition* |
|---|---|
| $x_{ni}$ | Observed flow on link *n*, for day *i*. |
| $l_n$ | Flow variable on link *n*. |
| $\mu$ | The location parameter for a lognormal distribution, which is also mean of the corresponding normal distribution. |

| | |
|---|---|
| $\sigma^2$ | The scale parameter for a lognormal distribution, which is also the variance of the corresponding normal distribution. |
| $m_T$ | The expected total demand- $E(T)$. |
| $v_T$ | The variance of the total demand- $Var(T)$. |
| $m_n$ | The expected link flow on link *n*- $E(l_n)$. |
| $v_n$ | The variance of link flow on link *n*- $Var(l_n)$. |
| $s_T$ | The standard deviation of the total demand- $Std(T)$. |
| $s_n$ | The standard deviation of flow on link *n*- $Std(l_n)$. |
| $P$ | The set of links with error. |
| $Q$ | The set of links without error. |
| $e_{ni}$ | The error of loop count on link *n*, day *i*. |
| $R_{()}$ | The estimation of a parameter when measurement error is considered. |
| $CV$ | The coefficient of variation, defined as the ratio of the standard deviation to the mean of a variable. |

In this chapter, two assumptions are made to guarantee consistency, uniqueness and computation simplicity:

(1) Each O-D pair demand is proportional to the total demand and the demand proportions are fixed. That is, the objective of this model is to scale each O-D demand distribution while preserving the demand proportions. Statistically, this means we have more confidence in the demand proportions but not the exact number of trips. Similar to the traditional O-D estimation approaches, the demand proportions can be regarded as prior information which is obtained from other transportation techniques such as gravity model and census data. These techniques can provide relatively satisfactory demand proportions, and the use of loop detector data in this model provides a way to calibrate the number of trips.

(2) The trip demand is assumed to follow a certain statistical distribution where previously a lognormal distribution has been used (Wen et al., 2014, Duell et al., 2014). Under the assumption of a log-normally distributed demand, this chapter focuses on estimating the demand distribution parameters. Note that other distributions can also be used if they do not change the convexity of the objective function, and preserve the positiveness. In addition, for the P-STRUE model a log-normal distributed total trip demand allows for a closed form expression of the probability density function, which helps to construct the model analytically.

### *6.4.1* The strategic user equilibrium assignment model for perfectly correlated O-D demands (P-STRUE)

The P-STRUE model is used as the assignment model for the O-D estimation, and details of the P-STRUE model can be found in (Dixit et al., 2013) and in section 2.3.1. The link travel time function is assumed to be the BPR function by the U.S. Bureau of Public Roads (U.S, 1964), as defined in Equation 3-1, due to its widespread use in transport planning models. The fraction of the total demand between O-D pair $m$, namely $q_m$ , can be obtained from the prior estimates, i.e. census data, gravity model. The total demand is assigned to each link by the link proportions:

$$l_n = f_n T \qquad n \in N \tag{6-9}$$

The link proportions can be obtained from the P-STRUE assignment model; therefore; it is known in the O-D matrix estimation problem, from equation 6-9, that each link flow is the link proportions multiplied by the total demand variable, according to the properties of lognormal distribution, if the total demand variable follows a lognormal distribution, then the link flow represented in the above equation also follows a lognormal distribution, which is related to the total demand distribution by the following equations:

$$m_n = f_n m_T \tag{6-10}$$

$$v_n = f_n^2 v_T \tag{6-11}$$

It is worth noting that the proof above is valid under the assumption that users will stick to their strategic link choice. From the above equations we can see the link flow will vary according to the demand volatility. In next section, a maximum likelihood method will be proposed to estimate the total demand and its variation.

### *6.4.2* The maximum likelihood O-D estimation incorporating the strategic user equilibrium (MLOD)

In MLOD, we firstly find the parametric expression of the probability density function of link flow distribution. The parameters for the link flow distribution can be obtained by the definition of lognormal distribution:

$$\mu_n = ln\, m_n - \frac{1}{2} ln(1 + \frac{v_n}{m_n^2}) \tag{6-12}$$

$$\sigma_n^2 = ln(1 + \frac{v_n}{m_n^2}) \tag{6-13}$$

Substitute 6-12 and 6-13 into Equations 6-10 and 6-11, we have the transformation of the total demand distribution to link flow distribution:

$$\sigma_n = \sigma_T \tag{6-14}$$

$$\mu_n = ln f_n + \mu_T \tag{6-15}$$

It is important to realize that $\mu$ and $\sigma^2$, which appear in the equations of the lognormal distribution, do not denote the mean and the variance of the lognormal distribution, but of the corresponding parameters of the normal distribution. The mean and the variance of the lognormal distribution are indicated in the following discussion by $m$ and $v$. Since each link flow follows a lognormal distribution, the probability of observing $x_n$ trips on link *n* is:

$$P(x_n) = \frac{1}{x_n \sigma_n \sqrt{2\pi}} e^{-\frac{(ln\, x_n - \mu_n)^2}{2\sigma_n^2}} \quad n \in N \qquad 6\text{-}16$$

Where, $x_n$ is the observed flow on link *n.*

The joint probability of observing a set of link flows can be obtained by the product of the probability density functions:

$$P([x_n]) = \prod_{1}^{n} \frac{1}{x_n \sigma_n \sqrt{2\pi}} e^{-\frac{(ln\, x_n - \mu_n)^2}{2\sigma_n^2}} \quad n \in N \qquad 6\text{-}17$$

Furthermore, we may collect more than one set of loop counts, namely the observed day-to-day link flows. It is therefore necessary to maximize the joint probability of observing all sets of link flows, in order to reduce the effect of noise and observation failure. Here the observed link flows are indicated by an *n*-by-*i* matrix, *n* is the number of links and *i* is the number of observations:

$$[x_{ni}] = \begin{bmatrix} x_{11} & \cdots & x_{1i} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{ni} \end{bmatrix} \qquad 6\text{-}18$$

The maximum likelihood method here is to maximize the joint probability of observing all sets of link flows, which is given by the following equation:

$$P([x_{ni}]) = \prod_{1}^{i}\prod_{1}^{n}\frac{1}{x_{ni}\sigma_n\sqrt{2\pi}}e^{-\frac{(\ln x_{ni}-\mu_n)^2}{2\sigma_n^2}} \qquad 6\text{-}19$$

Conventionally, we maximize the logarithm of the joint probability because taking logarithm of the function will not change its convexity. By plugging in Equations 6-14 and 6-15 into Equation 6-19, the objective function becomes:

$$max\text{: } \log[P([x_{ni}])] = \log(\prod_{1}^{i}\prod_{1}^{n}\frac{1}{x_{ni}\sigma_n\sqrt{2\pi}}e^{-\frac{(\ln x_{ni}-\mu_n)^2}{2\sigma_n^2}}) \qquad 6\text{-}20$$

Change signs and it then becomes a minimization problem:

$$min: \ z_1(\mu_T, \sigma_T) = \sum_{1}^{i}\sum_{1}^{n} ln(\,x_{ni}\sigma_T\sqrt{2\pi}) + \frac{(ln\frac{x_{ni}}{f_n} - \mu_T)^2}{2\sigma_T{}^2} \qquad 6\text{-}21$$

Subject to: $\sigma_T > 0$

To prove the convexity of the objective function, we only need to show that for an arbitrary $[x_{ni}]$, the function below is convex:

$$z_1(\mu_T, \sigma_T) = \sum_{1}^{n} ln(\,x_{ni}\sigma_T\sqrt{2\pi}) + \frac{(ln\, x_{ni} - \mu_T)^2}{2\sigma_T{}^2} \qquad 6\text{-}22$$

The Hessian matrix of $z_1(\mu_T, \sigma_T)$ can be found by taking second partial derivatives with respect to $\mu_T$ and $\sigma_T$:

$$H = \begin{bmatrix} {\sigma_T}^{-2} & 2\sigma_T^{-3}(lnx_{ni} - \mu_T) \\ 2\sigma_T^{-3}(lnx_{ni} - \mu_T) & \sigma_T^{-2} + 3\sigma_T^{-4}(lnx_{ni} - \mu_T)^2 \end{bmatrix} > 0 \quad 6\text{-}23$$

The Hessian matrix is positive definite, and hence the function is strictly convex. The sum of several convex functions is also a convex function; therefore, we have proved that our objective function is strictly convex, which has a global optimal solution. The optimal solutions can be found by taking the first derivative with respect to mean and variance of the total demand:

$$\mu_T = \frac{\sum_1^i \sum_1^n ln\frac{x_{ni}}{f_n}}{ni} \quad 6\text{-}24$$

$$\sigma_T^2 = \frac{\sum_1^i \sum_1^n (ln\frac{x_{ni}}{f_n} - \mu_T)^2}{ni} \quad 6\text{-}25$$

Sensitivity is a measurement of a model's robustness, in the proposed model, observed loop counts may have noise due to loop detector failure, measurement error and so forth, and such errors can be expressed as a matrix that has the same dimension as the observed loop counts:

$$[e_{ni}] = \begin{bmatrix} e_{11} & \cdots & e_{1i} \\ \vdots & \ddots & \vdots \\ e_{n1} & \cdots & e_{ni} \end{bmatrix} \quad 6\text{-}26$$

Therefore, the actual loop count matrix which includes error term is:

$$[r_{ni}] = [x_{ni}] + [e_{ni}] = \begin{bmatrix} x_{11} + e_{11} & \cdots & x_{1i} + e_{1i} \\ \vdots & \ddots & \vdots \\ x_{n1} + e_{n1} & \cdots & x_{ni} + e_{ni} \end{bmatrix} \quad 6\text{-}27$$

The corresponding estimated $\mu_T$ and $\sigma_T^2$ based on observed loop counts with error term are represented as $R\mu_T$ and $R\sigma_T^2$:

$$R\mu_T = \frac{\sum_1^i \sum_1^n ln\frac{x_{ni} + e_{ni}}{f_n}}{ni} \quad \text{6-28}$$

$$R\sigma_T^2 = \frac{\sum_1^i \sum_1^n (ln\frac{x_{ni} + e_{ni}}{f_n} - R\mu_T)^2}{ni} \quad \text{6-29}$$

From the definition of lognormal distribution, the corresponding expected total demand without and with error term are expressed as $m_T$ and $Rm_T$ respectively:

$$m_T = e^{\mu_T + 0.5\sigma_T^2} = e^{\frac{\sum_1^i \sum_1^n ln\frac{x_{ni}}{f_n} + 0.5\sum_1^i \sum_1^n (ln\frac{x_{ni}}{f_n} - \mu_T)^2}{ni}} \quad \text{6-30}$$

$$Rm_T = e^{R\mu_T + 0.5R\sigma_T^2} = e^{\frac{\sum_1^i \sum_1^n ln\frac{x_{ni}+e_{ni}}{f_n} + 0.5\sum_1^i \sum_1^n (ln\frac{x_{ni}+e_{ni}}{f_n} - R\mu_T)^2}{ni}} \quad \text{6-31}$$

Therefore, the sensitivity function of the expected total demand is:

$$Rm_T - m_T = e^{\frac{\sum_1^i \sum_1^n ln\frac{x_{ni}+e_{ni}}{f_n} + 0.5\sum_1^i \sum_1^n \left(ln\frac{x_{ni}+e_{ni}}{f_n} - R\mu_T\right)^2}{ni}} - e^{\frac{\sum_1^i \sum_1^n ln\frac{x_{ni}}{f_n} + 0.5\sum_1^i \sum_1^n (ln\frac{x_{ni}}{f_n} - \mu_T)^2}{ni}} \quad \text{6-32}$$

Also, the analytical solution of the standard deviation of the total demand without and with error term can be expressed as $s_T$ and $Rs_T$ respectively:

$$s_T = e^{\mu_T + 0.5\sigma_T^2}\sqrt{e^{\sigma_T^2} - 1}$$

$$= e^{\frac{\sum_1^i \sum_1^n ln\frac{x_{ni}}{f_n} + 0.5\sum_1^i \sum_1^n (ln\frac{x_{ni}}{f_n} - \mu_T)^2}{ni}} * \sqrt{e^{\frac{\sum_1^i \sum_1^n (ln\frac{x_{ni}}{f_n} - \mu_T)^2}{ni}} - 1} \quad 6\text{-}33$$

$$Rs_T = e^{R\mu_T + 0.5R\sigma_T^2}\sqrt{e^{R\sigma_T^2} - 1}$$

$$= e^{\frac{\sum_1^i \sum_1^n ln\frac{x_{ni}+e_{ni}}{f_n} + 0.5\sum_1^i \sum_1^n \left(ln\frac{x_{ni}+e_{ni}}{f_n} - R\mu_T\right)^2}{ni}} * \sqrt{e^{\frac{\sum_1^i \sum_1^n (ln\frac{x_{ni}+e_{ni}}{f_n} - R\mu_T)^2}{ni}} - 1} \quad 6\text{-}34$$

Therefore, sensitivity function of the standard deviation of total demand is:

$$Rs_T - s_T =$$

$$\left[e^{\frac{\sum_1^i \sum_1^n ln\frac{x_{ni}+e_{ni}}{fn} + 0.5\sum_1^i \sum_1^n \left(ln\frac{x_{ni}+e_{ni}}{fn} - \frac{\sum_1^i \sum_1^n ln\frac{x_{ni}+e_{ni}}{fn}}{ni}\right)^2}{ni}} * \right.$$

$$\left.\sqrt{e^{\frac{\sum_1^i \sum_1^n \left(ln\frac{x_{ni}+e_{ni}}{fn} - \frac{\sum_1^i \sum_1^n ln\frac{x_{ni}+e_{ni}}{fn}}{ni}\right)^2}{ni}} - 1}\right]$$

$$-\left[e^{\frac{\sum_1^i \sum_1^n \ln\frac{x_{ni}}{f_n} + 0.5\sum_1^i \sum_1^n (\ln\frac{x_{ni}}{f_n} - \mu_T)^2}{ni}} * \sqrt{e^{\frac{\sum_1^i \sum_1^n (\ln\frac{x_{ni}}{f_n} - \mu_T)^2}{ni}} - 1}\right] \quad 6\text{-}35$$

The analytical expression of the sensitivity function indicates some characteristics of the estimated results if we design the sensitivity analysis as following:

(1) The specific error- the proportion of the error term $e_{ni}$ to the actual flow $x_{ni}$ on link $n$:

$$e_{ni} = kx_{ni}, \qquad n \in N \tag{6-36}$$

(2) The systematic error- a number of links that are under a specific error:

$$e_{pi} = kx_{pi}, \qquad p \in P, P \cup Q = N \tag{6-37}$$

$$e_{qi} = 0, \qquad q \in Q, P \cup Q = N \tag{6-38}$$

Where, $P$ is the set of links with error and $Q$ is the set of links without error. The expected total demand is monotonically increasing with respect to the systematic error and specific error, but the estimated standard deviation of total demand is not monotonic with respect to the systematic error because from the analytical expression in Equation 6-35 we can see it is determined by several factors including link proportion, link flow and the specific error. The combined effect of these factors does not satisfy monotonicity.

### *6.4.3* The least squares O-D estimation incorporating the strategic user equilibrium (LSOD)

Here we use the same notations in the MLOD method but with a different statistical method, the LSOD objective function is the sum of the squared residual in mean and standard deviation of link flows:

$$min: \ z_2(m_T, s_T) = \sum_{1}^{n}(f_n m_T - m_n)^2 + \sum_{1}^{n}(f_n s_T - s_n)^2 \tag{6-39}$$

The first order derivative can be obtained by taking partial derivatives with respect to $m_T$ and $s_T$:

$$\frac{\partial z_2(m_T, s_T)}{\partial m_T} = \sum_{1}^{n} 2f_n(f_n m_T - m_n) \tag{6-40}$$

$$\frac{\partial z_2(m_T, s_T)}{\partial s_T} = \sum_{1}^{n} 2f_n(f_n s_T - s_n) \tag{6-41}$$

The Hessian matrix of the objective function is obtained by taking partial derivative with respect to $m_T$ and $s_T$:

$$H = \begin{bmatrix} \sum_{1}^{n} 2f_n^2 & 0 \\ 0 & \sum_{1}^{n} 2f_n^2 \end{bmatrix} > 0 \tag{6-42}$$

It is shown that the Hessian matrix is strictly positive, and therefore the objective function has unique optimal solution, which can be found when the first derivative of the objective function is equal to zero:

$$\frac{\partial z_2(m_T, s_T)}{\partial m_T} = 0 \rightarrow m_T = \frac{\sum_1^n f_n m_n}{\sum_1^n f_n^2} \tag{6-43}$$

$$\frac{\partial z_2(m_T, s_T)}{\partial s_T} = 0 \rightarrow s_T = \frac{\sum_1^n f_n s_n}{\sum_1^n f_n^2} \qquad 6\text{-}44$$

As the maximum likelihood method, the estimated mean and standard deviation of total demand with error term can be expressed as $Rm_T$ and $Rs_T$ respectively:

$$Rm_T = \frac{\sum_1^n f_n \frac{\sum_1^i (e_{ni} + x_{ni})}{i}}{\sum_1^n f_n^2} \qquad 6\text{-}45$$

$$Rs_T = \frac{\sum_1^n f_n Rs_n}{i * \sum_1^n f_n^2} \qquad 6\text{-}46$$

Therefore, the sensitivity expression of estimated results can be obtained:

$$Rm_T - m_T = \frac{\sum_1^n f_n \frac{\sum_1^i (e_{ni} + x_{ni})}{i}}{\sum_1^n f_n^2} - \frac{\sum_1^n f_n \frac{\sum_1^i x_{ni}}{i}}{\sum_1^n f_n^2} = \frac{\sum_1^n f_n \sum_1^i e_{ni}}{i * \sum_1^n f_n^2} \qquad 6\text{-}47$$

$$Rs_T - s_T = \frac{\sum_1^n p_n (Rs_n - s_n)}{i * \sum_1^n f_n^2}$$

$$= \frac{\sum_1^n f_n \left(\sqrt{\frac{\sum_1^i [x_{ni} + e_{ni} - \frac{\sum_1^i (x_{ni} + e_{ni})}{i}]^2}{i}} - \sqrt{\frac{\sum_1^i [x_{ni} - \frac{\sum_1^i x_{ni}}{i}]^2}{i}}\right)}{i * \sum_1^n f_n^2} \qquad 6\text{-}48$$

In LSOD method, the sensitivity function of both the mean and standard deviation of total demand illustrates its monotonicity with respect to systematic error and specific error. The sensitivity analysis will prove the monotonicity in both methods in the numerical analysis part.

If we compare the optimal solution of both MLOD and LSOD methods, since the logarithm calculus in MLOD method (See Equations 6-24 and 6-25) is only defined for strictly positive numbers, loop counts data needs to be filtered out when either loop counts or link proportions is zero. Therefore, MLOD may not be used if a network has many links with zero loop counts.

### *6.4.4* The Bi-level iterative process

Assuming that the P-STRUE model represents the route choice behaviour, we can formulate a bi-level programming problem, where the upper level is either MLOD or LSOD, and the lower level is the P-STRUE assignment problem. The objective functions of both the upper and the lower level are strictly convex; therefore, the model has a unique solution. A 4-step solution algorithm has been proposed to the bi-level programming:

(1) (Initialization) k=0. Start from the prior O-D matrix, obtain the fraction of total trips $q_m$ and initial values for the mean and the variance of the total demand. Using this demand as input, implement the P-STRUE model to get a set of link proportions $[f_n]_k$ (which provides the flows on each link). Note that $q_m$ will be kept invariant over the bi-level iterations while $\mu_T^k$ and $\sigma_T^k$ will be calibrated.

(2) (Optimization) Substituting the link-flow proportion matrix $[f_n]_k$, solve the upper-level to obtain $\mu_T^k$ and $\sigma_T^k$ of the total demand.

(3) (Simulation) Using $\mu_T^k$ and $\sigma_T^k$, apply the P-STRUE model to produce a new set of link flow proportions $[f_n]_{k+1}$.

(4) (Convergence test) Calculate the deviation between analytically estimated and observed link flows, and the deviation between analytically estimated total demand distributions of two consecutive iterations, when both of them have met the stopping criterions (the relative change is smaller than a critical value), stop.

## 6.5 Numerical results and analysis

### *6.5.1* Example on a moderate-scale network

The objective of the analysis is to test if the MLOD and LSOD can effectively estimate the total demand distribution from day-to-day observed link flows. The simulation approach consists of artificially determining the total demand distribution (in Sioux Falls network, the expected total demand is provided, and the standard deviation of simulated total demand is assumed here) and generating random link flow samples accordingly. The simulated link flows are used to represent observed day-to-day link flow discussed in the previous parts. The estimated total demand distribution should closely approximate the simulated total demand distribution: the link flow distribution produced by the P-STRUE model should also closely match the simulated link flows. The analysis reveals both proposed estimation methods will reproduce the desired total demand distribution from the random samples with perturbed prior estimates. The analysis also reflects the scalability of the both MLOD and LSOD to networks of substantial complexity.

Numerical tests are conducted on the Sioux Falls network (24 nodes and 76 links). The network properties are pre-defined (Bar-Gera, 2012b) (see Figure 3-1). The notations used in this section are defined in Table 6-2. Each O-D demand is specified as a proportion of the total network demand, and therefore the demand for a given O-D pair is the O-D proportion multiplied by the total demand. The BPR function parameters $\alpha$ and $\beta$ are set to 0.15 and 4.0, respectively.

The simulated link flows are generated by the following way:

1) The parameters $\mu_T, \sigma_T$ are determined for the simulated total demand.
2) We implement the P-STRUE based on the simulated total demand distribution and obtain a set of link proportions.
3) We generate 100 samples of the total demand from the lognormal distribution using $\mu_T, \sigma_T$ as parameters and each sample total demand is assigned to the network using the pre-calculated link proportions.

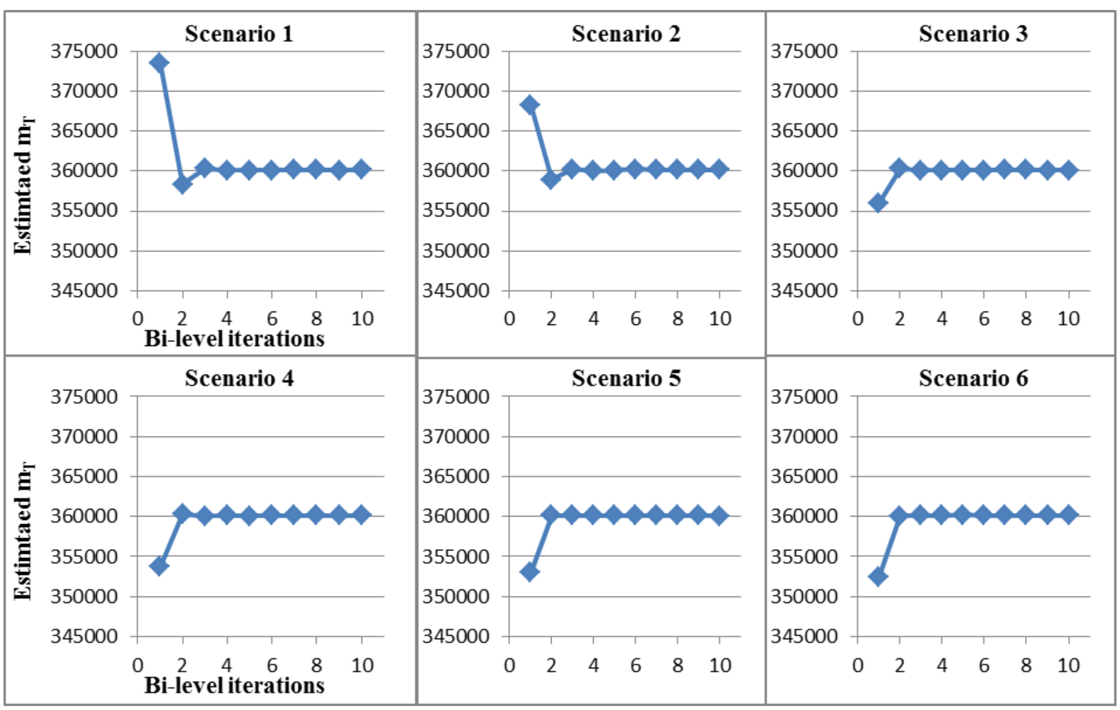
Note that the simulation method implies that travellers will exactly be represented by the STRUE. In the O-D estimation problem normally an assumption of route choice behavior is required, the role of the assignment model in the O-D estimation context is to provide users' route choice. The authors respect that which traffic assignment model can represent the real-world behavior is still an open question, but this discussion is beyond the scope of this chapter. Here the numerical analysis works on the premise that travelers' route choice behavior can be modelled by the STRUE.

The simulated expected total demand of the Sioux Falls network is $m_A = 360600$, and the coefficient of variation $CV$ is equal to 0.2, i.e. the standard deviation is 20% of the expected total demand. In Table 6-3, scenarios 1 to 6 represent different initial estimates of the total demand distribution. From the simulation, all the links are used.

Table 6-3 Different scenarios of initial estimation of O-D matrix.

| *Scenario* | *Scenario description* | $\boldsymbol{m_T}$ | $\boldsymbol{s_T}$ |
|---|---|---|---|
| *1* | $m_T = 0.8m_A$ *and* $CV = 0.1$ | *288480* | *28848* |
| *2* | $m_T = 0.8m_A$ *and* $CV = 0.3$ | *288480* | *86544* |
| *3* | $m_T = 1.2m_A$ *and* $CV = 0.1$ | *432720* | *43272* |
| *4* | $m_T = 1.2m_A$ *and* $CV = 0.3$ | *432720* | *129816* |
| *5* | $m_T = 1.5m_A$ *and* $CV = 0.1$ | *540900* | *54090* |
| *6* | $m_T = 1.5m_A$ *and* $CV = 0.3$ | *540900* | *162270* |
| *Simulated* | $m_A = 360600$ *and* $cov = 0.2$ | | |

In Figure 6-3 and Figure 6-4, the $x$-axis represents the number of iterations of the bi-level program. Figure 6-3 and Figure 6-4 illustrate the estimated mean and standard deviation of total demand in each iteration for the MLOD and LSOD methods, respectively. Each series represents an initial demand scenario. Both figures show that the estimated results converge to the simulated ones within 3 iterations. This indicates the model's robust performance against biased initial estimates, and demonstrates the efficiency in arriving at convergence. The estimated results of the first iteration in both figures are very different from the simulated ones. This is because the link proportions of the first iteration are obtained based on the initial demand scenario specified. The initial estimates in scenarios 1 and 2 are very biased, and as a result, the first iteration results are inaccurate. Both MLOD and LSOD provide a similar estimation of *E(T)* that is approximately equal to the simulated expected total demand. Note that MLOD always provide an overestimation in *Std(T)* as long as initial estimates are biased, but this overestimation is eliminated after the second or third iteration. It is therefore necessary to incorporate the Bi-level process to reduce the impact of biased initial estimates, especially due to the difficulty in obtaining the standard deviation of demand.



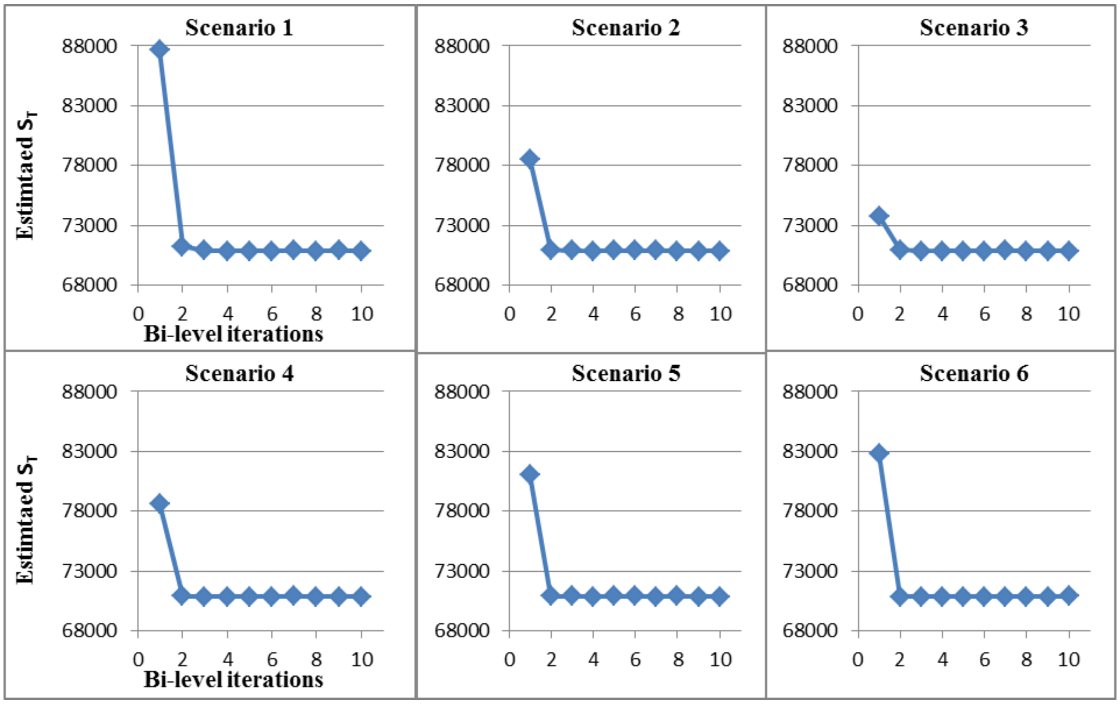
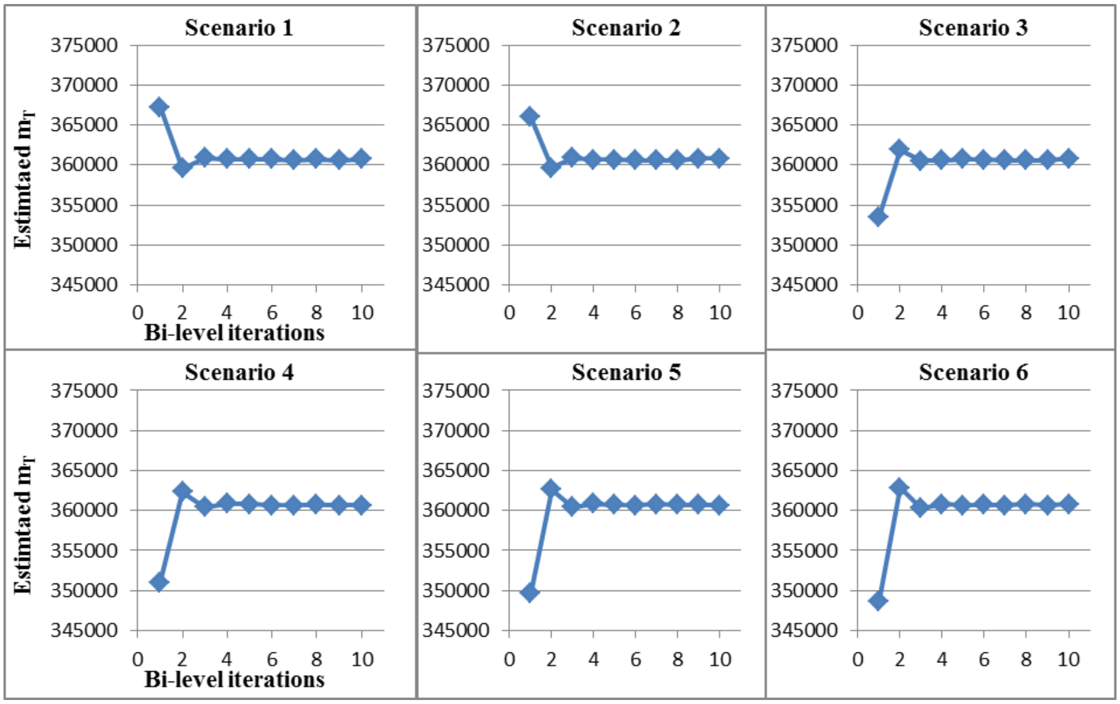



Figure 6-3 (a) Estimated expected total demand of MLOD under different scenarios of initial estimation; results of 10 bi-level iterations are presented. (b) Estimated standard deviation of total demand of MLOD under different scenarios of initial estimation; results of 10 bi-level iterations are presented.



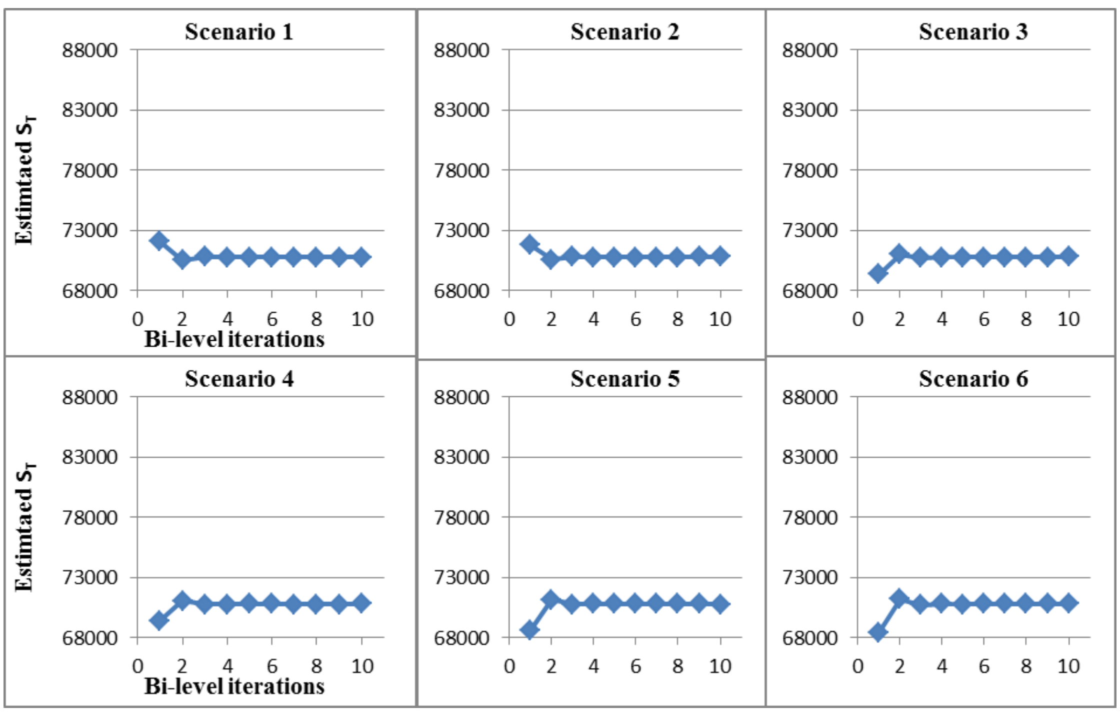
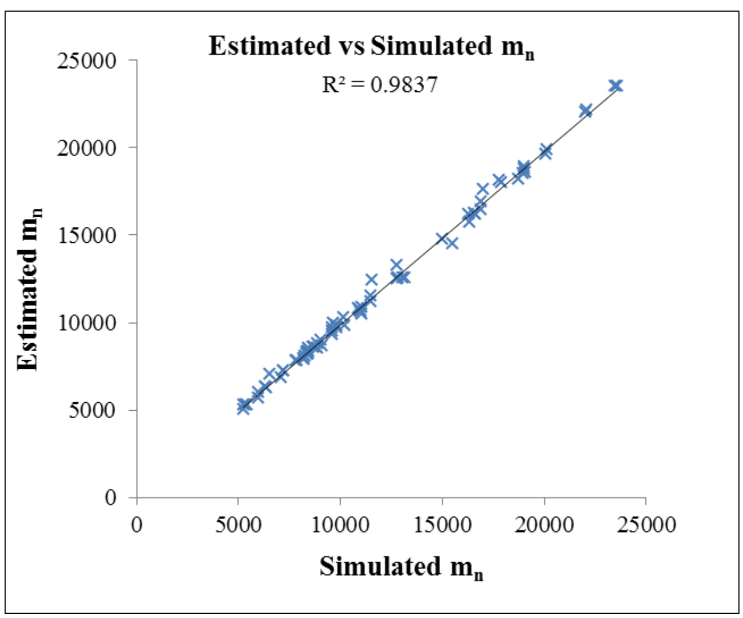




Figure 6-4 (a) Estimated expected total demand of LSOD under different scenarios of initial estimation; results of 10 bi-level iterations are presented. (b)

Estimated standard deviation of total demand of LSOD under different scenarios of initial estimation; results of 10 bi-level iterations are presented.

Figure 6-5 and Figure 6-6 compare the performance of the estimation methods at the link level for the simulated and analytical results. In Figure 6-5, the *x*-axis indicates the simulated expected link flow while the *y*-axis represents the estimated expected link flow. The estimated link flows are analytically produced by the P-STRUE model based on the total demand distribution after the convergence criterion has been met. The estimated expected link flows and the corresponding simulated expected link flows are sorted from the smallest to the largest. The R-squared values of both methods are 0.9837 and 0.9917 respectively, which are very close to 1, and it indicates that the estimated results closely approximate the simulated expected link flows.



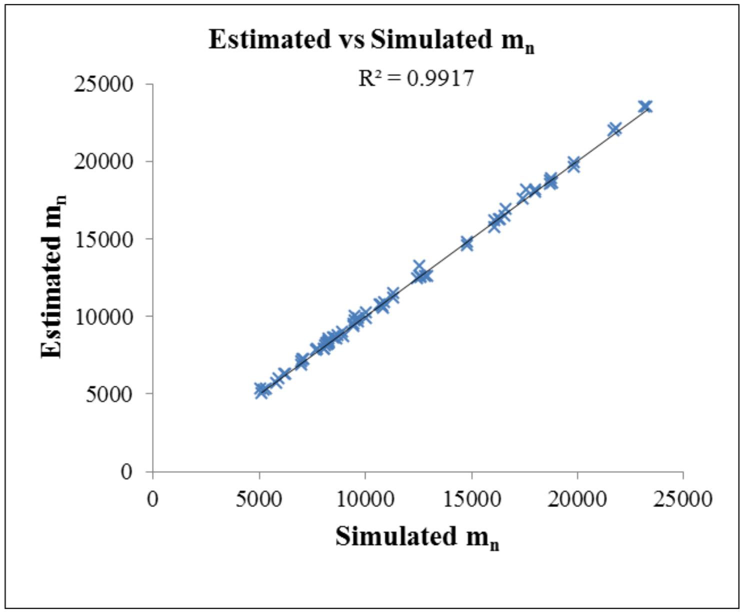
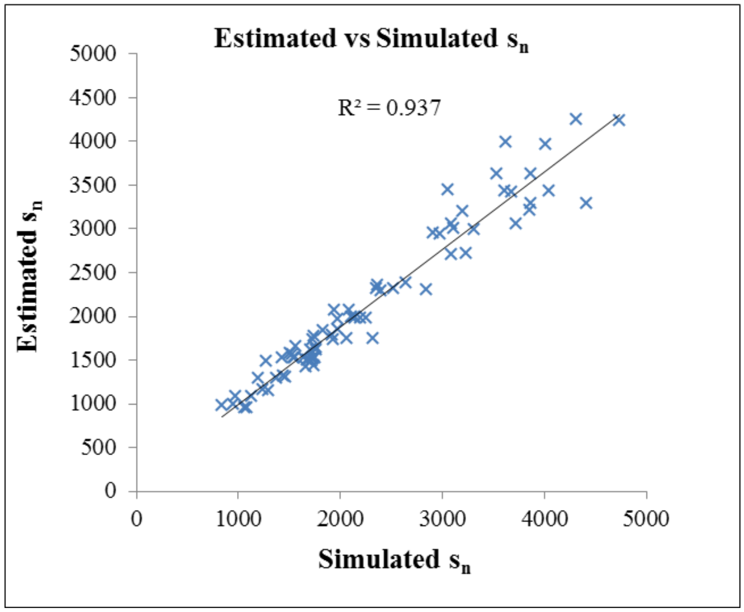



Figure 6-5 (a) The estimated and simulated expected link flow comparison of the MLOD method, estimated results are produced by the P-STRUE model based on estimated demand distribution. (b) The estimated and simulated expected link flow comparison of the LSOD method, estimated results are produced by the P-STRUE model based on estimated demand distribution.



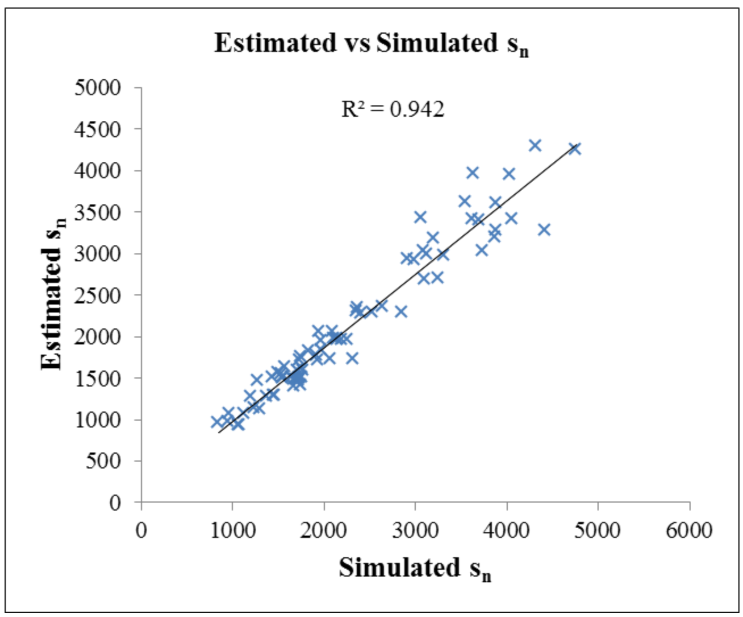




Figure 6-6 (a) The estimated and simulated standard deviation of link flow comparison of the MLOD method, estimated results are produced by the P-STRUE model based on estimated demand distribution. (b) The estimated and simulated standard deviation of link flow comparison of LSOD method, and estimated results are produced by the P-STRUE model based on estimated demand distribution.

A major strength of the proposed estimation methods, which is an artefact of using the P-STRUE model for traffic assignment, is the estimation of link flow variation. Since the total demand distribution is calibrated based on day-to-day simulated link flows, it is therefore necessary to compare the estimated standard deviation of link flow to the simulated one. In Figure 6-6, the estimated standard deviation of link flow is produced by the P-STRUE model based on the total demand distribution after the bi-level convergence criterion has been met. The *x*-axis denotes the simulated standard deviation of link flow while the *y*-axis indicates the estimated one. It is illustrated in Figure 6-6 that despite the fact that the R-squared value is smaller than that of the expected link flow analysis; the R-squared values of both methods still suggest a satisfying goodness of fit. Note that if the standard deviation of link flow is very high, the estimated results may be more than 20% different from the simulated ones. Such a heteroscedasticity is due to the fact that the higher the standard deviation, the higher the corresponding link proportion. However, both methods can reproduce the link flow distribution if provided an estimated total demand distribution, which indicates its applicability to traffic assignment model.

### 6.5.2 Model sensitivity analysis

Loop detector data is prone to error, and the error can be far more complicated in reality, for simplicity, here we design the systematic error and specific error as in Equations 6-36 to 6-38 to test the model's sensitivity. The robustness of the proposed estimation methods with respect to both error types is explored using sensitivity analysis. The specific error indicates the significance of the error such as the failure of loop detectors or lack of information on a link, while the latter one measures the scale of the error, i.e. which links have an error. In this analysis the specific error is set at 10%, 20% and 30% respectively, and systematic error is represented as a set of links that have error, which is shown in Table 6-4. The impact of different systematic error and specific error is illustrated, by providing the estimated mean and standard deviation of the total demand. Note that as this is a sensitivity analysis, the link proportions derived from the prior demand distribution are assumed to be identical to the simulated ones to eliminate the effect of biased prior estimates, i.e. link proportions are fixed, since we only focus on the impact of erroneous link flow observations in this part.

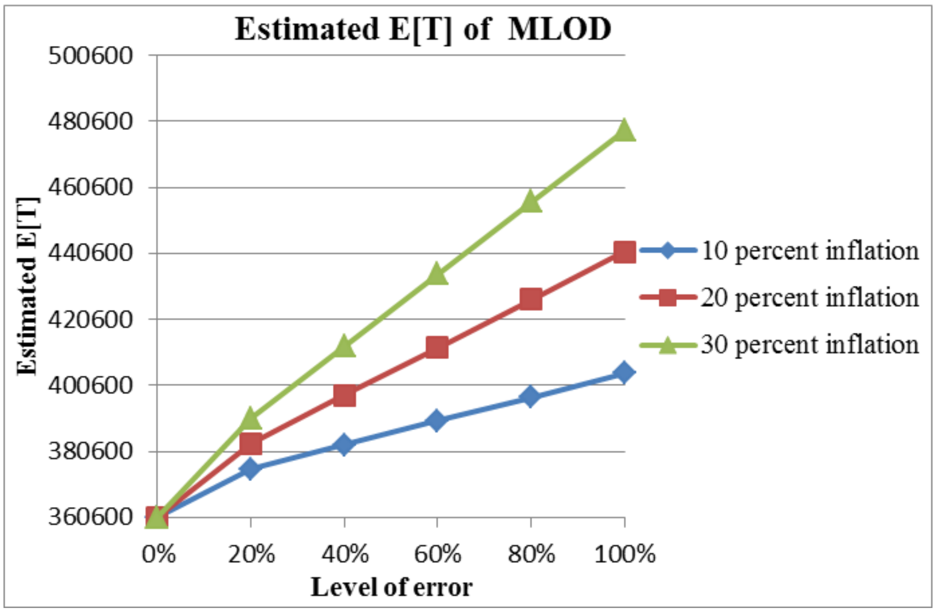
Table 6-4 Different systematic error and specific error.

| *Links with error term (Systematic error)* | *Total number of samples with error term* | *Specific error* |
|---|---|---|
| 1-76 | 7600 | 10%, 20%, 30% |
| 16-76 | 6100 | 10%, 20%, 30% |
| 31-76 | 4600 | 10%, 20%, 30% |

| | | |
|---|---|---|
| 46-76 | 3100 | 10%, 20%, 30% |
| 61-76 | 1600 | 10%, 20%, 30% |

In Figure 6-7, the x-axis represents the systematic error, from 0% (no link has an error) to 100% with an increment of 20%. Each series indicates a different specific error, from 10% to 30% inflation with an increment of 10%. The y-axis starts from the estimated expected total demand without error. It is demonstrated that the estimated expected total demand of both methods rises with the increase in systematic error and specific error; these two error categories have a moderate impact on the estimated expected total demand in both methods. Additionally, LSOD provides estimations closer to the simulated one under low systematic error (20%). Therefore, both systematic error and specific error should be treated equally, and under low systematic error, LSOD can potentially provide a better estimation of estimated expected total demand.

Interestingly, in Figure 6-8, there is a drop of *Std[T]* when the systematic error is high in the results of MLOD. This can be explained by the non-linearity characteristics in the analytical expression of sensitivity. Because from the analytical expression in Equation 6-35 we can see it is determined by several factors including link proportion, link flow and the specific error. The combined effect of these factors does not satisfy monotonicity. In LSOD method, *Std[T]* monotonically increase with systematic error and specific error, and it provides a better estimated *Std[T]* under all scenarios, because of its advantage in the estimation of an over-determined system.



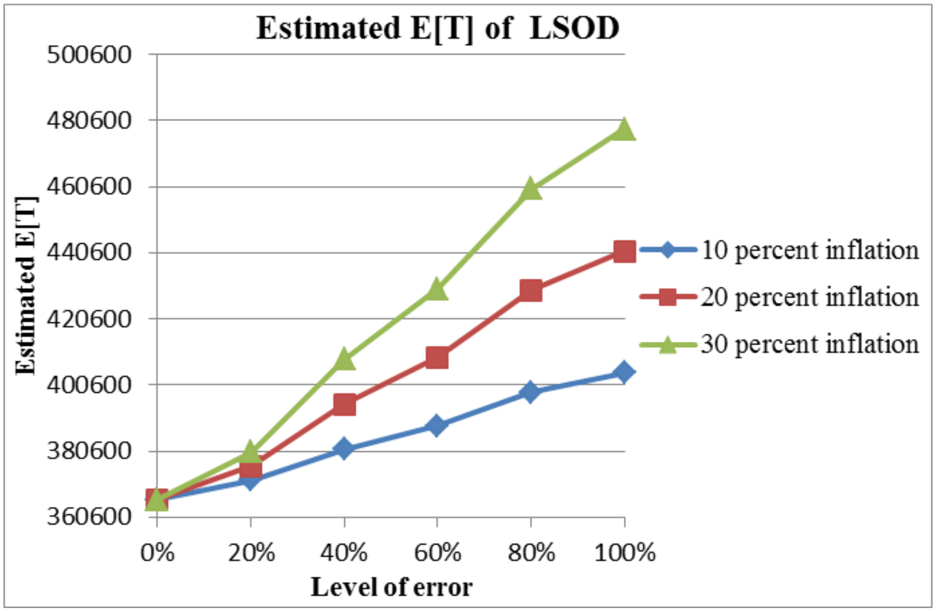



Figure 6-7 (a) The estimated expected demand of MLOD method under different systematic error and specific error. (b) The estimated expected demand of LSOD method under different systematic error and specific error.

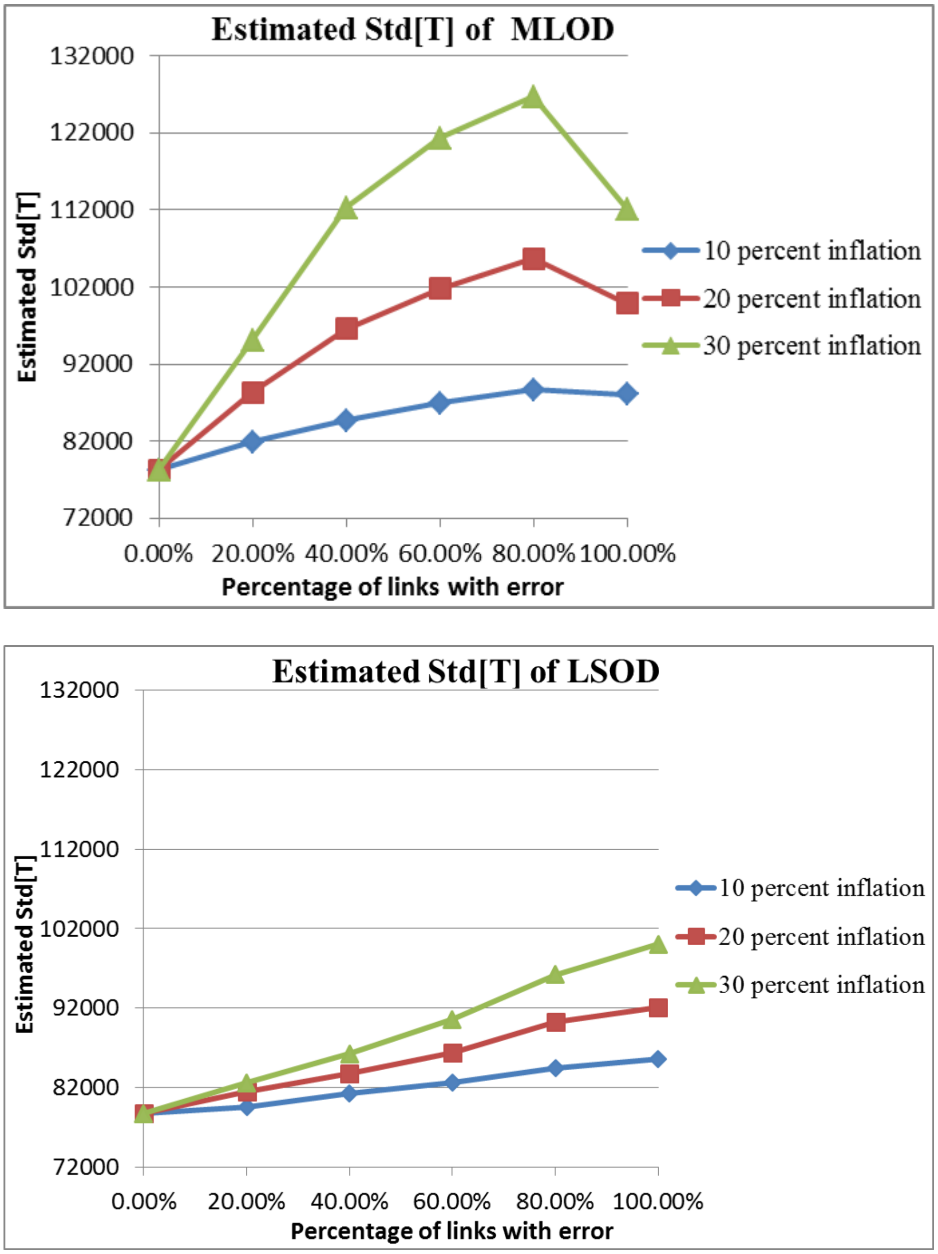



Figure 6-8 (a) The estimated standard deviation of total demand of MLOD method under different systematic error and specific error. (b) The estimated standard deviation of total demand of LSOD method under different systematic error and specific error.

### *6.5.3* Example on a medium-scale network: the Anaheim network

To demonstrate the proposed model's scalability, a numerical analysis is also conducted on the Anaheim network, which consists of 38 zones, 416 nodes and 914 links. The demand proportions, and network properties can be found in (Bar-Gera, 2012a). Units used in the network are length is in feet, free flow travel time in minutes, and speed in feet per minute. The BPR function parameters $\alpha$ and $\beta$ are set to 0.15 and 4.0, respectively. A similar method as in the Sioux Falls network example is used to generate the simulated link flows, while the actual expected total demand is computed by aggregating the trip table. Figure 6-9 presents a map of the Anaheim network.

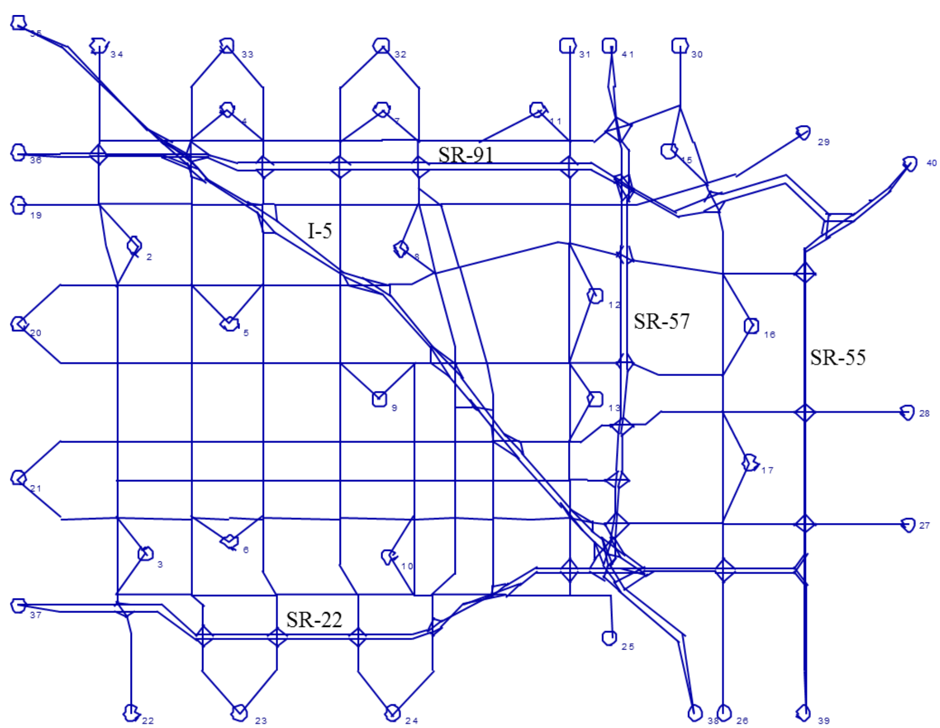



Figure 6-9 The Anaheim network

Results of both methods are illustrated in Table 6-5; the same performance measures are depicted as in the Sioux Falls network. In addition, to demonstrate the computation efficiency, computation time is recorded. The programming code is written in MATLAB, and the computer used here has the following configurations: CPU: Intel i7-3770 3.4G Hz quad core; Ram size: 16 GB, Software: Windows 7 enterprise version. The computation time may vary depending on computer configuration, software version and other factors. The estimated expected total demand of both methods is statistically indifferent to the actual expected demand. The MLOD method slightly underestimates the standard deviation of total demand, while LSOD overestimates the standard deviation of total demand, but the relative error is rather small (relative error is calculated by dividing absolute error by actual value). R-squared value on the link level demonstrates that the estimated total demand distribution can reproduce a set of link flow close to the simulated ones. Computing time of both methods is short, which proves the model's applicability on medium-scale networks. MLOD takes a moderately longer computation time, due to that some links have no flow in it, and some link proportions may become zero during the Bi-level process. In this case, this data needs to be filtered out, which increases the computation burden to some extent.

Table 6-5 Performance measures of the Anaheim network

| *Performance measures* | *MLOD* | *LSOD* |
|---|---|---|

| | | |
|---|---|---|
| *Estimated* $m_T$ | 104730 | 104890 |
| *Relative error of estimated* $m_T$ | 0.03% | 0.19% |
| *Estimated* $s_T$ | 20304 | 21433 |
| *Relative error of estimated* $s_T$ | 3.03% | 2.36% |
| $R^2$ *value of estimated and simulated expected link flow* | 0.991 | 0.993 |
| $R^2$ *value of estimated and simulated standard deviation of link flow* | 0.984 | 0.989 |
| *Computation time* | 179 seconds | 156 seconds |
| *Actual demand distribution* | $\mathrm{m_A} = 104690$ and $\mathrm{s_A} = 20939$ | |

## 6.6 Chapter summary

This chapter proposes two methods (MLOD and LSOD) to estimate the total traffic demand distribution based on day-to-day observed link flows. The model considers link flow variation when estimating the total demand distribution and the P-STRUE model can provide link flow variation by accounting for the volatility in demand. A bi-level programming method is included to reduce the impact of biased initial estimates of the total demand distribution, and both the upper level and the lower level problem are proved to be strictly convex. A sensitivity analysis is conducted for both MLOD and LSOD methods. Numerical analyses are conducted on both the Sioux Falls network and the Anaheim network. Results at the system level and the link level are similar, and demonstrate the robustness of both MLOD and LSOD methods. In general, both MLOD and LSOD demonstrate scalability with computation efficiency while providing a satisfying estimation of total demand distribution. However, the MLOD method requires non-zero link flow and link proportions, which may limit its applicability.

Sensitivity analysis shows that the impact of error is predictable. Under the proposed model the problem is over-determined (unlike the traditional O-D estimation problem), that is, the number of variables to be estimated is far smaller than the number of known constraints, and therefore the lack of information on several links will not limit the applicability. In addition, two consequences can be derived: 1), the advantage of the LSOD method is that it is less sensitive to detector error. This is because only the mean and standard deviation of link flows are considered; therefore, the impact of errors or outliers is averaged out and the estimation tends to be less sensitive to error; and 2), despite different objective functions for the two proposed methods, the estimated results without error are similar because in the proposed models the number of constraints greatly exceeds the parameters to be calibrated, namely the total demand distribution.

Lastly, it should be noted that the assumption of fixed O-D demand proportions may limit the model applicability. Therefore, one future research effort will be the generalization of the proposed model, especially on estimating O-D demand separately for each O-D pair. This requires extending the strategic user equilibrium for independently distributed O-D demands case. The key contribution will be to account for demand volatility in the assignment model while considering observed link flow variation in O-D demand estimation. This chapter already provides an insight in light of this. Also, it is valuable to investigate the use of the covariance of loop counts. This can potentially provide much more information than only the link flow distribution. Additionally, dynamic traffic assignment may be integrated into the proposed model. Generally, since the OD estimation problem is a combination of a statistical optimization model and a traffic assignment model, an improvement in either process warrants further research.

# Chapter 7

# Estimation of sparse O-D matrix accounting for demand volatility

## 7.1 Introduction

As a fundamental element of the transportation planning process, O-D trip matrix plays a principal role and can have a significant impact on the prediction results. O-D demand is inherently volatile and may vary day-to-day due to various factors. In Chapter 6, a model was introduced to calibrate the total demand and its variability, where each O-D pair demand is assumed to be perfectly-correlated with each other. In this chapter, a model is proposed to estimate the O-D matrix when each O-D demand is independently distributed. These two models have their own applicability contingent on different situations so both models should be closely examined. When each O-D pair is independent of each other and the demand proportions as illustrated in the previous chapter is not available, a challenge arises as a consequence: the number of O-D pair to be estimated is often much greater than the number of monitored links. This issue is denoted as ill-posedness or under-determination. Therefore, additional sources of information are required to mitigate the issue of under-determination. More background information on O-D estimation has been presented in Section 6.1.

In real life, it is commonly observed that some centroids tend to be more popular than others; in addition, only few trips are made for intro-zonal travel. Consequently, a large portion of trips will be made for a small portion of O-D pairs, that is, there are a lot of O-D pairs with only a few or even zero trips. Mathematically, this implies that the O-D matrix is sparse. In the proposed model, the assumption of sparse O-D matrix is represented by the L1 regularization, because minimization of the L1 regularization term induces a sparse solution. Previous researchers have used L1 regularization to account for network anomalies (Mardani and Giannakis, 2013, Chawla et al., 2012, Zhang et al., 2005), and the impact of path flow sparsity in the O-D estimation problem was also explored (Sanandaji and Varaiya, 2014). However, few researchers have considered that the O-D demands should be non-negative in the regularization problem, which is an important constraint in the O-D estimation process. (Menon et al., 2015) provides an in-depth discussion of the importance of non-negativity in O-D demands in ill-posed problems, and showed that it could be useful in providing a potentially unique solution. Some researchers have focused on the Lagrangian dual of the for L1 regularization, which regards the problem as an example of the basis pursuit principle. The advantage is to avoid tuning weight parameters for the regularization term, with a compromise of spending more time on the optimization procedure (Sanandaji and Varaiya, 2014, Chen et al., 2001, Cheman, 2006, Zou and Hastie, 2005).

Additionally, users' route choice information are often assumed known, or obtained with an elaborate computation. Besides, the inherent volatility in demand

and the resulted link flow, variation are often neglected, whilst both could influence users' route choice significantly. Therefore, in addition to the sparsity regularization, the strategic user equilibrium for independently distributed O-D demands (I-STRUE) proposed in Chapter 4 is also implemented to account for demand uncertainty. The I-STRUE is a user equilibrium assignment model which assumes that travellers choose a route to minimize their expected travel cost, where their decision is based on knowledge of a demand distribution, rather than a deterministic demand value. The I-STRUE is defined such that at equilibrium, all used paths have equal and minimal expected travel costs. The I-STRUE can account for demand volatility while maintaining computation efficiency.

Further, this model explicitly treats demand as a causal variable: the correlation and variation of link flow are caused by the demand volatility. This is a different interpretation of link flow variation, which was often explained as a measurement error in the generalized least squares method (Cascetta, 1984, Bell, 1991). Such a notion allows the utilization of link flow correlation, which can be obtained from loop detectors on multiple days, and could hence improve the estimation quality (Goel et al., 2005, Hazelton, 2001, Hazelton, 2003).

In general, sparsity regularization is combined with link flow correlation to provide additional inputs for the O-D estimation process. This could mitigate the issue of under-determination of the problem. The highlights of the proposed model are:

1) The non-negativity of O-D demands is considered in the optimization problem.
2) The sparsity of O-D matrix is accounted for and is used as a regularization to enhance estimation quality.
3) Link flow correlation is incorporated to improve more information for the under-determined O-D estimation problem.
4) A specific assignment model (I-STRUE) is implemented to address the impact of demand volatility in users' route choice, while maintaining computation efficiency.

In Section 7.2, the model is formulated as a convex generalized least squares problem with regularization. The usefulness of the sparsity assumption and link flow correlation are presented on the Sioux Falls network in Section 7.3. Section 7.4 provides a conclusion and possible future research direction.

## 7.2 Problem formulation

This section defines the mathematical formulation of our proposed model, and a summary of the notations used in the section is listed as follows:

| | |
|---|---|
| $N$ | Link (index) set. |
| $M$ | O-D pair (index) set. |

| | |
|---|---|
| $K_m$ | The path set for O-D pair *m*. |
| $V$ | The vector of link flow. |
| $\tilde{V}$ | The vector of the observed expected link flow. |
| $T$ | The vector of O-D travel demand. |
| $\tilde{T}$ | The vector of the prior estimate of O-D travel demand. |
| $A$ | The assignment map matrix, which represents the proportion of O-D pair demand $T_m$ traversing link *n*. |
| $Y$ | The covariance matrix of observed link flows. |
| $t_{nf}$ | The free flow travel time on link *n*. |
| $C_n$ | The capacity on link *n*. |
| $t_n()$ | The travel cost function for link *n*. |
| $d_n^m$ | Users' O-D specific link choice, which represents the proportion of O-D pair demand $T_m$ on link *n.* |
| $\overline{h_k^m}$ | The expected flow on path *k*, connecting O-D pair *m.* |

| | |
|---|---|
| $G(\,)$ | The probability density function of a variable. |
| $T_m$ | Demand variable for O-D pair *m.* |
| $\lambda_n$ | The parameter of the Poisson distribution for flow on link *n.* $[\lambda] = [\lambda_1 \quad \dots \quad \lambda_n]^{`}$. |
| $s_m$ | The parameter of the Poisson distribution for O-D pair demand $T^m$. |
| $\delta_{n,k}^m$ | Link-Path indicator variable. $\delta_{n,k}^m = \begin{cases} 1 & if\ link\ n\ is\ on\ path\ k\ between\ OD\ pair\ m \\ 0 & otherwise \end{cases}$ |
| $\delta_1$ | Weight parameter for the prior O-D estimation. |
| $\delta_2$ | Weight parameter of the L1 regularisation. |
| $p_k^m$ | Proportion of flow on path k, connecting OD pair m, must be non-negative. |

The use of the generalized least squares method has a long history. Traditionally, its application in transport O-D estimation problem is to find an O-D matrix that minimizes the squared Mahalanobis distance of two residual vectors: the vector of link flow and the vector of O-D demands. As demonstrated in Section 6.1.3.2, the function of such an objective is:

$$T^{GLS} = argmin: (T - \tilde{T})^{`} Z^{-1}(T - \tilde{T}) + (V - \tilde{V})^{`} Y^{-1}(V - \tilde{V}) \quad 7\text{-}1$$

Subject to:

$$T \geq 0 \quad 7\text{-}2$$

$$V = AT \quad 7\text{-}3$$

Where, $Y^{-1}$ indicates the inverse of the variance-covariance matrix of the 'errors' in observed link flows, $Z^{-1}$ indicates the inverse of the variance-covariance matrix of O-D demands, and $()^{`}$ means the transpose of a matrix. Since the O-D estimation problem is generally an ill-posed problem if no prior information is available, it is necessary to utilize as much information as possible. However, a majority of the previous literature has often neglected the correlation between link flows, that is, $Y$ is considered as a diagonal matrix whose diagonal elements represent the variance of the observation error. In addition, the non-negativity constraint is often ignored during the optimization process. To mitigate the under-determination of such a problem, these two issues are discussed and resolved here to provide a robust estimation of O-D matrix.

### 7.2.1 Correlation of link flows

Incorporating the correlation of link flows can potentially utilize more information from observed link flows, because given a network of $n$ links, $n$ pieces of information are input to the optimization problem if only the variance of link flows is considered, but $n^2$ pieces of information are utilized if the covariance matrix is incorporated. A key assumption is made here to account for the covariance of link flows:

**A1.** Each O-D pair demand follows the Poisson distribution and is independent of each other.

Such an assumption allows us to interpret the error term differently from other literature: the 'error' is actually the volatility of link flow due to the demand uncertainty, instead of the measurement errors. In other words, each O-D pair demand is a 'causal' variable which is projected to each link by the assignment map matrix $A$. More importantly, such an interpretation of the covariance matrix $Y$ allows us to scrutinize the correlation between each link. Thus, given the observed link flows on multiple days, we will be able to find the covariance matrix of link flows:

$$Obs = \begin{bmatrix} l_1^1 & \cdots & l_1^d \\ \vdots & \ddots & \vdots \\ l_n^1 & \cdots & l_n^d \end{bmatrix} \tag{7-4}$$

$$Y = Covariance(Obs) \tag{7-5}$$

Because each O-D pair demand and link flow follows a Poisson distribution, they can be defined by their corresponding parameters (for a Poisson distribution, the parameter is this variable's expectation):

$$T = \begin{bmatrix} s_1 \\ \vdots \\ s_m \end{bmatrix}, \quad V = \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix} \tag{7-6}$$

Therefore, the objective is to find a vector of demand parameter for the generalized least squares method. Before proceeding further in solving the objective function, we introduce another regularization term for our optimization problem.

### 7.2.2 Regularization inducing sparsity and non-negativity of O-D demand

It is a common phenomenon that only a small portion of O-D pairs will have a large number of trips, especially for commuter trips, which means the O-D matrix tends to be a sparse matrix. Hence, it is assumed here that the O-D matrix is sparse to some extent. In our optimization problem, the sparsity of an O-D matrix can be obtained by adding the L1 regularization to the objective function:

$$T^{GLS} = argmin\colon \left(AT - \tilde{V}\right)^{`} Y^{-1}\left(AT - \tilde{V}\right) + \delta_1\left(T - \tilde{T}\right)^{`}\left(T - \tilde{T}\right) + \delta_2\|T\|_1 \qquad 7\text{-}7$$

Where, $\|T\|_1$ represents the L1 norm of a vector, $Y^{-1}$ indicates the inverse of the variance-covariance matrix of link flows, $\delta_1$ and $\delta_2$ are weight parameters which are contingent on our belief of the corresponding regularization. The L1 norm here can be expressed as:

$$\|T\|_1 = \sum_{m \in M} |s_m| \qquad 7\text{-}8$$

To explain why minimization of the L1 norm induces sparsity, we start from the L0 norm of a vector, which is:

$$\|T\|_0 = \sum_{m \in M} \widehat{s_m} \tag{7-9}$$

$$\widehat{s_m} = \begin{cases} 1, & \gamma_m \neq 0 \\ 0, & \gamma_m = 0 \end{cases} \tag{7-10}$$

Clearly the L0 norm is the number of non-zero elements in the O-D matrix. Previous literature on the Lasso algorithm and on the compressed sensing has suggested that under some assumptions, minimizing L1 norm can approximate the minimization of L0 norm (Tibshirani, 1996, Donoho, 2006), that is, minimization of L1 norm induces a sparse O-D matrix.

The function of L1 norm is not differentiable everywhere; however, one may note that the parameter for each O-D pair is non-negative, which allows us to write the objective function as the following form:

$$\begin{aligned} z(T) = \underset{T \geq 0}{argmin} \left(AT - \tilde{V}\right)^{`} Y^{-1} \left(AT - \tilde{V}\right) + \delta_1 (T - T^p)^{`} (T - T^p) \\ + \delta_2 \sum_{m \in M} s_m \end{aligned} \tag{7-11}$$

Hence the objective function becomes differentiable everywhere. When $\delta_2$ equals 0, the problem is similar to the generalized least squares O-D estimation problem. The differentiability enables us to prove the convexity of the objective function when we take the second partial derivatives with respect to the demand vector, which provides us the Hessian matrix of the objective function:

$$\frac{\partial z(T)}{\partial T \partial T} = A^{`} Y^{-1} A + 2\delta_1 I \quad \text{7-12}$$

Where, $I$ is an identity matrix of dimension $m$ by $m$. Clearly the Hessian matrix is positive definite, hence the proposed model has a globally optimal solution. It is vital that the model can provide a unique estimation of O-D matrix, because it guarantees that the proposed model is applicable to a variety of transportation planning process.

The use of non-negative O-D parameters is consistent with our intuition- the number of trips made between each O-D pair should always be greater than or equal to zero. In addition, such a constraint allows the assumption of Poisson distribution. If the non-negativity constraint is not considered, we may obtain an analytical optimal solution by taking the first partial derivative. Many researchers have solved the generalized least squares O-D estimation by such a closed form update technique. Notwithstanding this, it is hard to interpret a negative O-D demand, so we avoid a negative solution in our formulation, the optimization problem with non-negative constraints can be solved by various methods such as gradient descent method.

### 7.2.3 The assignment map matrix and demand volatility

Due to the above interpretation that the link flow variation is caused by demand volatility, we need to account for these uncertainties in the assignment map matrix. That is, users should consider demand volatility when making their route choice decision. Such a goal can be achieved by applying the strategic user equilibrium for independently distributed O-D demands (I-STRUE). I-STRUE is defined such that the expected travel costs are equal on all used paths, and this commonly expected travel time is less than the actual expected travel time on any unused path. In other words, given user equilibrium expected path cost, any deviation from the existing expected path flows cannot reduce the expected path cost. The assignment problem has been formulated in Chapter 4 of the thesis.

In the model, the expected path flow is expressed as a proportion multiplied by the expected demand, that is,

$$\lambda_n = \sum_{m \in M} \sum_{k \in K_m} \overline{h_k^m} \delta_{n,k}^m = \sum_{m \in M} \sum_{k \in K_m} p_k^m \delta_{n,k}^m s_m \qquad \text{7-13}$$

Then, we can obtain the O-D specific link proportions, also known as the assignment map matrix, by the following equation:

$$d_n^m = \sum_{k \in K_m} p_k^m \delta_{n,k}^m \quad \forall m \in M, \forall n \in N \qquad \text{7-14}$$

$$A = \begin{bmatrix} d_1^1 & \dots & d_1^m \\ \vdots & \ddots & \vdots \\ d_n^1 & \dots & d_n^m \end{bmatrix} \qquad \text{7-15}$$

The assignment map matrix represents the proportion of O-D pair demand $T_m$ traversing link *n*, it indicates the proportion of link flow disaggregated by different O-D pairs, which is extremely important in many transportation applications such as O-D matrix estimation, emission analysis and network design problem. To summarize, the implementation of I-STRUE accounts for the demand uncertainty in users' routing mechanism while maintaining the computation simplicity under the classical user equilibrium formulation.

## 7.3 Numerical demonstration

The objective of the analysis is to test if the proposed model can effectively estimate the O-D trips from day-to-day observed link flows. This analysis is conducted on the Sioux Falls network (see Figure 3-1). Each O-D demand is assumed to follow a Poisson distribution and is independent of each other. The BPR function parameters $\alpha$ and $\beta$ are set to 0.15 and 4.0, respectively. The prior O-D matrix is assumed to be a ten percent overestimation of the simulated one, that is, the simulated O-D matrix will be inflated by ten percent to represent the prior O-D matrix, to demonstrate the fact that a prior O-D matrix is normally inaccurate.

To collate observed link flow data, a Monte-Carlo simulation is conducted. It consists of running the strategic user equilibrium model and generating random link flow samples accordingly. Firstly, we run the I-STRUE based on the trip table pre-defined in Bar-Gera (referred to as simulated O-D matrix in this section) assuming the demand follows Poisson distribution, and obtain the assignment map matrix. Then 10000 O-D matrices are sampled independently from its corresponding Poisson distribution. Finally, these sampled O-D matrices are assigned to each link according to the assignment map matrix. The resulted simulated link flows are used to represent observed day-to-day link flow discussed in Equations 7-4 and 7-5. The impact of the weight parameters $\delta_1$ and $\delta_2$ are explored, in conjunction with how the sparsity regularization will facilitate and improve the estimation. The estimated O-D matrix should closely approximate the simulated one, and the link flow distributions reproduced by the I-STRUE model based on estimated O-D matrix should also closely match the simulated link flows.

To evaluate the performance of the proposed model, the mean square error (MSE) is introduced here:

$$MSE = \sum_{m \in M} \frac{s_m - s_m^*}{|M|} \quad 7\text{-}16$$

Where, $s_m^*$ and $s_m$ denote the simulated O-D demand and estimated O-D demand respectively, $|M|$ is the number of O-D pairs. MSE indicates how the estimation deviates from the simulated O-D matrix. In Figure 7-1, the impact of the weight parameters is demonstrated. When $\delta_1$ is fixed to 0.25 or 0.3, as illustrated in the two series in the figure, the MSE of the estimated O-D demands drops with the increase of $\delta_2$. On the contrary, If $\delta_1$ is fixed to 0.1, 0.15 or 0.2, the curve is similar to parabola, and the MSE will climb up after the minimum is reached. The figure shows that the incorporation of sparsity regularization improves the estimated results in general; however, if we excessively amplify the importance of sparsity (that is, if we believe the O-D matrix is very sparse, which is not the case for the Sioux Falls network), the regularization may have detrimental impact on the estimation. Therefore, the choice of the weight parameters should be scrutinized according to different cases rather than apply a set of uniform values.

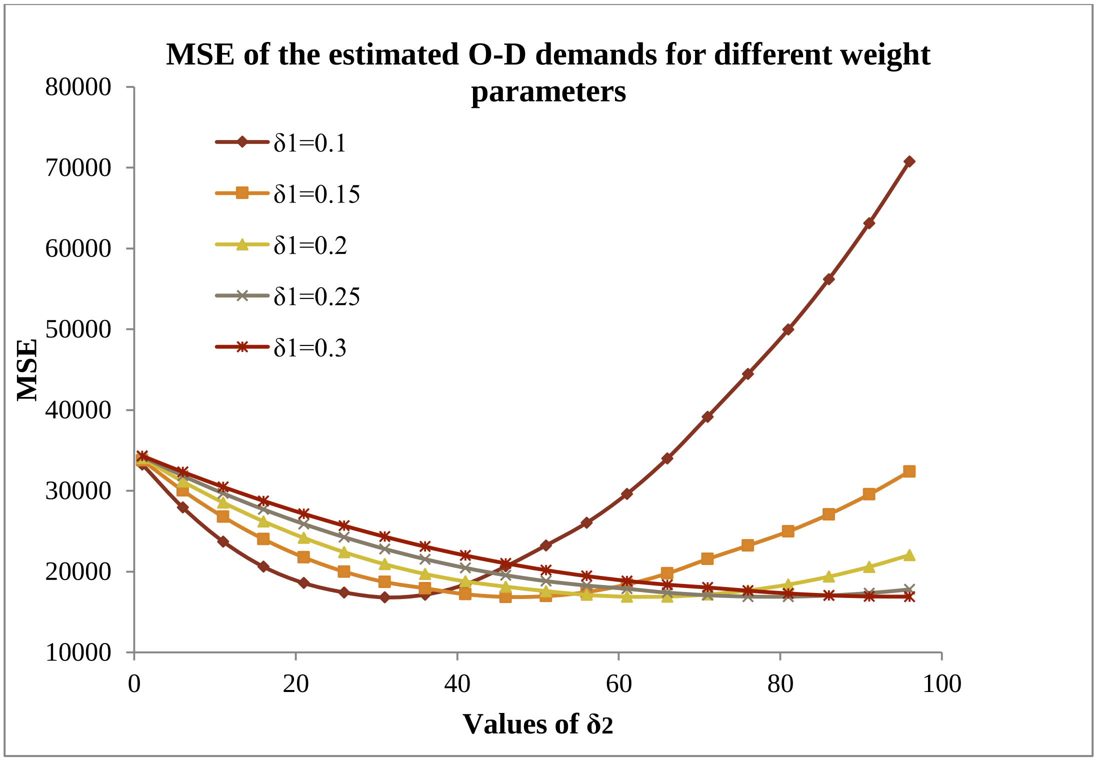



Figure 7-1 The impact of weight parameters on the estimated results.

In Figure 7-2, the simulated expected link flow and the corresponding estimated mean link flow are plotted from the smallest to the largest. The estimated expected link flows are produced by the I-STRUE model based on the estimated O-D matrix (when $\delta_1 = 0.1, \delta_2 = 31$). It is illustrated that the estimated expected link flow closely approximate the simulated expected link flow. Hence, the proposed model is capable of finding an estimated O-D matrix that produces a set of link flow similar to the simulated one while being as sparse as possible.

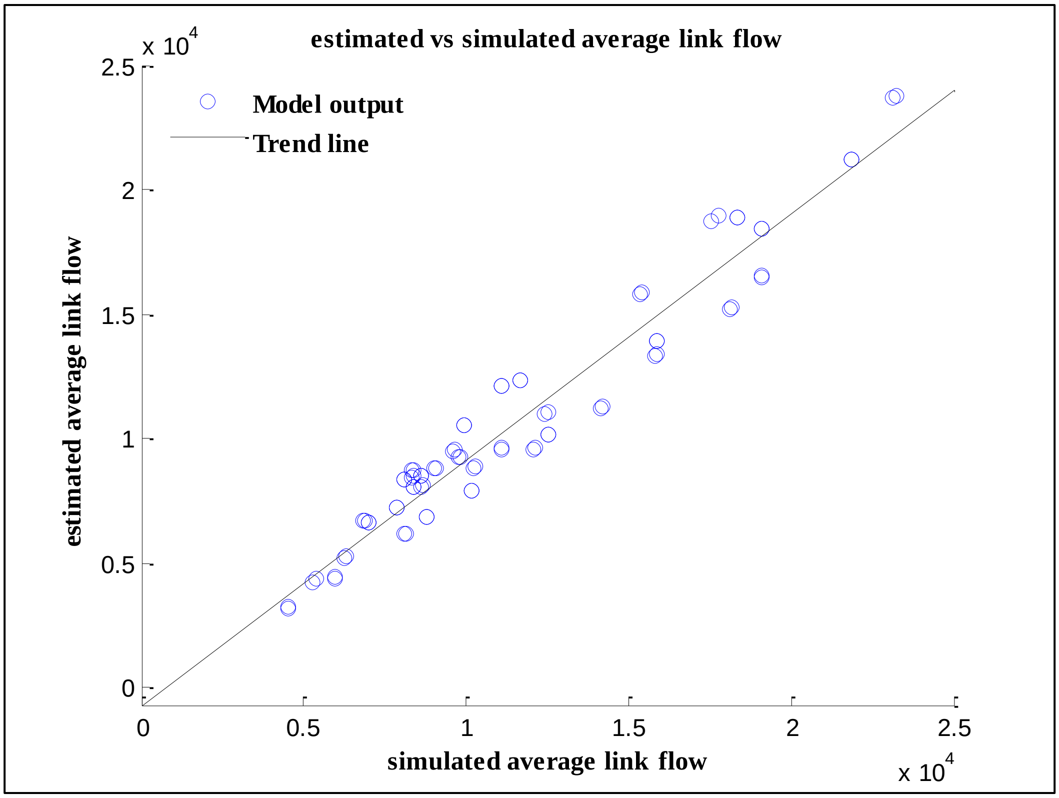



Figure 7-2 The regression analysis of estimated and simulated expected link flow.

Figure 7-3 demonstrates the sparsity level for different $\delta_2$. According to (Hurley and Rickard, 2009, Hoyer, 2004), the Hoyer's formula provides a scalable, normalized and generalized sparsity measure. It is therefore adopted to illustrate the sparsity level of the estimated O-D matrix:

$$H = \left(\sqrt{|M|} - \frac{\sum_{m \in M} s_m}{\sqrt{\sum_{m \in M} s_m^2}}\right) / (\sqrt{|M|} - 1) \qquad 7\text{-}17$$

The weight parameter $\delta_1$ is fixed to 0.1, and $\delta_2$ will vary from 1 to 96 with an increment of 5. It is presented that the estimated O-D matrix becomes sparser with the increase of $\delta_2$. Therefore, as discussed in the previous section, minimization of the L1 norm regularization can induce a sparse solution. Additionally, if it is believed that the true O-D matrix tends to be sparse, a higher weight should be imposed on $\delta_2$.

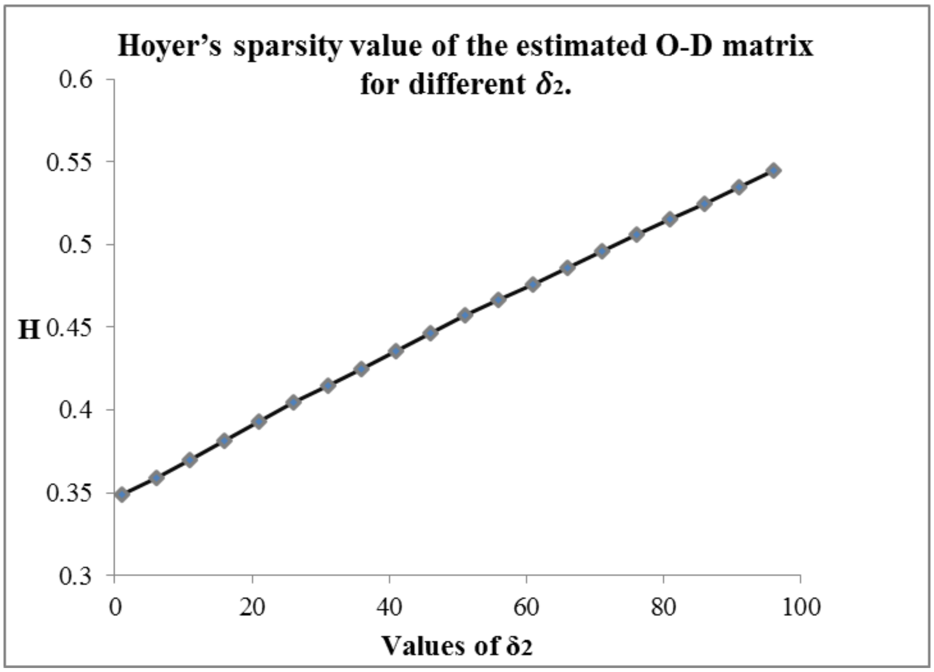



Figure 7-3 Hoyer's sparsity value of the estimated O-D matrix for different $\delta_2$.

## 7.4 Chapter summary

Under-determination is a common issue in O-D matrix estimation problem, to mitigate such an issue, this work incorporates sparsity regularization and link flow correlation in the generalized least squares method. In addition, a specific assignment model, I-STRUE, is implemented to provide users' route information while accounting for demand volatility. The solution to the proposed mathematical formulation has been proved to be unique. Numerical analysis suggests that sparsity regularization can improve estimation quality if treated properly, and the link flows produced based on the estimated O-D matrix can closely approximate the observed link flows. Therefore, by utilizing the features of the sparsity of O-D matrix, as well as the link flow correlation, the model is capable of providing a more robust estimation of O-D matrix.

However, every model has its limitation. The weight parameter for the sparsity regularization term needs to be tuned according to specific cases. Another limitation is that in the proposed model, it is assumed that each O-D demand follows a Poisson distribution, which is not always the case in real world. Hence, extending the model to other distributions can enhance the model's applicability. The partial correlation case may also be a future direction.

# Chapter 8

# Conclusion and research outlook

In the ancient times, a forest is the whole world for a primitive tribe- it is where they flourish, breed and evolve. However, for the people living in a metropolis nowadays, every major city around the world does not seem that far because of the convenience brought by the contemporary means of transportation. Nowadays, transportation plays an essential role in commuting, logistics and public transit in our daily life. Therefore, government and organizations are paying more attention to the transportation planning process which could directly affect the effectiveness and efficiency of their investment and policy.

A variety of planning models are proposed for the transportation planning process focusing on different aspects of transportation. Nowadays, with the increasing availability of data, network stochasticity such as demand volatility and capacity uncertainty continues to be a factor that should not be neglected in planning models. Moreover, the immense complexity when accounting for the stochasticity requires models that can maintain both tractability and computation simplicity.

## 8.1 Chapter summary and contributions

The thesis proposes methodologies to account for stochasticity in two aspects of the transportation planning process: traffic assignment and Origin-Destination travel demand estimation. They are closely related to each other, because a traffic assignment model provides users' route choice information and allocates traffic trips to different road segments of a network, while most statistical O-D matrix estimation models require users' route choice information to infer the O-D matrix. Therefore, improvements in either traffic assignment or O-D demand estimation can benefit the other one.

For the traffic assignment problem, a key notion called STRUE is introduced in the thesis, which assumes that travellers choose a route to minimize their expected travel cost, where their decision is based on knowledge of a demand distribution, rather than a deterministic demand value. They then stick to the route choice strategy regardless of the day-to-day realized demand. STRUE is defined such that at equilibrium, all used paths have equal and minimal expected travel costs, and no user can reduce his expected travel cost by unilaterally switching his routes. The original P-STRUE model assumes each O-D demand is perfectly-correlated with each other; in addition, capacity is assumed to be deterministic. These assumptions limit the applicability of the model; therefore, the thesis generalizes the model and proposed two extensions in Chapter 3 and Chapter 4.

Firstly, the impact of both demand and capacity uncertainties under the STRUE framework is explored in Chapter 3. The problem is formulated as a convex optimization problem. Numerical analysis reveals that both capacity and demand uncertainties have a significant impact on network performance, and capacity uncertainty plays an important role in the prediction of total system travel time variability. Therefore both uncertainties should be incorporated to provide a robust prediction.

Secondly, the assumption of perfectly-correlated O-D demands is relaxed to enhance the applicability of STRUE in Chapter 4, where the proposed I-STRUE model provides a computationally efficient methodology for computing users' O-D specific link choice while accounting for demand volatility. Both O-D specific link choice and user equilibrium link flow have been proved to be unique under the proposed model, which is extremely important to ensure their applicability in transportation planning applications. Both extensions are based on the STRUE framework, so the optimization process has been proved simple and efficient. This guarantees their application in larger scale networks.

An application of P-STRUE has been proposed in Chapter 5 which focuses on the short-term impact of new infrastructure developments. Users are assumed to adapt to the change in traffic demand by gradually learning from their day-to-day travel experience. Simulation results suggest that the users' level of confidence in their initial perception of traffic demand has a significant impact on the learning process, and the higher is the confidence, the shorter is the learning time. Hence information provision should be meticulously considered when new development is proposed.

In addition to the traffic assignment modelling, novel methodologies have been also proposed for Origin-Destination demand estimation. In Chapter 6, a Bi-level programming method has been proposed to calibrate the total demand and its variability by explicitly utilizing day-to-day link flow variation. The method calls on the P-STRUE model, and thus incorporates day-to-day demand volatility into the estimation process. Results on the system level and the link level have demonstrated the scalability, solution uniqueness and computation efficiency of the model. The model's capability to provide prediction error analytically has been also illustrated.

Unlike the total demand calibration, there is an additional challenge in the classic traffic count based O-D estimation problem: the number of demands to be estimated is often much greater than the number of monitored links. Sparsity regularization and link flow correlation have been integrated in the generalized least squares method, and I-STRUE has been implemented to provide users' route information while accounting for demand volatility. The proposed mathematical program can provide a unique estimation of O-D matrix. Numerical analysis has shown an improvement in estimation quality if the weight parameters are tuned properly.

## 8.2 Limitations and possible future research

As has been mentioned at the beginning of the thesis, every model has its applicability, so are the models proposed in this thesis. A number of assumptions are made in these models to ensure tractability, solution uniqueness and computation efficiency. These assumptions may impose limitations to the models, but can be addressed in the future. A key assumption made in the thesis is that the variables which represent the network uncertainties follow certain types of statistical probability distribution. For the C-STRUE model in Chapter 3, it is assumed that road capacity follows gamma distribution. While for the I-STRUE model in Chapter 4 and the O-D estimation model in Chapter 7, demand is assumed to follow Poisson distribution. For the total demand calibration model in Chapter 6 and the learning model in Chapter 5, demand is assumed to follow lognormal distribution. These probability distributions are often subject to their own mathematical limitations notwithstanding the convenience of using them. For example, Poisson distribution implies identical mean and variance, lognormal and gamma distribution may not always fit the empirical data. In this regard, it is interesting to investigate additional probability distributions to further facilitate the usefulness of the models. Furthermore, a generalized framework of STRUE that can accommodate different types of distributions would be a milestone for the research.

Also, the notion of STRUE is based on the assumption that network users tend to stick to the route choice they make based on the expected travel time and are knowledgeable of the demand distribution. However, not all users would make route decision solely on expected travel time. Travel time reliability, road conditions are so forth may also play an important role in their decision making. Further, the literature on travel time variability suggests multiple reasons for the variations on links. O-D demand volatility is one reason, for sure, but driver behaviour and external factors may not be neglected either. The thesis strives to find a compromise between complication and efficiency, so it only considers demand volatility as a source of travel time variation. But to the pursuit of a more accurate prediction, these factors should be taken into account.

Additionally, congestion effects are not accounted for in the assignment models, which may have an impact on model prediction. Hence, a dynamic or quasi-dynamic variation of both C-STRUE and I-STRUE is another area worthy of exploring.

Moreover, the rapid development of ridesharing services (such as UBER) and the potential deployment of connected and autonomous vehicles will bring many challenges in the road traffic system. These user classes may have different routing mechanism and perception on travel time. In the meantime, these technologies urge for transportation models that can fully utilize their advantages in information sharing.

Finally, real world data should be applied to validate the proposed models. The main novelty of the thesis is still on the methodological aspect, which needs to be validated with real world data on large scale networks. With the increasing availability of transport data, transport practitioners can use loop detector data, field survey data or plate number data as the metrics of model performance and validity.

The thesis aims at proposing methodologies which can fundamentally provide more useful insights for researchers and also improve prediction quality of the transportation planning process, so that more reliable decisions can be made by policy makers and investors.

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