COMPUTATIONAL FINANCE: 422

The Capital Asset Pricing Model

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(Slides courtesy of Daniel Kuhn)

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This Lecture

- The two-fund theorem
- Inclusion of a risk-free asset and the one-fund theorem
- The market portfolio
- The capital market line
- The Capital Asset Pricing Model (CAPM)
  - The beta of an asset/portfolio
  - The security market line
  - Systematic and unsystematic risk
- CAPM as a pricing formula

Further reading:

- D.G. Luenberger: *Investment Science*, Chapters 6 & 7

The Markowitz Model

Markowitz problem (with short selling allowed):

$$
\begin{aligned}
\text{minimize} \quad & \frac{1}{2}\boldsymbol{w}^\top \Sigma \boldsymbol{w} \\
\text{subject to} \quad & \boldsymbol{w}^\top \bar{\boldsymbol{r}} - \bar{r}_{\mathrm{P}} = 0 \\
& \boldsymbol{w}^\top \boldsymbol{e} - 1 = 0
\end{aligned}
$$

Lagrangian function:

$$
L(\boldsymbol{w}, \lambda, \mu) = \frac{1}{2}\boldsymbol{w}^\top \Sigma \boldsymbol{w} - \lambda \left(\boldsymbol{w}^\top \bar{\boldsymbol{r}} - \bar{r}_{\mathrm{P}} \right) - \mu \left(\boldsymbol{w}^\top \boldsymbol{e} - 1 \right) \ ,
$$

Optimality conditions:

$$
\Sigma \boldsymbol{w} - \lambda \bar{\boldsymbol{r}} - \mu \boldsymbol{e} = \boldsymbol{0}\,, \quad \bar{\boldsymbol{r}}^\top \boldsymbol{w} = \bar{r}_{\mathrm{P}}\,, \quad \boldsymbol{e}^\top \boldsymbol{w} = 1
$$

Solution of The Markowitz Model

The portfolio weights and the Lagrange multipliers of the optimal portfolio with expected return $\bar{r}_{\mathrm{P}}$ are given by:

$$\begin{pmatrix} \boldsymbol{w} \\ \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} \Sigma & -\bar{\boldsymbol{r}} & -\boldsymbol{e} \\ -\bar{\boldsymbol{r}}^\top & 0 & 0 \\ -\boldsymbol{e}^\top & 0 & 0 \end{pmatrix}^{-1} \begin{pmatrix} \boldsymbol{0} \\ -\bar{r}_{\mathrm{P}} \\ -1 \end{pmatrix} = f(\bar{r}_{\mathrm{P}}) \, .$$

$\Rightarrow$ f is an affine function (i.e., a linear function $+$ a constant) of the return target parameter $\bar{r}_{\mathrm{P}}$.

The Two-Fund Theorem

Theorem 0.1. *Let* $(\boldsymbol{w}_1, \lambda_1, \mu_1)$ *and* $(\boldsymbol{w}_2, \lambda_2, \mu_2)$ *be Markowitz solutions for* $\bar{r}_{\mathrm{P}}^1$ *and* $\bar{r}_{\mathrm{P}}^2$*, respectively. Then, the Markowitz solution for*

$$\bar{r}_{\mathrm{P}}^3 = \alpha \bar{r}_{\mathrm{P}}^1 + (1-\alpha)\bar{r}_{\mathrm{P}}^2$$

is given by

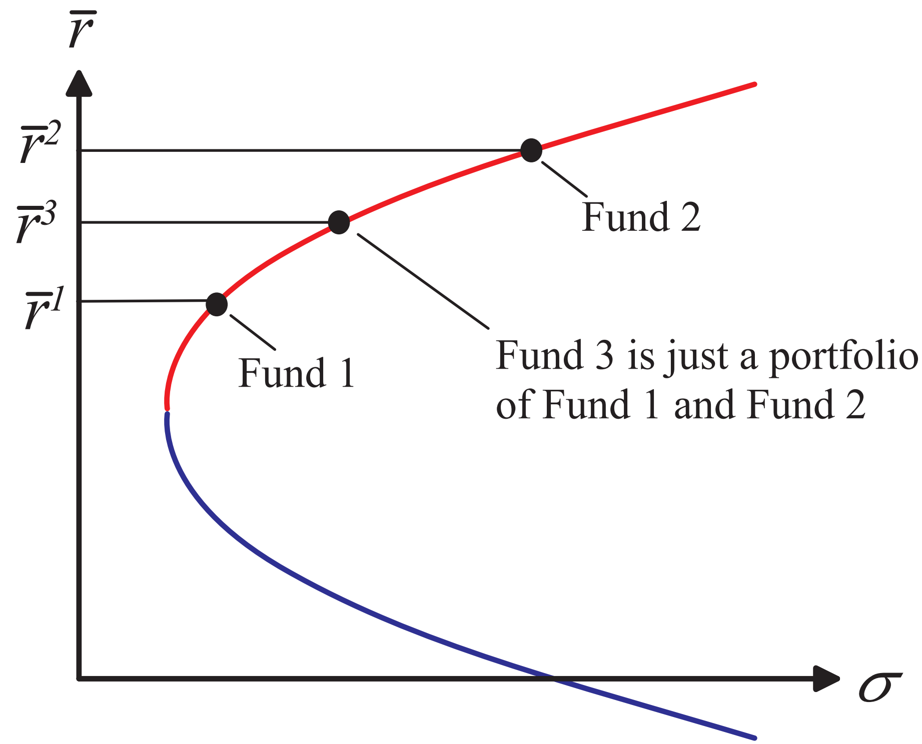
$$(\boldsymbol{w}_3, \lambda_3, \mu_3) = \alpha(\boldsymbol{w}_1, \lambda_1, \mu_1) + (1-\alpha)(\boldsymbol{w}_2, \lambda_2, \mu_2)\,.$$

Proof.

$$\begin{aligned}
(\boldsymbol{w}_3, \lambda_3, \mu_3) &= f(\bar{r}_{\mathrm{P}}^3) = f(\alpha \bar{r}_{\mathrm{P}}^1 + (1-\alpha)\bar{r}_{\mathrm{P}}^2) \\
&= \alpha f(\bar{r}_{\mathrm{P}}^1) + (1-\alpha) f(\bar{r}_{\mathrm{P}}^2) \\
&= \alpha(\boldsymbol{w}_1, \lambda_1, \mu_1) + (1-\alpha)(\boldsymbol{w}_2, \lambda_2, \mu_2)
\end{aligned}$$

$\square$

Importance of the Two-Fund Theorem



- Investors seeking efficient portfolios need only invest in combinations of two efficient funds.
- Under the assumptions of the mean-variance model, there is no need for anyone to buy individual stocks.

Inclusion of a Risk-Free Asset

- So far we have assumed that all assets are risky, i.e., they each have $\sigma > 0$.
- A risk-free asset has return r_f that is deterministic. It has $\sigma = 0$ and satisfies $r_f = \mathrm{E}(r_f)$.
- Consider any risky asset whose return r has mean value $\bar{r}$ and standard deviation $\sigma > 0$. The covariance of the risk-free return r_f with r must be zero:

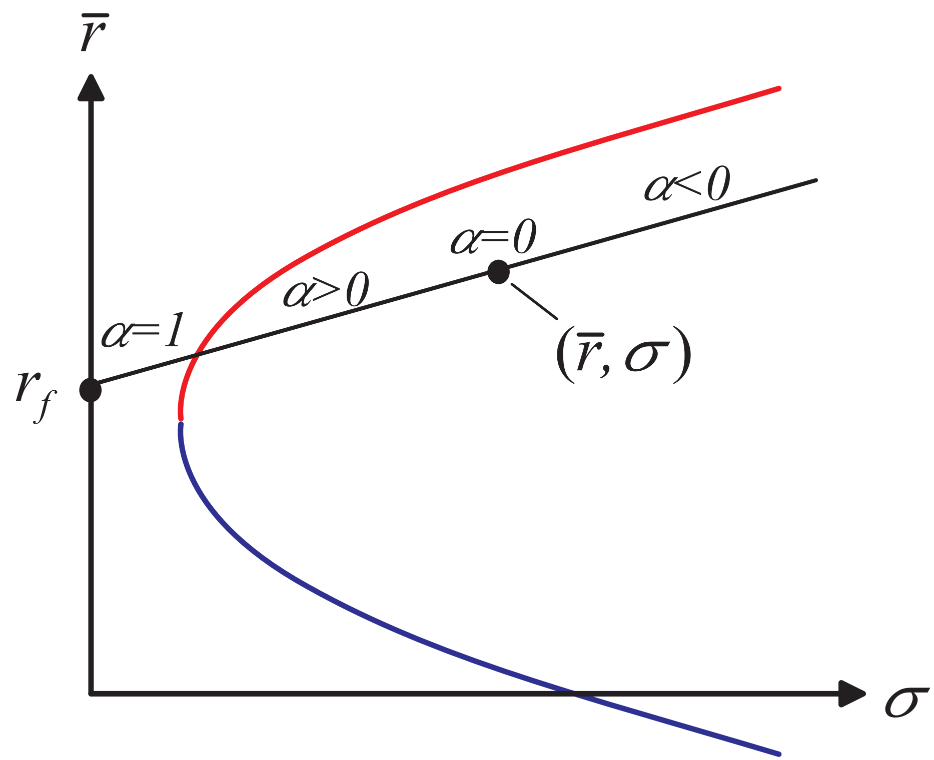
$$\operatorname{cov}(r, r_f) = \mathrm{E}[(r - \mathrm{E}[r])(r_f - \mathrm{E}[r_f])] = 0\,.$$

- Form a portfolio using weight α for the risk-free asset and $1 - \alpha$ for the risky asset.

$$\Rightarrow \quad \bar{r}_{\mathrm{P}} = \alpha r_f + (1-\alpha)\bar{r}\,, \quad \sigma_{\mathrm{P}}^2 = (1-\alpha)^2\sigma^2\,.$$

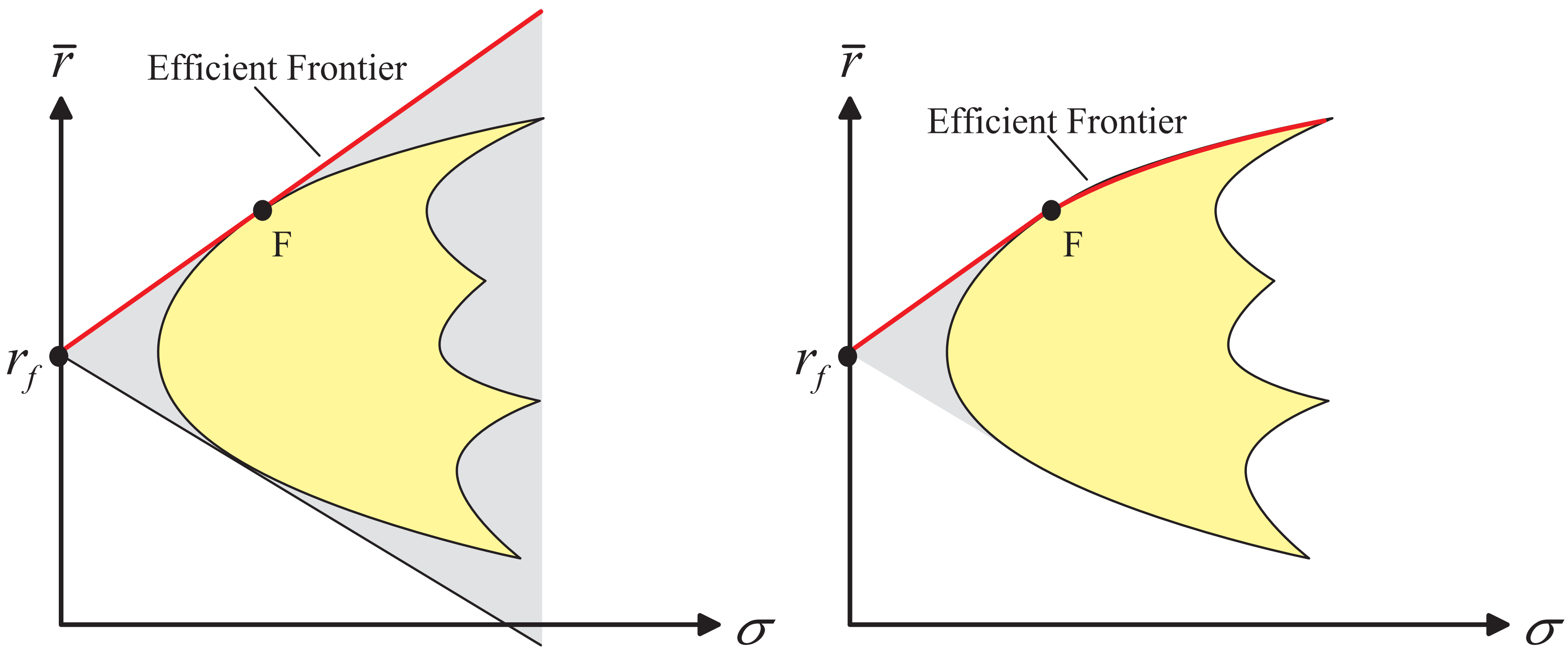
Inclusion of a Risk-Free Asset

- For this portfolio we have:
  - mean $= \alpha r_f + (1-\alpha)\bar{r}$
  - standard deviation $= (1-\alpha)\sigma$
- As we vary α, this maps out a straight line.



Expanded Feasible Set

- If both borrowing and lending are allowed, an infinite triangular region is obtained.
- If only lending is allowed, the region will have a triangular front end, but will curve for larger σ.



The One-Fund Theorem

Theorem 0.2. *When risk-free borrowing and lending are available, there is a single fund F of risky assets such that any efficient portfolio can be constructed as a combination of the fund F and the risk-free asset.*

Markowitz problem with a risk-free asset:

Let $\boldsymbol{w}$ be the weights of the risky assets and w_0 the weight of the risk-free asset.

$$\begin{aligned} \text{minimize} \quad & \frac{1}{2}\boldsymbol{w}^\top \Sigma \boldsymbol{w} \\ \text{subject to} \quad & w_0 r_f + \boldsymbol{w}^\top \bar{\boldsymbol{r}} = \bar{r}_{\mathrm{P}} \\ & w_0 + \boldsymbol{w}^\top \boldsymbol{e} = 1 \end{aligned}$$

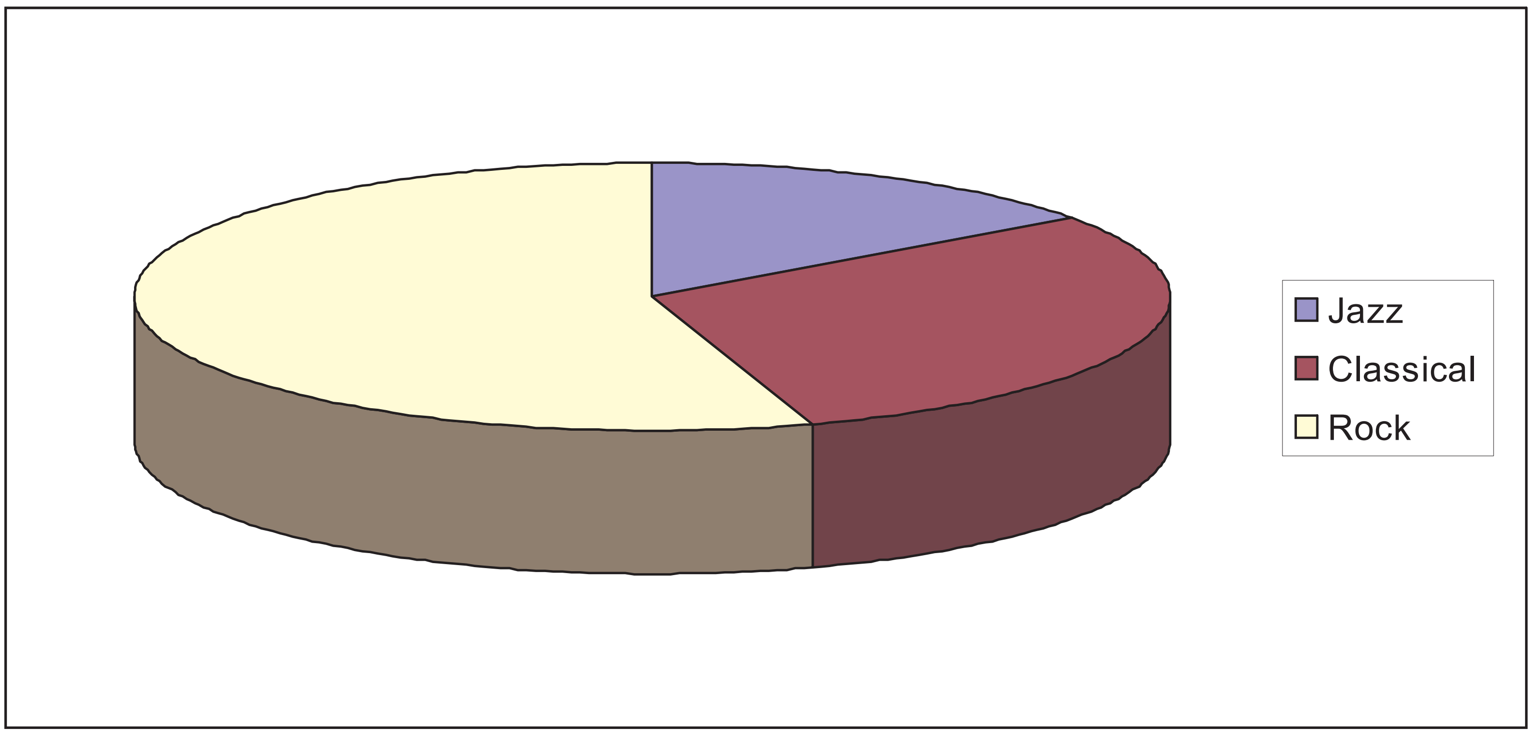
Market Portfolio

- Under the one fund theorem, every investor will buy a combination of the fund F and the risk-free asset.
  $\Rightarrow$ Everyone buys the same portfolio of risky assets.
- Equilibrium argument: F must be the market portfolio!
- The market portfolio is a portfolio of every stock in the market in proportion to its market capitalization.
- Asset i's weight in the market portfolio is

$$w_i = \frac{\text{total value of all assets of type } i}{\text{total value of all assets in the market}}.$$

Example: Market Portfolio

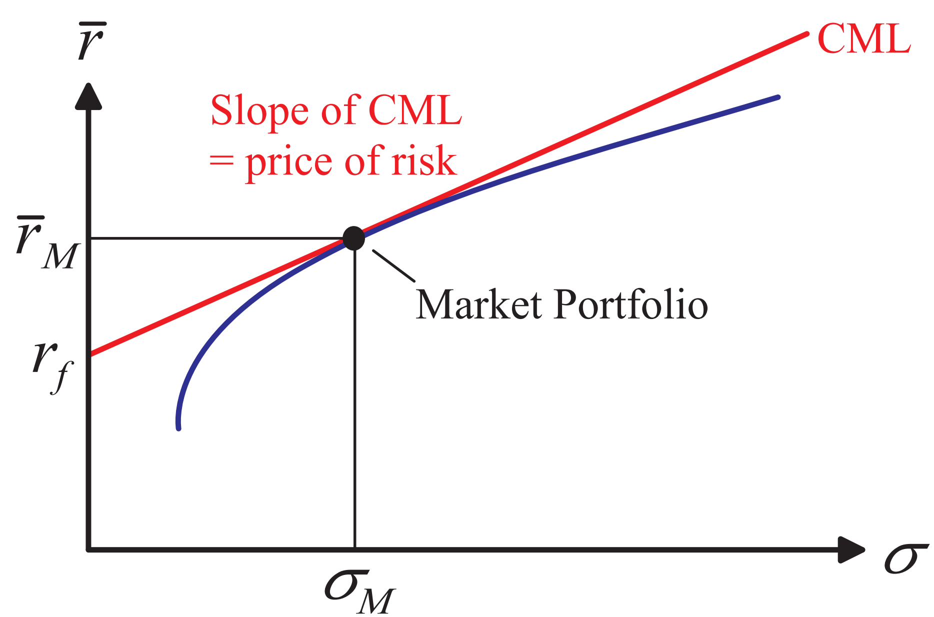
Security	Shares Outstanding	Price per Share (£)	Capitalization	Weight
Jazz	10'000.00	6	60'000.00	3/20
Classical	30'000.00	4	120'000.00	3/10
Rock	40'000.00	5.5	220'000.00	11/20
Total	80'000.00		400'000.00	1



The Capital Market Line

- In the presence of a risk-free asset, the efficient frontier is called capital market line (CML).
- Any asset on the CML satisfies

$$\bar{r} = r_f + \underbrace{\left(\frac{\bar{r}_M - r_f}{\sigma_M} \right)}_{\text{"price of risk"}} \sigma \, .$$



Example

- Given $r_f = 10\%$, $\bar{r}_M = 17\%$, $\sigma_M = 12\%$.
- You would like to earn 33% on average. What is the minimal risk (standard deviation) of your portfolio?
- For the risk to be minimal, your portfolio should lie on the capital market line. Thus

$$
\begin{aligned}
\bar{r} &= r_f + \left(\frac{\bar{r}_M - r_f}{\sigma_M} \right) \sigma \\
\Rightarrow \quad 0.33 &= 0.10 + \frac{0.17 - 0.10}{0.12} \sigma \,.
\end{aligned}
$$

The solution of this equation is $\sigma = 40\%$.

The Capital Asset Pricing Model

Theorem 0.3 (Capital Asset Pricing Model (CAPM))**.** *Assume that*

- *all investors are Markowitz mean-variance investors;*
- *short selling is allowed;*
- *there is a risk-free asset;*
- *the investors share the same predictions of means, variances, and covariances.*

If the market portfolio M is efficient, the expected return $\bar{r}_i$ *of any asset* i *satisfies*

$$\bar{r}_i - r_f = \beta_i \left(\bar{r}_M - r_f \right) ,$$

where

$$\beta_i = \frac{\sigma_{iM}}{\sigma_M^2} .$$

Proof of the CAPM (I)

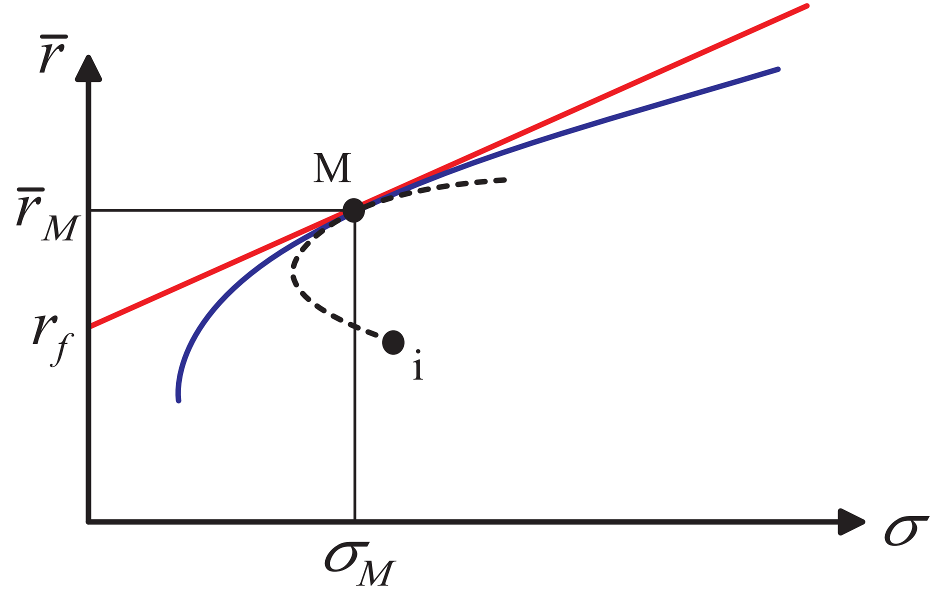
Consider a portfolio consisting of a portion α invested in asset i and a portion $1-\alpha$ invested in the market portfolio.

This portfolio has

$$\text{mean return:} \quad \bar{r}_\alpha = \alpha \bar{r}_i + (1-\alpha)\bar{r}_M$$

$$\text{standard deviation:} \quad \sigma_\alpha = \sqrt{\alpha^2\sigma_i^2 + 2\alpha(1-\alpha)\sigma_{iM} + (1-\alpha)^2\sigma_M^2}$$

As α varies, $\bar{r}_\alpha$ and σ_α trace out a curve in the $\bar{r}$-σ diagram.



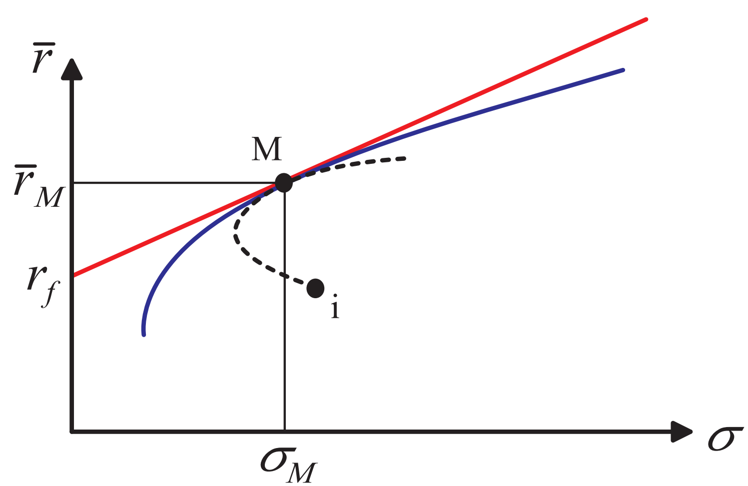
Proof of the CAPM (II)

At $\alpha = 0$, the curve and the CML are tangent.

$\Rightarrow$ They must have the same slope!

- Slope of the CML: $\dfrac{\bar{r}_M - r_f}{\sigma_M}$
- Slope of the curve at $\alpha = 0$:

$$\left.\frac{\mathrm{d}\bar{r}_\alpha}{\mathrm{d}\sigma_\alpha}\right|_{\alpha=0} = \left.\left(\frac{\mathrm{d}\bar{r}_\alpha}{\mathrm{d}\alpha}\right)\left(\frac{\mathrm{d}\sigma_\alpha}{\mathrm{d}\alpha}\right)^{-1}\right|_{\alpha=0}$$



Proof of the CAPM (III)

Necessary derivatives:

$$\frac{\mathrm{d}\bar{r}_\alpha}{\mathrm{d}\alpha} = \bar{r}_i - \bar{r}_M$$

$$\frac{\mathrm{d}\sigma_\alpha}{\mathrm{d}\alpha} = \frac{\alpha\sigma_i^2 + (1-2\alpha)\sigma_{iM} + (\alpha-1)\sigma_M^2}{\sigma_\alpha}$$

$$\Rightarrow \quad \left.\frac{\mathrm{d}\sigma_\alpha}{\mathrm{d}\alpha}\right|_{\alpha=0} = \frac{\sigma_{iM} - \sigma_M^2}{\sigma_M}$$

$\Rightarrow$ slope of curve at $\alpha = 0$: $\left.\frac{\mathrm{d}\bar{r}_\alpha}{\mathrm{d}\sigma_\alpha}\right|_{\alpha=0} = \frac{(\bar{r}_i - \bar{r}_M)\sigma_M}{\sigma_{iM} - \sigma_M^2}$.

Proof of the CAPM (IV)

Equality of the two slopes implies:

$$\frac{\bar{r}_M - r_f}{\sigma_M} = \frac{(\bar{r}_i - \bar{r}_M)\sigma_M}{\sigma_{iM} - \sigma_M^2}\,.$$

We now solve for $\bar{r}_i$ to obtain the final result:

$$\bar{r}_i = r_f + \left(\frac{\bar{r}_M - r_f}{\sigma_M^2}\right)\sigma_{iM} = r_f + \beta_i(\bar{r}_M - r_f)$$

where

$$\beta_i = \frac{\sigma_{iM}}{\sigma_M^2}\,.$$

$\square$

Example

- Given $r_f = 10\%$, $\bar{r}_M = 17\%$, $\sigma_M = 12\%$.
- What is the expected return of an asset whose covariance with the market is 0.0288?
- Solution: use CAPM!

Compute beta: $$\beta_i = \frac{\sigma_{iM}}{\sigma_M^2} = \frac{0.0288}{(0.12)^2} = 2\,.$$

Thus:

$$\bar{r}_i = r_f + \beta_i(\bar{r}_M - r_f) = 0.10 + 2(0.17 - 0.10) = 24\%\,.$$

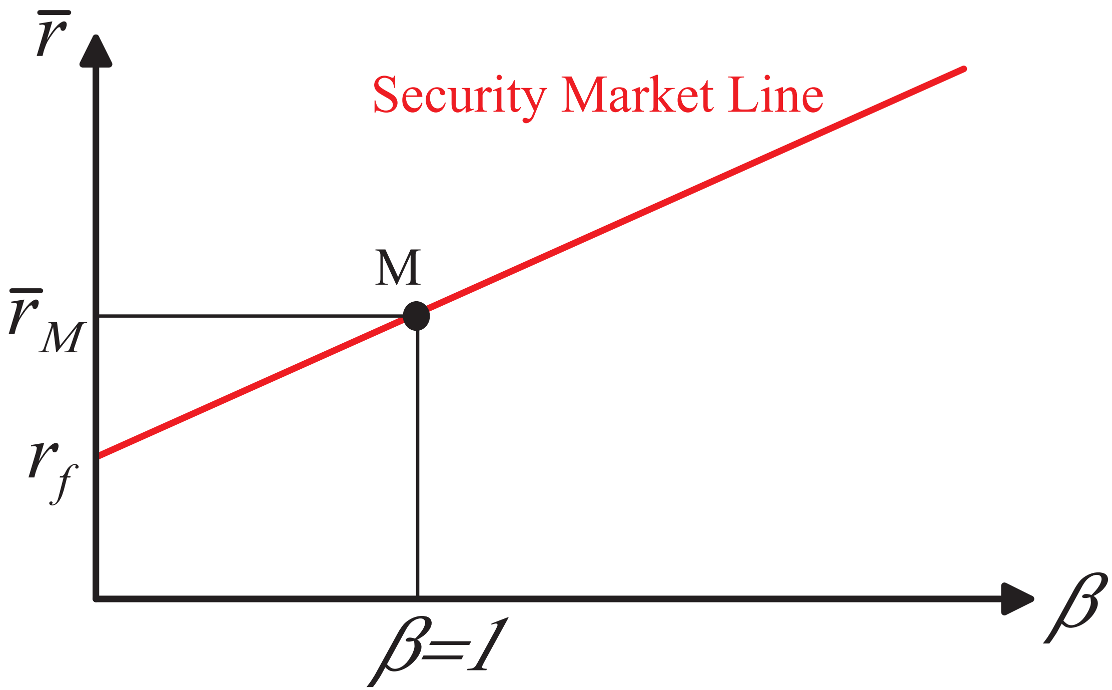
Beta of a Portfolio

- It is easy to calculate the beta of a portfolio in terms of the betas of the assets in the portfolio:

$$
\begin{aligned}
\beta_{\mathrm{P}} &= \frac{\mathrm{cov}(r_{\mathrm{P}}, r_M)}{\mathrm{var}(\mathrm{r_M})} \\
&= \frac{\mathrm{cov}(\sum_{i=1}^{n} w_i r_i, r_M)}{\mathrm{var}(\mathrm{r_M})} \\
&= \frac{\sum_{i=1}^{n} w_i \mathrm{cov}(r_i, r_M)}{\mathrm{var}(\mathrm{r_M})} \\
&= \sum_{i=1}^{n} w_i \beta_i \,.
\end{aligned}
$$

The Security Market Line

CAPM: $$\bar{r}_i = r_f + \beta_i(\bar{r}_M - r_f)\,, \quad \beta_i = \frac{\sigma_{iM}}{\sigma_M^2}\,.$$



Under the equilibrium conditions assumed by the CAPM, any asset should fall on the security market line (SML).

Importance of the CAPM

- The correlation with the market (β) determines the expected excess rate of return of an asset (w.r.t. r_f):
  - $\beta = 0 \Rightarrow \bar{r}_i = r_f + 0(\bar{r}_M - r_f) = r_f$: risk-free rate
  - $\beta = 1 \Rightarrow \bar{r}_i = r_f + 1(\bar{r}_M - r_f) = \bar{r}_M$: market return
  - $\beta = 2 \Rightarrow \bar{r}_i = r_f + 2(\bar{r}_M - r_f) = 2\bar{r}_M - r_f$

- Note that
  - the CML relates the expected rate of return of an efficient portfolio to its standard deviation/risk;
  - the SML relates the expected rate of return of an individual asset to its beta/systematic risk.

Risk and CAPM

- Expected return: $\bar{r}_i = r_f + \beta_i(\bar{r}_M - r_f)$
- Therefore, we can write

$$r_i = r_f + \beta_i(r_M - r_f) + \varepsilon_i\,, \qquad (*)$$

where the random var. ε_i is chosen to make $(*)$ true.

- CAPM implies that $\mathrm{E}(\varepsilon_i) = 0$.
- CAPM also implies that $\mathrm{cov}(\varepsilon_i, r_M) = 0$:

$$\begin{aligned}
\mathrm{cov}(r_i, r_M) &= \mathrm{cov}(r_f + \beta_i(r_M - r_f) + \varepsilon_i, r_M) \\
&= \mathrm{cov}(r_f, r_M) + \beta_i \mathrm{cov}(r_M, r_M) - \beta_i \mathrm{cov}(r_f, r_M) + \mathrm{cov}(\varepsilon_i, r_M) \\
&= \beta_i \mathrm{cov}(r_M, r_M) + \mathrm{cov}(\varepsilon_i, r_M) = \mathrm{cov}(r_i, r_M) + \mathrm{cov}(\varepsilon_i, r_M)
\end{aligned}$$

$\Rightarrow \mathrm{cov}(\varepsilon_i, r_M) = 0\,,$ i.e., ε_i is uncorrelated with the market!

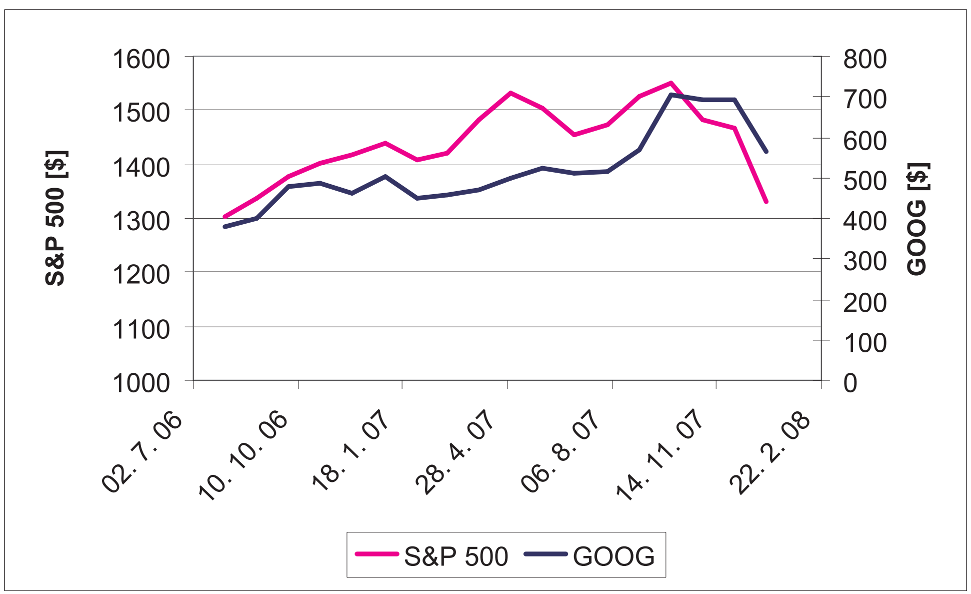
Google Inc. (GOOG): $\beta = 0.74$

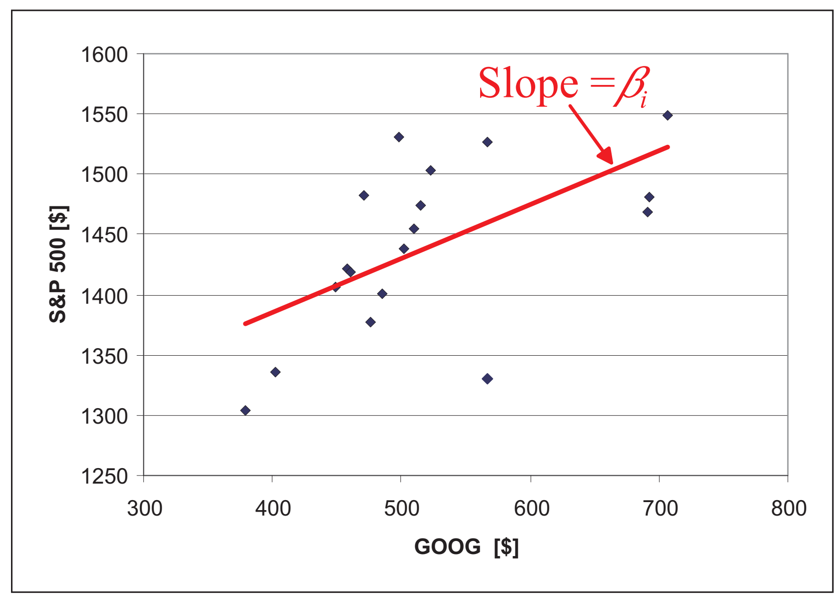
- We have

$$r_i = r_f + \beta_i(r_M - r_f) + \varepsilon_i \, ,$$

where ε_i has zero mean and is uncorrelated with r_M.

- β_i determines the expected movement with the market!





Systematic Risk (I)

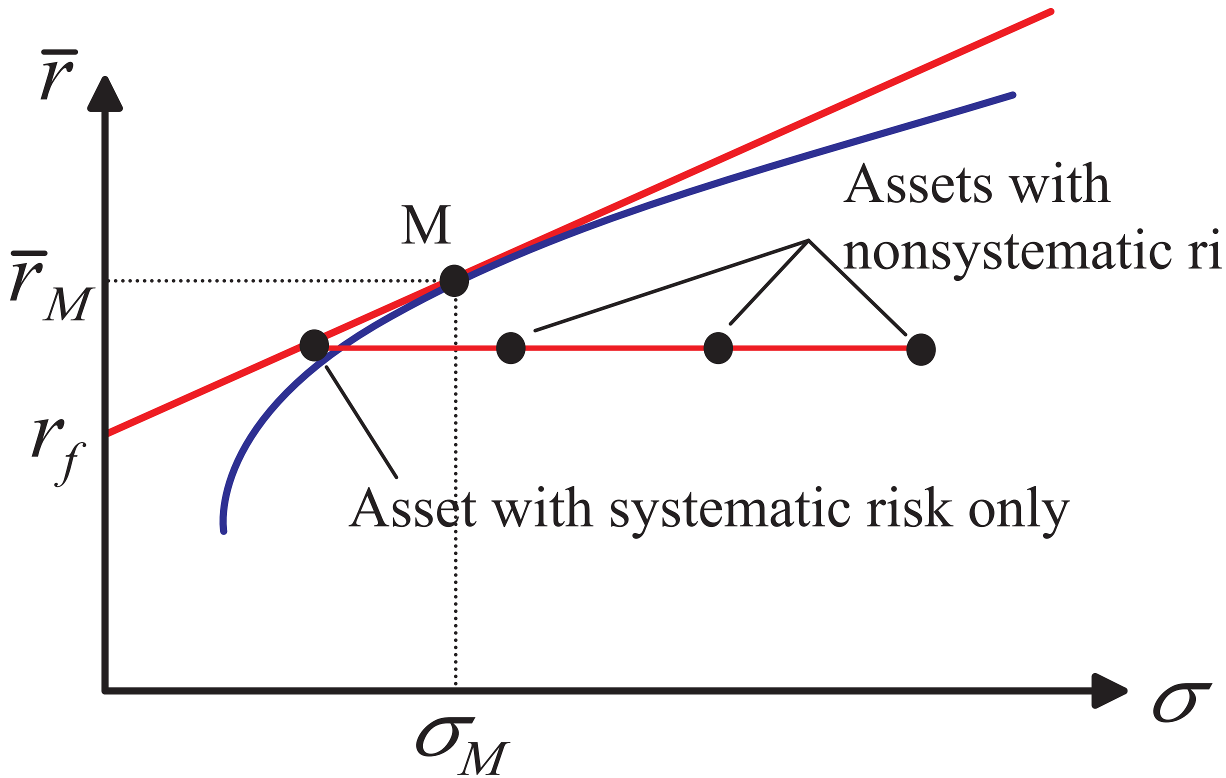
- The variance of an asset is commonly viewed as its risk:

$$
\begin{aligned}
\sigma_i^2 &= \operatorname{cov}(r_i, r_i) \\
&= \operatorname{cov}(r_f + \beta_i(r_M - r_f) + \varepsilon_i, r_f + \beta_i(r_M - r_f) + \varepsilon_i) \\
&= \beta_i^2 \sigma_M^2 + \operatorname{var}(\varepsilon_i)\,.
\end{aligned}
$$

- The risk in r_i is the sum of two terms:
  - $\beta_i^2 \sigma_M^2 =$ systematic risk
    - ∗ associated with the market as a whole
    - ∗ can not be diversified
  - $\operatorname{var}(\varepsilon_i) =:$ nonsystematic, idiosyncratic, specific risk
    - ∗ uncorrelated with the market
    - ∗ can be reduced by diversification

Systematic Risk (II)

- If $\varepsilon_i = 0$, then this asset has only systematic risk
  $\Rightarrow$ it lies on the CML.
- If $\varepsilon_i \neq 0$, then this asset has also nonsystematic risk
  $\Rightarrow$ it falls to the right of the CML.



Summary of CAPM (I)

- Returns:

$$r_i = r_f + \beta_i(r_M - r_f) + \varepsilon_i \, ,$$

  where ε_i has zero mean and is uncorrelated with r_M.

- Expected returns:

$$\bar{r}_i = r_f + \beta_i(\bar{r}_M - r_f)$$

- Variances:

$$\sigma_i^2 = \beta_i^2 \sigma_M^2 + \mathrm{var}(\varepsilon_i)$$

The expected return of an asset/portfolio is not determined by its variance, but only by its beta, which measures the amount of risk from the market portfolio.

Summary of CAPM (II)

- You are only rewarded (expected return) for risk that you cannot diversify away.

- Risk is measured by β, not the variance of your asset.

- The return of an asset i is determined by how it fits into the market portfolio, not by its characteristics alone.

CAPM as a Pricing Formula

- CAPM is a pricing formula.
- The standard CAPM formula only contains expected rates of return.
- Suppose an asset is bought at a (fixed) price P and later sold at a (random) price Q. The rate of return is

$$r = \frac{Q - P}{P}\,.$$

- The CAPM formula yields:

$$\frac{\bar{Q} - P}{P} = r_f + \beta(\bar{r}_M - r_f) \;\Rightarrow\; P = \frac{\bar{Q}}{1 + r_f + \beta(\bar{r}_M - r_f)}\,.$$

Example

- Consider an oil well with an expected payoff of $£1,000$. The standard deviation of this payoff is 0.40.

- The β of the oil well is 0.6, the risk-free rate is 10%, and the return on the market is 17%.

- According to CAPM, what is the price of the oil well?

- Answer:

$$P = \frac{\bar{Q}}{1+r_f+\beta(\bar{r}_M-r_f)} = \frac{£1{,}000}{1+0.1+0.6(0.17-0.1)} = £867 \,.$$

(Note that the standard deviation of 0.40 was not needed in the calculation)

Certainty Equivalent Form

- The formula

$$P = \frac{\bar{Q}}{1+r_f+\beta(\bar{r}_M-r_f)} \qquad (*)$$

looks like a we discount the expected payoff $\bar{Q}$ at a risk-adjusted interest rate $r_f + \beta(\bar{r}_M - r_f)$.

- The formula $(*)$ is equivalent to

$$P = \frac{1}{1+r_f}\left(\bar{Q} - \frac{\operatorname{cov}(Q, r_M)}{\sigma_M^2}(\bar{r}_M - r_f)\right) .$$

This looks like we discount the risk-adjusted payoff,

$$\left(\bar{Q} - \frac{\operatorname{cov}(Q, r_M)}{\sigma_M^2}(\bar{r}_M - r_f)\right) ,$$

at the risk-free rate r_f.

Proof of Certainty Equivalent Formula

We start from the formula

$$P = \frac{\bar{Q}}{1+r_f+\beta(\bar{r}_M - r_f)} . \qquad (*)$$

Using the fact that $r = Q/P - 1$, the value of beta becomes

$$\beta = \frac{\mathrm{cov}(Q/P - 1, r_M)}{\sigma_M^2} = \frac{\mathrm{cov}(Q/P, r_M)}{\sigma_M^2} = \frac{\mathrm{cov}(Q, r_M)}{P\sigma_M^2} .$$

Substituting this into $(*)$ gives

$$P = \frac{\bar{Q}}{1 + r_f + \frac{\mathrm{cov}(Q, r_M)}{P\sigma_M^2}(\bar{r}_M - r_f)} .$$

Proof of Certainty Equivalent Formula

Multiplying this formula with the denominator of the rhs,

$$P(1 + r_f) + \frac{\mathrm{cov}(Q, r_M)}{\sigma_M^2}(\bar{r}_M - r_f) = \bar{Q}\,,$$

and solving for P yields the desired result

$$P = \frac{1}{1 + r_f}\left(\bar{Q} - \frac{\mathrm{cov}(Q, r_M)}{\sigma_M^2}(\bar{r}_M - r_f)\right),$$

which is linear in Q. The risk-adjusted payoff

$$\left(\bar{Q} - \frac{\mathrm{cov}(Q, r_M)}{\sigma_M^2}(\bar{r}_M - r_f)\right)$$

is the certainty equivalent for the random payoff Q.

NPV Using CAPM

- A firm can use the CAPM to decide which projects it should carry out.
- It is natural to define the NPV of a project that costs P and generates a random payoff Q as

$$\text{NPV} = -P + \frac{1}{1+r_f}\left(\bar{Q} - \frac{\text{cov}(Q, r_M)}{\sigma_M^2}(\bar{r}_M - r_f)\right).$$

- It is appropriate for the firm to select the group of projects that maximize NPV.