

COMPUTATIONAL FINANCE: 422

The Basic Theory of Interest

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(Slides courtesy of Daniel Kuhn)

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This Lecture

- The time value of money
 - Compounding
 - Present and future value
 - Net present value as a decision criterion
- The term structure of interest rates
 - Spot rates
 - Forward rates
 - Expectation dynamics

Further reading:

- D.G. Luenberger: *Investment Science*, Chapters 2,4

Principal and Interest

Example: if you invest \$1.00 in a bank account that pays 8% **interest** per year, then **at the end of 1 year** you will have in your account \$1.08.

- **Principal:** amount invested (W).
- **Interest:** 'rent' paid on investment (I).
- **Interest rate:** interest per unit of currency invested (r).

$$\Rightarrow I = W \times r$$

Account holdings:

- **Initial wealth** (today): $W_0 = W$;
- **Terminal wealth** (after one year): $W_1 = W(1 + r)$.

Compound Interest I

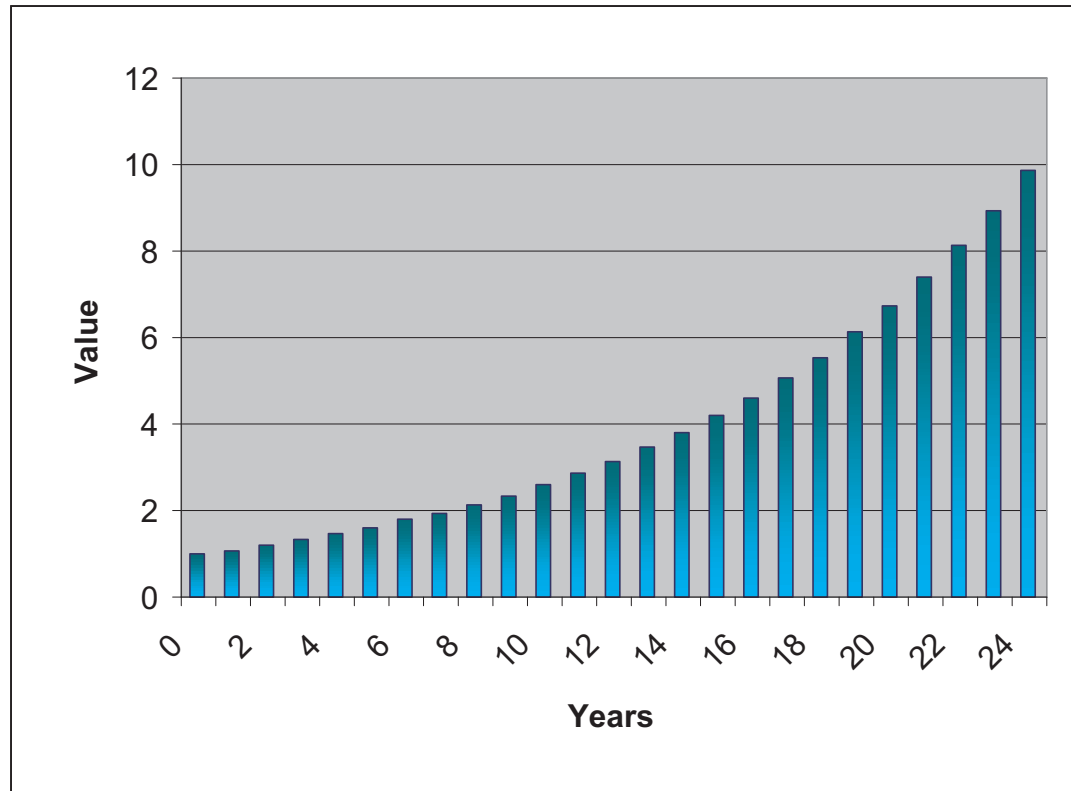
Consider a situation in which money is invested in a bank account over **several periods**. Assume that the **interest rate** in the n th year is r_n for $n = 1, 2, 3, \dots$. We obtain the following **account holdings**:

- **today**: $W_0 = W$;
- **after 1 year**: $W_1 = W(1 + r_1)$;
- **after 2 years**: $W_2 = W_1(1 + r_2) = W(1 + r_1)(1 + r_2)$;
- **after n years**: $W_n = W_{n-1}(1 + r_n) = W \prod_{i=1}^n (1 + r_i)$.

If the **interest rate is constant**, i.e., $r_n = r$, then

$$W_n = W(1 + r)^n \quad \Rightarrow \quad r = \left(\frac{W_n}{W_0} \right)^{1/n} - 1.$$

Compound Interest II



The seven-ten rule:

- Money invested at 7% doubles in about 10 years;
- Money invested at 10% doubles in about 7 years (Figure).

Compounding at Various Intervals

It is traditional to quote the interest rate on a yearly basis but then apply the appropriate proportion of that interest rate over each compounding period. Divide a year in m equally spaced compounding periods.

- Nominal interest rate: r
- Length of a compounding period: $1/m$ [years]
- Interest rate for each of the m periods: r/m
- Growth of the account over k periods: $[1 + r/m]^k$
- Growth of the account over 1 year: $[1 + r/m]^m$
- The effective interest rate is the number r_{eff} such that

$$1 + r_{\text{eff}} = [1 + r/m]^m .$$

Continuous Compounding I

Increasing the number of compounding intervals per year **infinitely** leads to the idea of **continuous compounding**.

- Time measured in **years**: t
- Time measured in **# compounding intervals**: $k = tm$

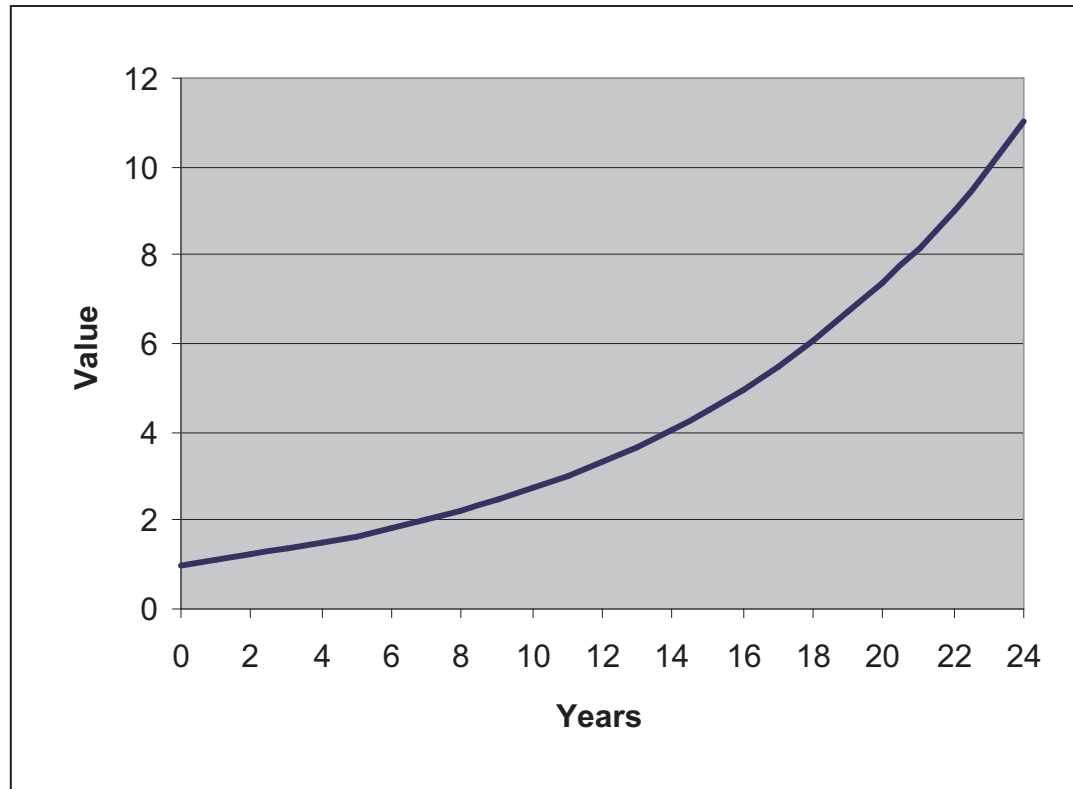
If m is very large, then we can assume that $k \in \mathbb{N}$.

If m tends to infinity, then the **growth** of an account with (nominal) interest rate r over t years becomes:

$$[1 + r/m]^k = [1 + r/m]^{mt} = ([1 + r/m]^m)^t \rightarrow e^{rt}.$$

The last expression corresponds to **continuous compounding** (in the limit $m \rightarrow \infty$) \Rightarrow this leads to the familiar **exponential growth curve**.

Continuous Compounding II



Under **continuous compounding** at 10% the value of \$1

- **doubles** in about 7 years;
- grows by a **factor of 8** in about 20 years.

Debt

- A bank deposit grows over time due to interest compounding.
- If I borrow money from the bank at an interest rate r and make no payments, then my debt increases over time according to the same formulas.

Time Value of Money

- Money **invested/borrowed** today leads to **increased value/debt** in the future as a result of **interest**.
- The **compounding formulas** of the previous slides show how to calculate this **future value**.
- We can use the same formulas to determine the **present value** that should be assigned to money that is to be **received at a later time**.

Present Value

Suppose that the annual interest rate r is compounded m times per year. The following are equivalent:

- receive an amount A after k compounding periods;
- receive an amount $d_k A$ today, where

$$d_k = \frac{1}{(1 + r/m)^k} < 1$$

denotes the discount factor corresponding to period k .

In fact, if we deposit $d_k A$ in a bank account today, then we receive A after k compounding periods.

$\Rightarrow d_k A$ is the present value of A .

The Ideal Bank

Def.: An **ideal bank**:

- applies the same interest rate to **deposits** and **loans**.
- has **no service charges** or **transaction costs**.
- has the same interest rate **for any size of principal**.

Interest rates for different transactions may be different:

- a 2-year **certificate of deposit** (CD) might offer a higher rate than a 1-year CD.

Def.: If an **ideal bank** has an interest value that is **independent** of the length of time for which it applies, it is called a **constant ideal bank**.

Future and Present Value of Streams I

- Consider a **cash flow stream** $x_0, x_1, x_2, \dots, x_n$.
- x_k occurs at the end of period k .
- We can use a **constant ideal bank** to move all cash flows to the **end of period n** or to the **present time**.

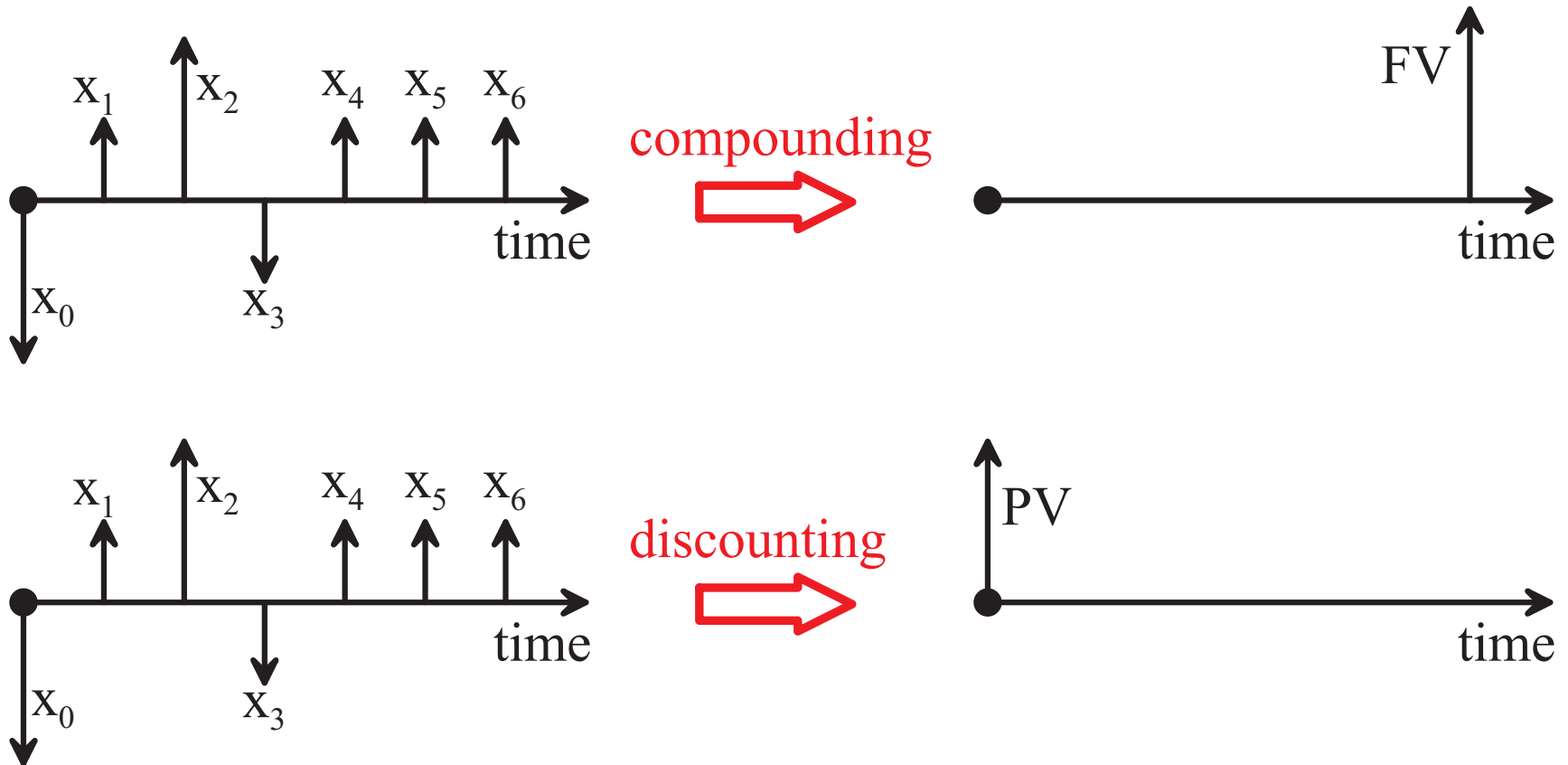
Def.: The **future value** of the stream is

$$\text{FV} = \sum_{k=0}^n x_k (1 + r/m)^{n-k} \quad \leftarrow \quad \text{'compounding'}$$

Def.: The **present value** of the stream is

$$\text{PV} = \sum_{k=0}^n \frac{x_k}{(1 + r/m)^k} \quad \leftarrow \quad \text{'discounting'}$$

Future and Present Value of Streams II



Present Value and an Ideal Bank

Def.: Two CF streams are **equivalent** if they can be transformed into each other by an **ideal bank**.

Example: A **10% bank** can change

- $(1, 0, 0)$ to $(0, 0, 1.21)$ by receiving a **deposit** of \$1 now and **paying principal and interest** of \$1.21 in 2 years;
- $(0, 0, 1.21)$ to $(1, 0, 0)$ by issuing a **loan** for \$1 now.

Theorem: The **CF streams** x_0, x_1, \dots, x_n and y_0, y_1, \dots, y_n are **equivalent** for a constant ideal bank with interest rate r iff their PVs are **equal**.

\Rightarrow Evaluate **CF streams** only on the basis of their **PVs**.

Net Present Value

- Different choices can lead to different CF streams.
- PV can be used to rank these choices:
the higher the PV, the more desirable the choice.
- Here, one must include all cash flows associated with an investment, both positive and negative.
- In this case, PV is termed net present value (NPV).

When to Cut a Tree?

You want to plant trees in order to sell lumber:

- buy seedlings today: initial cost of 1;
- two options as to when to harvest:
 - (a) after 1 year: early moderate revenues of 2;
 - (b) after 2 years: later but higher revenues of 3 (due to additional growth).

Net present values for $r = 10\%$:

- (a) $NPV = -1 + 2/1.1 = 0.82$;
- (b) $NPV = -1 + 3/(1.1)^2 = 1.48$.

⇒ it is best to cut later.

When to Cut a Tree?

Assume that the proceeds of a harvest can be used to plant additional trees \Rightarrow the business has **several cycles**.

Reconsider the two options:

- (a) **cut early**: money is **doubled every year**;
- (b) **cut later**: money is tripled every 2 years \Rightarrow in average, money **grows by a factor $\sqrt{3}$ per year**.

\Rightarrow it is best to cut early.

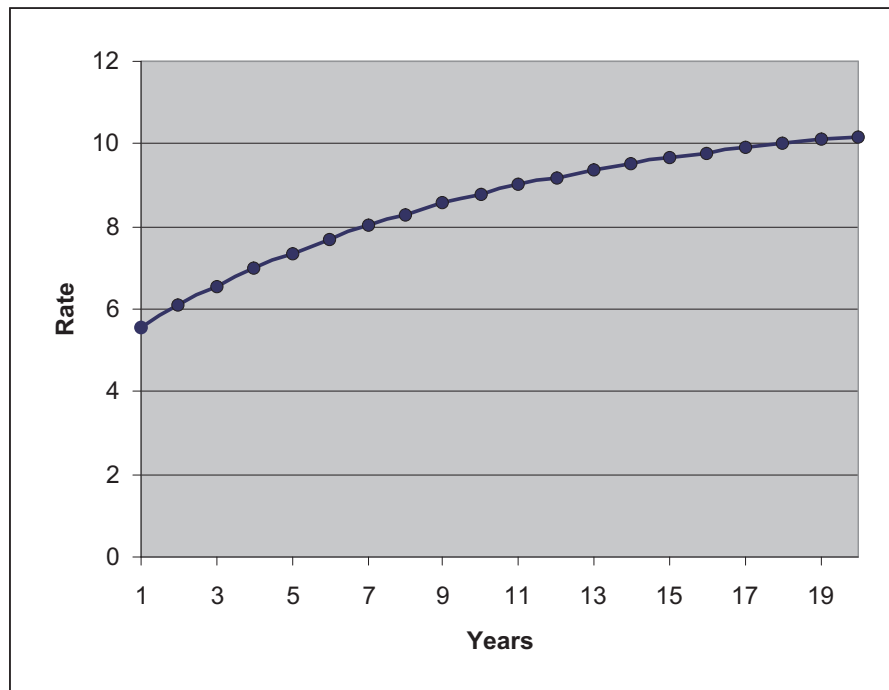
Repeatable activities must be compared over **the same time horizon**, e.g., 2 years in the tree cutting example:

$$\text{NPV(a)} = -1 + 4/(1.1)^2 = 2.31 > \text{NPV(b)} = -1 + 3/(1.1)^2 = 1.48$$

The Term Structure of Interest Rates

In reality, there is a **whole family of interest rates** at any point in time — a different rate for each **maturity time**.

Def.: The **spot rate s_t** is the **annualized interest rate** charged for money held from the present until time t .



Compounding Conventions

Under different **compounding conventions**, the spot rate s_t is defined as follows:

- **yearly compounding**: s_t is defined such that

$$(1 + s_t)^t$$

is the growth factor of a deposit held for t years ($t \in \mathbb{N}$);

- **m compounding periods/year**: s_t is defined such that

$$(1 + s_t/m)^{mt}$$

is the corresponding growth factor ($t \in \frac{1}{m}\mathbb{N}$);

- **continuous compounding**: s_t is defined such that $e^{s_t t}$ is the corresponding growth factor ($t \in \mathbb{R}_+$).

Properties of Spot Rate Curves

- Long commitments tend to offer higher interest rates than short commitments.
 - ⇒ Spot rate curves are normally upward sloped.
- The spot rate curve undulates around in time (like a branch in the wind).
- The spot rate curve is called
 - normally shaped: if it is increasing;
 - inverted: if it is decreasing.^a
- The spot rate curve is smooth.

^aThe inverted shape occurs when short-term rates increase rapidly, and investors believe, that the rise is temporary.

Discount Factors

For a given set of spot rates, we can define the corresponding **discount factors** d_t :

- **yearly compounding:**

$$d_t = \frac{1}{(1 + s_t)^t} \quad t \in \mathbb{N};$$

- **m compounding periods/year:**

$$d_t = \frac{1}{(1 + s_t/m)^{mt}} \quad t \in \frac{1}{m}\mathbb{N};$$

- **continuous compounding:**

$$d_t = e^{-s_t t} \quad t \in \mathbb{R}.$$

Present Value

Given any CF stream $x_0, x_1, x_2, \dots, x_n$, the present value relative to the prevailing spot rates is

$$PV = x_0 + d_1x_1 + d_2x_2 + \dots + d_nx_n .$$

Note that:

- d_t acts like a price for cash received at time t ;
- PV is the sum of 'price times quantity' for all cash components.

Example: Price of a 10-year Bond

Consider an 8% bond maturing in 10 years:

- the bond pays \$8 at the end of the years 1, 2, ..., 9 and \$108 at the end of year 10.
- the end-of-year discount factors for years 1, 2, ..., 10 can be calculated from a given spot rate curve.
- We take the products of the cash flows with the corresponding discount factors and sum.

⇒ The value of the bond is \$97.34.

Year	1	2	3	4	5	6	7	8	9	10	Total PV
Spot Rate	5.571	6.088	6.555	6.978	7.361	7.707	8.020	8.304	8.561	8.793	
Discount	0.947	0.889	0.827	0.764	0.701	0.641	0.583	0.528	0.477	0.431	
Cash Flow	8	8	8	8	8	8	8	8	8	108	
PV	7.58	7.11	6.61	6.11	5.61	5.12	4.66	4.23	3.82	46.50	97.34

Forward Rates I

Forward rates are interest rates for money to be borrowed between two dates **in the future**, but under **terms agreed upon today**.

Example: Assume that you **commit today** to deposit \$1 in a bank account for 1 year, **starting in 1 year from now**. That loan will accrue interest at a prearranged rate f (**agreed upon now**).

f is the **forward rate** for money to be lent in this way.

f can be determined from the current spot rates.

Forward Rates II

Two possibilities to invest \$1 over a period of two years:^a

1. Leave \$1 in a 2-year account.

⇒ After 2 years you obtain $\$(1 + s_2)^2$.

2. Place \$1 in a 1-year account and make arrangements that the proceeds $\$(1 + s_1)$ will be lent for 1 year starting a year from now.

⇒ After 2 years you obtain $\$(1 + s_1)(1 + f)$.

Comparison principle: $(1 + s_2)^2 \stackrel{!}{=} (1 + s_1)(1 + f)$

$$\Rightarrow f = \frac{(1 + s_2)^2}{1 + s_1} - 1$$

^aYearly compounding.

Forward Rates III

General forward rate definition: The forward rate between times t_1 and t_2 ($t_1 < t_2$) is denoted by f_{t_1, t_2} . It is the interest rate charged for borrowing money at t_1 which is to be repaid (with interest) at t_2 . f_{t_1, t_2} is agreed on today ($t = 0$).

The forward rate $f_{i,j}$ satisfies (yearly compounding)

$$(1 + s_j)^j = (1 + s_i)^i (1 + f_{i,j})^{j-i} \quad \Rightarrow \quad f_{i,j} = \left[\frac{(1 + s_j)^j}{(1 + s_i)^i} \right]^{1/(j-i)} - 1.$$

- This is called the implied forward rate.
- It may be slightly different from the market forward rate due to market imperfections.

Different Compounding Conventions

● **Yearly compounding:** $(1 + s_j)^j = (1 + s_i)^i (1 + f_{i,j})^{j-i}$

$$\Rightarrow f_{i,j} = \left[\frac{(1 + s_j)^j}{(1 + s_i)^i} \right]^{1/(j-i)} - 1$$

● **m periods/year:** $(1 + s_j/m)^j = (1 + s_i/m)^i (1 + f_{i,j}/m)^{j-i}$

$$\Rightarrow f_{i,j} = m \left[\frac{(1 + s_j/m)^j}{(1 + s_i/m)^i} \right]^{1/(j-i)} - m$$

● **Continuous compounding:** $e^{s_{t_2} t_2} = e^{s_{t_1} t_1} e^{f_{t_1, t_2} (t_2 - t_1)}$

$$\Rightarrow f_{t_1, t_2} = \frac{s_{t_2} t_2 - s_{t_1} t_1}{t_2 - t_1}$$

Spot Rate Forecasts I

The forward rate $f_{1,2}$ is

- the **implied rate** for money loaned for 1 year, a year from now;
- the **market expectation** of what the 1-year spot rate will be next year.

The same argument applies to all other rates, too.

⇒ The **current spot rate curve** s_1, s_2, \dots, s_n implies a set of **forward rates** $f_{1,2}, f_{1,3}, \dots, f_{1,n}$, which define the **expected spot rate curve** $s'_1, s'_2, \dots, s'_{n-1}$ for next year:

$$s'_{j-1} = f_{1,j} = \left[\frac{(1 + s_j)^j}{1 + s_1} \right]^{1/(j-1)} - 1 \quad \text{for } j = 2, 3, \dots, n \quad (1)$$

Spot Rate Forecasts II

The entity of all **future expected spot rate curves** implied by an initial curve can be displayed as follows:

$$\begin{array}{ccccccc}
 f_{0,1} & f_{0,2} & f_{0,3} & \cdots & f_{0,n-2} & f_{0,n-1} & f_{0,n} \\
 f_{1,2} & f_{1,3} & f_{1,4} & \cdots & f_{1,n-1} & f_{1,n} & \\
 f_{2,3} & f_{2,4} & f_{2,5} & \cdots & f_{2,n} & & \\
 \vdots & \vdots & & & & & \\
 f_{n-2,n-1} & f_{n-2,n} & & & & & \\
 f_{n-1,n} & & & & & &
 \end{array}$$

The transformation (1) of the spot rate curve is termed **expectation dynamics**.

Discount Factors

We denote by d_{t_1, t_2} the **discount factor** to discount cash received at time t_2 back to time t_1 where $t_1 < t_2$.

- **Yearly compounding:**

$$d_{i,j} = \frac{1}{(1 + f_{i,j})^{j-i}}$$

- **m periods/year:**

$$d_{i,j} = \frac{1}{(1 + f_{i,j}/m)^{j-i}}$$

- **Continuous compounding:**

$$d_{t_1, t_2} = e^{-f_{t_1, t_2}(t_2 - t_1)}$$

Running Present Value I

For any $i < j < k$ we have (yearly compounding)

$$d_{i,k} = d_{i,j} d_{j,k} .$$

The **present value** $PV(0)$ of a CF stream x_0, x_1, \dots, x_n is

$$\begin{aligned} PV(0) &= x_0 + d_1 x_1 + d_2 x_2 + \dots + d_n x_n \\ &= x_0 + d_1 (x_1 + d_{1,2} x_2 + \dots + d_{1,n} x_n) \\ &= x_0 + d_1 PV(1) , \end{aligned}$$

where $PV(1)$ is the **present value** of the stream x_1, \dots, x_n as viewed at time 1. The values $d_{1,k}$, $k = 2, 3, \dots, n$, are the **discount factors 1 year from now** under an assumption of expectation dynamics.

Running Present Value II

Define now the **time k present value** as

$$\text{PV}(k) = x_k + d_{k,k+1}x_{k+1} + d_{k,k+2}x_{k+2} \cdots + d_{k,n}x_n.$$

The relations

$$d_{k,k+j} = d_{k,k+1}d_{k+1,k+j} \quad \text{for } j = 1, 2, \dots, n - j$$

imply that the **present values** $\text{PV}(k)$ satisfy the **recursion**

$$\text{PV}(k) = x_k + d_{k,k+1}\text{PV}(k+1).$$

\Rightarrow $\text{PV}(0)$ can be calculated by means of a **backward recursion** starting with $\text{PV}(n) = x_n$.