

COMPUTATIONAL FINANCE: 422

Fixed-Income Securities

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This Lecture

- Basic terminology
- Examples of fixed-income securities
 - Annuities
 - Bonds
- Valuation of fixed-income securities
- Risk management of fixed-income securities
 - Credit and interest rate risk
 - Yield
 - Duration
 - Immunization

Further reading:

- D.G. Luenberger: *Investment Science*, Chapters 3,5

Definitions

- **Financial instrument**: a legal obligation or claim having monetary value.
 - Examples: stocks, bonds, mortgages, futures, insurance, etc.
 - **Security**: a tradable financial instrument satisfying legal and regulatory requirements.
 - **Fixed-income security**: security that promises a fixed (that is, definite) income over a span of time.
 - Examples: bonds, mortgages, annuities, etc.
- ⇒ A fixed-income security represents the ownership of a definite cash flow stream.

Remarks

The issuer of a fixed-income security could **default** (by, say, **going bankrupt**).

⇒ There is a (typically small) chance that the promised income is **discontinued** or **delayed**.

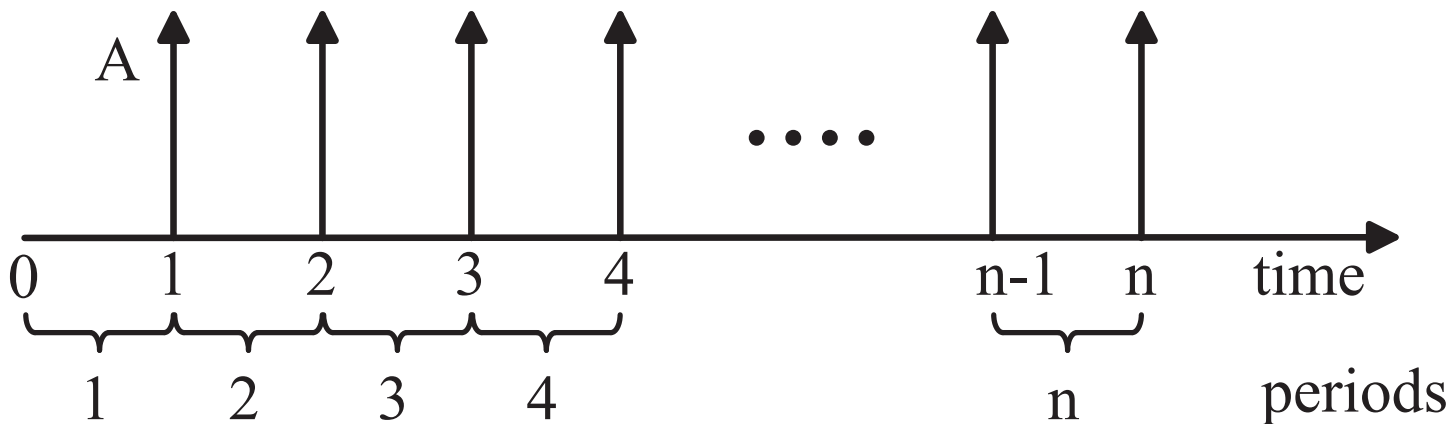
Some securities whose cash flows depend on **contingencies** or **fluctuating indices** are also called fixed-income:

- **Adjustable-rate mortgages**
- **Callable bonds**
- **etc.**

General idea: a fixed-income security's **cash flow stream** is **fixed** except in **well-defined contingent circumstances**.

Annuities

- **Annuity**: a contract that pays the holder money periodically according to a fixed schedule.
- **Example**: pension benefits



The **present value** of an annuity that pays a **fixed amount** A at the end of each of n **equally spaced periods** is

$$PV = \sum_{k=1}^n d_k A .$$

Geometric Series

Calculate $S_n = \sum_{k=0}^n x^k$ for $n \in \mathbb{N}$. We use the recursions:

$$S_{n+1} = 1 + x(1 + x + x^2 + \cdots + x^n) = 1 + xS_n \quad (1)$$

$$S_{n+1} = (1 + x + x^2 + \cdots + x^n) + x^{n+1} = S_n + x^{n+1} \quad (2)$$

By combining (1) and (2), we obtain

$$S_n = \frac{1 - x^{n+1}}{1 - x}.$$

For $x < 1$, we can calculate $S_\infty = \sum_{k=0}^\infty x^k$ to be

$$S_\infty = \lim_{n \rightarrow \infty} \frac{1 - x^{n+1}}{1 - x} = \frac{1}{1 - x}.$$

Annuity Valuation

- If the **spot rate curve is constant**, $s_k = r$ for all $k = 1, 2, \dots, n$, then the value of the annuity becomes

$$\begin{aligned} \text{PV} &= \sum_{k=1}^n \frac{A}{(1+r)^k} = A \left(\frac{1 - \frac{1}{(1+r)^{n+1}}}{1 - \frac{1}{1+r}} - 1 \right) \\ &= \frac{A}{r} \left(1 - \frac{1}{(1+r)^n} \right). \end{aligned}$$

Equivalently, we have

$$A = \frac{r(1+r)^n \text{PV}}{(1+r)^n - 1}.$$

Tables give PV/A as a function of r and n .

Amortization

- You have borrowed \$1,000 at 12% interest compounded monthly, and you must **repay this loan with equal monthly payments over 5 years.**
- How much are the monthly payments?

Given: $PV = \$1,000$

$r = 12\%/12 = 1\%$ per month

$n = 5 \times 12 = 60$

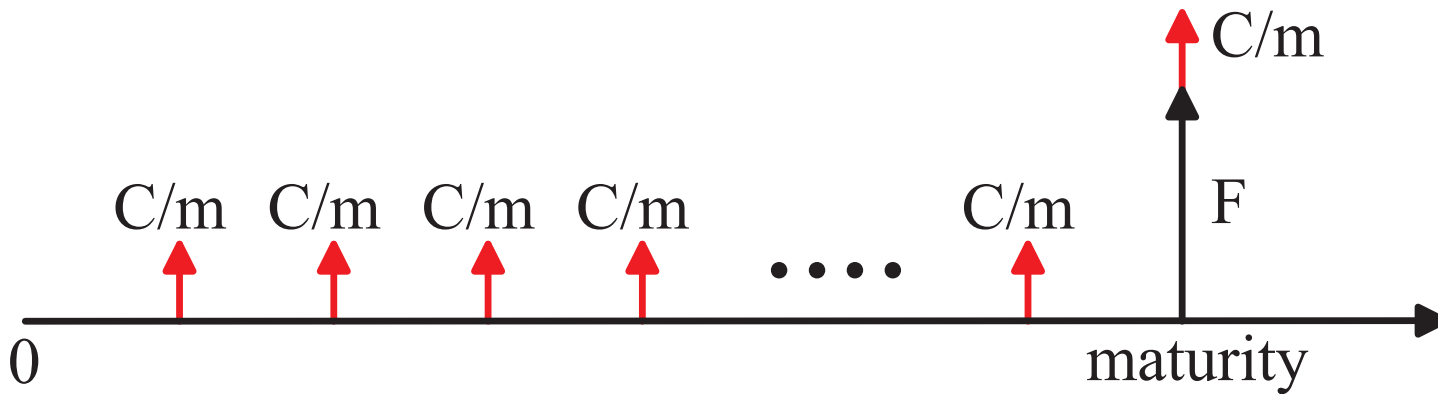
$$\Rightarrow A = \frac{r(1+r)^n PV}{(1+r)^n - 1} = \$22.24 \text{ per month}$$

Replacing a **current obligation** by **periodic payments** is called **amortization**.

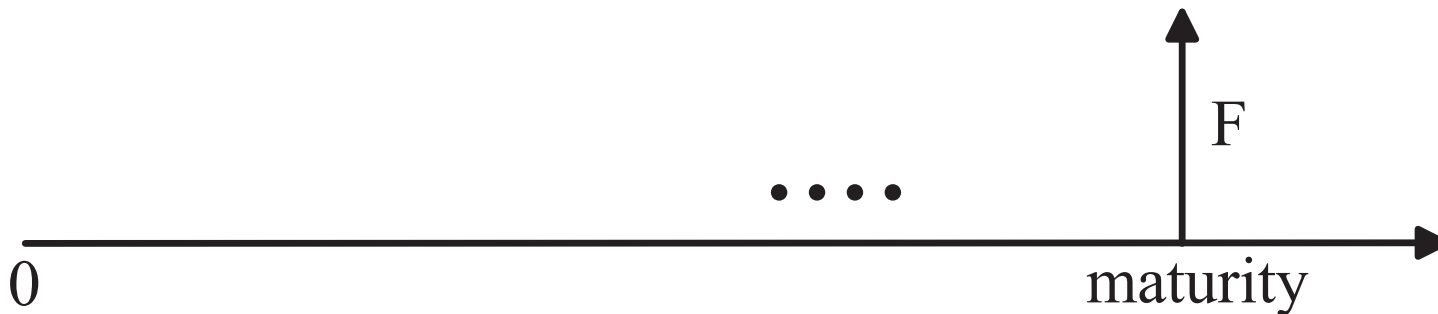
Bonds

- **Bond**: an agreement to pay money according to the rules of the issue.
- Bond specifications:
 - **maturity date**: time of the last payment
 - **face value** or **par value**: an amount to be paid at the maturity date
 - **coupon payment**: an amount paid **periodically** expressed as a **percentage** of the face value (the last coupon is paid at maturity)
- $F = \text{face value}$, $r_C = \text{coupon rate}$
 $\Rightarrow C = r_C F = \text{yearly coupon amount}$;
if there are m coupon payments per year, then each **subannual coupon payment** amounts to C/m .

Payoff Structure of a Bond



A **zero coupon bond** has no coupon payments; it only pays the **face value** at maturity.



Example: Price of a 10-year Bond

Consider an 8% bond maturing in 10 years:

- the bond pays \$8 at the end of the years 1, 2, ..., 9 and \$108 at the end of year 10.
- the end-of-year discount factors for years 1, 2, ..., 10 can be calculated from a given spot rate curve.
- We take the products of the cash flows with the corresponding discount factors and sum.

⇒ The value of the bond is \$97.34.

Year	1	2	3	4	5	6	7	8	9	10	Total PV
Spot Rate	5.571	6.088	6.555	6.978	7.361	7.707	8.020	8.304	8.561	8.793	
Discount	0.947	0.889	0.827	0.764	0.701	0.641	0.583	0.528	0.477	0.431	
Cash Flow	8	8	8	8	8	8	8	8	8	108	
PV	7.58	7.11	6.61	6.11	5.61	5.12	4.66	4.23	3.82	46.50	97.34

Bond Terminology

- **corporate bond**: issued by a corporation
- **municipal bond**: issued by a municipality
- **treasury bond**: issued by government, maturity more than 10 years, 2 coupon payments per year
- **treasury note**: similar to treasury bond, but issued for 1-10 years
- **treasury bill**: similar to treasury bond, no coupon payments, matures in 3, 6, or 9 months

Quality Ratings I

- Bonds offer principally a **deterministic income stream**. However, they are subject to **default** if the issuer falls into **bankruptcy**.
- **Rating classifications** are published by Moody's and Standard & Poor's.
- **Treasury securities** are not rated, since they are considered to be essentially **free of default risk**.
- A bond with a low rating will have a **lower price than a comparable bond with a higher rating**.

Quality Ratings II

	Moody's	S&P	
Investment grade	Aaa	AAA	Best quality, smallest credit risk
	Aa	AA	High grade
	A	A	High to medium grade
	Baa	BBB	Medium grade
Speculative grade (junk bonds)	Ba	BB	Judged to be speculative
	B	B	Increasingly speculative
	Caa	CCC	Danger of default
	Ca	CC	High chance of default
	C	C	Small chance of no default
	D	D	In default

Bond Quotes (finance.yahoo.com)

BOND SCREENER RESULTS									
Type	Issue	State	Price	Coupon(%)	Maturity▲	YTM(%)	Current Yield(%)	Fitch Ratings	Callable
Corp	EXPORT IMPORT BK KOREA	-	102.17	5.125	14-Feb-2011	-52.654	5.016	A	No
Corp	BANK AMER CORP SUB INTNTS BE	-	102.05	6.500	15-Feb-2011	-44.685	6.369	A	No
Corp	FEDERAL NATL MTG ASSN	-	102.22	4.500	15-Feb-2011	-38.219	4.402	AAA	No
Corp	NORTHROP GRUMMAN CORP	-	102.24	7.125	15-Feb-2011	-47.904	6.969	BBB	No
Corp	TYCO INTL GROUP S A	-	102.24	6.750	15-Feb-2011	-48.290	6.602	A	No
Corp	WAL MART STORES INC	-	102.15	4.125	15-Feb-2011	-49.144	4.038	AA	No
Muni	ALABAMA WTR POLLUTION CTL AUTH REF BDS	AL	104.12	4.000	15-Feb-2011	-91.731	3.842	Not Rated	No
Muni	SAN DIEGO CALIF PUB FACS FING REF LEASE REV BDS	CA	104.18	5.000	15-Feb-2011	-91.594	4.799	A	No
Zero	U S TREAS SEC STRIPPED INT PMT 15-Feb-2011	-	100.03	0.000	15-Feb-2011	-0.677	0.000	AAA	No
Treas	T-NOTE 5.000 15-Feb-2011	-	100.26	5.000	15-Feb-2011	0.021	4.987	AAA	No
Corp	FEDERAL NATL MTG ASSN MTN	-	102.21	4.125	17-Feb-2011	-34.320	4.036	AAA	No
Corp	CISCO SYS INC	-	102.29	5.250	22-Feb-2011	-32.493	5.133	A	No
Corp	FEDERAL HOME LOAN BANKS	-	102.00	0.250	25-Feb-2011	-24.964	0.245	AAA	No
Treas	T-NOTE 4.500 28-Feb-2011	-	100.39	4.500	28-Feb-2011	0.041	4.482	AAA	No
Corp	HERTZ CORP	-	102.55	7.400	1-Mar-2011	-21.916	7.216	B	No

Remark: The **current yield** of a bond is the ratio of the annual coupon payment and the bond's current price.

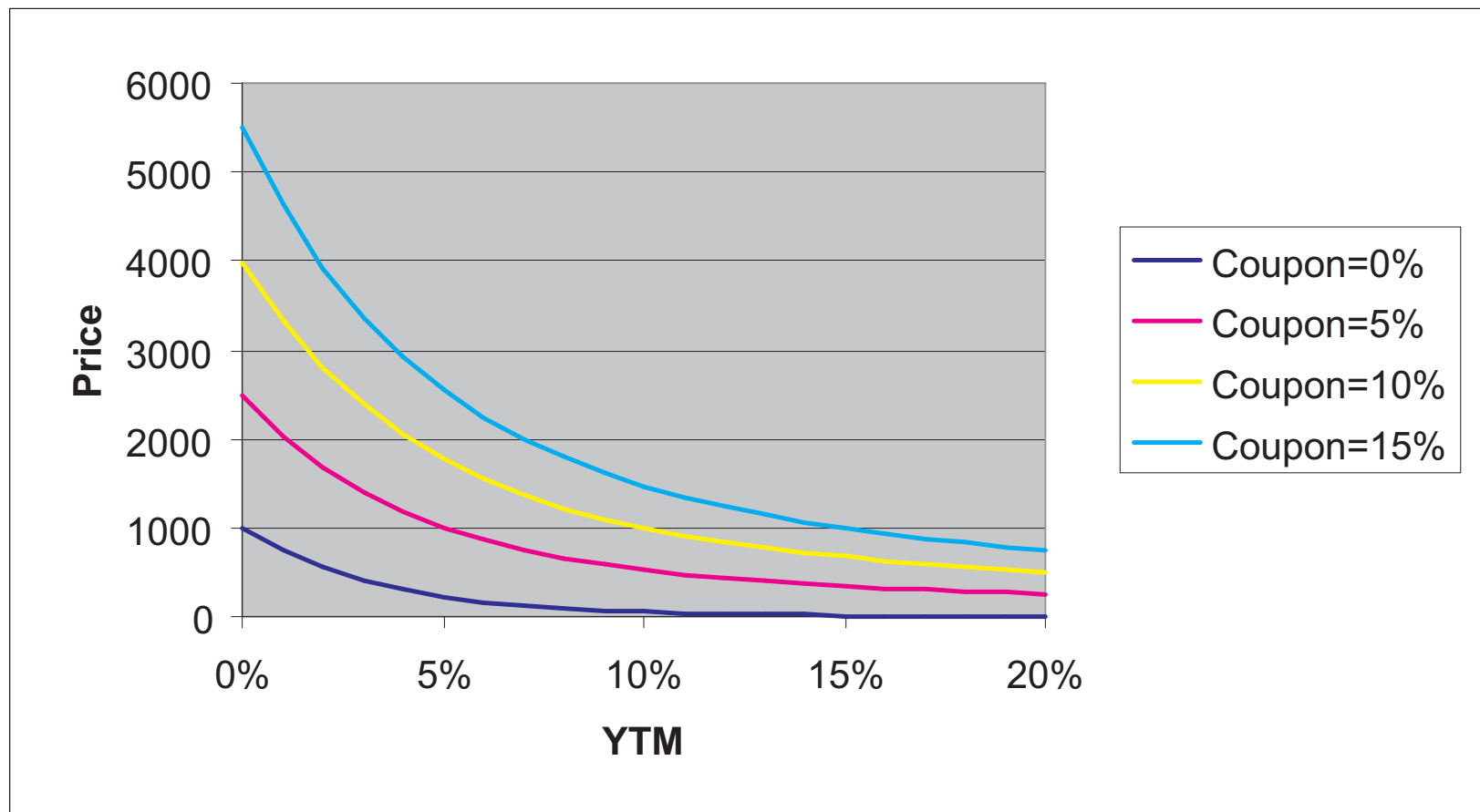
Yield

- A bond's **yield** is the interest rate at which the **PV of the stream of payments** (coupon payments & face value redemption) is **equal to the current price**.
- More properly, it is called **yield to maturity (YTM)**.
- Consider a bond with **price** P and **face value** F that makes m coupon payments of C/m per year, and there are n remaining periods. The bond's **YTM** λ is such that

$$\begin{aligned} P &= \frac{F}{[1 + (\lambda/m)]^n} + \sum_{k=1}^n \frac{C/m}{[1 + (\lambda/m)]^k} \\ &= \frac{F}{[1 + (\lambda/m)]^n} + \frac{C}{\lambda} \left\{ 1 - \frac{1}{[1 + (\lambda/m)]^n} \right\}. \end{aligned}$$

Price-Yield Curve and Coupon Rate

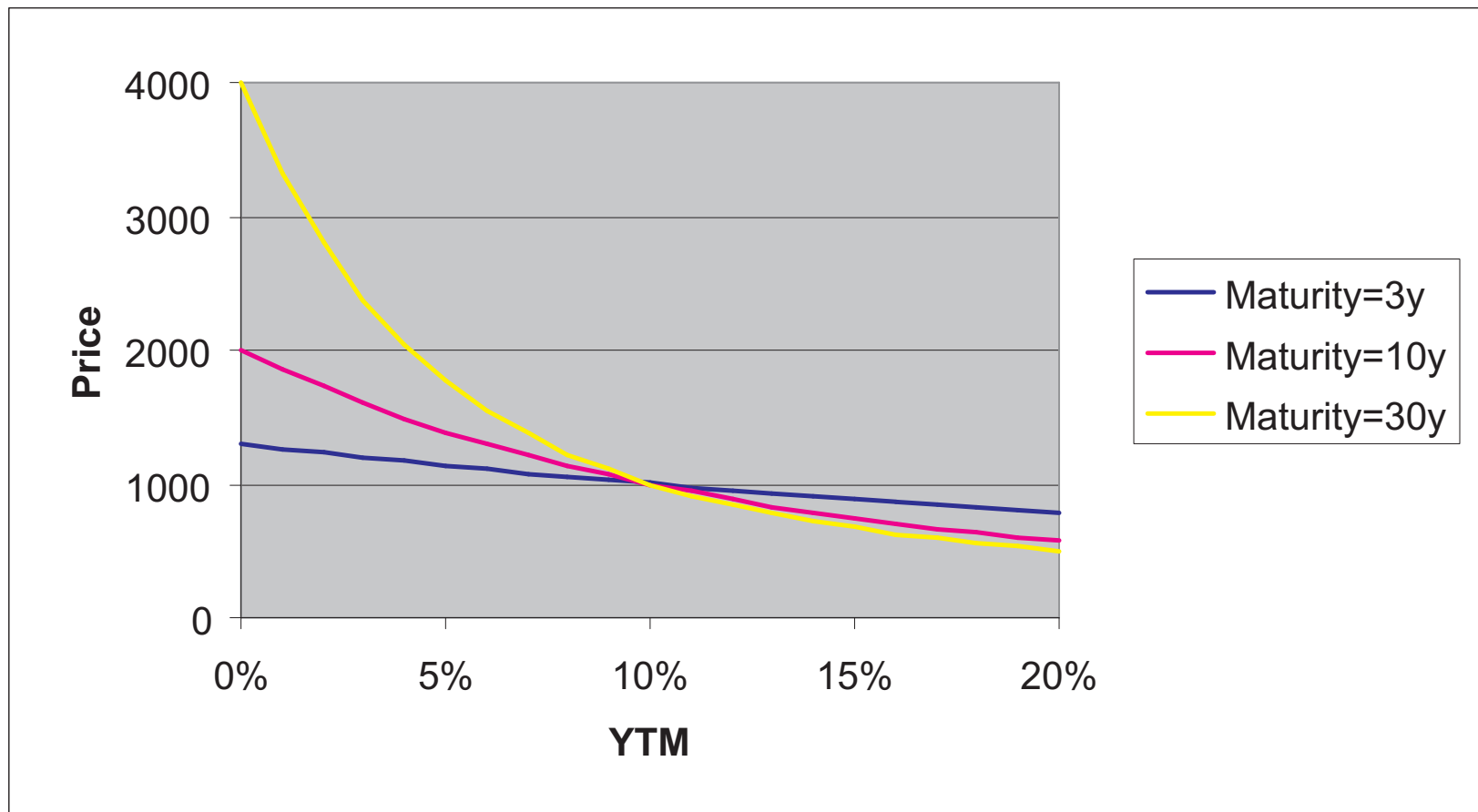
● Bond data: $F = 1000$, $m = 2$, $n = 60$ (i.e., 30 years).



⇒ Inverse dependence between price and yield!

Price-Yield Curve and Maturity

● Bond data: $F = 1000$, $m = 2$, Coupon = 10%.



⇒ The longer the time to maturity, the more sensitive is the price of the bond to the yield.

Interest Rate Risk

- The **price-yield curve** describes the **interest rate risk** associated with a bond.
- The **yield** can roughly be identified with the **market rate** for the underlying bond.
- Suppose you bought a 10% bond with $F = 1000$, $m = 2$, $n = 60$, when the yield was 10% \Rightarrow **price = 1000**.
- If the yield rises to 11%, then the price of your bond drops to 913 \Rightarrow **a 8.72% loss!**

Duration

- Long bonds are more sensitive to interest rate changes than short bonds.
- When there are coupons, maturity time does not exactly correspond to sensitivity.
- Another measure of time length termed duration does give a quantitative measure of interest rate sensitivity.

Macauley Duration I

- Suppose that cash flows are received at times t_0, t_1, \dots, t_n . The **Macauley duration** of this stream is

$$D = \frac{PV(t_0)t_0 + PV(t_1)t_1 + \dots + PV(t_n)t_n}{PV(t_0) + PV(t_1) + \dots + PV(t_n)},$$

where $PV(t_k)$ denotes the present value of the cash flow that occurs at time t_k **computed w.r.t. the yield**.

- ⇒ **Weighted average of the cash flow times**, where the weight of time t_k is given by $w_k = PV(t_k)/PV_{\text{tot}}$ with $PV_{\text{tot}} = PV(t_0) + PV(t_1) + \dots + PV(t_n)$.

Macaulay Duration II

- Macaulay duration is a **weighted average of times**
⇒ it is quoted in **years**.
- In practice, when people refer to duration, they often mean Macaulay duration.
- By definition of D , the following relations hold.
 - **Zero-coupon bond:**
Macaulay duration = maturity date
 - **Coupon bond:**
Macaulay duration < maturity date

Macauley Duration III

● Consider a bond with the following specifications:

c - coupon rate per year

λ - yield

m - periods per year

n - remaining periods

⇒ This bond's Macauley duration is given by

$$D = \frac{1 + \lambda/m}{\lambda} - \frac{1 + \lambda/m + n(c/m - \lambda/m)}{c[(1 + \lambda/m)^n - 1] + \lambda}.$$

Modified Duration I

- Let $P(\lambda)$ be the price of a bond as a function of yield, and denote by λ_0 the current yield.
- Def.:** The **modified duration** D_M of this bond is

$$D_M = -\frac{1}{P(\lambda_0)} \left. \frac{dP(\lambda)}{d\lambda} \right|_{\lambda=\lambda_0}.$$

Intuition:

- sensitivity = derivative**;
- the minus sign ensures that $D_M \geq 0$;
- dividing by $P(\lambda_0)$ makes D_M a **relative sensitivity**.

Modified Duration II

- Modified duration determines the **percentage change** in the price of a bond given a change in yield:

$$D_M = -\frac{1}{P(\lambda_0)} \left. \frac{dP(\lambda)}{d\lambda} \right|_{\lambda=\lambda_0} \approx -\frac{1}{P} \frac{\Delta P}{\Delta \lambda}$$

$$\Rightarrow \Delta P \approx -D_M P \Delta \lambda.$$

- If the yield increases by 1%, **what percentage change will occur in a bond price with modified duration of 5?**

Answer: **The price will decrease by 5%!**

Relation between D and D_M I

Theorem 0.1 *If λ is the yield and m the number of compounding periods per year, then*

$$D_M = \frac{D}{1 + \lambda/m}.$$

\Rightarrow In the case of *continuous compounding* $D_M = D$.

Proof. For the stream c_0, c_1, \dots, c_n we have

$$PV_k = \frac{c_k}{(1 + \lambda/m)^k}, \quad PV_{\text{tot}} = \sum_{k=0}^n \frac{c_k}{(1 + \lambda/m)^k}, \quad D = \sum_{k=0}^n \frac{k}{m} \frac{PV_k}{PV_{\text{tot}}}.$$

$$\begin{aligned} \Rightarrow \quad \frac{dPV_{\text{tot}}}{d\lambda} &= \sum_{k=0}^n \frac{-(k/m)c_k}{(1 + \lambda/m)^{k+1}} = -\frac{1}{1 + \lambda/m} \sum_{k=0}^n \frac{(k/m)c_k}{(1 + \lambda/m)^k} \\ &= -\frac{1}{1 + \lambda/m} \sum_{k=0}^n \frac{k}{m} PV_k = -\frac{PV_{\text{tot}} D}{1 + \lambda/m} \end{aligned}$$

Relation between D and D_M II

Thus, we find

$$D_M = -\frac{1}{PV_{\text{tot}}} \frac{dPV_{\text{tot}}}{d\lambda} = \frac{D}{1 + \lambda/m}.$$

This observation completes the proof. □

Example: A 10%, 30 year bond with semiannual coupons is selling at par and has Macaulay duration $D = 9.94$.

By how many percent will the bond's price change if yield increases by 1%?

$$\frac{\Delta P}{P} \approx -D_M \Delta\lambda = -\frac{D}{1 + \lambda/m} \Delta\lambda = -\frac{9.94}{1 + 10\%/2} 1\% = -9.47\%$$

⇒ A 1% increase in yield causes a 9.47% drop in price!

Duration of a Portfolio I

- Consider a set of m fixed-income securities with prices P_i and durations D_i for $i = 1, 2, \dots, m$, all computed at a common yield.
- The portfolio consisting of the aggregate of these securities has price P and duration D given by

$$P = P_1 + P_2 + \dots + P_m$$

$$D = w_1 D_1 + w_2 D_2 + \dots + w_n D_n$$

where $w_i = P_i/P$, $i = 1, 2, \dots, m$.

- The formula for D is true for the Macaulay duration as well as for the modified duration!

Duration of a Portfolio II

Consider two cash flow streams A and B with

$$D^A = \sum_{k=0}^n \frac{t_k \text{PV}_k^A}{P^A} \quad \text{and} \quad D^B = \sum_{k=0}^n \frac{t_k \text{PV}_k^B}{P^B}.$$

The Macaulay duration of the aggregate stream $A + B$ is

$$\begin{aligned} D^{A+B} &= \sum_{k=0}^n \frac{t_k (\text{PV}_k^A + \text{PV}_k^B)}{P^A + P^B} = \sum_{k=0}^n \frac{t_k \text{PV}_k^A}{P^A + P^B} + \sum_{k=0}^n \frac{t_k \text{PV}_k^B}{P^A + P^B} \\ &= \frac{P^A}{P^A + P^B} \sum_{k=0}^n \frac{t_k \text{PV}_k^A}{P^A} + \frac{P^B}{P^A + P^B} \sum_{k=0}^n \frac{t_k \text{PV}_k^B}{P^B} \\ &= \frac{P^A}{P^A + P^B} D^A + \frac{P^B}{P^A + P^B} D^B. \end{aligned}$$

Immunization I

- Problem of major practical value:
 - Construct a portfolio which is protected (immunized) against changes in interest rates.
- Example:
 - You have an obligation to pay £1,000 in 2 years, but you can only buy bonds of maturities 1 or 5 years.
 - Buying 1 year bonds: you face reinvestment risk as you do not know the bond prices in 1 year.
 - Buying 5 year bonds: you may fail to meet your obligation when interest rates change, i.e., you may not be able to sell the bond after 1 year at the desired price.

Immunization II

- Assume that you face a series of **cash obligations** and you wish to acquire a **portfolio** that you will use to **pay these obligations**.
 - The **stream of obligations** has present value P and duration D .
 - You can invest in two **bonds** with present values P_1 and P_2 and durations D_1 and D_2 , respectively.
- ⇒ **Construct a portfolio whose present value and duration equal those of your obligations.**

$$\left. \begin{aligned} P &= x_1 P_1 + x_2 P_2 \\ D &= \frac{x_1 P_1}{P} D_1 + \frac{x_2 P_2}{P} D_2 \end{aligned} \right\} \text{Solve for } x_1 \text{ and } x_2!$$

Immunization III

- **Obligation:** £1,000 in 2 years.
- You can **invest** in a 1 year and 5 year zero coupon bond.

	Obligation	Bond 1	Bond 2
Face value	1000	100	100
Maturity	2	1	5
Coupon	0	0	0
Yield	0.09	0.09	0.09
Mac. Duration	2	1	5
Price	841.68	91.74	64.99

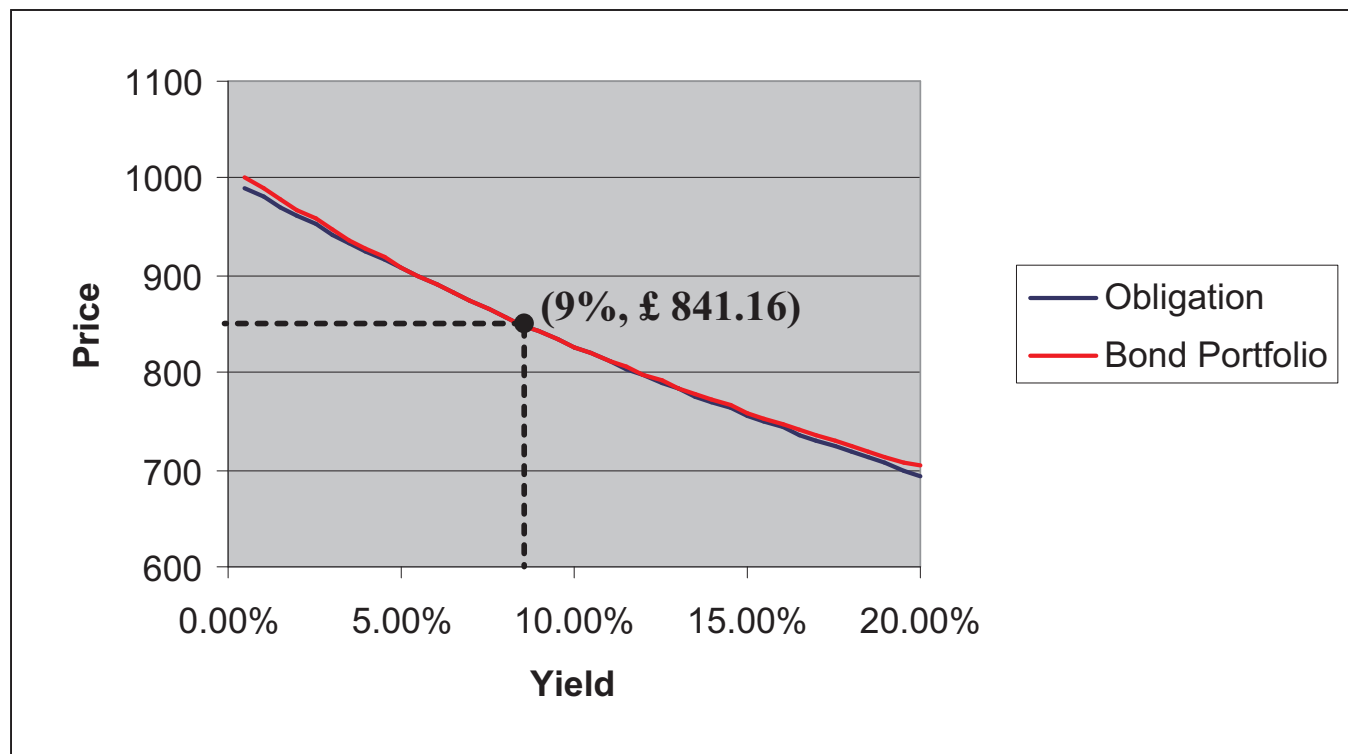
- Form a **portfolio** with the same price and duration as the obligation.
- Let x_1 and x_2 be the **numbers of bond 1 and 2** held in the portfolio, respectively.

Immunization IV

● Match prices: $91.74x_1 + 64.99x_2 = 841.68$

● Match durations: $\frac{91.74x_1}{841.68}1 + \frac{64.99x_2}{841.68}5 = 2$

$\Rightarrow x_1 = 6.88$ and $x_2 = 3.24$



Convexity I

- Idea behind immunization: Taylor expansion!
- λ_0 is the current value of the yield.
- We can expand a fixed-income security's price as a function of yield around λ_0 :

$$P(\lambda) = P(\lambda_0) + P'(\lambda_0)(\lambda - \lambda_0) + \frac{1}{2}P''(\lambda_0)(\lambda - \lambda_0)^2 + \dots$$

- $P(\lambda_0)$ is the price.
 - $P'(\lambda_0) = -D_M P(\lambda_0)$ is the (unnormalized) duration.
 - $P''(\lambda_0) = P(\lambda_0)C$ is the (unnormalized) convexity.
- We can match as many terms as we like to adjust the Taylor series of our portfolio to that of our obligation.

Convexity II

- The **Convexity** of a fixed-income security is defined as

$$C = \frac{1}{P(\lambda_0)} \left. \frac{d^2 P(\lambda)}{d\lambda^2} \right|_{\lambda=\lambda_0} .$$

- If $\Delta\lambda$ is a small **change in yield** and ΔP is the corresponding **change in price**, then

$$\Delta P \approx -D_M P \Delta\lambda + \frac{PC}{2} (\Delta\lambda)^2 .$$

This is a **second-order** approximation to the **price-yield curve**.

⇒ **Convexity can be used to improve immunization!**