# Lecture 6

Muna Naik

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## Canonical co-ordinate neighbourhoods

We have shown in propositon 4 of the previous lecture that $exp$ map is a local diffeomorphism from the lie algebra to the lie group. Suppose $G$ is the lie group and $\mathfrak{g}$ its lie algebra(as always!) and let $U_0, V_e$ be neighbourhoods of $0 \in \mathfrak{g}, e \in G$ respectively such that $exp|_{U_0} : U_0 \to V_e$ is a diffeomorphism. Then it is clear that the inverse map $\xi : V_e \to U_0$ given by $exp(\vec{x}) \mapsto \vec{x}$ is a co-ordinate system on $V_e$. Such co-ordinate systems are called canonical co-ordinate systems.

## Matrix exponentials

Suppose $G = GL_{n\times n}(\mathbb{C})$ and $\mathfrak{g} = \backslash$. It has been shown in the presentations that the map given by the power series

$$\theta(A) = \sum_{k\geq 0} \frac{A^k}{k!}$$

is smooth and satisfies $exp(X) = \theta(X)$. We will see a different approach to some of the results.

**Proposition 1**

The map $\theta$ has the following properties.

1. $\theta$ is continuous.
2. $AB = BA \implies \theta(A+B) = \theta(A)\theta(B)$.
3. $\theta((s+t)A) = \theta(sA)\theta(tA)$
4. $\theta(-A) = \theta(A)^{-1}$.

*Proof.* Note that $A \mapsto A^k/k!$ is continuous with $\|A^k/k!\| \leq \|A\|^k/k!$. Thus

$$\|\theta(A)\| \leq \sum_{k\geq 0} \frac{\|A\|^k}{k!} \leq e^{\|A\|}$$

So for all $A$ such that $\|A\| \leq R$ we can use the wierstrass M-test to show that $\theta$ is continuous. As $R$ goes to infinity we see that $\theta$ is continuous everywhere. Note that since the series are absolutely convergent we can use an argument identical to that of Theorem 3.50 in Rudin's Principles of Mathematical Analysis to argue that

$$\theta(A)\theta(B) = \left(\sum_{k\geq 0} \frac{A^k}{k!}\right)\left(\sum_{l\geq 0} \frac{B^l}{l!}\right) = \sum_{n\geq 0} \sum_{k+l=n} \frac{A^k B^l}{k!l!}$$

Given that $AB = BA$. Now we can further manipulate the series using binomial theorem to arrive at the following.

$$\theta(A)\theta(B) = \sum_{n\geq 0} \frac{(A+B)^n}{n!} = \theta(A+B)$$

The 3rd property follows from the second by setting $A = sA, B = tA$. The fourth property follows from the second by setting $B = -A$. $\square$

The 3rd property shows that $\alpha(t) := \theta(tH)$ is a group homomorphism from $\mathbb{R}$ to $GL_{n\times n}(\mathbb{C})$. Now we will show that it is differentiable.

**Proposition 2**
Let $A \in M_n(\mathbb{C})$. Let $\alpha : \mathbb{C} \to \backslash$ given by

$$\alpha(t) = e^{tA}$$

is once differentiable and $\alpha'(t) = \alpha(t)A$.

*Proof.* Observe that

$$\alpha(t+s) - \alpha(t) = \theta(tA+sA) - \theta(tA) = \theta(tA)(\theta(sA) - I) = \theta(tA)\left(sA + \sum_{k\geq 2} \frac{s^k A^k}{k!}\right)$$

Note that $k! \geq 2^{k-1}$ for all $k \geq 2$. Thus

$$\|\alpha(t+s) - \alpha(t) - s\theta(tA)A\| \leq \sum_{k\geq 2} \frac{\|sA\|^k}{k!} \leq \theta(tA)\sum_{k\geq 2} \frac{\|sA\|^k}{2^{k-1}} = \theta(tA)|s|^2\|A\|^2 \frac{\|sA\|^k}{2^k}$$

For $s$ sufficiently small $\|sA\|$ is sufficiently small so that the series in the end is a convergent geometric series. Thus

$$\frac{\|\alpha(t+s) - \alpha(t) - s\theta(A)A\|}{|s|} \leq |s|M$$

for some $M$ and sufficiently small $s$. Thus $\alpha$ is differentiable at $t$ and $\alpha'(t) = \alpha(t)A$. $\square$

Note that this shows that $\alpha'(t) = \alpha(t)A$. But the map $X \mapsto \alpha(t)X$ is a linear map and thus its own derivative. So we have $\alpha'(t) = dl_{\alpha(t)}A$. Thus $\alpha$ is an integral curve of the left invariant vector field corresponding to the tangent vector $A$ at $I$ of the lie group $GL_{n\times n}(\mathbb{C})$. But by uniqueness of integral curves and definition of exponentials we have $\alpha(t) = \gamma_{\tilde{A}}(t) = exptA$. (The notation $\tilde{A}$ is used to denote the left invariant vector field corresponding to $A$). The following is an exersice regarding a mistake in one of the presentations.

**Exercise.** *Show that the set of positive definite matrices is open in the space of all Hermitian matrices.*

*Proof.* Let $A \in Herm_n(\mathbb{C})$ be positive definite. Set $v \in \mathbb{C}^n - 0$. Let $H$ be a Hermitian matrix. Since $A$ is positive definite the quantity $\langle v, Av\rangle$ has a positive minimum $C$ on the unit sphere in $\mathbb{C}^n$. Thus

$$\langle v, Av\rangle = \|v^2\|\langle \frac{v}{\|v\|}, A\frac{v}{\|v\|}\rangle > C\|v\|^2$$

Also observe that $|\langle v, Hv\rangle| \leq \|v\|\|Hv\| \leq \|v\|^2\|H\|_{op}$. So for $\|H\|_{op} \leq C/2$ we have

$$\langle v, (A+H)v\rangle = \langle v, Av\rangle + \langle v, Hv\rangle$$

$$\implies \langle v, (A+H)v\rangle \geq C\|v\|^2 - \frac{C}{2}\|v\|^2 > 0$$

Since $v$ was arbitrary we have that $A+H$ is positive definite. So we have found an open ball around $A$ in the space of Hermitian matrices that is contained in the set of positive definite matrices and thus the set of positive definite matrices is open. □

# Topological groups

We continue with the digression onto topological groups started in the previous lecture. With the notation from the previous lecture we observe the following.

**Exercise.** *$G/G_0$ with quotient topology is a topological group*

*Proof.* Let $\mu : G \times G \to G$ be given by $\mu(x,y) = xy^{-1}$. And $\tilde{m}u : G/G_0 \times G/G_0 \to G/G_0$ given by $(xG_0) \cdot (yG_0) \to xy^{-1}G_0$. Then we have the following diagram.

$$\begin{array}{ccc} G \times G & \xrightarrow{\mu} & G \\ {\scriptstyle \pi\times\pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ G/G_0 \times G/G_0 & \xrightarrow{\tilde{\mu}} & G/G_0 \end{array}$$

The maps $\mu, \pi, \pi \times \pi$ are all continuous. The diagram is commutatuive simply by definition of quotient groups. We will be done once we show that $\tilde{\mu}$ is continuous. Set $p = (p_1G_0, p_2G_0) \in G/G_0 \times G/G_0$ and $U$ an open neighbourhood of $\tilde{\mu}(p)$. Since $\mu' := \pi \circ \mu$ is continuous and $\mu'(p_1, p_2) = \tilde{m}u(p)$ we have $V_1, V_2$ open neighbourhoods of $p_1, p_2$ respectively such that $\mu'(V_1 \times V_2) \subseteq U$. Note that this means $\mu'(V_1G_0 \times V_2G_0) \subseteq U$ and by definition of quotient map $\pi$, $N_1 = \pi(V_1G_0)$ is open and so is $N_2 = \pi(V_2G_0)$. So

$$\tilde{m}u(N_1 \times N_2) = \tilde{m}u \circ (\pi \times \pi)(V_1 \times V_2) = \mu'(V_1 \times V_2) \subseteq U$$

And clearly $p \in N_1 \times N_2$. This shows that $\tilde{m}u$ is continuous at $p$ and since $p$ was arbitrary we have that $\tilde{m}u$ is continuous. □

**Exercise.** *$G$ is hausdorff $\implies$ $G/G_0$ is hausdorff.*

*Proof.* $G_0$ is closed since it is a connected component. Hence $G - G_0$ is open. Note that $G - G_0 = \pi^{-1}(G/G_0 - eG_0)$. By definition of quotient topology we get $G/G_0 - eG_0$ is open. This shows that the identity is closed in $G/G_0$. Since left multiplications are homeomorphisms this argument easily shows that all points are closed in $G/G_0$. We show that following claim.

**Claim.** *In any topological group $G$ if all points are closed then $G$ is hausdorff.*

*Proof.* Suppose $e \neq y \in G$. Then there must be $U$ a neighbourhood of $e$ such that $y \notin U$ since $y$ is closed. By a result in the previous lecture we have a symmetric neighbourhood $V$ of $e$ such that $V^2 \subseteq U$. Note that if $z \in V \cap yV$ then we have $z = x_1y = x_2$ for some $x_1, x_2 \in V$. Thus $y = x_1^{-1}x_2 \in V^2 \subseteq U$ which is a contradiction. Thus $V \cap yV = \emptyset$. So $V$ and $yV$ are disjoint neighbourhoods of $e$ and $y$ respectively. For arbitrary $x, y \in G$ we can find disjoint neighbourhoods of $e$ and $x^{-1}y$ and then apply the homeomorphism $l_x$ to get the desired. □

□

**Exercise.** *If $H$ is an open subgroup of a connected topological group $G$ then $G = H$.*

*Proof.* We have shown that open subgroups are automatically closed. Since $G$ is connected and $H$ is non-empty we have that $G = H$. □

Now consider the $exp$ map from $\mathfrak{g} \to G$ for a lie group $G$. Since $\mathfrak{g}$ is connected we have that $exp(\mathfrak{g}) \subseteq G_0$. Now consider a canonical neighbourhood $U_0$ of 0 which is connected in $\mathfrak{g}$. Then it is clear that $exp(U_0)$ is open in $G$ and is contained in $G_0$. By the definition of subspace topology we have that $exp(U_0)$ is an open neighbourhood of identity in $G_0$. Then we have that $\langle U_0 \rangle$ is open in $G_0$ by the last exercise in lecture 5 and thus $G_0 = \langle exp(U_0) \rangle$ by the exercise above. As a corollary(again of the last exercise in the previous lecture) we have

$$g \in G_0 \implies \exists X_1, ..., X_k \in U_0 : g = exp(X_1)exp(X_2)...exp(X_k)$$

We have the following proposition as a corollary of the above discussion.

**Proposition 3**

Let $G$ be a lie group with lie algebra $\mathfrak{g}$. Then

1. $G_0$ is open in $G$.
2. Any $g \in G_0$ can be written as $exp(X_1)exp(X_2)...exp(X_n)$ for some $\{X_k\} \subseteq \mathfrak{g}$.
3. The group $G/G_0$ has a discrete structure.

*Proof.* Since $G$ is a manifold it is locally connected and this means that connected components are open $\implies G_0$ is open. The second point follows from the discussion above. For the third point note that for a point $xG_0 \in G/G_0$ we have $\pi^{-1}(xG_0) = xG_0 = l_x(G_0)$. Since $G_0$ is open and $l_x$ is a homeomorphism we have $\pi^{-1}(xG_0)$ is open. Which means $\{xG_0\}$ is open in $G/G_0$. Thus $G/G_0$ has the discrete topology. □

We have seen that given any homomorphism of lie groups we get a corresponding homomorphism of lie algebras. We now show that this association is unique for connected lie groups.

**Proposition 4**

Let $G$ be a connected lie group with lie algebra $\mathfrak{g}$ and $H$ be a lie group with lie algebra $\mathfrak{h}$. Let $\phi, \psi : G \to H$ be homomorphisms of lie groups. Then if $d\phi_0 = d\psi_0$ then $\phi = \psi$.

*Proof.* Since $G = G_0$ we use the previous proposition to write $g = exp(X_1)exp(X_2)...exp(X_n)$. Then

$$\begin{aligned}
\phi(g) &= \phi(exp(X_1)exp(X_2)...exp(X_n)) \\
&= \phi(exp(X_1))\phi(exp(X_2))...\phi(exp(X_n)) \\
&= exp(d\phi_0(X_1))exp(d\phi_0(X_2))...exp(d\phi_0(X_n)) \\
&= exp(d\psi_0(X_1))exp(d\psi_0(X_2))...exp(d\psi_0(X_n)) \\
&= \psi(exp(X_1))\psi(exp(X_2))...\psi(exp(X_n)) \\
&= \psi(exp(X_1)exp(X_2)...exp(X_n)) \\
&= \psi(g)
\end{aligned}$$

In steps 2,4 we have used Theorem 6 from lecture 5. □