COMPUTATIONAL FINANCE: 422

Asset Price Dynamics

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This Lecture

True multiperiod investments fluctuate in value/pay random dividends $\Rightarrow$ we need suitable mathematical models.

- Binomial lattices
  - conceptually simple
  - useful for numerical calculations
- Ito processes
  - more realistic than binomial lattices
  - useful for analytical (and numerical) calculations

Further reading:

- D.G. Luenberger: *Investment Science*, Chapter 11
- D.J. Higham: *Financial Option Valuation*, Chapters 5–7

Binomial Lattice Model

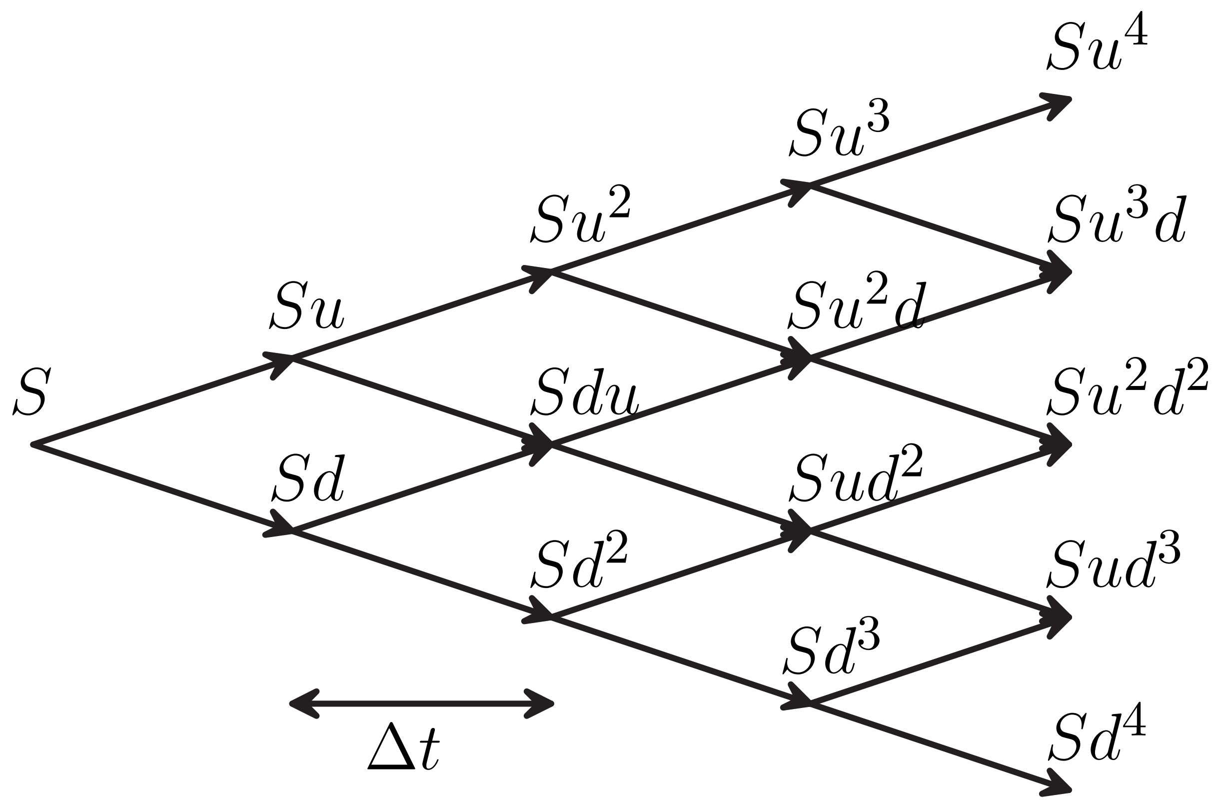
A simple model for the price of a non-dividend paying stock: the binomial lattice model.

- period length Δt
- if the price at the beginning of a period is S, then the price at the beginning of the next period is:
  - Su, $u > 1$, with probability p,
  - Sd, $d < 1$, with probability $1 - p$.

In this multiplicative model, the price will never drop below zero if $u > 0$ and $d > 0$.

Binomial Lattice Model II

We obtain a lattice since an up-movement followed by a down is the same as a down followed by an up:



The Additive Model

We now look at a discrete-time model in which the stock price ranges over a continuum. We assume that

- $k = 0, 1, \ldots, N$ represent $N + 1$ time points
- $S(k)$ denotes the stock price at time k
- recursive construction of stock prices:

$$S(k+1) = a\, S(k) + u(k)\,, \quad \text{for} \quad k = 0, 1, \ldots, N$$

- we require $a \geq 0$ and $u(k)$, $k = 0, 1, \ldots, N-1$ are independent random variables
- we interpret the $u(k)$ as ‘shocks’ or ‘disturbances’ that cause the price to fluctuate

Normal Price Distribution

An explicit calculation yields:

$$
\begin{aligned}
S(1) &= aS(0) + u(0) \\
S(2) &= a^2 S(0) + au(0) + u(1) \\
&\vdots \\
S(k) &= a^k S(0) + a^{k-1}u(0) + a^{k-2}u(1) + \cdots + u(k-1)
\end{aligned}
$$

Assume that the disturbances are independent and identically normally distributed, that is, $u(k) \sim \mathcal{N}(0, \sigma^2)$.

$$\Rightarrow \mathrm{E}[S(k)] = a^k S(0)$$

Thus, for $a > 0$ the expected value of the price increases exponentially over time.

Deficiencies of the Additive Model

The additive model lacks realism since:

- normal random variables can adopt negative values; however, stock prices are always positive
- the variability σ of the price shocks does not depend on the price level; however, price shocks are expected to be proportional to the stock price

$\Rightarrow$ we need a better approach: the multiplicative model

Multiplicative Model

Assume that:

$$S(k+1) = u(k)S(k) \quad \text{for} \quad k = 0, 1, \ldots, N-1$$

where the disturbances $\{u(k)\}_{k=0}^{N-1}$ are again mutually independent random variables.

The disturbance $u(k) = S(k+1)/S(k)$ describes the relative change in price between times k and $k+1$.

The multiplicative model reduces to the additive model if we take logarithms:

$$\underbrace{\ln S(k+1)}_{=:X(k+1)} = \underbrace{\ln S(k)}_{=:X(k)} + \underbrace{\ln u(k)}_{=:w(k)} \quad \text{for} \quad k = 0, 1, \ldots, N-1$$

Multiplicative Model II

⇒ Express the multiplicative disturbances in terms of the additive disturbances:

$$u(k) = e^{w(k)} \quad \text{for} \quad k = 0, 1, \ldots, N-1\,.$$

⇒ If $w(k) \sim \mathcal{N}(\nu, \sigma^2)$, then the $u(k)$ are lognormal random variables (their logarithm is normally distributed).

⇒ Under the multiplicative model, prices cannot become negative, and price fluctuations are proportional to the current price level.

Lognormal Price Distribution

An explicit calculation yields (recall that $X(k) := \ln S(k)$):

$$\begin{aligned} X(1) &= X(0) + w(0) \\ X(2) &= X(0) + w(0) + w(1) \\ &\vdots \\ X(k) &= X(0) + \textstyle\sum_{i=0}^{k} w(i) \end{aligned}$$

Since the $w(k) \sim \mathcal{N}(\nu, \sigma^2)$ are mutually independent, we find

$$X(k) \sim \mathcal{N}\left(X(0) + k\nu, k\sigma^2\right)$$

$\Rightarrow$ The expected value and the variance of the log-price increase linearly with time, while the stock price $S(k) = e^{X(k)}$ is lognormally distributed.

Justification of Lognormal Model

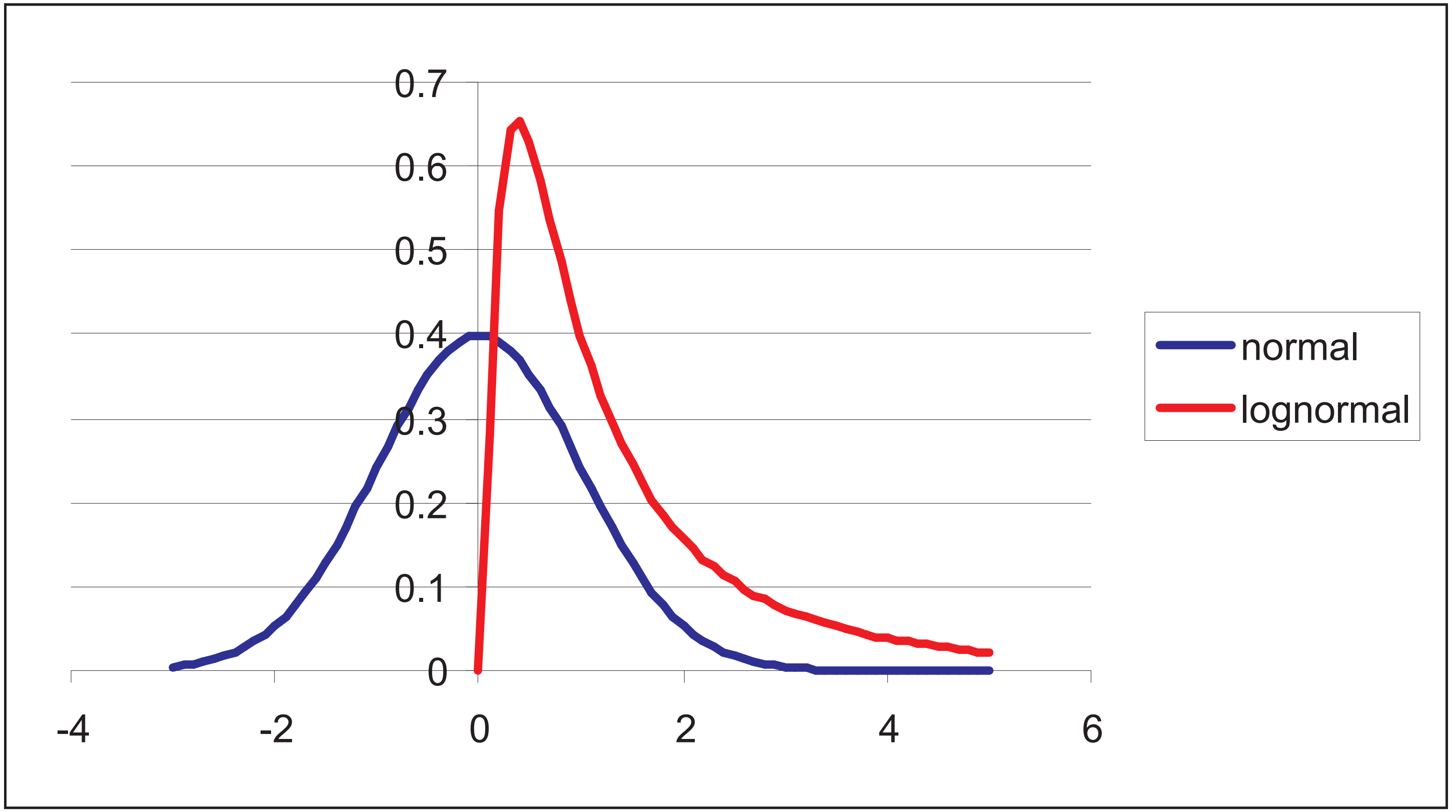
The price of an asset reflects the investor's belief about the value of the underlying company. Thus, it depends heavily on news, rumors, and 'information shocks'.

The disturbances $u(k) = e^{w(k)}$, $k = 0, 1, \dots, N-1$, are:

- independent (since new information is impossible to forecast by definition)
- identically distributed (since there are many information shocks that are aggregated to a time-invariant quantity)
- subject to finite variance (since historical price relatives have finite variance)

$\Rightarrow$ By the central limit theorem, $\ln S(k) = \ln S(0) + \sum_{i=0}^{k} w(i)$ is approximately normally distributed.

Lognormal Random Variables I



Probability density functions of a normal random variable $w \sim \mathcal{N}(0, 1)$ and lognormal random variable $u = e^{w}$.

Lognormal Random Variables II

If u is a lognormal variable, then $w = \ln u$ is normal, e.g., $w \sim \mathcal{N}(\nu, \sigma^2)$. The expected value of u amounts to

$$\begin{aligned} \mathrm{E}(u) \;=\; \mathrm{E}(e^w) \;&=\; \int_{-\infty}^{+\infty} e^w \, \frac{e^{-\frac{(w-\nu)^2}{2\sigma^2}}}{\sqrt{2\pi}\,\sigma} \, \mathrm{d}w \\ &=\; e^{\nu+\frac{\sigma^2}{2}} \int_{-\infty}^{+\infty} \frac{e^{-\frac{(w-\nu-\sigma^2)^2}{2\sigma^2}}}{\sqrt{2\pi}\,\sigma} \, \mathrm{d}w \;=\; e^{\nu+\frac{\sigma^2}{2}}, \end{aligned}$$

which increases with σ. A similar calculation yields

$$\mathrm{var}(u) = e^{(2\nu+\sigma^2)}(e^{\sigma^2} - 1).$$

Products and powers of jointly lognormal variables are lognormal since sums and multiples of jointly normal variables are normal, respectively.

Random Walk I

Eventually, we let $\Delta t \downarrow 0 \Rightarrow$ continuous time model.

Consider an additive process over N periods of length Δt:

$$z(t_{k+1}) = z(t_k) + \epsilon(t_k)\sqrt{\Delta t}\,, \quad t_{k+1} = t_k + \Delta t\,, \quad 0 \leq k < N\,,$$

where $z(0) = 0$, and the disturbances $\epsilon(t_k) \sim \mathcal{N}(0,1)$ are independent for different k. This is called a random walk.

For $j < k$ we find that $z(t_k) - z(t_j)$ is normally distributed:

$$z(t_k) - z(t_j) = \sum_{i=j}^{k-1} \epsilon(t_i)\sqrt{\Delta t} \quad \Rightarrow \quad \mathrm{E}[z(t_k) - z(t_j)] = 0\,.$$

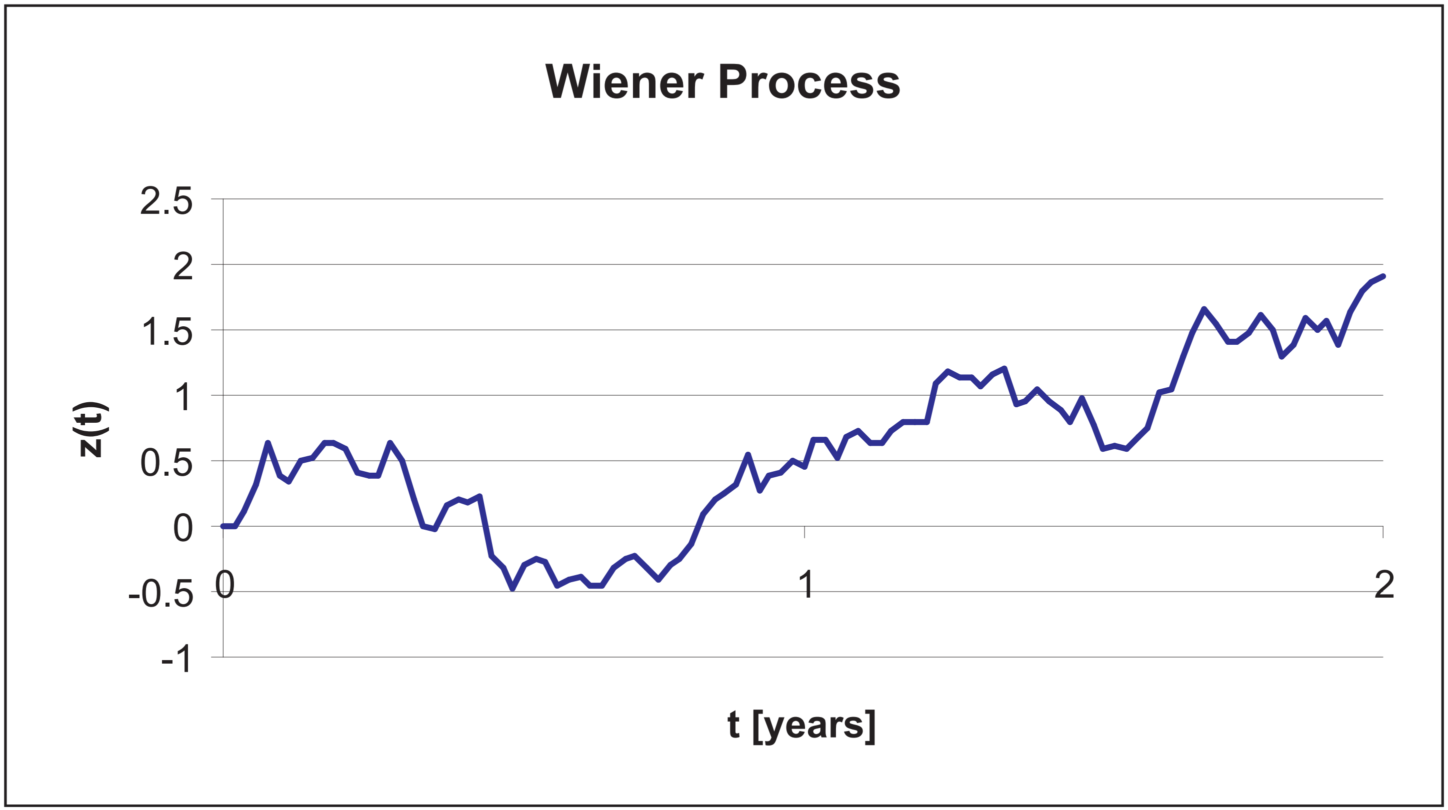
Random Walk II

By independence of the $\epsilon(t_k)$'s, we have

$$
\begin{aligned}
\operatorname{var}[z(t_k) - z(t_j)] &= \mathrm{E}\left[\sum_{i=j}^{k-1} \epsilon(t_i)\sqrt{\Delta t}\right]^2 \\
&= \mathrm{E}\left[\sum_{i=j}^{k-1} \epsilon(t_i)^2 \Delta t\right] = (k-j)\Delta t = t_k - t_j \, .
\end{aligned}
$$

Note that $z(t_{k_2}) - z(t_{k_1})$ and $z(t_{k_4}) - z(t_{k_3})$ are independent if $t_{k_1} < t_{k_2} \leq t_{k_3} < t_{k_4}$ since each of these differences is made up of ϵ's that are themselves independent.

Limit $\Delta t \downarrow 0$



For $\Delta t \downarrow 0$, we obtain a Wiener process.

Wiener Process I

A Wiener process is obtained by taking the limit of the random walk process as $\Delta t \downarrow 0$. We write

$$\mathrm{d}z = \epsilon(t)\sqrt{\mathrm{d}t}\,,$$

where dz denotes an infinitesimal increment of the Wiener process, and dt denotes an infinitesimal time interval. Each $\epsilon(t)$ is a $\mathcal{N}(0, 1)$ random variable, while $\epsilon(t')$ and $\epsilon(t'')$ are independent for all $t' \neq t''$.

Note that this description of a Wiener process is not rigorous since we have not defined appropriate limiting operations. This is merely an intuitive description.

Wiener Process II

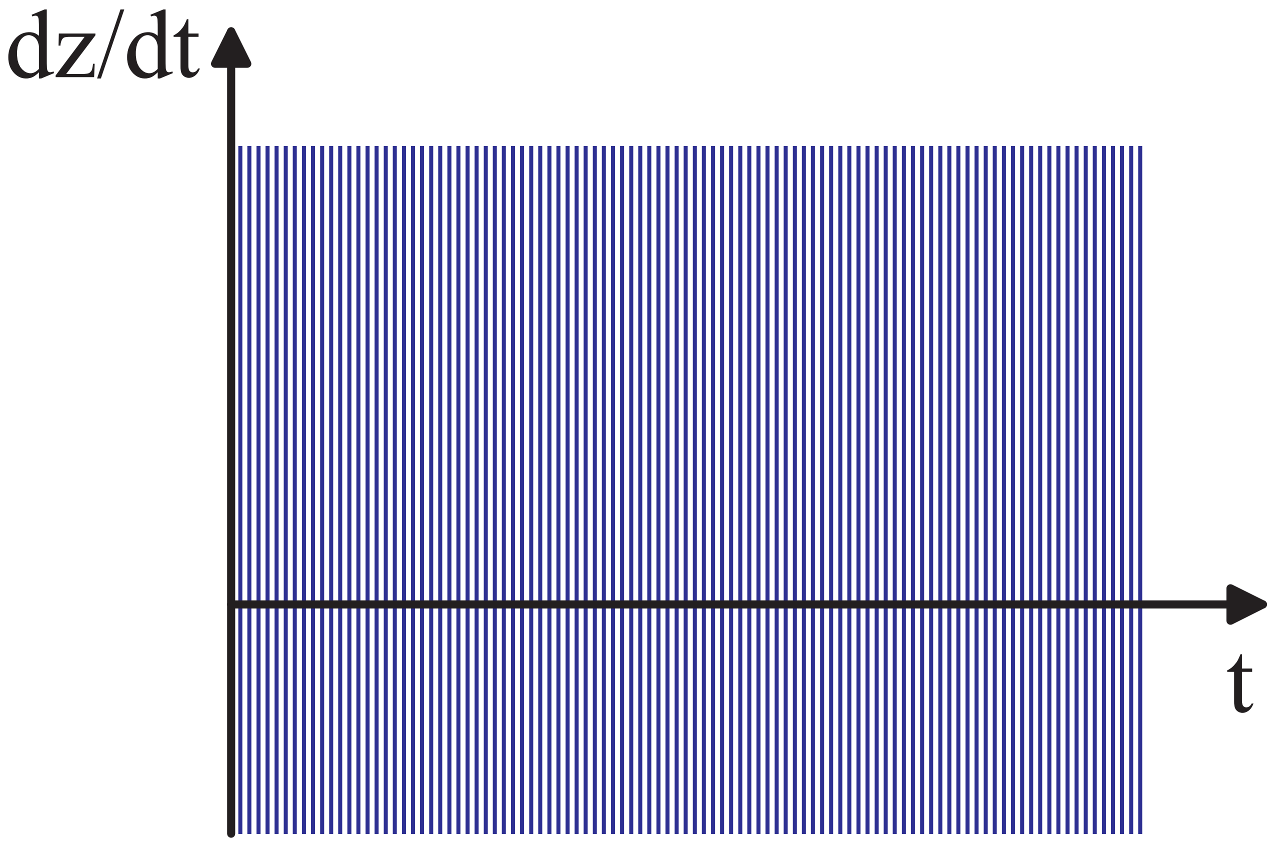
Alternatively, we can define a Wiener process $z(t)$ (also called Brownian Motion) by listing its properties:

1. For all $s < t$ we have $z(t) - z(s) \sim \mathcal{N}(0, t - s)$;
2. For all $t_1 < t_2 \leq t_3 < t_e$ the random variables $z(t_2) - z(t_1)$ and $z(t_4) - z(t_3)$ are independent;
3. $z(0) = 0$ with probability 1.

With probability 1, a Wiener process $z(t)$ is nowhere differentiable with respect to time. Intuition:

$$\mathrm{E}\left[\frac{z(t+s) - z(t)}{s}\right]^2 = \frac{s}{s^2} = \frac{1}{s} \to \infty \quad \text{for} \quad s \downarrow 0$$

White Noise



White noise is the ‘derivative’ of a Wiener process.

More Stochastic Processes

By inserting white noise into ordinary differential equations, we can construct a family of new stochastic processes.

- Generalized Wiener process: for given $a, b \in \mathbb{R}$ and a Wiener process z we can define a generalized Wiener process x through:

$$\mathrm{d}x(t) = a\mathrm{d}t + b\mathrm{d}z \quad \Rightarrow \quad x(t) = x(0) + at + bz(t)\,.$$

- Ito process: if $a(x,t)$ and $b(x,t)$ are functions of (x,t) and z is a Wiener process, then we can define an Ito process x through

$$\mathrm{d}x(t) = a(x,t)\mathrm{d}t + b(x,t)\mathrm{d}z\,.$$

Ito processes have no analytical solution in general.

Geometric Brownian Motion I

Recall that the multiplicative model is

$$X(k+1) - X(k) = w(k)$$

where $X(k) = \ln S(k)$ and the $w(k) \sim \mathcal{N}(\nu \Delta t, \sigma^2 \Delta t)$ are independent for different k. For $\Delta t \downarrow 0$ we obtain

$$\mathrm{d}X(t) = \nu \mathrm{d}t + \sigma \mathrm{d}z \,.$$

The (known) solution for X gives a solution for S:

$$S(t) = e^{X(t)} = \exp[X(0) + \nu t + \sigma z(t)] = S(0) e^{\nu t + \sigma z(t)}.$$

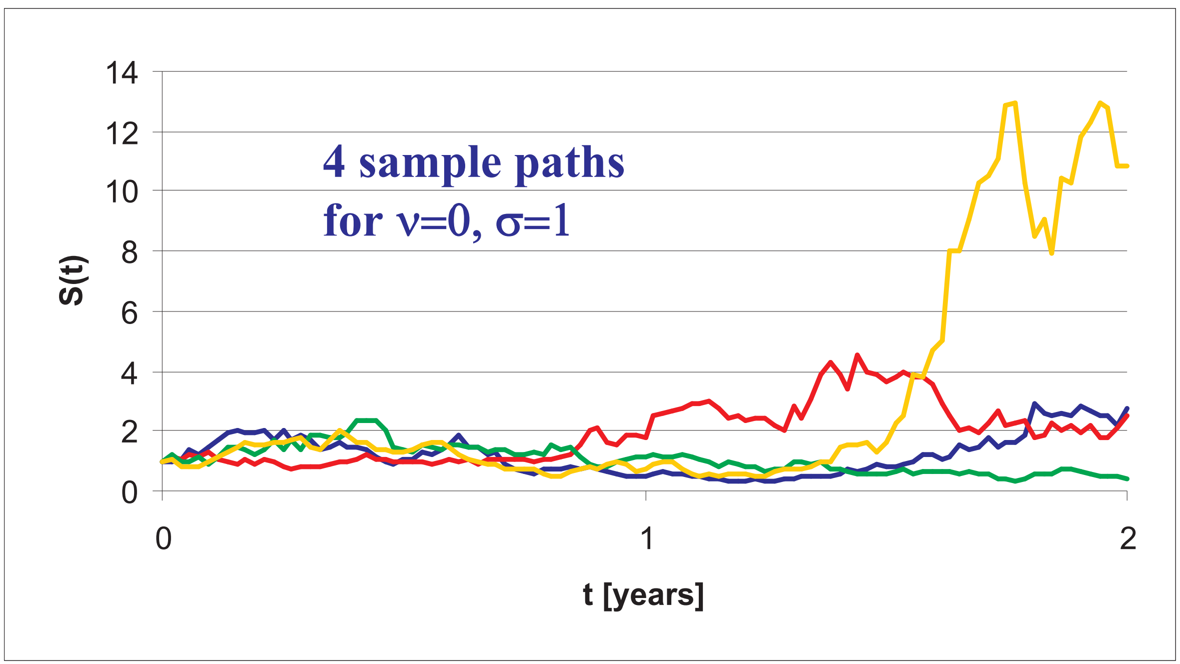
$\Rightarrow \ln S(t) \sim \mathcal{N}(\ln S(0) + \nu t, \sigma^2 t)$.

The process S is called a geometric Brownian motion.

Geometric Brownian Motion II

For a geometric Brownian motion process S the random variable $S(t)$ is lognormal for each $t \geq 0$. Thus, we have:

$$\begin{aligned} \mathrm{E}[S(t)] &= S(0) e^{(\nu + \frac{\sigma^2}{2})t} \\ \mathrm{var}[S(t)] &= S(0)^2 e^{(2\nu + \sigma^2)t} \left(e^{\sigma^2 t} - 1 \right) \end{aligned}$$



Ito Calculus I

The chain rule of ordinary calculus suggests that

$$\mathrm{d}\ln[S(t)] = \frac{\mathrm{d}S(t)}{S(t)}\,.$$

From our previous calculations, we know that $\mathrm{d}\ln[S(t)] =$ $= \mathrm{d}X(t) = \nu\mathrm{d}t + \sigma\mathrm{d}z$. Thus, we are tempted to write

$$\frac{\mathrm{d}S(t)}{S(t)} = \nu\mathrm{d}t + \sigma\mathrm{d}z\,.$$

However, this formula is not correct since the chain rule of stochastic calculus has an additional term. The correct formula reads:

$$\frac{\mathrm{d}S(t)}{S(t)} = \left(\nu + \tfrac{1}{2}\sigma^2\right)\mathrm{d}t + \sigma\mathrm{d}z\,.$$

Ito Calculus II

- The correction term $\frac{1}{2}\sigma^2$ required when transforming the equation for $\ln S(t)$ to $S(t)$ is a special case Ito's lemma, which applies to transformations of Ito processes.
- With $\mu = \nu + \frac{1}{2}\sigma^2$ we find

$$\frac{\mathrm{d}S(t)}{S(t)} = \mu \mathrm{d}t + \sigma \mathrm{d}z\,.$$

- The term $\mathrm{d}S(t)/S(t) = S(t+dt)/S(t) - 1$ can be thought of as the instantaneous rate of return over a time interval of length $\mathrm{d}t$.

Simulation I

To simulate a geometric Brownian motion, we select a small period length Δt, let $t_k = k\Delta t$ for $k = 0, 1, 2, \ldots$, and fix $S(0)$. One possible simulation equation is

$$S(t_{k+1}) - S(t_k) = \mu S(t_k)\Delta t + \sigma S(t_k)\epsilon(t_k)\sqrt{\Delta t}\,,$$

where the $\epsilon(t_k) \sim \mathcal{N}(0, 1)$ are independent for different k.

$$\Rightarrow \quad S(t_{k+1}) = \left[1 + \mu\Delta t + \sigma\epsilon(t_k)\sqrt{\Delta t}\right] S(t_k) \tag{1}$$

Another possible approach is to use the equation

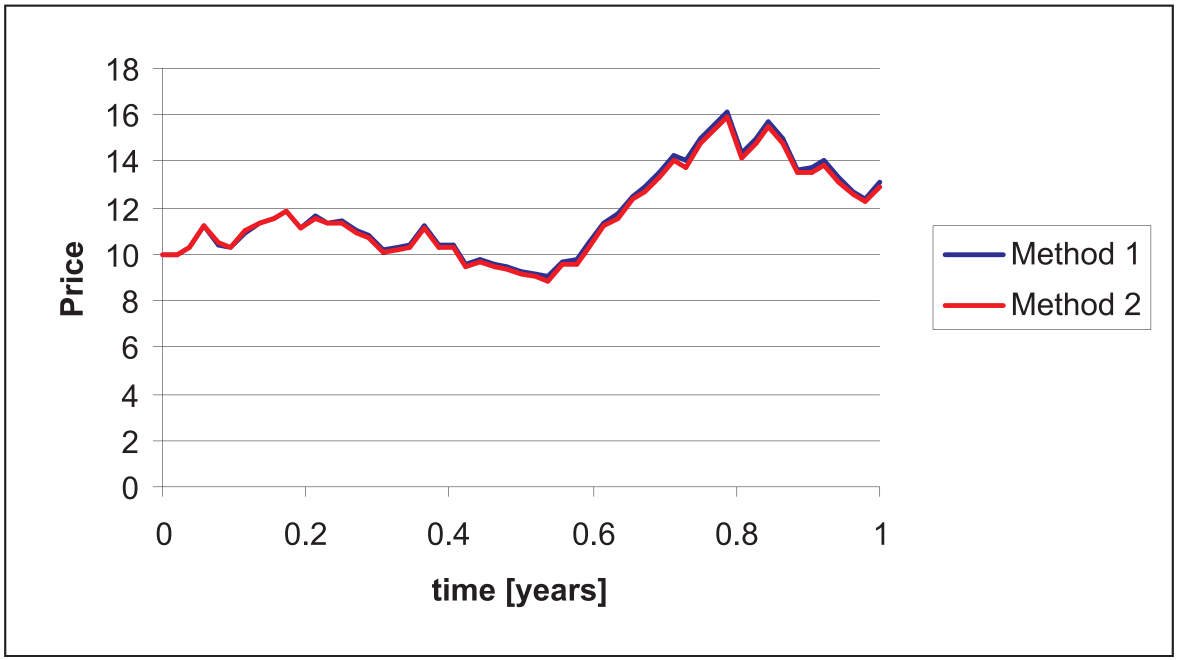
$$\ln S(t_{k+1}) - \ln S(t_k) = \nu\Delta t + \sigma\epsilon(t_k)\sqrt{\Delta t}\,,$$

$$\Rightarrow \quad S(t_{k+1}) = e^{\nu\Delta t + \sigma\epsilon(t_k)\sqrt{\Delta t}} S(t_k) \tag{2}$$

Simulation II

The simulation equations (1) and (2) are not equal, but the differences cancel in the long run. In practice, either method can be used.

Example: Set $S(0) = 10$, $\nu = 15\%$, $\sigma = 40\%$, and $\Delta t = 1/52$ (one week). We simulate the stock price over one year.



Ito's Lemma

The chain rule of ordinary calculus needs to be generalized since the differentials of Ito processes have order $\sqrt{\mathrm{d}t}$
$\Rightarrow$ their squares produce first-order effects.

Lemma 1 (Ito's lemma)**.** *Consider an Ito process defined through*

$$\mathrm{d}x(t) = a(x,t)\mathrm{d}t + b(x,t)\mathrm{d}z\ ,$$

where z is a standard Wiener process. Define a new process $y(t) = F(x(t),t)$. Then $y(t)$ satisfies the Ito equation

$$\mathrm{d}y(t) = \left(\frac{\partial F}{\partial x}a + \frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}b^2\right)\mathrm{d}t + \frac{\partial F}{\partial x}b\mathrm{d}z\ . \tag{3}$$

Ordinary calculus would give (3) without the $\frac{1}{2}$ term.

Sketch of the Proof of Ito's Lemma I

Expand y w.r.t. a change Δy. In this Taylor expansion, we keep all terms up to first order in Δt. Since Δx is of order $\sqrt{\Delta t}$, we must expand to second order in Δx:

$$\begin{aligned} y + \Delta y &= F(x,t) + \frac{\partial F}{\partial x}\Delta x + \frac{\partial F}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(\Delta x)^2 \\ &= F(x,t) + \frac{\partial F}{\partial x}(a\Delta t + b\Delta z) + \frac{\partial F}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(a\Delta t + b\Delta z)^2 \end{aligned}$$

The last square term becomes $a^2(\Delta t)^2 + 2ab\Delta t\Delta z + b^2(\Delta z)^2$, whose first two summands are of order higher than 1 in Δt and can thus be neglected. The remaining term $b^2(\Delta z)^2$ has expected value $b^2\Delta t$ and a variance of order 2 in Δt. Thus, it may be approximated by $b^2\Delta t$, which is nonstochastic.

Sketch of the Proof of Ito's Lemma II

Substituting this into the last equation and rearranging terms yields

$$y + \Delta y = F(x,t) + \left(\frac{\partial F}{\partial x} a + \frac{\partial F}{\partial t} + \frac{1}{2} \frac{\partial^2 F}{\partial x^2} b^2 \right) \Delta t + \frac{\partial F}{\partial x} b \Delta z \,.$$

By using $y = F(x,t)$ and taking the limit $\Delta t \downarrow 0$, we obtain Ito's equation (3). $\square$

Example

Assume that $S(t)$ follows a geometric Brownian motion

$$\mathrm{d}S = \mu S \mathrm{d}t + \sigma S \mathrm{d}z \,.$$

Ito's lemma gives the equation for $F(S(t)) = \ln S(t)$.

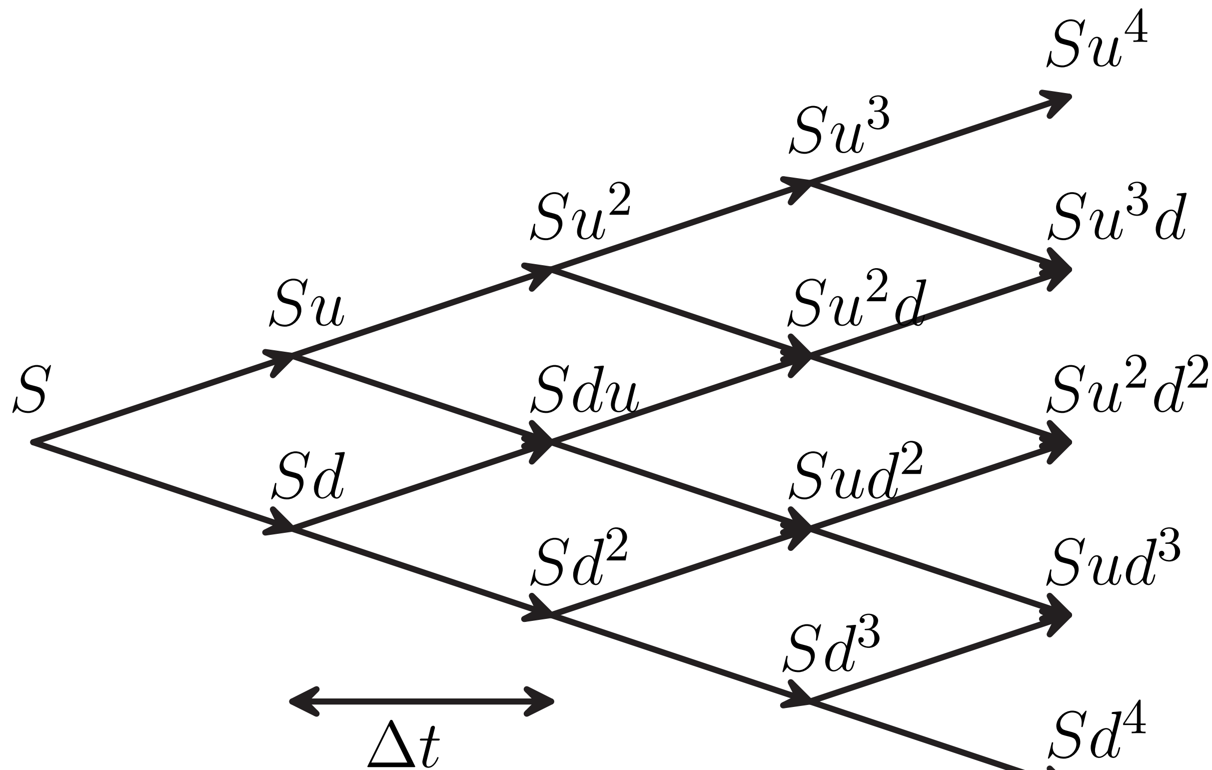
In this example, we have $a = \mu S$, $b = \sigma S$, $\partial F / \partial S = 1/S$, and $\partial^2 F / \partial S^2 = -1/S^2$. Therefore, according to (3), we find

$$\mathrm{d} \ln S = \left(\frac{a}{S} - \frac{1}{2} \frac{b^2}{S^2} \right) \mathrm{d}t + \frac{b}{S} \mathrm{d}z = \left(\mu - \frac{1}{2} \sigma^2 \right) \mathrm{d}t + \sigma \mathrm{d}z \,.$$

This is the well-known formula the we derived earlier.

Binomial Lattice Revisited I

Consider again the binomial lattice model considered earlier:



Binomial Lattice Revisited II

This model is similar to the multiplicative model since in each step the price is multiplied by a random variable:

- binomial model: the random variable takes the value u with probability p and the value d with probability $1 - p$;
- multiplicative model: the random variable is given by e^w for $w \sim \mathcal{N}(\nu \Delta t, \sigma^2 \Delta t)$.

By choosing suitable values for u, d, and p, we can match the binomial to the multiplicative model!

This is done by matching the expectation and the variance of the logarithm of a price change.

Binomial Lattice Revisited III

A direct calculation yields:

$$
\begin{aligned}
\mathrm{E}(w) &= p \ln u + (1-p) \ln d \\
\mathrm{var}(w) &= p(\ln u)^2 + (1-p)(\ln d)^2 - [p \ln u + (1-p) \ln d]^2 \\
&= p(1-p)(\ln u - \ln d)^2 \,.
\end{aligned}
$$

Defining $U = \ln u$ and $D = \ln d$, the parameter matching equations become

$$
\begin{aligned}
pU + (1-p)D &= \nu \Delta t \\
p(1-p)(U-D)^2 &= \sigma^2 \Delta t \,.
\end{aligned}
$$

Note that we have three parameters U, D, and p, but only two equations. We are thus free to set $D = -U \Leftrightarrow d = 1/u$.

Binomial Lattice Revisited IV

With $D = -U$, the parameter matching equations read

$$\begin{aligned} (2p-1)U &= \nu \Delta t \\ 4p(1-p)U^2 &= \sigma^2 \Delta t\,. \end{aligned}$$

The solutions of these equations are

$$\begin{aligned} p &= \frac{1}{2} + \frac{1/2}{\sqrt{\sigma^2/(\nu^2 \Delta t) + 1}} \approx \frac{1}{2} + \frac{1}{2}\left(\frac{\nu}{\sigma}\right)\sqrt{\Delta t} \\ \ln u &= \sqrt{\sigma^2 \Delta t + (\nu \Delta t)^2} \approx \sigma \sqrt{\Delta t} \\ \ln d &= -\sqrt{\sigma^2 \Delta t + (\nu \Delta t)^2} \approx -\sigma \sqrt{\Delta t} \end{aligned}$$

and the approximations hold for small values of Δt.