Lecture 2

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1 Lie subgroups

Proposition 1

Let G be a lie group. Let H be an abstract subgroup of G, such that

1. H is a submanifold of G
2. H is a topological group.

Then H is a liegroup.

Proof. It suffices to show that the μ map given by $(x, y) \mapsto x \cdot y^{-1}$ is smooth. The fact that H is a topological group implies that this map is continuous. Theorem 9 of the previous lecture implies that the map is smooth. $\square$

Definition 2

An abstract subgroup satisfying the proberites of the proposition is called a **Lie subgroup** of G.

Proposition 3

If G_1, G_2 are two lie groups, then the product $G_1 \times G_2$ is a lie group under the product manifold structure.

Proof. exercise. $\square$

We will now show that $O_{n\times n}(\mathbb{R})$ is a lie group. First we will show that it is an embedded manifold in $GL_{n\times n}(\mathbb{R})$. Consider the map $f(A) = AA^T$. Then,

$$f'(A)(H) = AH^T + HA^T$$

Observe that $\mathrm{Im}(f') \subseteq S(n)$, the vector space of all symmetric matrices. Conversely, if $M \in S(n)$ we have,

$$f'(A)(\frac{SA}{2}) = A\frac{A^TS}{2} + \frac{SA}{2}A^T = S$$

This tells us that we shouls consider f as a map from $GL_{n\times n}(\mathbb{R})$ to $S(n)$. Then I is a regular value of f, meaning that $f^{-1}(I)$ is an embedded submanifold of $GL_{n\times n}(\mathbb{R})$. It is clearly a topological group, as the μ map is simply obtained by retriction from $GL_{n\times n}(\mathbb{R})$ and thus proposition 1 applies.

2 Lie algebras

Definition 4

Let V be a vector space over a field $\mathbb{K}$. V is said to be a **lie algebra** if it has a bilinear map satisfying,

1. $[X, X] = 0$
2. $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$

For all $X, Y, Z \in V$.

Remark. *If* $\operatorname{char} \mathbb{K} \neq 2$, *then the first condition is equivalent to* $[X, Y] = -[Y, X]$.

In this course, $\mathbb{K}$ will always be $\mathbb{R}$ or $\mathbb{C}$.

Example 5

We give a list of them.

1. Any vector space with the bracket $[X, Y] = 0$ is a lie algebra.
2. $(\mathbb{R}^{\nVdash}, \times)$ is a lie algebra.
3. $(M_{n\times n}(\mathbb{R}), [A, B] = AB - BA)$ is a lie algebra.
4. The same can be done for any associative algebra over $\mathbb{K}$.
5. $sl_n(\mathbb{K}) = \{A \in M_{n\times n}(\mathbb{R}) : \operatorname{tr}(A) = 0\}$ is a lie sub-algebra.
6. $so_n(\mathbb{R}) = \{A \in M_{n\times n}(\mathbb{R}) : A^T = -A\}$ is a lie sub-algebra.
7. For any smooth manifold, $\mathcal{D}^1(M)$ is a lie algebra under the bracket $[X, Y] = XY - YX$.

Definition 6

Let $\phi : M \to N$ be a smooth map between manifolds. Let $X \in \mathcal{D}^1(M)$ and $Y \in \mathcal{D}^1(N)$. We say X, Y are **ϕ-related** and write $X \sim_\phi Y$ or $d\phi(X) = Y$ if for all smooth functions f on N, we have $X(f \circ \phi) = Y(f) \circ \phi$. Or equivalently, at each point $p \in M$, we have $d\phi_p(X_p) = Y_{\phi(p)}$.

Proposition 7

Let $\phi : M \to N$ be a smooth map.

(a) Let X_1, X_2 be two vector fields on M and Y_1, Y_2 be two vector fields on N. If $d\phi(X_1) = Y_1$ and $d\phi(X_2) = Y_2$ then,

  (1) $d\phi([X_1, X_2]) = [Y_1, Y_2]$

  (2) $d\phi(X_1 + X_2) = Y_1 + Y_2$

(b) Let $X \in \mathcal{D}^1(M)$, $Y \in \mathcal{D}^1(N)$ and $d\phi(X) = Y$. Then, $d\phi(aX) = aY$ for all $a \in \mathbb{R}$.

Proof.

$$\begin{aligned}
[X_1, X_2](f \circ \phi) &= X_1(X_2(f \circ \phi)) - X_2(X_1(f \circ \phi)) \\
&= X_1(Y_2(f) \circ \phi) - X_2(Y_1(f) \circ \phi) \\
&= Y_1(Y_2(f)) \circ \phi - Y_2(Y_1(f)) \circ \phi \\
&= [Y_1, Y_2](f) \circ \phi
\end{aligned}$$

As desired. The others follow from linearity of derivatives. □

Definition 8

Let G be a lie group. We say that $X \in \mathcal{D}^1(G)$ is left invariant if for all $g \in G$, we have $dl_g(X) = X$. Similarly, it is right invariant if $dr_g(X) = X$.

Exercise. *Let G be $\mathbb{R}^n$. Let $X \in \mathcal{D}^1(G)$. Let $\{u_1, .., u_n\}$ be co-ordinate functions. Let $X(p) = \sum_{i=1}^{n} a_i(p) \frac{\partial}{\partial u_i}$. Show that X is left invariant if and only if a_i are constants.*

Remark. *In the abelian case, left and right invariant vector fields are the same, and we shall simply call them invariant vector fields.*

We will use the notation $\mathcal{D}^1_L(G), \mathcal{D}^1_R(G)$ for the set of left and right invariant vector fields respectively.

Proposition 9

Let G be a lie group. Then, $\mathcal{D}^1_L(G)$ and $\mathcal{D}^1_R(G)$ are lie sub-algebras of $\mathcal{D}^1(G)$.

Proof. The last proposition makes this trivial. Suppose $X, Y \in \mathcal{D}^1_L(G)$. Then,

$$\begin{aligned}
dl_g(X + Y) &= dl_g(X) + dl_g(Y) = X + Y \\
dl_g(aX) &= a dl_g(X) = aX \\
dl_g([X, Y]) &= [dl_g(X), dl_g(Y)] = [X, Y]
\end{aligned}$$

Same proof for the right invariant case. □

Observe that if $X \in \mathcal{D}^1_L(G)$, then $X(a) = (dl_a)_e(X(e))$ And thus X is completely determined by the value at identity. This leads us to th following map, $\theta_L : \mathcal{D}^1_L(G) \to T_eG$ given by $X \mapsto X(e)$ It is easy to see that this map is linear and injective. And we prove the following natural claim.

Claim. *θ_L is onto.*

Proof. Let $v \in T_eG$. Define $(X^v)_g = (dl_g)_e(v)$. We need to check it is smooth. It is equivalent to showing, for any smooth f, $a \mapsto (X^v)_a(f)$ is smooth. This follows from the observation that, if v corresponds to differentiation along the smooth curve $\gamma(t)$, then $(X^v)_a(f) = [(dl_a)_e(v)](f) = v(f \circ l_a) = \frac{d}{dt}|_{t=0} f(a\gamma(t))$. Consider the maps,

$$G \times (-\epsilon, \epsilon) \to G \times G \to G \to \mathbb{R}$$

$$(a, \epsilon) \mapsto (a, \gamma(t)) \mapsto a \cdot \gamma(t) \mapsto f(a \cdot \gamma(t))$$

The smoothness of the composite map is evident. Now consider a product co-ordinate neighbourhood of $(a, 0)$ in $G \times (-\epsilon, \epsilon)$. As the map is smooth, its derivative with respect to t is also smooth. In particular, smooth in a. Now we check that it is left invariant.

$$(dl_a)_g(X^v)_g = (dl_a)_g \circ (dl_g)_e(v) = (dl_a \circ dl_g)_e(v) = (dl_{ag})(v) = (X^v)_{ag}$$

as desired. □

Since we have shown that θ_L is an isomorphism, and that $\mathcal{D}^1_L(G)$ is a lie algebra, we have a natural lie algebra structure on T_eG.

Definition 10

Given a lie group G, we define the **lie algebra of G** is $T_e(G)$ with the lie algebra structure derived above.

Exercise 11. *Show that the lie algebra of $GL_{n\times n}(\mathbb{R})$ is isomporphic to $M_{n\times n}(\mathbb{R})$.*