Lecture 3

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Recap

We defined a map $\theta_L : \mathcal{D}^1_L(G) \to T_e(G)$ given by just $\theta(X) = X_e$ which was shown to be a linear isomorphism, and this induced a lie-algebra structure on $T_e(G)$ given by

$$[v, w] := \theta_L([\theta_L^{-1}(v), \theta_L^{-1}(w)])$$

And this will be called the lie algebra of the group G and will be denoted as Lie(G). We can also define an anlalogous map $\theta_R : \mathcal{D}^1_R(G) \to T_e(G)$ given by $\theta_R(X) = X_e$ and this will also be a linear isomorphism. And we get a different lie algebra structure on $T_e(G)$.

Exercise. *Describe the relation between the two lie algebra structures on $T_e(G)$.*

Proof. Define the dual group G^* with the same manifold structure but $a * b := b \cdot a$. It is elemntary to verify that this relation satisfies group axioms. Now define the map $\phi : G \to G^*$ by $\phi(a) = a^{-1}$. This is a group isomorphism. Now note that the proposition 4 proven below shows that $d\phi_e$ is a homomorphism of lie algebras. But since ϕ is an involution, $d\phi_e$ is an isomorphism. Hence the two lie algebra structures are isomorphic.
Now observe that If a vector field is left invariant in G^*, then it must be right invariant in G since left multiplication in G^* is right multiplication in G. Also, the lie bracket comes only from the smooth structure. Thus are the same for both G, G^*. Hence we have the desired result. In fact, the derivative of the inverse map is an isomorphism. $\square$

Typically the Lie group is denoted in capital letter and the lie algebra is denoted in the corresponding german gothic letters($\mathfrak{g}$).

1 Homomorphisms

Definition 1

Let $\mathfrak{g}, \mathfrak{h}$ be two lie algebras. A linear map $T : \mathfrak{g} \to \mathfrak{h}$ is called a **lie algebra homomorphism** if

1. ϕ is linear.
2. $T([X, Y]) = [T(X), T(Y)]$ for all $X, Y \in \mathfrak{g}$

Such a homomorphism is called an **isomorphism** if T is bijective, and if there is an isomorphism between $\mathfrak{g}$ and $\mathfrak{h}$ then they are said to be isomorphic. Written as $\mathfrak{g} \cong \mathfrak{h}$

Definition 2

Let G and H be two lie groups and a map $\phi : G \to H$ is called a **lie group homomorphism** if

1. ϕ is a group homomorphism.
2. ϕ is smooth.

Such a homomorphism is called an **isomorphism** if ϕ is a diffeomorphism, and if there is an isomorphism between G and H then they are said to be isomorphic. Written as $G \cong H$

Proposition 3

Let G and H be two lie groups and $\phi : G \to H$ be a lie group homomorphism. Let $X \in \mathcal{D}_L^1(G)$ be such that $X_e = v$. Then $d\phi_e(v) \in T_e(H)$ And let Y be the corresponding left invariant vector field on H. Then $X \sim_\phi Y$

Proof. Let $f \in C^\infty(H)$. We need to show that, $X(f \circ \phi)(a) = Y(f)(\phi(a))$. But we have

$$\begin{aligned}
X(f \circ \phi)(a) &= (d\phi)_a(X_a)(f) \\
&= (d\phi)_a \circ (dl_a)_e(v)(f) \\
&= d(\phi \circ l_a)_e(v)(f) \\
&= d(l_{\phi(a)} \circ \phi)_e(v)(f) \\
&= (dl_{\phi(a)})_e(d\phi_e(v))(f) \\
&= Y_{\phi(a)}(f) \\
&= Y(f)(\phi(a))
\end{aligned}$$

As was desired. □

Let $\mathfrak{g} = Lie(G)$ and $\mathfrak{h} = Lie(H)$ be the lie algebras of G and H respectively. Suppose $\phi : G \to H$ is a lie group homomorphism and $u, v \in \mathfrak{g}$ also let $X^u, X^v \in \mathcal{D}_L^1(G)$ be such that $X_e^u = u$ and $X_e^v = v$. Then $d\phi_e(u), d\phi_e(v) \in \mathfrak{h}$. Let $Y^{d\phi_e u}, Y^{d\phi_e v} \in \mathcal{D}_L^1(H)$ be the corresponding left invariant vectorfields on H. Then $X^u \sim_\phi Y^{d\phi_e u}$ and $X^{d\phi_e v} \sim_\phi Y^v$ by the above proposition. And hence by the proposition 7 of lecture 2 we have $[X^u, X^v] \sim_\phi [Y^{d\phi_e u}, Y^{d\phi_e v}]$. And hence we have $d\phi_e([u, v]) = [d\phi_e(u), d\phi_e(v)]$. Consequently we have the following proposition.

Proposition 4

Let G and H be two lie groups and $\phi : G \to H$ be a lie group homomorphism. Then $d\phi_e : \mathfrak{g} \to \mathfrak{h}$ is a lie algebra homomorphism.

Question. *Let G be $GL_{n\times n}(\mathbb{R})$ and $Lie(G) = \mathfrak{g} = (M_{n\times n}(\mathbb{R}), [,])$. But $M_{(\times(}(\mathbb{R})n)$ has another lie algebra structure given by $[[A, B]] = AB - BA$. But is $[[A, B]] = [A, B]$?*

We intend to show that the answer is yes. First we gather the following facts which will simplify the proof. Suppose E is an open subset of $\mathbb{R}$ and let $(u_1, u_2, .., u_n)$ be the euclidean co-ordinates on E.

1. Any tangent vector $V = \sum_{i=1}^n v_i \frac{\partial}{\partial u_i}$ at a point $a \in E$ can be identified with the vector $(v_1, v_2, .., v_n)$, which is actually an isomorphism between $T_a(E)$ and $\mathbb{R}^n$.
2. If $f : E \to E$ is a linear map restricted to E, then the derivative map when expressed under the isomorphism above is actually f itself.

3. Similarly, if f is a linear functional restricted to E then its derivative under the isomorphism in (1) is again itself.

Now note that $GL_{n\times n}(\mathbb{R})$ is an open subset of $M_{n\times n}(\mathbb{R}) \cong \mathbb{R}^{n^2}$. Denote the orresponding isomorphism between $T_e(GL_{n\times n}(\mathbb{R}))$ and $\mathbb{R}^{n^2}$ by mat . Also for any matrix A, the map $B \mapsto AB$ is a linear map on $M_{n\times n}(\mathbb{R})$. Then, if $a, b \in \mathrm{Lie}(GL_{n\times n}(\mathbb{R}))$, and X^a, X^b are the corresponding left invariant fields,

$$\mathrm{mat}([a,b])_{ij} = [a,b](u_{ij}) = [X^a, X^b]_e(u_{ij})$$

From (1) we know that $\mathrm{mat}[(dl_g)_e(a)] = \mathrm{mat}(g)\mathrm{mat}(a)$. Thus $(X^a)_g(u_{ij}) = [\mathrm{mat}(g)\mathrm{mat}(a)]_{ij}$ And this is a linear map on $\mathbb{R}^{n^2}$ as a function of $\mathrm{mat}(g)$. This allows us to apply (3). Culminating these observations,

$$(X^a)_e(X^b(u_{ij})) = D_{\mathrm{mata}}([\mathrm{mat}(g)\mathrm{mat}(a)]_{ij}) = [\mathrm{mat}(b)\mathrm{mat}(a)]_{ij}$$

Hence,

$$[X^a, X^b]_e(u_{ij}) = [\mathrm{mat}(a)\mathrm{mat}(b) - \mathrm{mat}(b)\mathrm{mat}(a)]_{ij}$$

Hence the mat map is an isomorphism of these lie algebras as was desired.

Exercise. *Read the proof of the same concept from Kumaresan's book.*

2 The determinant map

Question. *What is the derivative of the determinant map?*

We give two approaches to this problem.

Approach 1:
Let $X = (x_{ij})$. Then $\det(X) = \sum_{i=1}^{n} x_{1i}c_{1i}$.Then we have $\frac{\partial \det(X)}{\partial x_{ij}} = c_{1j}$. Thus $\det{}'(X)(H) = \sum_{1\le i,j\le n} c_{ij}h_{ij} = \mathrm{tr}(\mathrm{Adj}(X)H)$.

Approach 2:
First we can compute the derivative at the identity by computing the directional derivative.

$$\det(I + tH) = 1 + t\,\mathrm{tr}(H) + o(t^2)$$

$$\implies \det{}'(I)(H) = \frac{d}{dt}\det(I + tH) = \mathrm{tr}(H)$$

Now if A is invertible, we see that

$$\det(A + tH) = \det(A)\det(I + tA^{-1}H) = \det(A) + t\,\mathrm{tr}(\det(A)A^{-1}H) + o(t^2)$$

$$\implies \det{}'(A)(H) = \frac{d}{dt}\det(I + AH) = \mathrm{tr}(\det(A)A^{-1}H) = \mathrm{tr}(\mathrm{Adj}(A)H)$$

Now note that $GL_{n\times n}(\mathbb{R})$ is a dense subset of $M_{n\times n}(\mathbb{R})$ and the determinant map is smooth. Hence the derivative of the determinant map is the same as the above expression for all $A \in M_{n\times n}(\mathbb{R})$.

Remark. *Another approach to the same result is possible simply using the fact that the determinant map is a multilinear map.*

Exercise. *Show that for any lie group $m : G \times G \to G$, the multiplication map, is a sumbersion.*

Proof. Pick a point $(a, b) \in G \times G$. Note that the tangent space at (a, b) is isomorphic to $T_a(G) \times T_b(G)$. So, it suffices to compute the action of $dm_{(a,b)}$ on each of these spaces individually. Now if $(0, v) \in T_a(G) \times T_b(G)$, then there must be a curve (say γ), fully contained in $a \times G$ such that the tangent vector to the curve at 0 is $(0, v)$. Then the multiplication map is basically the left multiplication by a on γ. Hence the derivative is $dl_a(v)$. Applying the same argument to the other side, we get

$$dm_{(a,b)}(u, v) = dl_a(v) + dr_b(u)$$

Since dl_a is itself surjective, it is clear that the total map is surjective as well. $\square$