Predicate Logic

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Example: MSc regulations

pe: pass exams
pc: pass courseworks
pp: pass projects
re: retake exams
ce: cheat in exams

In propositional logic:

$\mathbf{pe \wedge pc \wedge pp \rightarrow pm}$

$\mathbf{(\neg pc \vee ce) \rightarrow (\neg\, pm \wedge \neg\, re)}$

Not expressive enough if we want to consider individual students, to check who has passed the MSc, and who has not, for example.

Example

John:
passes the coursework
cheats in exams

Mary:
passes the coursework
passes exams
passes projects

Who passes the MSc?

Increase the expressive power of the formal language by adding

- predicates
- variables
- quantification.

E.g.

For all individuals X:

$\mathbf{pe(X) \wedge pc(X) \wedge pp(X) \rightarrow pm(X)}$

For all individuals X:

$\mathbf{(\neg pc(X) \vee ce(X)\,) \rightarrow (\neg\, pm(X) \wedge \neg\, re(X)\,)}$

Now given:

pc(john)
ce(john)

pc(mary)
pe(mary)
pp(mary)

We can conclude:

¬ pm(john)
¬ re(john)

pm(mary)

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More formal expression of the MSc regulations

$\forall$**X (pe(X) $\wedge$ pc(X) $\wedge$ pp(X) $\rightarrow$ pm(X))**

$\forall$**X((¬pc(X) $\vee$ ce(X)) $\rightarrow$ (¬ pm(X) $\wedge$ ¬ re(X)))**

$\forall$: **Universal Quantifier**

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Another example

Every student has a tutor.
for all X
(if X is a student then
there is a Y such that Y is tutor of X)

$\forall$**X (student(X) $\rightarrow \exists$ Y tutor(Y,X))**

$\exists$: **Existential Quantifier**

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The Predicate Logic Language Alphabet:

- **Logical connectives (same as propositional logic): $\wedge\ \vee\ \neg\ \rightarrow\ \leftrightarrow$**
- **Predicate symbols (as opposed to propositional symbols):a set of symbols each with an associated arity>=0.**
- **A set of constant symbols.**
  E.g. mary, john, 101, 10a, peter_jones
- **Quantifiers $\forall\ \exists$**
- **A set of variable symbols. E.g. X, Y, X1, YZ.**

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Arity

In the previous examples:

Predicate Symbol	Arity
student	1
tutor	2
pm	1
pp	1

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A predicate symbol with
arity = 0 is called a **nullary predicate**,
arity = 1 is called a **unary predicate,**
arity = 2 is called a **binary predicate.**

A predicate symbol with arity=n (usually n>2) is called an **n-ary** predicate.

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Definition:

A **Term** is any constant or variable symbol.

Syntax of a grammatically correct sentence (wff) in predicate logic

- **p(t1,..., tn) is a wff if p is an n-ary predicate symbol and the ti are terms.**
- **If W, W1, and W2 are wffs then so are the following:**

  $\neg$ W   W1 $\wedge$ W2   W1 $\vee$ W2

  W1 $\rightarrow$ W2   W1 $\leftrightarrow$ W2

  $\forall$X(W) $\exists$X(W)

  where X is a variable symbol.
- **There are no other wffs.**

From the description above you can see that propositional logic is a special case of predicate logic.

Convention used in most places in these notes:

- Predicate and constant symbols start with lower case letters.
- Variable symbols start with upper case letters.

Examples

The following are wffs:

1. **$\neg$ married(john)**
2. **$\forall$X($\neg$ married(X) $\rightarrow$ single(X) $\vee$ divorced(X) $\vee$ widowed(X))**
3. **$\exists$X (bird(X) $\wedge$ $\neg$ fly(X))**

The following are not wffs:

4. $\neg$ X
5. single(X) $\rightarrow$ $\forall$Y
6. $\forall\exists$X (bird(X) $\rightarrow$ feathered(X))

Exercise
which of the following are wffs?

1. $\forall$X p(X)
2. $\forall$X p(Y)
3. $\forall$X $\exists$Y p(Y)
4. q(X,Y,Z)
5. p(a) $\rightarrow$ $\exists$q(a,X,b)
6. p(a) $\vee$ p(a,b)

7. ¬ ¬ ∀X r(X)
8. ∃X ∃Y p(X,Y)
9. ∃X,Y p(X,Y)
10. ∀X (¬∃Y)
11. ∀x (¬∃Y p(x,Y))

Exercise

Formalise the following in predicate logic using the following predicates (with their more or less obvious meaning):

lecTheatre/1, office/1, contains/2, lecturer/1, has/2, same/2, phd/1, supervises/2, happy/1, completePhd/1.

1. 311 is a lecture theatre and 447 is an office.
2. Every lecture theatre contains a projector.
3. Every office contains a telephone and either a desktop or a laptop computer.
4. Every lecturer has at least one office.
5. No lecturer has more than one office.

6. No lecturers share offices with anyone.
7. Some lecturers supervise PhD students and some do not.
8. Each PhD student has an office, but all PhD students share their office with at least one other PhD student.

9. A lecturer is happy if the PhD students he/she supervises successfully complete their PhD.
10. Not all PhD students complete their PhD.

Note:

∃X p(X) states that there is **at least** one X such that p is true of X.

E.g. **∃X father(X, john)**

says John has **at least** one father (assuming *father(X,Y)* is to be read as X is father of Y).

Exercise

Assuming a predicate *same(X,Y)* that expresses that X and Y are the same individual, express the statement that John has exactly one father. You may also assume a binary predicate "father" as above.

Some useful equivalences

All propositional logic equivalences hold for predicate logic wffs.

E.g. $\neg(A \wedge B) \equiv \neg A \vee \neg B$

So

$\neg(\mathrm{academic(john)} \wedge \mathrm{rich(john)}) \equiv$

$\neg \mathrm{academic(john)} \vee \neg \mathrm{rich(john)}$

Another instance of the same equivalence:

$\neg((\forall X\ \mathrm{employed}(X)) \wedge \mathrm{inflation(low)}) \equiv$

$\neg \forall X\ \mathrm{employed}(X) \vee \neg \mathrm{inflation(low)}$

Some other equivalences in predicate logic

- $\forall X p(X) \equiv \neg\exists X \neg p(X)$

  all true, none false
- $\forall X \neg p(X) \equiv \neg\exists X\ p(X)$

  all false - none true
- $\exists X p(X) \equiv \neg\forall X \neg p(X)$

  at least one true - not all false
- $\exists X \neg p(X) \equiv \neg\forall X\ p(X)$

  at least one false - not all true

Some other equivalences in predicate logic

Suppose W1, W2 are wffs.

If W1 can be transformed to W2 by a consistent renaming of variables, then W1 and W2 are equivalent.

E.g.

$\forall X\ p(X) \equiv \forall Y\ p(Y)$

Some other equivalences in predicate logic

$\forall X\ \exists Y\ (p(X,Y) \rightarrow q(Y, X)) \equiv$

$\forall Z\ \exists W\ (p(Z,W) \rightarrow q(W, Z)) \equiv$

$\forall Y\ \exists X\ (p(Y, X) \rightarrow q(X,Y))$

But

$\forall X\ \exists Y\ (p(X,Y) \rightarrow q(Y, X))$ is not equivalent to

$\forall Z\ \exists W\ (p(Z,W) \rightarrow q(Z, Z))$

Some other equivalences in predicate logic

If two wffs differ only in the order of two adjacent quantifiers of the same kind, then they are equivalent.

E.g.

$\forall X\ \forall Y\ p(X,Y) \equiv \forall Y\ \forall X\ p(X,Y)$

But

$\forall X\ \exists Y\ p(X,Y)$ is not equivalent to $\exists Y\ \forall X\ p(X,Y)$

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Some notes on quantifiers

1. Free and Bound variables:

   An occurrence of a variable in a wff is bound if it is within the scope of a quantifier in that sentence. It is free if it is not within the scope of any quantifier in that wff.

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$$\forall X\ (p(X) \rightarrow q(Y,X))$$

Both occurrences of X in the above sentence are bound (they are both within the scope of the $\forall$.)

The occurrence of Y is free (it is not within the scope of any quantifier.)

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$$(\forall X\ p(X)) \wedge (\exists X q(X))$$

In the sentence above, both occurrences of X are bound, the first by the $\forall$, the second by the $\exists$.

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$$(\forall X\ p(X)) \wedge (\exists Y q(X,Y))$$

In the sentence above, the first occurrence of X is bound, the second is free. The occurrence of Y is bound.

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2. A particular occurrence of a variable is bound by the closest quantifier which can bind it.

E.g.

$\forall X\ (p(X) \rightarrow \forall X\ q(X)) \equiv$
$\forall X\ (p(X) \rightarrow \forall Y\ q(Y))$

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3. Law of vacuous quantification

$\forall X\ W \equiv W$ if W (a wff) contains no free occurrences of X.

E.g.

$\forall X\ (p(a) \rightarrow q(a)) \equiv p(a) \rightarrow q(a)$

$\forall X\ \exists X\ p(X) \equiv \exists X\ p(X)$

$\forall X \forall X\ (p(X,X) \rightarrow q(X)) \equiv \forall X\ (p(X,X) \rightarrow q(X))$

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Definition.

If a wff contains no free occurrences of variables it is said to be **closed**, otherwise it is said to be **open**.

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Rules of Inference
Natural Deduction

All inference rules for propositional logic + 4 new rules to deal with the quantifiers.

1. ∀-elimination (∀E)

$$\frac{\forall X\ p(X)}{p(a)}$$

where a is any constant.

The constant a must replace every free occurrence of X in P(X).

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E.g.

From $\forall X$ beautiful(X) we can conclude beautiful(quasimodo).

From

$\forall X$ (lion(X)$\rightarrow \exists Y$(lioness(Y)$\wedge$provides_food(Y,X)))

We can infer

lion(shere_khan)$\rightarrow$ $\exists Y$(lioness(Y) $\wedge$ provides_food(Y,shere_khan))

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Exercise

Bankers and judges are rich.

Martin is either a banker or a judge.

So Martin is rich.

Formalise the above argument and show that it is valid.

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2. **∀-Introduction (∀I)**

(Universal generalisation)

If we know the ground terms and there are a small number of them, e.g. $a_1, \ldots, a_n$, then to show $\forall X\ p(X)$ we show $p(a_1), \ldots, p(a_n)$.

But this is not practical in general.

So to show $\forall X\ p(X)$, we show p(a) for an arbitrary constant a on which there are no constraints.

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∀-Introduction (∀I)

$$\frac{p(a)}{\forall X\ p(X)}$$

provided the following conditions are met:

i. a is an arbitrary constant.
ii. There are no assumptions involving a, left undischarged, used to obtain p(a).
iii. Substitution of X for a in p(X) is uniform, i.e. X is substituted for every occurrence of a.

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E.g.

From

$\forall Y\ (q(a,Y) \rightarrow \exists Z\ (r(Z) \wedge t(Z,Y,a)))$

we can infer

$\forall X \forall Y\ (q(X,Y) \rightarrow \exists Z\ (r(Z) \wedge t(Z,Y,X)))$

provided a is an arbitrary constant.

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Note:

To be on the safe side:

Make sure there is no variable clash when applying the rule.

The safest is to introduce a new variable, i.e. one that does not occur in the original wff.

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Exercise

All fruits are rich in vitamins.

Everything that is rich in vitamins is good for you.

So fruits are good for you.

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Exercise

Given

1. $\forall X\ (p(X) \rightarrow \exists Y\ q(X,Y))$
2. $\forall Z\ (\exists X\ q(Z,X) \wedge r(a) \rightarrow s(Z,a))$
3. $r(a)$

show

$\forall X\ (\neg p(X) \vee s(X,a))$

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3. ∃ -Introduction (∃I)

$$\frac{p(t)}{\exists X\ p(X)}$$

where t is any term, and X does not clash with any occurrence of X in p(t).

X is substituted for one or more occurrences of t in p(t).

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Example

Given Dudley More is a pianist and an actor,

p(dm)∧a(dm)

we can derive each of the following by an application of the ∃I rule.

∃X (p(X) ∧ a(X))

∃X (p(X) ∧ a(dm))

∃X (p(dm) ∧ a(X)).

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Beware clash of variables:

Example:

There is a course that Mary likes.

∃X (course(X) ∧ likes(mary,X))

We can derive:

∃Y ∃X(course(X) ∧ likes(Y,X))

but not

∃X ∃X(course(X) ∧ likes(X,X))

(There is a course that likes itself!)

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4. ∃-Elimination (∃E)

$$\frac{\exists X p(X),\ \forall X\ (p(X) \to W)}{W}$$

provided X does not occur as a free variable in W.

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From

1. ∃X mad_cow(X)
2. ∀X (mad_cow(X) → meat_industry_in_trouble)

we can immediately derive

meat_industry_in_trouble

by an application of ∃E .

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Alternative rule for ∃-Elimination

p(a) assume

..

..

$$\frac{\exists X p(X),\quad W}{W}$$

where W is any wff, provided the following conditions are met:

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i. a is an arbitrary constant.

ii. In proving W from p(a) the only assumption left undischarged in which a occurs is p(a).

iii. a does not occur in W or in ∃X p(X).

Note:

p(a) is an assumption, which is discharged by the application of ∃E rule, above.

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Example

There is an exciting film.

All exciting films make a lot of money.

So there is a film that makes a lot of money.

1. $\exists X$ (film(X) $\wedge$ exciting(X)) given
2. $\forall X$(film(X)$\wedge$exciting(X)$\rightarrow$makes_money(X)) given
   3. film(a) $\wedge$ exciting(a) assume
   4. film(a)$\wedge$exciting(a)$\rightarrow$makes_money(a) 2, $\forall$E
   5. makes_money(a) 3,4, $\rightarrow$E
   6. film(a) 3, $\wedge$E
   7. film(a) $\wedge$ makes_money(a) 5,6, $\wedge$I
   8. $\exists X$ (film(X) $\wedge$ makes_money(X)) 7, $\exists$I
9. $\exists X$ (film(X) $\wedge$ makes_money(X)) 1,3,8, $\exists$E

Compare with:

Air Force 1 is an exciting film.

All exciting films make a lot of money.

So there is a film that makes a lot of money.

1. film(af1) $\wedge$ exciting(af1)) given
2. $\forall X$(film(X)$\wedge$exciting(X)$\rightarrow$makes_money(X)) given
3. film(af1)$\wedge$exciting(af1)$\rightarrow$makes_money(af1) 2,$\forall$E
4. makes_money(af1) 3,1, $\rightarrow$E
5. film(af1) 1, $\wedge$E
6. film(af1) $\wedge$ makes_money(af1) 5,4, $\wedge$I
7. $\exists X$ (film(X) $\wedge$ makes_money(X)) 6, $\exists$I

Exercise

Formalise the following argument and show that it is valid.

Someone murdered Andrew.

Anyone who murders someone is either a psychopath or hates the person he murders.

So there is someone who is either a psychopath or hates Andrew.

Be careful!

When applying the inference rules identify the dominant connective/quantifier correctly. Example:

From **$\forall X$ (p(X) $\wedge$ q(X))**

we can derive **p(a)$\wedge$q(a)** by $\forall$E.

But from **$\neg\forall X$ (p(X) $\wedge$ q(X))**

we cannot derive **$\neg$(p(a)$\wedge$q(a))** by $\forall$E.

Be careful!

From $\neg p(a)$

we can derive $\exists X \neg p(X)$ by $\exists$I.

But from $\neg p(a)$

we cannot derive $\neg\exists X\ p(X)$ by $\exists$I.

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Soundness and Completeness

Predicate logic is sound and complete.

Decidability

Definition:

A logical system is **decidable** iff it is possible to have an effective method (an algorithm) that is guaranteed to recognise correctly whether a wff is a theorem of the system or not. In other words, a logical system is decidable if it satisfies conditions 1 and 2 below.

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1) If $\models$ W then there is an algorithm that recognises that W is a theorem.

2) If it is not the case that $\models$W then there is an algorithm that recognises that W is not a theorem.

Propositional logic is decidable.

Predicate logic is not - it is semi-decidable, that is, it satisfies condition 1, above, but not condition 2.

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The Equality Relation (=)

Example:

T:

doctor_jekyl = mr_hyde

$\exists$X murdered(doctor_jekyl, X)

$\forall$X ($\exists$Y murdered(X,Y) $\rightarrow$ criminal(X))

We can show:

T |- criminal(doctor_jekyl)

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We would also like to be able to show:

T |- criminal(mr_hyde)

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Reasoning with equality

Axiom: $\forall$X X=X

Inference rules: Rules of substitution:

$$\frac{a=b,\ p(a)}{p(b)}\quad \textbf{eqsub.}$$

p(a): any predicate logic wff that contains "a " as a term, where a is not a bound variable.

p(b): wff p(a) with some (or all) occurrences of "a" replaced by "b".

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Similarly

a=b, p(b) **eqsub.**
p(a)

Exercises

1. Show that "=" is
   i. symmetric, i.e.
      $\forall X\ \forall Y\ (X=Y \rightarrow Y=X)$
   ii. transitive, i.e.
      $\forall X\ \forall Y\ \forall Z\ (X=Y \wedge Y=Z \rightarrow X=Z)$
2. Give two equivalent predicate logic representations of the following sentence, both involving equality.
   "There is exactly one spy."

Functions

The language of predicate logic can be augmented with functions. To do so we have to make the following changes.

Add the following to the alphabet of predicate logic:

Function symbols:

a set of symbols, each with an associated arity >=0.

Function symbols must be different from predicate symbols.

(constants can now be thought of as nullary functions.)

Change the definition of term as follows:

Definition:

A **term** is one of the following:

- any constant symbol
- any variable symbol
- of the form $f(t_1, ..., t_n)$ where f is any n_ary function symbol, n>0, and the t_i are terms.

Examples:

1. Mary's father is rich.

   Not using functions:

   $\exists X\ (father(X,mary) \wedge rich(X))$

   Using functions:

   rich(f(mary))

2. Mary and John have the same father.

f(mary) = f(john)

3. Mary's father is father of Jan's father.

f(mary) = f(f(jan))

The only other change necessary is to the ∀ ∀E **rule of inference, as follows:**

<u>∀X p(X)</u>

p(t)

where t is any gound term.

Definition.

A **ground term** is a term that contains no variables, e.g. a, f(a), g(f(a), f(b)).