

*Interactive Computer Graphics:
Lecture 12*

Introduction to Surface Construction

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Non Parametric Surface

- Surfaces can be constructed from Cartesian equations directly, and this is acceptable for specific applications, usually involving interpolation.
- As before, using a simple polynomial surface is a quick and easy approach.

Non Parametric Polynomial Surface

$$(x \ y \ z \ 1) \begin{pmatrix} a & b & c & d \\ b & e & f & g \\ c & f & h & j \\ d & g & j & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = 0$$

- which multiplies out to:

$$ax^2 + ey^2 + hz^2 + 2bxy + 2cxz + 2fyz + 2dx + 2gy + 2jz + 1 = 0$$

- Because of the symmetry there are 9 scalar unknowns in the equation
- So we need to specify nine points through which the surface will pass

As Before

- This formulation suffers the same problems as the non-parametric spline curve. It is a fixed surface for a given set of nine points.
- We need more flexibility for the design of surfaces.

Simple Parametric surfaces

- We can extend the formulation to simple parametric surfaces using the vector equation:

$$\mathbf{P}(\mu, \nu) = (\mu, \nu, 1) \begin{pmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \\ \mathbf{b} & \mathbf{d} & \mathbf{e} \\ \mathbf{c} & \mathbf{e} & \mathbf{f} \end{pmatrix} \begin{pmatrix} \mu \\ \nu \\ 1 \end{pmatrix}$$

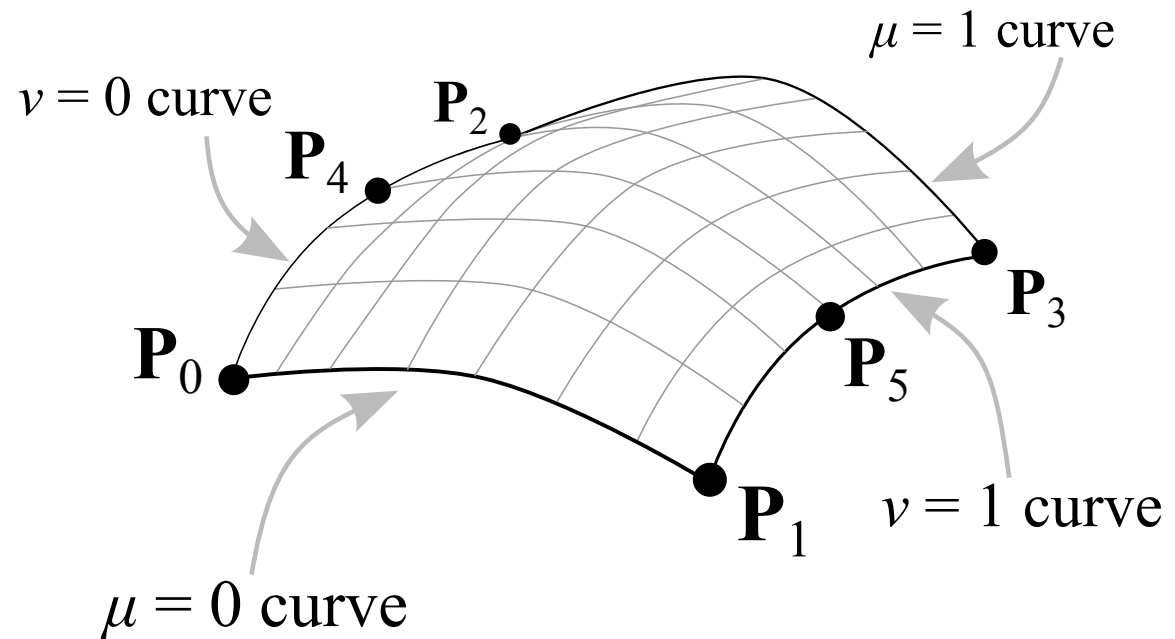
$$\mathbf{P}(\mu, \nu) = \mathbf{a}\mu^2 + \mathbf{d}\nu^2 + 2\mathbf{b}\mu\nu + 2\mathbf{c}\mu + 2\mathbf{e}\nu + \mathbf{f}$$

- There are six unknown parameter vectors $\{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}, \mathbf{f}\}$

Associating points and parameters

- We can solve for the six vector unknowns by substituting in six points at known values of μ and ν .
- We might have an arrangement such as:

	μ	ν
\mathbf{P}_0	0	0
\mathbf{P}_1	0	1
\mathbf{P}_2	1	0
\mathbf{P}_3	1	1
\mathbf{P}_4	1/2	0
\mathbf{P}_5	1/2	1



Surface parameter equations

- Substituting these values of μ and ν into the patch equation gives us these six equations

$$\mathbf{P}_0 = \mathbf{f}$$

$$\mathbf{P}_1 = \mathbf{d} + 2\mathbf{e} + \mathbf{f}$$

$$\mathbf{P}_2 = \mathbf{a} + 2\mathbf{c} + \mathbf{f}$$

$$\mathbf{P}_3 = \mathbf{a} + 2\mathbf{b} + 2\mathbf{c} + \mathbf{d} + 2\mathbf{e} + \mathbf{f}$$

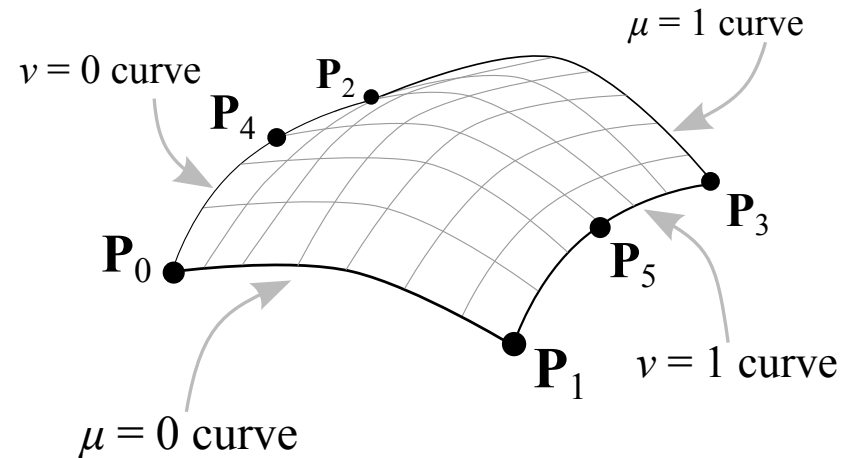
$$\mathbf{P}_4 = \mathbf{a}/4 + \mathbf{c} + \mathbf{f}$$

$$\mathbf{P}_5 = \mathbf{a}/4 + \mathbf{b} + \mathbf{c} + \mathbf{d} + 2\mathbf{e} + \mathbf{f}$$

- The \mathbf{P} 's are known and we can solve for the unknowns $\{\mathbf{a}, \dots, \mathbf{f}\}$ using standard methods

Getting the edges from the surface equation

μ and ν are in the range $[0, 1]$.
Thus the contours that bound the patch can be found by substituting 0 or 1 for one of μ or ν in the patch equation.



$$\mathbf{P}(0, \nu) = \mathbf{d}\nu^2 + 2\mathbf{e}\nu + \mathbf{f}$$

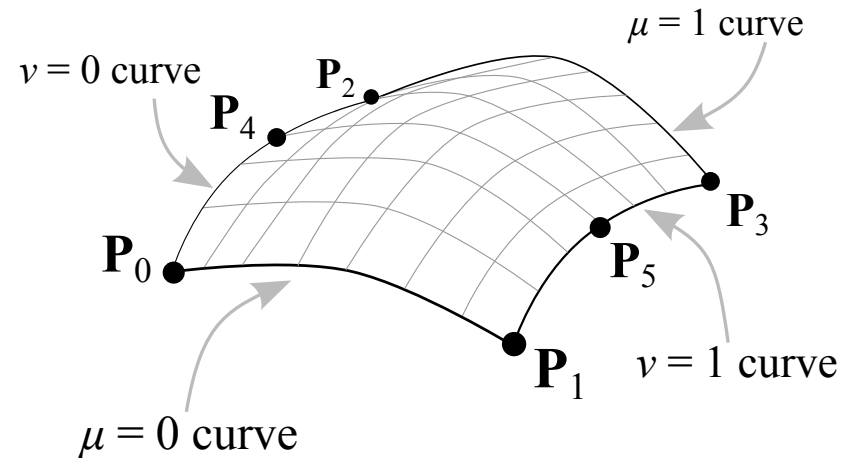
$$\mathbf{P}(1, \nu) = \mathbf{a} + 2(\mathbf{b} + \mathbf{e})\nu + 2\mathbf{c} + \mathbf{d}\nu^2 + \mathbf{f}$$

$$\mathbf{P}(\mu, 0) = \mathbf{a}\mu^2 + 2\mathbf{c}\mu + \mathbf{f}$$

$$\mathbf{P}(\mu, 1) = \mathbf{a}\mu^2 + 2(\mathbf{b} + \mathbf{c})\mu + \mathbf{d} + 2\mathbf{e} + \mathbf{f}$$

The resulting surface

The boundaries are all second order curves and so will be nice and smooth



There is quite a lot of flexibility in this formulation, but it is still only suitable for simple surfaces.

We can use higher orders

E.g. using the tensor product

$$\mathbf{P}(\mu, \nu) = \begin{pmatrix} \mu^3 & \mu^2 & \mu & 1 \end{pmatrix} \begin{pmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} & \mathbf{d} \\ \mathbf{b} & \mathbf{e} & \mathbf{f} & \mathbf{g} \\ \mathbf{c} & \mathbf{f} & \mathbf{h} & \mathbf{j} \\ \mathbf{d} & \mathbf{g} & \mathbf{j} & \mathbf{k} \end{pmatrix} \begin{pmatrix} \nu^3 \\ \nu^2 \\ \nu \\ 1 \end{pmatrix}$$

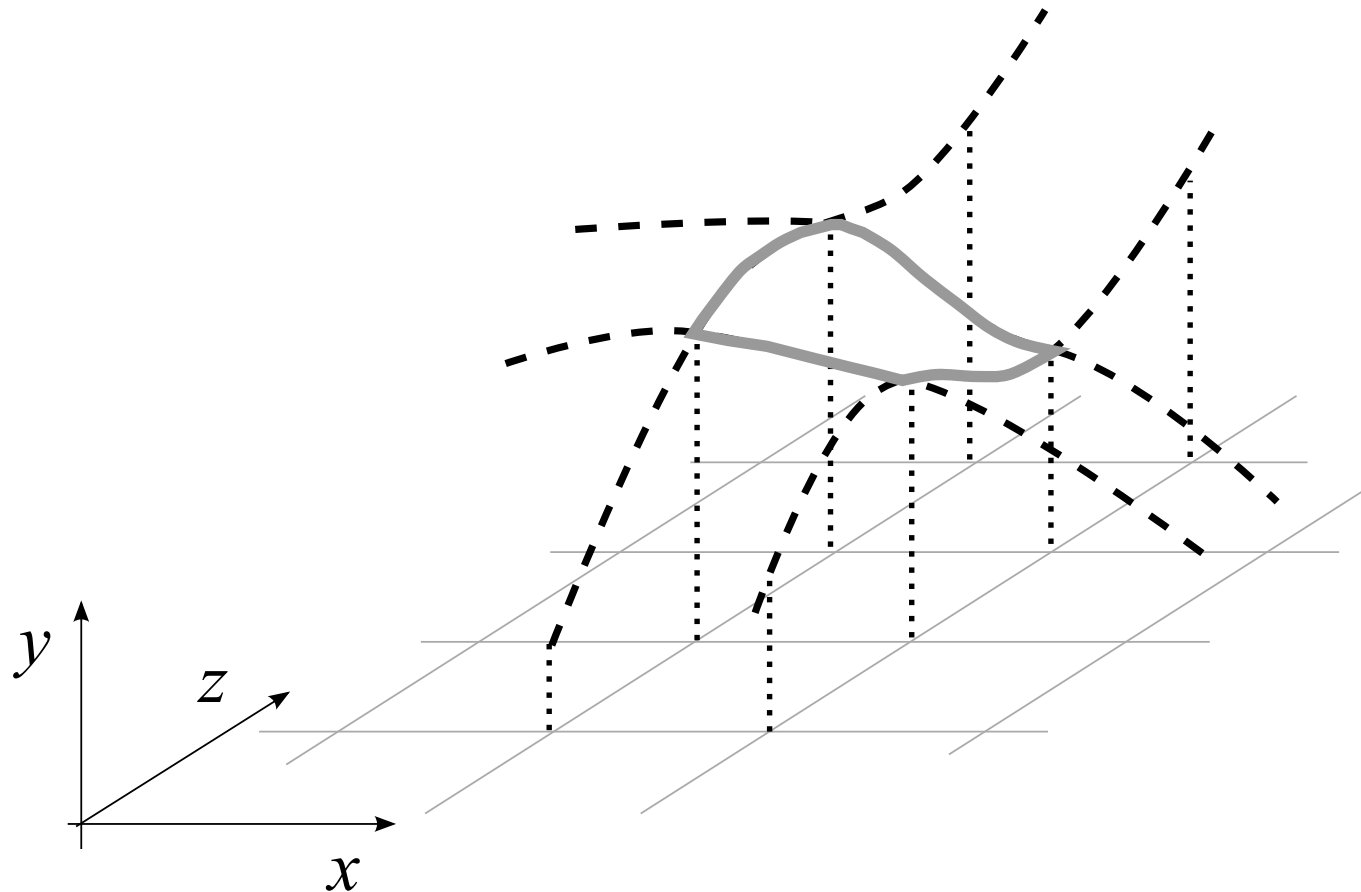
Using higher orders gives more variety in shape and better control

But the method is hard to apply and generalise, and so is not usually done

Cubic Spline Patches

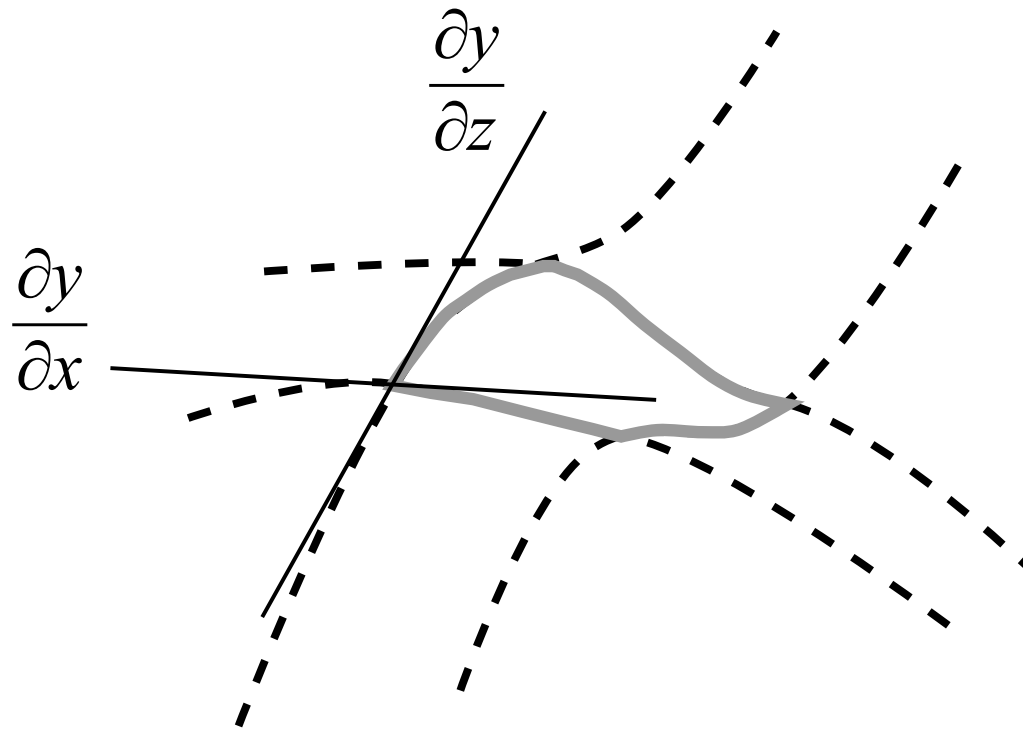
- The patch method is generally effective in creating more complex surfaces.
- The idea is, as in the case of the curves, to create a surface by joining a lot of simple surfaces continuously.

Cartesian surface patches - terrain map



Points and Gradients

- At each corner of the patch we need to interpolate the points and set the gradients to match the adjacent patch.
- There are two gradients



Parametric patches

- In practice we use the more general parametric patch formulation with two parameters μ and ν .
- The terrain map can be modelled with parametric patches.
- We need to match three values at each corner

$$\mathbf{P}(\mu, \nu) \quad \frac{\partial \mathbf{P}(\mu, \nu)}{\partial \mu} \quad \frac{\partial \mathbf{P}(\mu, \nu)}{\partial \nu}$$

Corners

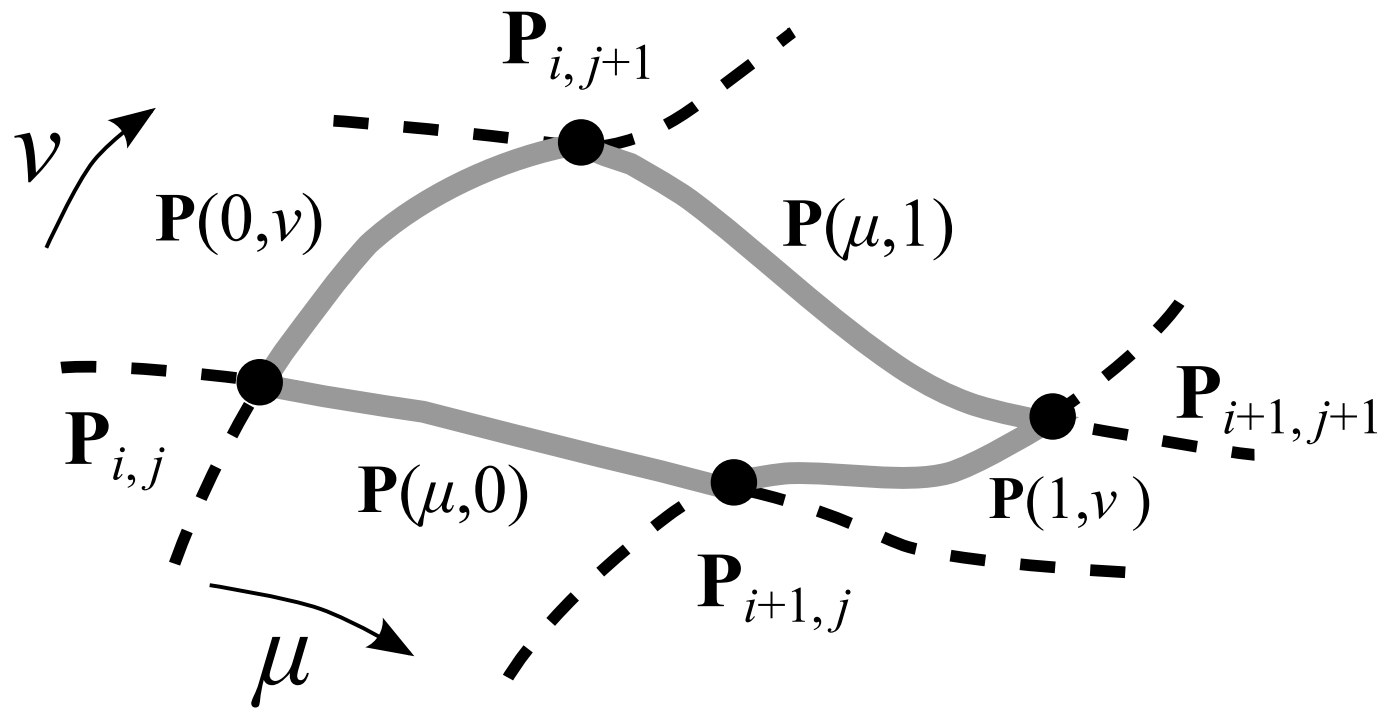
- As usual we adopt the convention that the corners are at parameter values $(0, 0)$, $(0, 1)$, $(1, 0)$ and $(1, 1)$
- We need to ensure that the patch joins its neighbours exactly at the edges.
- Hence we ensure that the edge contours are the same on adjacent patches

Edges

- We do this by designing the edge curves in an identical manner to the cubic spline curve patch.

Edge curve	Points joined	
$\mathbf{P}(0, \nu)$	$\mathbf{P}_{i,j}$	$\mathbf{P}_{i,j+1}$
$\mathbf{P}(1, \nu)$	$\mathbf{P}_{i+1,j}$	$\mathbf{P}_{i+1,j+1}$
$\mathbf{P}(\mu, 0)$	$\mathbf{P}_{i,j}$	$\mathbf{P}_{i+1,j}$
$\mathbf{P}(\mu, 1)$	$\mathbf{P}_{i,j+1}$	$\mathbf{P}_{i+1,j+1}$

A parametric spline patch



As long as the gradients are the same for the four patches that meet at a point the surface will join seamlessly

The Coons patch

To define the internal points we linearly interpolate the edge curves:

$$\begin{aligned}\mathbf{P}(\mu, \nu) = & \mathbf{P}(\mu, 0)(1 - \nu) + \mathbf{P}(\mu, 1)\nu + \\ & \mathbf{P}(0, \nu)(1 - \mu) + \mathbf{P}(1, \nu)\mu - \\ & \mathbf{P}(0, 1)(1 - \mu)\nu - \mathbf{P}(1, 0)\mu(1 - \nu) - \\ & \mathbf{P}(0, 0)(1 - \mu)(1 - \nu) - \mathbf{P}(1, 1)\mu\nu\end{aligned}$$

Substituting values of 0 or 1 for μ and/or ν we can easily verify that the equation fits the edge curves.

Rendering a patch: Polygonisation

To render (draw) a spline patch we can simply transform it into polygons.

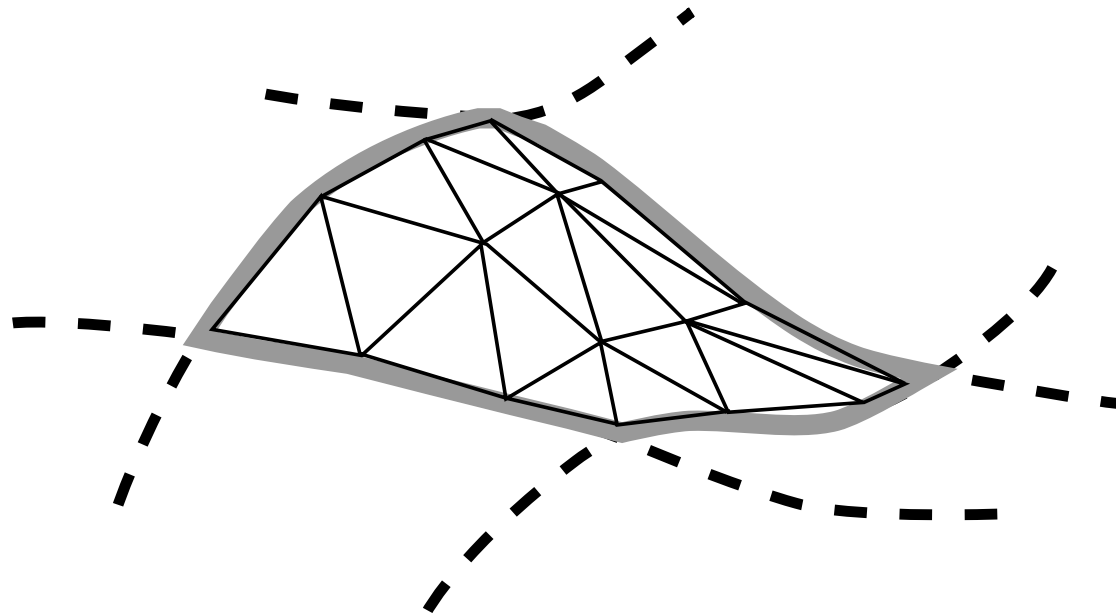
We select a grid of points, e.g.:

$$\mu = \{0.0, 0.1, 0.2, \dots 1.0\}$$

$$\nu = \{0.0, 0.1, 0.2, \dots 1.0\}$$

and triangulate to that grid.

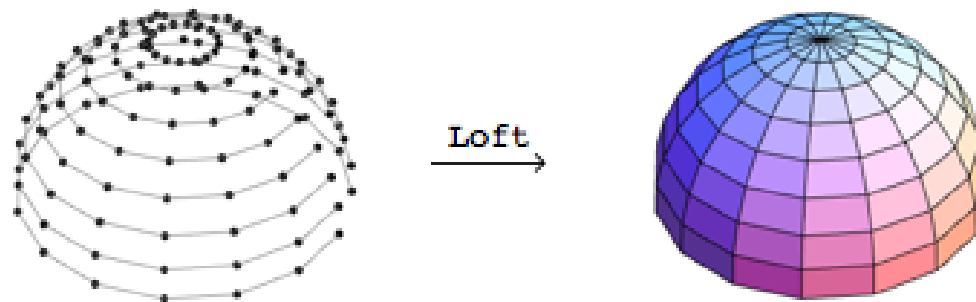
Rendering a patch: Polygonisation



- For speed we can use large polygons with Gouraud or Phong shading.
- For accuracy we use small polygons, chosen to match the pixel size.

Rendering a patch: Lofting

- Surfaces can also be drawn by a technique called lofting (now really obsolete).
- This means drawing contours of constant μ and/or of constant ν
- Algorithms for eliminating the hidden parts have been devised.



Rendering a patch: Ray tracing

- The patch equation is fourth order

$$\begin{aligned}\mathbf{P}(\mu, \nu) = & \mathbf{P}(\mu, 0)(1 - \nu) + \mathbf{P}(\mu, 1)\nu + \\ & \mathbf{P}(0, \nu)(1 - \mu) + \mathbf{P}(1, \nu)\mu - \\ & \mathbf{P}(0, 1)(1 - \mu)\nu - \mathbf{P}(1, 0)\mu(1 - \nu) - \\ & \mathbf{P}(0, 0)(1 - \mu)(1 - \nu) - \mathbf{P}(1, 1)\mu\nu\end{aligned}$$

- Hence no closed form solution exists for a ray patch intersection
- Can use numeric algorithms but computation can be costly

Rendering a patch: Ray tracing

- Numerical Ray-Patch algorithm
 1. Polygonise the patch at a low resolution (say 4 x 4)
 2. Calculate the ray intersection with the 32 triangles and find the nearest intersection.
 3. Polygonise the immediate area of the intersection and calculate a better estimate of the intersection
 4. Continue until the best estimate is found

Rendering a patch: Ray tracing

- Numerical Ray-Patch algorithm
 - May be multiple intersections between the ray and the surface
 - Algorithm will find an intersection, but not necessarily the nearest.
 - If the object is relatively smooth it should work well in most cases.
 - Note that it will be necessary to do a ray intersection with each patch of the object to find the nearest intersection.

Example of Using a Coons Patch

- Part of a terrain map defined on a regular x - y grid is as follows:

		$y, \nu \rightarrow$					
		2	3	4	5	6	7
x, μ ↓	7
	8	.	.	10	9	.	.
	9	.	14	12	11	10	.
	10	.	15	13	14	10	.
	11	.	.	10	11	.	.

- Find the Coons patch on the centre four points

Corners

- The corners at $\mu, \nu = 0, 1$ are defined directly in the question:

$$\mathbf{P}(0, 0) = (9, 4, 12)$$

$$\mathbf{P}(1, 0) = (10, 4, 13)$$

$$\mathbf{P}(0, 1) = (9, 5, 11)$$

$$\mathbf{P}(1, 1) = (10, 5, 14)$$

		$y, \nu \rightarrow$					
		2	3	4	5	6	7
x, μ \downarrow	7
	8	.	.	10	9	.	.
	9	.	14	12	11	10	.
	10	.	15	13	14	10	.
	11	.	.	10	11	.	.

Gradients in the x / μ direction

Example

$$\left. \frac{\partial \mathbf{P}}{\partial \mu} \right|_{(0,0)} = \frac{(10, 4, 13)^T - (8, 4, 10)^T}{2} = \begin{pmatrix} 1 \\ 0 \\ 1.5 \end{pmatrix}$$

		$y, \nu \rightarrow$					
		2	3	4	5	6	7
x, μ \downarrow	7
	8	.	.	10	9	.	.
	9	.	14	12	11	10	.
	10	.	15	13	14	10	.
	11	.	.	10	11	.	.

Gradients in the x / μ direction

$$\left. \frac{\partial \mathbf{P}}{\partial \mu} \right|_{(0,0)} = \frac{(10, 4, 13)^T - (8, 4, 10)^T}{2} = (1, 0, 1.5)^T$$

$$\left. \frac{\partial \mathbf{P}}{\partial \mu} \right|_{(1,0)} = \frac{(11, 4, 10)^T - (9, 4, 12)^T}{2} = (1, 0, -1)^T$$

$$\left. \frac{\partial \mathbf{P}}{\partial \mu} \right|_{(0,1)} = \frac{(10, 5, 14)^T - (8, 5, 9)^T}{2} = (1, 0, 2.5)^T$$

$$\left. \frac{\partial \mathbf{P}}{\partial \mu} \right|_{(1,1)} = \frac{(11, 5, 11)^T - (9, 5, 11)^T}{2} = (1, 0, 0)^T$$

Gradients in the y / v direction

$$\left. \frac{\partial \mathbf{P}}{\partial \nu} \right|_{(0,0)} = \frac{(9, 5, 11)^T - (9, 3, 14)^T}{2} = (0, 1, -1.5)^T$$

$$\left. \frac{\partial \mathbf{P}}{\partial \nu} \right|_{(1,0)} = \frac{(10, 5, 14)^T - (10, 3, 15)^T}{2} = (0, 1, -0.5)^T$$

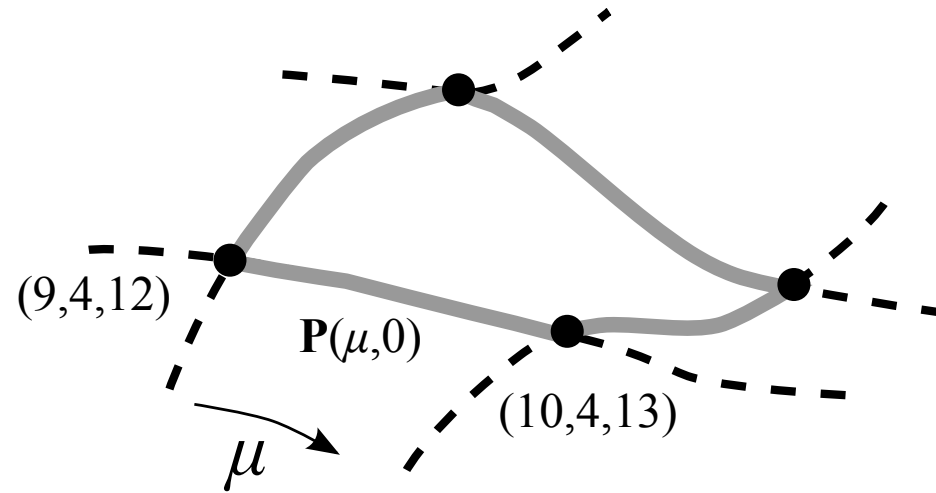
$$\left. \frac{\partial \mathbf{P}}{\partial \nu} \right|_{(0,1)} = \frac{(9, 6, 10)^T - (9, 4, 12)^T}{2} = (0, 1, -1)^T$$

$$\left. \frac{\partial \mathbf{P}}{\partial \nu} \right|_{(1,1)} = \frac{(10, 6, 10)^T - (10, 4, 13)^T}{2} = (0, 1, -1.5)^T$$

Finding the boundary curves

E.g. Finding curve $\mathbf{P}(\mu, 0)$

$$\mathbf{P}(\mu, 0) = \mathbf{a}_3\mu^3 + \mathbf{a}_2\mu^2 + \mathbf{a}_1\mu + \mathbf{a}_0$$



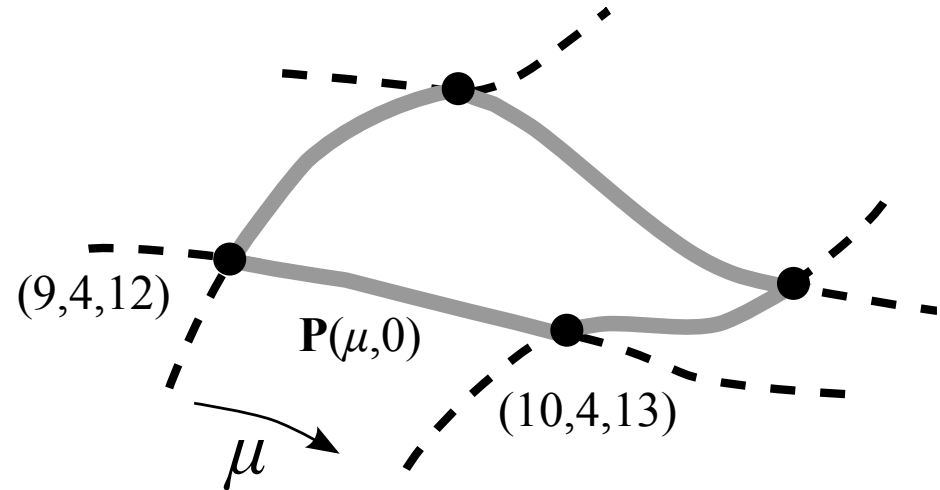
$$\begin{pmatrix} \mathbf{a}_0 \\ \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3 & -2 & 3 & -1 \\ 2 & 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{P}_0 \\ \mathbf{P}'_0 \\ \mathbf{P}_1 \\ \mathbf{P}'_1 \end{pmatrix}$$

- see cubic spline patch equation (previous lecture)

Finding the boundary curves

E.g. Finding curve $\mathbf{P}(\mu, 0)$

$$\mathbf{P}(\mu, 0) = \mathbf{a}_3\mu^3 + \mathbf{a}_2\mu^2 + \mathbf{a}_1\mu + \mathbf{a}_0$$



$$\begin{pmatrix} \mathbf{a}_0 \\ \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3 & -2 & 3 & -1 \\ 2 & 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} 9 & 4 & 12 \\ 1 & 0 & 1.5 \\ 10 & 4 & 13 \\ 1 & 0 & -1 \end{pmatrix}$$

After substituting in $\mathbf{P}(0, 0)$, $\left. \frac{\partial \mathbf{P}}{\partial \mu} \right|_{(0,0)}$, $\mathbf{P}(1, 0)$, $\left. \frac{\partial \mathbf{P}}{\partial \mu} \right|_{(1,0)}$

Finding the boundary curve $\mathbf{P}(\mu, 0)$

- Calculating the constant vectors \mathbf{a}_3 , \mathbf{a}_2 , \mathbf{a}_1 and \mathbf{a}_0

$$\mathbf{a}_0 = \mathbf{P}_0 = (9, 4, 12)$$

$$\mathbf{a}_1 = \mathbf{P}'_0 = (1, 0, 1.5)$$

$$\begin{aligned}\mathbf{a}_2 &= -3\mathbf{P}_0 - 2\mathbf{P}'_0 - 3\mathbf{P}_1 - \mathbf{P}'_1 \\ &= -3 \times (9, 4, 12) - 2 \times (1, 0, 1.5) + 3 \times (10, 4, 13) - (1, 0, 1) \\ &= (0, 0, 1)\end{aligned}$$

$$\begin{aligned}\mathbf{a}_3 &= 2\mathbf{P}_0 + \mathbf{P}'_0 - 2\mathbf{P}_1 + \mathbf{P}'_1 \\ &= 2 \times (9, 4, 12) + (1, 0, 1.5) - 2 \times (10, 4, 13) + (1, 0, 1) \\ &= (0, 0, 0.5)\end{aligned}$$

Finding the boundary curves $\mathbf{P}(\mu, 1)$, $\mathbf{P}(0, \nu)$, $\mathbf{P}(1, \nu)$

- These curves are found identically to $\mathbf{P}(\mu, 0)$.

- We now have all the individual bits:

$\mathbf{P}(\mu, 0)$: a cubic polynomial in μ

$\mathbf{P}(\mu, 1)$: a cubic polynomial in μ

$\mathbf{P}(0, \nu)$: a cubic polynomial in ν

$\mathbf{P}(1, \nu)$: a cubic polynomial in ν

$\mathbf{P}(0, 0)$, $\mathbf{P}(0, 1)$, $\mathbf{P}(1, 0)$ and $\mathbf{P}(1, 1)$: the corner points

- Given values of μ and ν , we can calculate each of these eight points

So, for any given value for μ and ν ...

... we can evaluate the coordinate on the Coons patch:

$$\begin{aligned}\mathbf{P}(\mu, \nu) = & \mathbf{P}(\mu, 0)(1 - \nu) + \mathbf{P}(\mu, 1)\nu + \\ & \mathbf{P}(0, \nu)(1 - \mu) + \mathbf{P}(1, \nu)\mu - \\ & \mathbf{P}(0, 1)(1 - \mu)\nu - \mathbf{P}(1, 0)\mu(1 - \nu) - \\ & \mathbf{P}(0, 0)(1 - \mu)(1 - \nu) - \mathbf{P}(1, 1)\mu\nu\end{aligned}$$