Logic and AI Programming:
Introduction to Logic
Introduction to Prolog

F. SADRI

Autumn Term

October – December

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INTRODUCTION TO LOGIC

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Course Material

Course material will be available via CATE.

Please print notes for the rest of the course and tutorial exercises as they appear on CATE.

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CONTENTS

Introduction to logic and some of its applications in computer science

- Propositional logic
  - Syntax
  - Semantics (Truth Tables)
  - Rules of inference (Natural Deduction)
- Predicate logic
  - Syntax
  - Informal semantics
  - Rules of inference (Natural Deduction)
- Prolog programming
- Time permitting: Abduction

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Books
background reading on logic

- Any book on logic will have useful examples.
- Richard Spencer-Smith, Logic and Prolog, Harvester Wheatsheaf, (The library has a number of copies)
- Jim Woodcock and Martin Loomes, Software Engineering Mathematics", Pitman Publishing

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Books
Prolog

- Ivan Bratko, "Prolog programming for artificial intelligence“, Addison-Wesley, Third Edition, 2001 and later.

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Logic has many applications in computing

- Good mathematical foundation for reasoning about computer and information systems, and behaviour of programs
- Software engineering: Formal specifications and formal verification of programs.
- Basis of a family of programming languages, e.g. Prolog
- Logic-based Learning course in term 2

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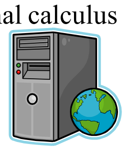
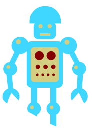
Logic has many applications in computing

- Database languages, e.g. relational calculus and some features of SQL



- Artificial intelligence



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Logic is also useful more generally in life

- Clear thinking
- Judging validity of arguments and justification of conclusions
- Spotting inconsistencies
- Awareness and avoidance of ambiguities of natural language

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Which of the following arguments are valid?

1) Advertisement for a computing book: If you don’t use computers you don’t need this book. But you are a person who uses computers. So you need this book.
2) If you work hard you will succeed. So if you do not succeed you have not worked hard.

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Which of the following arguments are valid?

3) Heard in a radio interview with a well-known politician: All our problems have come from the EU. So nothing good has resulted for us from our membership of the EU.
4) We cannot win the war on poverty without spending money. So if we do spend money we will conquer poverty.

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Another argument – Is it valid?

5) One of the old arguments of tobacco spokesmen against the claim that smoking causes lung cancer:

Lung cancer is more common among male smokers than it is among female smokers. If smoking were the cause of lung cancer, this would not be true. The fact that lung cancer is more common among male smokers means that it is caused by something in the male make-up. If follows that lung cancer is not caused by smoking, but something in the male make-up.

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Propositional Logic

- A good place to start.
- All logics are based on propositional logic.

Propositional Logic

Examples:

Passing the exams, the courseworks and the projects implies passing the MSc.

(pass_exams $\wedge$ pass_cwks $\wedge$ pass_projs) $\rightarrow$ pass_MSc

$\wedge$: and

$\rightarrow$: implies (if-then)

You cannot retake the exams if you have not passed the courseworks or if you have cheated in the exams.

($\neg$ pass_cwks $\vee$ cheat) $\rightarrow$ $\neg$ retake_exams

$\neg$: **not** sometimes written as ~

$\vee$: **or**

Components of a logic

- Language:
  - alphabet : symbols
  - syntax : rules for putting together the symbols to make grammatically correct sentences.
- Semantics:

  meaning of the symbols and the sentences.
- Inference rules

The Propositional Language: Alphabet

- **Propositional symbols**
  e.g. pass_exams, p, q, r, s, p1
- **Logical connectives:**
  $\wedge$: **and (conjunction)**
  $\vee$: **(inclusive) or (disjunction)**
  $\neg$: **not (negation)**
  $\rightarrow$: **implication (if-then)**
  $\leftrightarrow$: **double implication (if and only if)**

The Propositional Language:
Syntax of a grammatically correct sentence

(well formed formula, wff)

- A propositional symbol is a wff.
- If W, W1 and W2 are wffs then so are
  $\neg$ (W)
  (W1 $\wedge$ W2)
  (W1 $\vee$ W2)
  (W1 $\rightarrow$ W2) (W2 $\leftarrow$ W1)
  (W1 $\leftrightarrow$W2)
- There are no other wffs.

Examples

$(p \rightarrow q)$ is a wff

$((p \rightarrow q)\vee((p \wedge r) \rightarrow \neg(s)))$ is a wff

$(r \wedge \vee t)$ is not a wff

$(p \neg \rightarrow q)$ is not a wff

Exercise:

Formulate arguments (1) and (2) at the beginning of the notes.

Some notes on simplifying syntax

- To avoid ending up with a large number of brackets one can drop the outermost brackets.

Examples:

$(p \rightarrow q)$ can be written as **$p \rightarrow q$**

$((p \rightarrow q) \vee r)$ can be written as **$(p \rightarrow q) \vee r$**.

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- "$\neg$ " binds more closely than the other connectives. This can be used to drop some brackets.

Example

$(\neg (p) \wedge q) \rightarrow t$ can be written as **$(\neg p \wedge q) \rightarrow t$**

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- $\wedge$ and $\vee$ bind more closely than $\rightarrow$ and $\leftrightarrow$. This can be used to drop some brackets.

Examples:

$(\neg p \wedge q) \rightarrow t$ can be written as **$\neg p \wedge q \rightarrow t$**.

$(p \wedge q) \rightarrow (r \vee s)$ can be written as **$p \wedge q \rightarrow r \vee s$**.

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Binding Strength of the Connectives

To avoid having to use many brackets, there is a convention of ordering the connectives.

Also :

- Order of precedence
- Binding priority

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Binding Strength of the Connectives

Strongest $\neg$

$\wedge$

$\vee$

$\rightarrow$

Weakest $\leftrightarrow$

Binding conventions: Examples

- $p \vee q \wedge r$ is understood as $p \vee (q \wedge r)$
- $\neg p \vee q$ is understood as $(\neg p) \vee q$
- $p \rightarrow q \leftrightarrow r$ is understood as $(p \rightarrow q) \leftrightarrow r$

I prefer the first and third bracketed versions. They are more clear, and having a few brackets is not much of a burden! Please don't write unreadable formulas like

$p \vee \neg q \rightarrow \neg r \leftrightarrow \neg\neg s \wedge t \vee \neg u$

Use brackets to remove ambiguity

Example:

$P \rightarrow Q \rightarrow R$

is ambiguous.

In general

$P \rightarrow (Q \rightarrow R)$

and

$(P \rightarrow Q) \rightarrow R$

are not equivalent (do not have the same meaning).

Binding conventions

So $p \rightarrow q \rightarrow r$

is a problem.

It needs brackets to disambiguate it.

But $p \wedge q \wedge r$ and $p \vee q \vee r$

are fine (to be discussed later).

Use brackets to remove ambiguity

Example:

Go to work and go to dinner or go to the cinema.

Exercise:

Spot the ambiguity.

Give two possible formulations of the sentence.

Go to work and go to dinner, or go to the cinema.

Go to work, and go to dinner or go to the cinema.

Exercise: Which of the following are wffs?

Assume
p, q, r, sad, happy,
tall, rich, work_hard,
steal, borrow, and possess
are propositional symbols.

- $\text{rich} \rightarrow \text{happy}$
- $(p \vee q) \wedge (r \rightarrow p)$
- $p \vee \rightarrow q$
- $\text{sad} \rightarrow \neg\text{happy}$
- $\neg\text{happy} \leftarrow \text{sad}$

- rich $\rightarrow$ $\neg\neg$happy
- rich $\leftrightarrow$ (work_hard $\vee$ steal)
- (steal $\wedge$ $\vee$ borrow) $\rightarrow$ possess
- (steal $\vee$ borrow) $\rightarrow$ possess
- steal $\vee$ borrow $\rightarrow$ possess

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- $(p \wedge q \rightarrow r) \wedge (\neg p \rightarrow \neg q)$
- $p \rightarrow \neg p$
- $p \wedge \neg p$

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We will look at

- *Parse trees*
- *Principle connectives*
- *Subformulas*

in the lecture.

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Notes on terminology

- $\neg$ is a **unary** operator.
- The other connectives are **binary** operators.
- $X \vee Y$ is called the **disjunction of X and Y**.
- $X \vee Y$ X and Y are **disjuncts.**
- $X \wedge Y$ is called the **conjunction of X and Y**.
- $X \wedge Y$ X and Y are **conjuncts.**
- $\neg X$ is called the **negation of X.**

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Notes on terminology cntd.

- $A \rightarrow B$ is called an implication.
  A is called the antecedent,
  B is called the consequent.

A *Literal* is a proposition or the negation of a proposition.

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Semantics

Provides

- The meaning of the simple (atomic) units
- Rules for putting together the meaning of the atomic units to form the meaning of the complex units (sentences).

Semantics specifies under what circumstances a sentence is *true* or *false*.

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Example

John is not happy, but he is comfortable.

Represent as $\neg h \wedge c$

Four possible cases

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h	**c**	**$\neg$ h**	**$\neg$ h $\wedge$c**
T	T	F	F
T	F	F	F
F	T	T	T
F	F	T	F

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T (truth) and
F (falsity)
are known as **truth values.**

Constructing Truth Tables for the connectives

A	$\neg$ A
T	F
F	T

A	B	$A \wedge B$
T	T	T
T	F	F
F	T	F
F	F	F

A	B	$A \vee B$
T	T	T
T	F	T
F	T	T
F	F	F

A	B	$A \rightarrow B$
T	T	T
T	F	F
F	T	T
F	F	T

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A	B	$A \leftrightarrow B$
T	T	T
T	F	F
F	T	F
F	F	T

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The truth value of a wff is uniquely determined by the truth value of its components.

Example:

The truth table for the wff $(p \vee q) \wedge (r \rightarrow p)$ is as follows:

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p	q	r	$p \vee q$	$r \rightarrow p$	$(p \vee q) \wedge (r \rightarrow p)$
T	T	T	T	T	T
T	T	F	T	T	T
T	F	T	T	T	T
T	F	F	T	T	T
F	T	T	T	F	F
F	T	F	T	T	T
F	F	T	F	F	F
F	F	F	F	T	F

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Exercise:

How many rows will there be in a truth table for a wff containing n propositional symbols?

Exercise:

Go to work and go to dinner or go to the cinema.

The two wffs formulating the two interpretations of the above sentence are:

$\mathbf{w \wedge (d \vee c)}$

$\mathbf{(w \wedge d) \vee c}$

Under which interpretation(s) do the truth values of the two wffs differ?

Notes

The connective "$\vee$" stands for *inclusive* or, i.e.

$p \vee q$ is interpreted as true when either proposition is true or both are.

Often in English when we use "or" we intend *exclusive* or, e.g.

- I'll go shopping *or* I'll stay at home.
- Sit down and be quiet *or* leave the room.

Notes cntd.

In this propositional language there is no connective for exclusive or, but we can still express the concept, e.g.

(go-shopping $\vee$ stay-home) $\wedge$
$\neg$(go-shopping $\wedge$ stay-home)

In general "p exor q" can be represented as:
$(p \vee q) \wedge \neg(p \wedge q)$

Exercise:

Draw the truth table of the first wff above.

Notes cntd.

- Law of excluded middle:

A proposition (and consequently a wff) is either true or false – there is no middle ground, no "unknown".

- A proposition (and consequently a wff) cannot be both true and false.

Exercise:

Draw the truth table for $A \wedge \neg A$.

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Notes cntd.

- The interpretation of "$\rightarrow$" may be unintuitive sometimes.

Don't blame the "$\rightarrow$", think about the content of logic sentences you write!

$A \rightarrow B$ is simply the same as $\neg A \vee B$.

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Notes cntd.

- For any wffs A and B, "$A \leftrightarrow B$" is true exactly when A and B have the same truth values, i.e. when they are both true or both false.

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Some definitions

Definition :

A wff which evaluates to true in every interpretation of its constituent parts is called a **tautology**.

Example $\quad$ $\mathbf{A \vee \neg A}$

$\mathbf{A \rightarrow A}$

The two wffs above represent the **Law of excluded middle**.

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Definition

A wff which evaluates to false in every interpretation of its constituent parts is called an **inconsistency (contradiction)**, or is said to be **inconsistent**.

- Example $A \wedge \neg A$

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Definition

A wff which is neither a tautology, nor an inconsistency is a **contingency**, or is said to be **contingent**.

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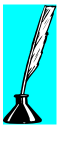
Exercise

For each of the following determine if it is a tautology, inconsistency or contingency by drawing the truth table.



a. $P \wedge (P \vee Q)$
b. $(P \vee Q) \wedge (P \rightarrow Q)$
c. $Q \wedge \neg P \wedge (P \vee (Q \rightarrow P))$
d. $(P \wedge (Q \vee P)) \leftrightarrow P$
e. $(P \rightarrow Q) \rightarrow (\neg P \vee Q)$
f. $((P \rightarrow Q) \wedge (R \rightarrow S) \wedge (P \vee R)) \rightarrow (Q \vee S)$

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Definition: Semantic Consequence

Let

S be a set of wffs, and

W be a wff.

If whenever all the wffs in S are true W is also true, then W is a **semantic consequence** of S.

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Semantic Consequence cntd.

Denoted as

S $\models$ W

"|=" is the **semantic turnstile** (a metasymbol).

We also say W is **semantically entailed** by S.

If W is a tautology then

$\models$ **W**.

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Exercise

Show the following:

a. $A \wedge B \models A \vee B$

b. cold, summer $\rightarrow \neg$cold $\models \neg$summer

c. Go back to one of the argument at the beginning of the notes you think is valid and show that the conclusion of the argument is semantically entailed by the premises.

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Definitions:Valid, Satisfiable

- *Valid* is just another name for tautology.

So a formula is valid if it is true in every interpretation.

|= A if A is valid.

- A formula is *satisfiable* if it is true in at least one interpretation.

	Validity	Satisfiability
A |= B	?? valid	?? unsatisfiable
A |= B and B |= A	?? valid	?? unsatisfiable

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Definition: Equivalence

Two wffs are **equivalent** iff their truth values are the same under every interpretation.

A is equivalent to B is represented as

A ≡ B .

"≡" is the metasymbol for equivalence.

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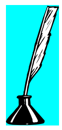
Exercise



Show
$A \to B \equiv \neg A \lor B \equiv \neg(A \land \neg B)$.

Example
work_hard → pass_exams ≡
¬work_hard ∨ pass_exams ≡
¬(work_hard ∧ ¬pass_exams)

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Some useful equivalences

Commutative Rules

$A \land B \equiv B \land A$

$A \lor B \equiv B \lor A$

Associative Rules

$(A \land B) \land C \equiv A \land (B \land C)$

$(A \lor B) \lor C \equiv A \lor (B \lor C)$

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Some useful equivalences

Idempotence

$A \land A \equiv A$

$A \lor A \equiv A$

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Some useful equivalences

Distributive Rules

$A \lor (B \land C) \equiv (A \lor B) \land (A \lor C)$

$A \land (B \lor C) \equiv (A \land B) \lor (A \land C)$

De Morgan's Rules

$\neg(A \lor B) \equiv \neg A \land \neg B$

$\neg(A \land B) \equiv \neg A \lor \neg B$

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Some useful equivalences

Double Negation Rule

$\neg\neg A \equiv A$

Implication Rule

$A \rightarrow B \equiv \neg A \vee B$

Some useful equivalences

$A \leftrightarrow B \equiv$
$(A \rightarrow B) \wedge (B \rightarrow A)$

Inference

Example: Given

(pass_exams ∧ pass_cwks ∧ pass_projects) → pass_MSc

pass_exams

pass_cwks

pass_projects

one can infer (conclude)

pass_MSc.

Example: Given

winter → (cold ∧ wet ∧ windy)

¬cold

Can you infer

¬winter ?

The "elections" example

Given

- If there are national elections then either the Tory party wins or the Labour party wins.
- If the unions do not support the Labour party then it does not win.
- There are national elections.

Can you infer

If the Tory party does not win then the unions support the Labour party?

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The "elections" example: Formalisation in logic

Given

Elections $\rightarrow$Labour_wins $\vee$ Tory_wins

$\neg$Unions_support_Labour $\rightarrow$ $\neg$Labour_wins

Elections

can you infer

$\neg$Tory_wins $\rightarrow$ Unions_support_Labour ?

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The "elections" example: Abbreviation

Premise:

1. **$E \rightarrow L \vee T$**
2. **$\neg U \rightarrow \neg L$**
3. **E**

Conclusion:

$\neg T \rightarrow U$

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You can try to use truth tables to see if the conclusion is semantically entailed by the premises.

How many rows?

Too many!

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E	L	T	U	$L \vee T$	$E \to L \vee T$	$\neg U$	$\neg L$	$\neg U \to \neg L$	$\neg T$	$\neg T \to U$
T	T	T	T	T	T	F	F	T	F	T

……..

..

..

..

.

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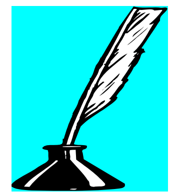
Can you give an informal proof of the conclusion from the premises without using truth tables?



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Performing inferences is very important in many applications of logic

Argument: Premise ⇒ Conclusion

Modelling : Theory ⇒ Consequences

Programming:

- Specification ⇒ Program
- Specification ⇒ Properties

Prolog: Program ⇒ Answers to queries

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Rules of Inference
Natural Deduction

(Reasoning purely at the syntactic level)

$\wedge$-elimination ($\wedge$E)

$$\frac{X \wedge Y}{X} \qquad \frac{X \wedge Y}{Y}$$

$\wedge$-introduction ($\wedge$I)

$$\frac{X, Y}{X \wedge Y} \qquad \frac{X, Y}{Y \wedge X}$$

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∨-elimination (∨E)

$$\frac{X \vee Y,\ \neg X}{Y} \qquad \frac{X \vee Y,\ \neg Y}{X}$$

∨-introduction (∨I)

$$\frac{X}{X \vee Y} \qquad \frac{X}{Y \vee X}$$

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→-elimination (→E) (Modus Ponens)

$$\frac{X,\ X \rightarrow Y}{Y}$$

→-introduction (→I)

X assume

.

.

$$\frac{Y}{X \rightarrow Y}$$

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¬-elimination and ¬-introduction (Reductio Ad Absurdum) (RAA) (Proof by contradiction)

¬X assume	X assume
.	.
.	.
.	.
$\frac{Y,\ \neg Y}{X}$	$\frac{Y,\ \neg Y}{\neg X}$

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Note: X and Y may be the same wff.

¬X assume	**X assume**
.	.
.	.
.	.
$\frac{\neg X,\ X}{X}$	$\frac{X,\ \neg X}{\neg X}$

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↔-introduction (↔I)

$$\frac{X \rightarrow Y,\ Y \rightarrow X}{X \leftrightarrow Y}$$

↔-elimination (↔E)

$$\frac{X \leftrightarrow Y}{X \rightarrow Y} \qquad \frac{X \leftrightarrow Y}{Y \rightarrow X}$$

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Note: In all the inference rules X and Y stand for any wffs. So the following, for example, is an application of the →elimination rule:

Given **$A \wedge (B \vee C)$** and

$(A \wedge (B \vee C)) \rightarrow ((A \rightarrow D) \vee (\neg E \wedge F))$

we can infer

$(A \rightarrow D) \vee (\neg E \wedge F)$

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Example

You may be wondering why we need the ∨I rule. Here is an example that uses it.

Example:

If there is a shortage of petrol or the tax on petrol is high then people are angry. There is a shortage of petrol.

So people are angry.

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Premise

1. **$(SP \vee HT) \rightarrow$ Anger**
2. **SP**

we want to conclude

Anger.

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1. $(SP \vee HT) \rightarrow Anger$ given
2. SP given

?

Anger

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1. $(SP \vee HT) \rightarrow Anger$ given
2. SP given

?

Anger $\rightarrow E$

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1. $(SP \vee HT) \rightarrow Anger$ given
2. SP given

?

$SP \vee HT$

Anger $\rightarrow E$

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Proof

1. $(SP \vee HT) \rightarrow Anger$ given
2. SP given
3. $SP \vee HT$ $2, \vee I$
4. Anger $1,3, \rightarrow E$

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Example

Derive $P \vee Q$ from $P \wedge Q$.

1. $P \wedge Q$ given

??????

$P \vee Q$

Derive $P \vee Q$ from $P \wedge Q$.

1. $P \wedge Q$ given

??????

$P \vee Q$ $\vee$I

Example

Derive $P \vee Q$ from $P \wedge Q$.

1. $P \wedge Q$ given
2. P 1, $\wedge$E
3. $P \vee Q$ 2, $\vee$I

Example

Derive R from $P, Q, (P \wedge Q) \rightarrow R$.

1. P given
2. Q given
3. $(P \wedge Q) \rightarrow R$ given

??????

R

1. P given
2. Q given
3. $(P \wedge Q) \rightarrow R$ given
   ??????
   R $\rightarrow$E

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1. P given
2. Q given
3. $(P \wedge Q) \rightarrow R$ given
4. $P \wedge Q$ 1,2, $\wedge$I
5. R 3,4, $\rightarrow$E

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Example

Derive $Q \rightarrow R$ from $P, (P \wedge Q) \rightarrow R$.

1. P given
2. $(P \wedge Q) \rightarrow R$ given
   ?????
   ?????
   ?????
   $Q \rightarrow R$

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1. P given
2. $(P \wedge Q) \rightarrow R$ given
   ?????
   ?????
   ?????
   $Q \rightarrow R$ $\rightarrow$I

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1. P given
2. $(P \wedge Q) \to R$ given
Q assume
????
R
$Q \to R$ $\to$I

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1. P given
2. $(P \wedge Q) \to R$ given
Q assume
????
R $\to$E
6. $Q \to R$ 3, 5, $\to$I

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1. P given
2. $(P \wedge Q) \to R$ given
3. Q assume
4. $P \wedge Q$ 1, 3, $\wedge$I
5. R 2, 4, $\to$E
6. $Q \to R$ 3, 5, $\to$I

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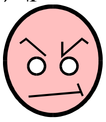
Example

Derive $\neg R$ from $P \wedge Q$, $R \to \neg P$.

1. $P \wedge Q$ given
2. $R \to \neg P$ given

????
????
????
$\neg R$



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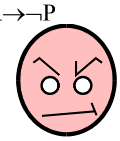
1. $P \wedge Q$ given
2. $R \rightarrow \neg P$ given

????
????
????



$\neg R$ RAA

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1. $P \wedge Q$ given
2. $R \rightarrow \neg P$ given

R assume
????
????

$\neg R$ RAA

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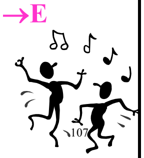
1. $P \wedge Q$ given
2. $R \rightarrow \neg P$ given
3. P 1, $\wedge$E
4. R assume
5. $\neg P$ 2, 4, $\rightarrow$E
6. $\neg R$ 3, 4, 5, RAA



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Some useful derived inference rules

Double negation elimination ($\neg\neg$E)

$$\frac{\neg\neg X}{X}$$

Double negation introduction ($\neg\neg$I)

$$\frac{X}{\neg\neg X}$$

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Law of excluded middle

$$\frac{}{X \vee \neg X}$$

Proof by cases

$$\frac{X \vee Y,\ X \to Z,\ Y \to Z}{Z}$$

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Modus Tollens

$$\frac{X \to Y,\ \neg Y}{\neg X}$$

Contraposition

$$\frac{X \to Y}{\neg Y \to \neg X}$$

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Dilemma

$$\frac{X \to Y,\ \neg X \to Y}{Y}$$

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Exercises

- Give a formal proof for the "elections" example.
- Using inference rules 1-9, show that inference rules 10-16 hold.

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Definition

Formal system = **Formal language + semantics + inference rules**

Propositional logic is a formal system.

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Definition

A **derivation** or **proof** of a wff W in propositional logic from a given set P of wffs, called premises, is a finite sequence of wffs such that the last wff is W and each wff in the sequence is one of the following:

- a premise, i.e. a wff in P
- an immediate consequence of one or more wffs preceding it in the sequence, as determined by one of the inference rules of propositional logic.
- An assumption (that is later discharged by an application of →I or RAA).

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P ├ W

denotes W is (syntactically) derivable from P.

├ is called the syntactic turnstile. It is a symbol in the metalanguage.

Example:

In the "elections" example:

E→L∨T, ¬U→¬L, E ├ ¬T→U

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Notes

1. **|-W** denotes W is derivable from an empty set of premises.
2. Let A, B be wffs.

If **A ├ B** then **├ A → B.**

If **├ A → B** then **A ├ B.**

In general if

P is a set of wffs, and

P' is a conjunction of the wffs in P, and

W is a wff then **P ├ W** iff **├ P' →W**

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3. Proofs (derivations) are independent of the "meaning" of the propositional symbol. So a proof is still valid if the symbols are replaced consistently.

Example

If we have a proof for

$\vdash \mathbf{X} \vee \neg \mathbf{X}$

replacing X by any wff W provides a proof for

$\vdash \mathbf{W} \vee \neg \mathbf{W}$.

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For example:

$\vdash$ **ToBe $\vee$ ¬ToBe**

$\vdash$ (fever$\rightarrow$flu) $\vee$ ¬(fever$\rightarrow$flu)

The last two derivations are **instances** of the first.

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4. For convenience, in a derivation we can use instances of previous derivations. That is, if we have previously shown

S|-W, and we are now attempting a new proof for another wff, but we have so far shown an instance of S, then we can write down the same instance of W in the derivation without reproducing its entire proof.

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Exercise A

Show

$P \rightarrow Q,\ R \rightarrow S \vdash (P \vee R) \rightarrow (Q \vee S)$.

Exercise B

Show

$(P \vee Q) \vee R \vdash P \vee (Q \vee R)$ and

$P \vee (Q \vee R) \vdash (P \vee Q) \vee R$.

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Exercise C

Formalise the following argument and show that it is valid. You may use the theorems in A and B, above.

In Britain one of the three parties, Tory, Labour or Liberal Democrat, is in power.

If the Tories are in power the government may support cuts in public spending.

If Labour is in power the government may support tax increases.

If the Liberal Democrats are in power the government may support proportional representation.

So in Britain the government may support cuts in public spending or tax increases or proportional representation.

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Be careful when you use assumptions in a derivation

Show $\vdash \neg(\neg A \wedge \neg B) \rightarrow (A \vee B)$

1. $\neg(\neg A \wedge \neg B)$	assume
2. $\neg(A \vee B)$	assume
3. $\neg A$	assume
4. $\neg B$	assume
5. $\neg A \wedge \neg B$	3,4,$\wedge$I
6. B	4,5,1,RAA
7. $A \vee B$	6,$\vee$I
8. A	3,2,7,RAA
9. $A \vee B$	8,$\vee$I
10. $A \vee B$	2,9,RAA
11. $\neg(\neg A \wedge \neg B) \rightarrow (A \vee B)$	1,10,$\rightarrow$I

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Notes

- The only inference rules that make use of assumptions are RAA and $\rightarrow$I.
- It is very important to be clear about the scope of assumptions.
- Any assumption made during a derivation will remain in force, and ultimately count as one of the premises for the conclusion, unless it gets discharged before the conclusion is reached in the proof.

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Exercises

Show

$A\rightarrow B, B\rightarrow C \vdash A\rightarrow C.$

(Transitivity of the implication.)

Show

$\vdash ((P\wedge Q) \vee (\neg P\wedge R)) \rightarrow (Q\vee R)$

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Exercises

Show
$\vdash (P \rightarrow Q) \leftrightarrow (\neg P \vee Q)$

Show
$P, \neg P \vdash Q$

Note:
The last exercise shows that anything can be derived from an inconsistent set of premises.

For example if we are given that
London is crowded and London is not crowded.
We can conclude
Everyone will get distinction in their results this year.
This may seem unintuitive, but think of it this way:
If you are prepared to believe a contradiction you are prepared believe anything!

Exercise

Consider the following derivation:

1. A	**assume**
2. $A \rightarrow A$	**1, 1, $\rightarrow$I**
3. A	**1, 2, $\rightarrow$E**

Is there anything wrong with it?
If so, what?

Soundness and Completeness

Propositional logic is both **sound** and **complete**.
Let S be any set of wffs, and let W be a wff.

Soundness means the following:
If $S \vdash W$ then $S \models W$.

Completeness means the following:
If $S \models W$ then $S \vdash W$.

So in propositional logic we are justified in switching between syntactic proofs and semantic consequences.

Example

$A \equiv B$ iff

$A \models B$ and $B \models A$ iff

$A \vdash B$ and $B \vdash A$ iff

$A \models B$ and $B \vdash A$

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Exercises

Given the equivalence

$A \wedge (B \wedge C) \equiv (A \wedge B) \wedge C$ Show that

$(P \wedge Q) \rightarrow R,\ \neg(R \wedge S) \vdash \neg(P \wedge (Q \wedge S))$.

Using the equivalences

$A \rightarrow B \equiv \neg A \vee B$ and

$\neg(A \vee B) \equiv \neg A \wedge \neg B$

or otherwise show

$\vdash (P \rightarrow Q) \vee (Q \rightarrow P)$.

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