Lecture 7

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We will gather certain elementary properties of left invariant fields on a Lie group G which will be used later. Throughout let G be a Lie group with Lie algebra $\mathfrak{g}$ and given $X \in T_e(G)$ we denote by $\tilde{X}$ the left invariant vector field on G corresponding to X.

Proposition 1

Say $f \in C^\infty(G, \mathbb{R}), g \in G, X \in T_e(G)$ we have

$$(\tilde{X}f)(g) = \left(\frac{d}{dt}\right)_{t=0} f(g\exp(tX))$$

And

$$(\tilde{X}f)(gexp(uX)) = \frac{d}{du} f(g\exp(uX))$$

Proof. Define $\gamma(t) = f(g\exp(tX))$ then observe that $\gamma = f \circ l_g \circ \exp(tX)$. And thus by chain rule we have

$$d\gamma_{t=0}\left(\frac{\partial}{\partial t}\right)_{t=0} = Df_g\left(dl_g(\exp(tX)'(0))\right) = Df_g\left(dl_g(X)\right) = Df_g\left(\tilde{X}_g\right)$$

Suppose y is some co-ordinate on $\mathbb{R}$ then we have with the co-ordinate representation of γ, f

$$\left(\frac{d}{dt}\right)_{t=0} f(g\exp(tX)) = \left(\frac{d}{dt}\right)_{t=0} (y \circ \gamma) = d\gamma_{t=0}\left(\frac{\partial}{\partial t}\right)_{t=0} (y) = Df_g(\tilde{X_g})(y) = \tilde{X}_g(y \circ f) = \tilde{X}(f \circ y)(g)$$

Let y be the identity map and we are done.From the above it follows that

$$\tilde{X}f(g\exp(uX)) = \left(\frac{d}{dt}\right)_{t=0} f(g\exp(uX)\exp(tX)) = \left(\frac{d}{dt}\right)_{t=0} f(g\exp((u+t)X)) = \frac{d}{du} f(g\exp(uX))$$

By the definition of derivatives. □

Proposition 2

Let $\{X_k\}_{k\in\mathbb{N}} \subseteq \mathfrak{g}$. Then the following hold

1. $(\tilde{X}_1..\tilde{X}_k f)(g) = \left(\frac{\partial^k}{\partial t_1 \ldots \partial t_k}\right)_{t_1=..=t_k=0} f(g\exp(t_1X_1)..\exp(t_kX_k))$
2. $(\tilde{X}_1^n f)(g) = \left(\frac{d^n}{dt^n}\right)_{t=0} f(g\exp(tX_1))$

Proof. We prove the first part by induction. The base case is the previous proposition. Suppose the first part holds for $k-1$ then we have

$$\begin{aligned}
(\tilde{X}_1..\tilde{X}_k f)(g) &= (\tilde{X}_1(\tilde{X}_2..\tilde{X}_k f))(g) \\
&= \left(\frac{d}{dt}\right)_{t=0} (\tilde{X}_2..\tilde{X}_k f)(g\exp(tX_1)) \\
&= \left(\frac{d}{dt}\right)_{t=0} \left(\left(\frac{\partial^{k-1}}{\partial t_2 ... \partial t_k}\right)_{t_2=..=t_k=0} f(g\exp(tX_1)\exp(t_2X_2)..\exp(t_kX_k))\right) \\
&= \left(\frac{\partial^k}{\partial t_1 ... \partial t_k}\right)_{t_1=..=t_k=0} f(g\exp(t_1X_1)..\exp(t_kX_k))
\end{aligned}$$

For the second part we again induct on n. The base case is again the previous proposition. Suppose the second part holds for $n-1$ then we have

$$(\tilde{X}_1^n f)(g) = [\tilde{X}_1^{n-1}(\tilde{X}_1 f)](g) = \left(\frac{d^n}{dt^{n-1}}\right)_{t=0} (\tilde{X}_1 f)(g\exp(tX_1)) = \left(\frac{d^n}{dt^n}\right)_{t=0} f(g\exp(tX_1))$$

□

Remark. *All the above propositions work for f being smooth in a neighbourhood of g. Also one can deal with f taking values in $\mathbb{R}^k$ by defining $\tilde{X}f = (\tilde{X}f_1, .., \tilde{X}f_k)$ where $f = (f_1, .., f_k)$ and $\tilde{X}$ is any vector field on G.*

Let $f \in C^\infty(G)$. Let $X_1, .., X_k \in \mathfrak{g}$ and define $F : \mathbb{R}^k \to \mathbb{R}$ by

$$F(t_1, .., t_k) = f(g\exp(t_1X_1)..\exp(t_kX_k))$$

Note that $F(0) = f(e)$. Also for $1 \le i < j \le k$ we have

$$\frac{\partial F}{\partial t_i}(0) = \left(\frac{d}{dt}\right)_{t=0} f(g\exp(tX_i)) = (\tilde{X}_i f)(e)$$

$$\frac{\partial F}{\partial t_i \partial t_j}(0) = \left(\frac{\partial}{\partial t_i \partial t_j}\right)_{t_i=t_j=0} f(g\exp(X_it)\exp(X_jt)) = (\tilde{X}_i\tilde{X}_j f)(e)$$

$$\frac{\partial^2 F}{\partial t_i^2}(0) = \left(\frac{d^2}{dt^2}\right)_{t=0} f(g(exp(tX_i))) = (\tilde{X}_i^2 f)(e)$$

All from proposition 2. Now note that being a composition of smooth functions F is itself smooth. Thus we have by Taylor's theorem if $h = (h_1, .., h_k) \in \mathbb{R}^k$,

$$F(h) = F(0) + \sum_{1 \le i \le n} h_i \partial_i F + \frac{1}{2} \sum_{1 \le i,j \le n} h_i h_j \partial_i \partial_j F(0) + o(|h|^3)$$

By substituting the derivatives obtained above we get

$$f(exp(h_1X_1)exp(h_2X_2)..exp(h_kX_k)) = f(e) + \sum_{1 \le i \le k} h_i \tilde{X}_i f(e) + \frac{1}{2} \sum_{1 \le j \le k} h_i^2 \tilde{X}_i^2 f(e) + \sum_{1 \le i < j \le k} h_i h_j (\tilde{X}_i \tilde{X}_j f)(e) + o(|h|^3)$$

We have arrived at the following proposition.

Proposition 3

Let U be an open neighbourhood of e in G and let $f \in C^\infty(U, \mathbb{R}^n)$. Let $X_1, .., X_k \in \mathfrak{g}$ the for all sufficiently small t we have

$$f(exp(tX_1)exp(tX_2)..exp(tX_k)) = f(e) + t \sum_{1 \le i \le k} \tilde{X}_i f(e) + \frac{t^2}{2}\left(\sum_{i=1}^{k} \tilde{X}_i^{\,2} f(e) + 2 \sum_{1 \le i < j \le k} \tilde{X}_i \tilde{X}_j f(e)\right) + o(|h|^3)$$

Towards Cartan's theorem

The purpose of the discussion above is to derive certain formulas which are weaker versions of the Baker-Campbell-Hausdorff formula which will be sufficient to prove the Cartan's theorem(In the following lectures) and for all our purposes in this course. Let U_0, V_e be canonical neighbourhoods of $0 \in \mathfrak{g}, e \in G$ respectively. The exp map is a diffeomorphism from U_0 to V_e. Let $\exp^{-1}$ denote the inverse of exp on V_e. We want to set f to be exp^{-1} in proposition 3. Note that $exp^{-1}(e) = 0$ and by proposition 2 if $X, Y \in \mathfrak{g}$ are distinct elements,

$$\tilde{X}^n \exp^{-1}(e) = \left(\frac{d^n}{dt^n}\right)_{t=0} exp^{-1}(exp(tX)) = \begin{cases} 0 & \text{if } n \ge 2 \\ X & \text{if } n = 1 \end{cases}$$

Note that $2\tilde{X}\tilde{Y} - [\tilde{X}, \tilde{Y}] = \tilde{X}\tilde{Y} + \tilde{Y}\tilde{Y} = (\tilde{X} + \tilde{Y})^2 - \tilde{X}^2 - \tilde{Y}^2$. So the above means

$$2\tilde{X}\tilde{Y} f(e) - [\tilde{X}, \tilde{Y}] f(e) = 0$$

$$\implies \tilde{X}\tilde{Y} f(e) = \frac{1}{2}[\tilde{X}, \tilde{Y}] f(e) = \frac{1}{2}[X, Y]$$

With these observations proposition 3 gives us

$$\exp^{-1}(\exp(tX_1)\exp(tX_2)\ldots\exp(tX_k)) = t(\sum_{1 \le i \le k} X_i) + \frac{t^2}{2} \sum_{1 \le i < j \le k} [X_i, X_j] + o(t^3)$$

Proposition 4

In the set up above

$$\exp(tX_1)\exp(tX_2)\ldots\exp(tX_k) = \exp\left(t(\sum_{1 \le i \le k} X_i) + \frac{t^2}{2} \sum_{1 \le i < j \le k} [X_i, X_j] + o(t^3)\right)$$

We have the following important corollary

Lemma 5

Say $X, Y \in \mathfrak{g}$. For sufficiently small t we have

1. $\exp(tX)\exp(tY) = \exp(tX + tY + \frac{t^2}{2}[X, Y] + o(t^3))$
2. $\exp(tX)\exp(tY)\exp(-tX)\exp(-tY) = \exp(t^2[X, Y] + o(t^3))$

Proof. Just substitute appropriately and simplify in the LHS of proposition 4. □