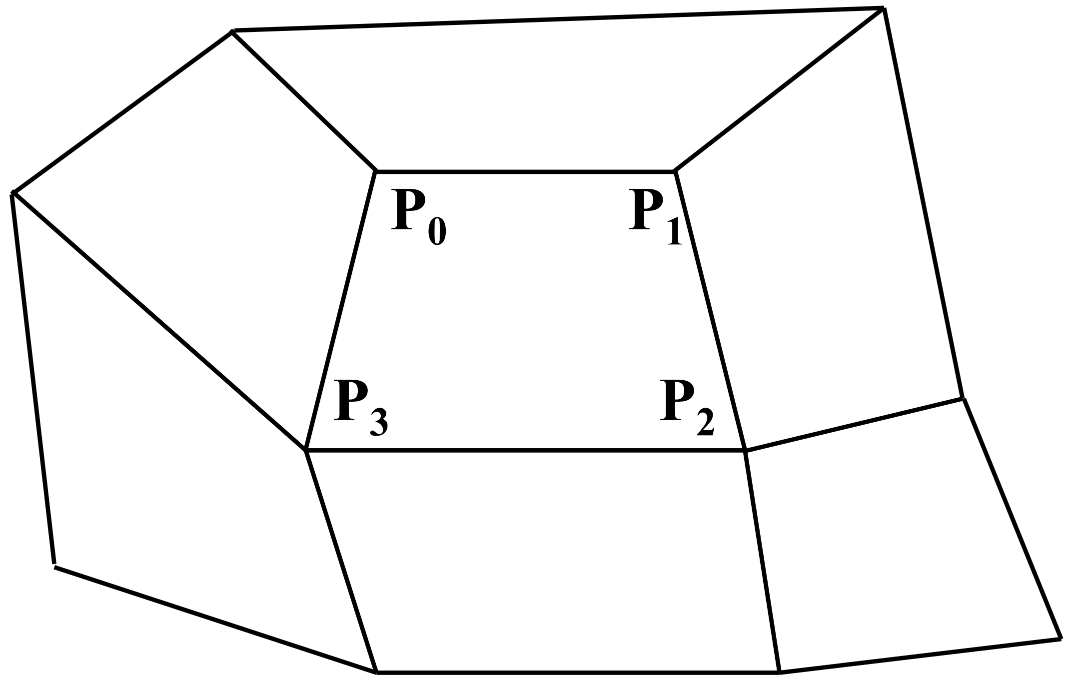
Interactive Computer Graphics: Revision Lecture

Shading

A scene is defined as a regular array of planar quadrilaterals part of which is shown in the figure.



	x	y	z
P_0	1	6	10
P_1	6	6	10
P_2	7	0	12
P_3	0	2	12

The scene is illuminated by one light source which is located at the origin. The viewpoint is also at the origin and the view direction is along the positive z axis. Only diffuse shading is used.

Shading

a Outline the different steps necessary to calculate the shading value for each pixel inside the projected quadrilateral using Gouraud shading.

b Explain how your answer to part a must be modified to compute the shading value for each pixel inside the projected quadrilateral using Phong shading instead of Gouraud shading.

c What effects can be demonstrated with Phong shading that cannot be created by Gouraud shading?

d Calculate the outer surface normal vector (i.e. the one that is directed towards the viewpoint) for the central quadrilateral.

e Assume that only flat diffuse shading is being used, the constant for diffuse reflection is $k_d = 1$ and that the light source is a white light with the following intensity: $r = 255$, $g = 255$, $b = 255$. Calculate the shading value at the centre of the quadrilateral.

Shading

First find two vectors on the plane:

$$\mathbf{p}_1 - \mathbf{p}_0 = [6, 6, 10] - [1, 6, 10] = [5, 0, 0]$$

$$\mathbf{p}_0 - \mathbf{p}_3 = [1, 6, 10] - [0, 2, 12] = [1, 4, -2]$$

Find the cross product: $[0, 10, 20]$ *or* $[0, 1, 2]$ *for simplicity, however this is facing away from the viewpoint (since* $[0, 1, 2] \cdot [0, 0, 1] = 2 > 0$*), so the answer is* $[0, -1, -2]$ *or* $[0, -1/\sqrt{5}, -2/\sqrt{5}]$.

Shading

i) *Average the vertices to find a central point* $[3.5, 3.5, 11.25]$.

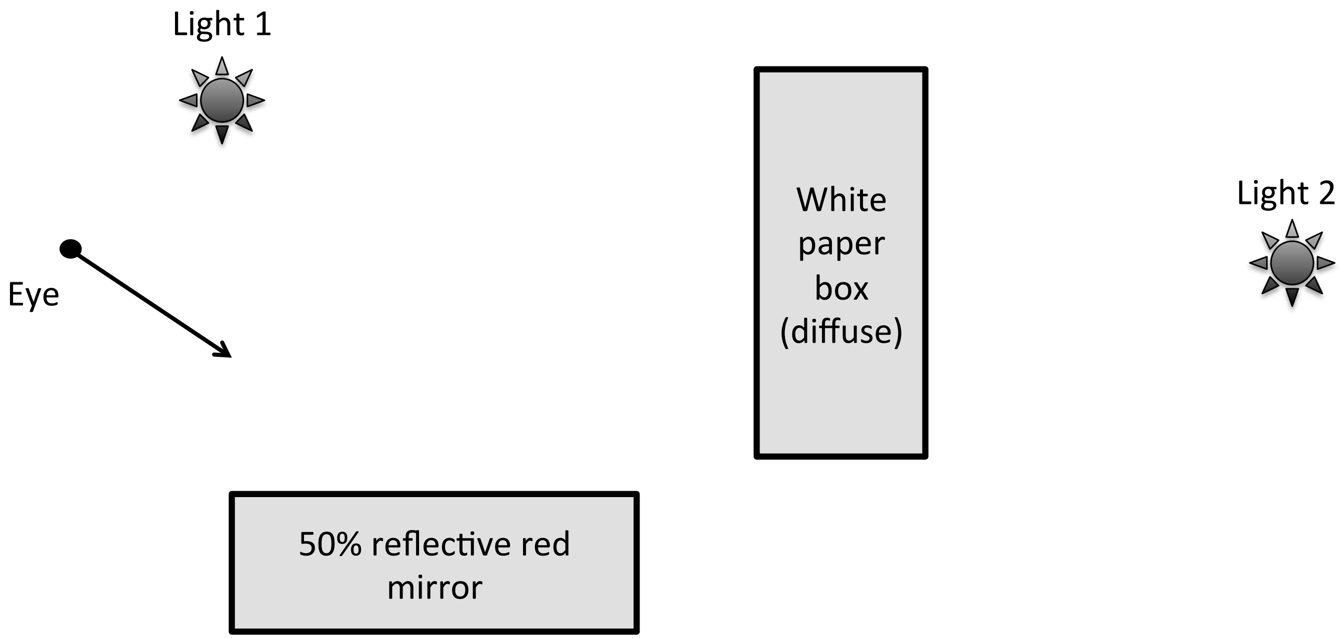
ii) *Find the light source direction* $[-3.5, -3.5, -11.25]$.

iii) *Compute* $\mathbf{n} \cdot \mathbf{l} = (0 + 3.5 + 22.5)/(|\mathbf{n}||\mathbf{l}|) = 24/(\sqrt{5}\sqrt{151.06}) = 0.95$

iv) *The final shading value is*
$0.95 \times (255, 255, 255) = (242.25, 242.25, 242.25)$.

Ray tracing

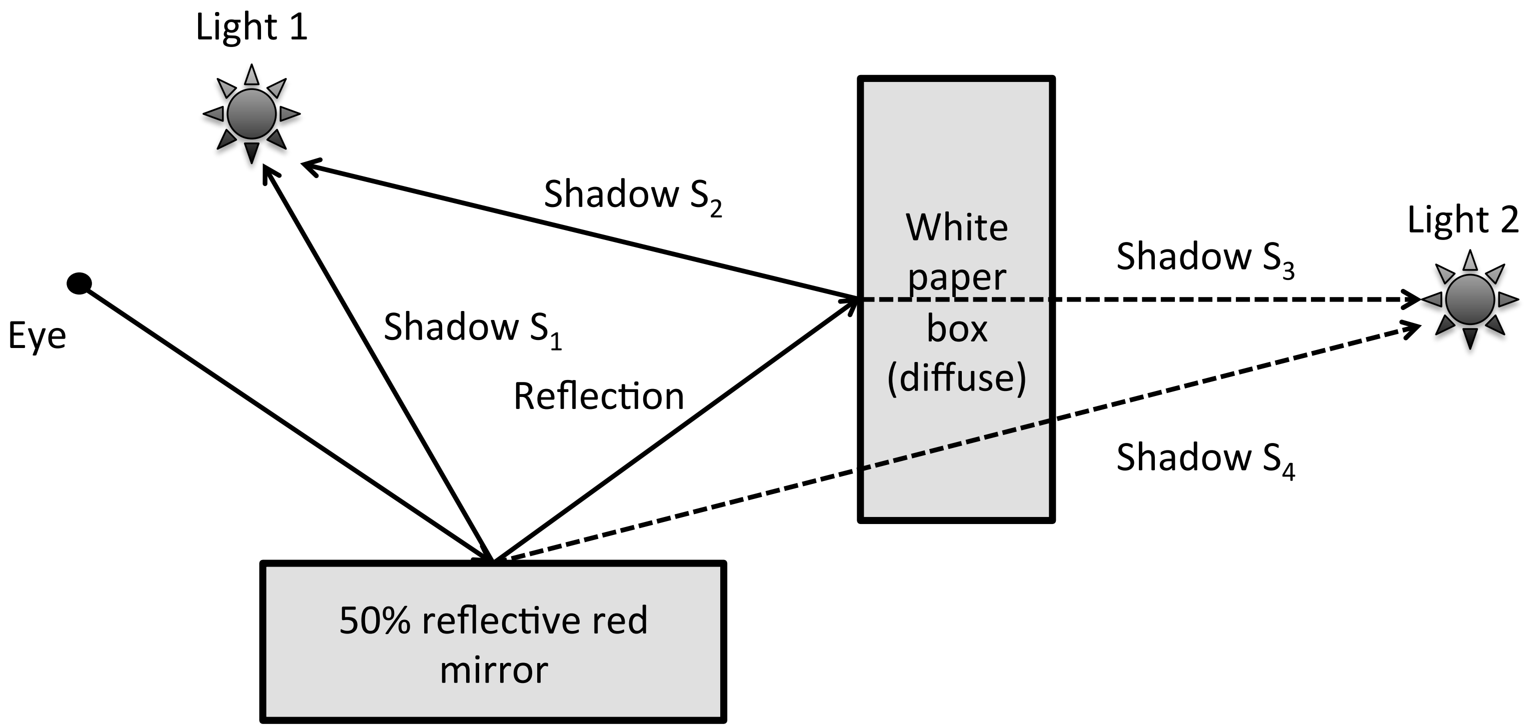
The scene in the figure below has a mirror, a white paper box and two lights:



i) Starting from the eye ray drawn, draw all additional reflection, refraction and shadow rays needed to compute the colour of the eye ray. Label each ray clearly.

ii) Write down a formula which describes the computation of the colour for the ray. Label your drawing from a part i) or add annotations to your terms in your equation so that it is clear what is being computed.

Ray tracing


Light 1
Shadow S_2
White
paper
box
(diffuse)
Light 2
Shadow S_3
Eye
Shadow S_1
Reflection
Shadow S_4
50% reflective red
mirror

Ray tracing

b Briefly explain which illumination effects can be achieved with ray tracing and how these effects are achieved. What is the key difference between ray tracing and radiosity?

c A ray originates at point $\mathbf{V}$ and is parallel with direction vector $\mathbf{d}$. A right-angled triangle is given by three points $\mathbf{P}_1$, $\mathbf{P}_2$ and $\mathbf{P}_3$ (the right angle is at point $\mathbf{P}_1$). Show in detail how you can calculate the intersection between the ray and the face defined by the triangle.

d In a concrete example, a ray starts at $\mathbf{V} = (7, 7, 0)$ and has a direction vector $\mathbf{d} = (0, 0, 1)$. The points of the triangle are given as $\mathbf{P}_1 = (6, 6, 10)$, $\mathbf{P}_2 = (12, 6, 10)$, and $\mathbf{P}_3 = (6, 10, 5)$. Calculate whether the ray intersects the face defined by the triangle.

Ray tracing:

c) *A point in the square triangle is given by*

$$\mathbf{P} = w_1\mathbf{u}_1 + w_2\mathbf{u}_2 + \mathbf{P}_1$$

where $\mathbf{u}_1 = \mathbf{P}_2 - \mathbf{P}_1$, $\mathbf{u}_2 = \mathbf{P}_3 - \mathbf{P}_1$ *and*

$$0 \leq w_1 \leq 1$$
$$0 \leq w_2 \leq 1$$
$$0 \leq w_1 + w_2 \leq 1$$

We first determine the intersection of the ray with the plane defined by the triangle. The normal of the plane is given by

$$\mathbf{n} = \pm(\mathbf{u}_1 \times \mathbf{u}_2)/(|\mathbf{u}_1 \times \mathbf{u}_2|)$$

The equation of the ray can be written as $\mathbf{V} + \alpha\mathbf{d}$. *The position of the intersection* q *is given by*

$$\mathbf{P}_1 + \mathbf{P} = \mathbf{V} + \alpha\mathbf{d}$$
$$\mathbf{P} = \mathbf{V} - \mathbf{P}_1 + \alpha\mathbf{d}$$

Ray tracing

Since $\mathbf{P} \cdot \mathbf{n} = 0$*:*

$$0 = (\mathbf{V} - \mathbf{P}_1)\mathbf{n} + \alpha\mathbf{dn}$$

$$\alpha = -(\mathbf{V} - \mathbf{P}_1)\mathbf{n}/(\mathbf{dn})$$

Now, we need to test whether the intersection point $\mathbf{P}$ *is inside the triangle. Si* $\mathbf{u}_1$ *and* $\mathbf{u}_2$ *are orthogonal, taking the dot product with* $\mathbf{u}_1$ *and* $\mathbf{u}_2$ *yields*

$$w_1 = (\mathbf{P}\mathbf{u}_1 - \mathbf{P}_1\mathbf{u}_1)/(\mathbf{u}_1\mathbf{u}_1)$$

$$w_2 = (\mathbf{P}\mathbf{u}_2 - \mathbf{P}_1\mathbf{u}_2)/(\mathbf{u}_2\mathbf{u}_2)$$

For a valid intersection the following conditions must be met:

$$0 \leq w_1 \leq 1$$

$$0 \leq w_2 \leq 1$$

$$0 \leq w_1 + w_2 \leq 1$$

Ray tracing

d)

First, we calculate the normal of the plane defined by the triangle:

$\mathbf{u}_1 = (6, 0, 0)$

$\mathbf{u}_2 = (0, 4, -5)$

$\mathbf{n} = (0, 30, 24) or (0, 0.78, 0.62)$

Calculating the intersection of the ray with the plane yields

$$\alpha = -((7, 7, 0) - (6, 6, 10))(0, 0.78, 0.62)/0.62 = (0.78 - 6.2)/0.62 = 8.74$$

and

Ray tracing

$\mathbf{P} = (7, 7, 8.74)$

Calculating the barycentric coordinates yields:

$$w_1 = ((7, 7, 8.74)(6, 0, 0) - (6, 6, 10)(6, 0, 0))/36 = (42 - 36)/36 = 0.16$$

$$w_2 = ((7, 7, 8.74)(0, 4, -5) - (6, 6, 10)(0, 4, -5))/41 = 0.25$$

$0 \leq w_1 \leq 1$

$0 \leq w_2 \leq 1$

$0 \leq w_1 + w_2 \leq 1$

So the intersection point is inside the triangle.

Splines

A cubic spline patch is defined by the parametric equation:

$$\mathbf{P}(\mu) = \mathbf{a}_3\mu^3 + \mathbf{a}_2\mu^2 + \mathbf{a}_1\mu + \mathbf{a}_0$$

where $\mathbf{P}(\mu)$ is a point that traces the locus of the curve, with $0 < \mu < 1$ and $\mathbf{a}_3$, $\mathbf{a}_2$, $\mathbf{a}_1$ and $\mathbf{a}_0$ are vector constants that define the shape of the curve.

a Given that the patch is to be drawn between two points $\mathbf{P}_i$ and $\mathbf{P}_{i+1}$ and the gradients at the ends are to be $\mathbf{P}'_i$ and $\mathbf{P}'_{i+1}$ respectively, write down four equations connecting $\mathbf{P}_i$ $\mathbf{P}_{i+1}$ $\mathbf{P}'_i$ $\mathbf{P}'_{i+1}$ and $\mathbf{a}_3$ $\mathbf{a}_2$ $\mathbf{a}_1$ $\mathbf{a}_0$.

b Solve the above equations to find the values in matrix $\mathcal{A}$ in the equation:

$$\begin{pmatrix} \mathbf{a}_0 \\ \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} = \mathcal{A} \begin{pmatrix} \mathbf{P}_0 \\ \mathbf{P}_1 \\ \mathbf{P}_2 \\ \mathbf{P}_3 \end{pmatrix}$$

Splines

a)

$$\begin{aligned} \mathbf{P}_i &= \mathbf{a}_0 \\ \mathbf{P}'_i &= \mathbf{a}_1 \\ \mathbf{P}_{i+1} &= \mathbf{a}_3 + \mathbf{a}_2 + \mathbf{a}_1 + \mathbf{a}_0 \\ \mathbf{P}'_{i+1} &= 3\,\mathbf{a}_3 + 2\,\mathbf{a}_2 + \mathbf{a}_1 \end{aligned}$$

b) *The equations in part a can be written*

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} \mathbf{a}_0 \\ \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} = \begin{pmatrix} \mathbf{P}_0 \\ \mathbf{P}_1 \\ \mathbf{P}_2 \\ \mathbf{P}_3 \end{pmatrix}$$

and the above matrix needs to be inverted to give the matrix $\mathcal{A}$*:*

$$\mathcal{A} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3 & -2 & 3 & -1 \\ 2 & 1 & -2 & 1 \end{pmatrix}$$

Splines

c Given that the spline patch is one of several making up a more complex curve, and the points are defined as

$$\mathbf{P}_{i-1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \mathbf{P}_{i} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad \mathbf{P}_{i+1} = \begin{pmatrix} 5 \\ 3 \end{pmatrix} \quad \mathbf{P}_{i+2} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

calculate the values of the gradients $\mathbf{P}'_i$ and $\mathbf{P}'_{i+1}$ using the central difference approximation and hence calculate the values of $\mathbf{a}_3$ $\mathbf{a}_2$ $\mathbf{a}_1$ $\mathbf{a}_0$.

d Calculate the co-ordinate and the gradient direction at the midpoint of the spline patch.

Splines

c)

$$\mathbf{P}'_i = \frac{1}{2}\left[\begin{pmatrix}5\\3\end{pmatrix} - \begin{pmatrix}0\\0\end{pmatrix}\right] = \begin{pmatrix}5/2\\3/2\end{pmatrix}$$

$$\mathbf{P}'_{i+1} = \frac{1}{2}\left[\begin{pmatrix}4\\2\end{pmatrix} - \begin{pmatrix}1\\3\end{pmatrix}\right] = \begin{pmatrix}3/2\\-1/2\end{pmatrix}$$

Using the entries in the matrix $\mathcal{A}$ from part b, we have

$$\mathbf{a}_0 = \begin{pmatrix}1\\3\end{pmatrix}$$

$$\mathbf{a}_1 = \begin{pmatrix}5/2\\3/2\end{pmatrix}$$

$$\begin{aligned}\mathbf{a}_2 &= -3\begin{pmatrix}1\\3\end{pmatrix} - 2\begin{pmatrix}5/2\\3/2\end{pmatrix} + 3\begin{pmatrix}5\\3\end{pmatrix} - \begin{pmatrix}3/2\\-1/2\end{pmatrix}\\ &= \begin{pmatrix}11/2\\-5/2\end{pmatrix}\end{aligned}$$

$$\begin{aligned}\mathbf{a}_3 &= 2\begin{pmatrix}1\\3\end{pmatrix} + \begin{pmatrix}5/2\\3/2\end{pmatrix} - 2\begin{pmatrix}5\\3\end{pmatrix} + \begin{pmatrix}5/2\\3/2\end{pmatrix}\\ &= \begin{pmatrix}-4\\1\end{pmatrix}\end{aligned}$$

Splines

d)

Putting $\mu = 2$ into

$$\mathbf{P}(\mu) = \mathbf{a}_3\mu^3 + \mathbf{a}_2\mu^2 + \mathbf{a}_1\mu + \mathbf{a}_0$$

gives the coordinate as

$$\mathbf{P}\left(\frac{1}{2}\right) = \frac{1}{8}\mathbf{a}_3 + \frac{1}{4}\mathbf{a}_2 + \frac{1}{2}\mathbf{a}_1 + \mathbf{a}_0$$

and substituting the values found for $\mathbf{a}_3$ $\mathbf{a}_2$ $\mathbf{a}_1$ $\mathbf{a}_0$ gives

$$\mathbf{P}\left(\frac{1}{2}\right) = \frac{1}{8}\begin{pmatrix} -4 \\ 1 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 11/2 \\ -5/2 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 5/2 \\ 3/2 \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 25/8 \\ 26/8 \end{pmatrix}$$

The gradient is given by

$$\mathbf{P}'(\mu) = 3\mathbf{a}_3\mu^2 + 2\mathbf{a}_2\mu + \mathbf{a}_1$$

which, at $\mu = 1/2$, is

$$\mathbf{P}'\left(\frac{1}{2}\right) = \frac{3}{4}\mathbf{a}_3 + \mathbf{a}_2 + \mathbf{a}_1$$

and substituting the values found for $\mathbf{a}_3$ $\mathbf{a}_2$ $\mathbf{a}_1$ $\mathbf{a}_0$ gives

$$\mathbf{P}'\left(\frac{1}{2}\right) = \frac{3}{4}\begin{pmatrix} -4 \\ 1 \end{pmatrix} + \begin{pmatrix} 11/2 \\ -5/2 \end{pmatrix} + \begin{pmatrix} 5/2 \\ 3/2 \end{pmatrix} = \begin{pmatrix} 5 \\ -1/4 \end{pmatrix}$$

Transformations

a In a computer graphics animation scene an object is defined as a planar polyhedron. The centre of the object is located at position $\mathbf{P} = (10,\, 0,\, 10)$, and the scene is drawn, as usual, in perspective projection with the viewpoint at the origin and the view direction along the z-axis.

Calculate the transformation matrix $\mathcal{M}$ that will shrink the object in size by a factor of 0.9 towards its centre point.

Transformations

In order, the transformation can be achieved in three steps:

i) *Translate the object to the origin*

ii) *Perform a shrink relative to the origin*

iii) *Translate the object back to its original centre*

In matrix terms:

$$\mathbf{x} \rightarrow \mathcal{M}\mathbf{x}$$

where

$$\mathcal{M} = \begin{pmatrix} 1 & 0 & 0 & 10 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 10 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0.9 & 0 & 0 & 0 \\ 0 & 0.9 & 0 & 0 \\ 0 & 0 & 0.9 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & -10 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -10 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Multiplying out, we get

$$\begin{pmatrix} 0.9 & 0 & 0 & 1 \\ 0 & 0.9 & 0 & 0 \\ 0 & 0 & 0.9 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

b In a different animation, the object defined in part a is required to rotate around an axis parallel to the z-axis and shrink. The rotation is anti-clockwise when viewed along the axis.

In each successive frame it should rotate by 10° and shrink to 0.9 of its original size. As before, the shrinkage is towards the object's centre.

Assume that $\cos(10^\circ) = 0.98$ and $\sin(10^\circ) = 0.17$. What is the transformation matrix that will achieve this animation?

Transformations

In order, the transformation can be achieved in four steps:

i) *Translate the object to the origin*

ii) *Perform a shrink relative to the origin*

iii) *Perform a rotaion around the z-axis*

iv) *Translate the object back to its original centre*

In fact the order of the shrink and the rotation, in this case, do not matter and reversing them would give the same result.

Relative to the origin, the matrix for the scaling is the same as in part a:

$$\begin{pmatrix} 0.9 & 0 & 0 & 0 \\ 0 & 0.9 & 0 & 0 \\ 0 & 0 & 0.9 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Transformations

A rotation around the z-axis by an angle θ is given by

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

So the complete sequence of transformations in matrix form is

$$\begin{pmatrix} 1 & 0 & 0 & 10 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 10 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0.9 & 0 & 0 & 0 \\ 0 & 0.9 & 0 & 0 \\ 0 & 0 & 0.9 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0.98 & -0.17 & 0 & 0 \\ 0.17 & 0.98 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & -10 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -10 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

which evalutes to

$$\begin{pmatrix} 0.8820 & -0.1530 & 0 & 1.1800 \\ 0.1530 & 0.8820 & 0 & -1.5300 \\ 0 & 0 & 0.9000 & 1.0000 \\ 0 & 0 & 0 & 1.0000 \end{pmatrix}$$

Transformations

c For another sequence the object is to shrink, as defined in part a, and to drop vertically downwards by 1 unit each frame.

If the animation sequence is made up of a number of frames, numbered consecutively from zero, what is the transformation matrix that should be applied at frame n?

Transformations

Frame n *is obtained after* n *translations downwards by one unit. This means the centre of the object is now at* $(10, -n, 10)$.

The matrix for the transformation can therefore be obtained by multiplying:

$$\begin{pmatrix} 1 & 0 & 0 & 10 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 10 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 10 \\ 0 & 1 & 0 & -n \\ 0 & 0 & 1 & 10 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0.9 & 0 & 0 & 0 \\ 0 & 0.9 & 0 & 0 \\ 0 & 0 & 0.9 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & -10 \\ 0 & 1 & 0 & n \\ 0 & 0 & 1 & -10 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

or, equivalently,

$$\begin{pmatrix} 1 & 0 & 0 & 10 \\ 0 & 1 & 0 & -n-1 \\ 0 & 0 & 1 & 10 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0.9 & 0 & 0 & 0 \\ 0 & 0.9 & 0 & 0 \\ 0 & 0 & 0.9 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & -10 \\ 0 & 1 & 0 & n \\ 0 & 0 & 1 & -10 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Transformations

Working from the right:

$$\begin{pmatrix} 1 & 0 & 0 & 10 \\ 0 & 1 & 0 & -n-1 \\ 0 & 0 & 1 & 10 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0.9 & 0 & 0 & -9 \\ 0 & 0.9 & 0 & 0.9n \\ 0 & 0 & 0.9 & -9 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

which gives a single matrix:

$$\begin{pmatrix} 0.9 & 0 & 0 & 1 \\ 0 & 0.9 & 0 & -0.1n-1 \\ 0 & 0 & 0.9 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Transformations

d The scene is to be viewed from a moving viewpoint specified by its position **C** and a left handed viewing coordinate system $\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$.

At one point in the animation the view direction is $\boldsymbol{w} = (1, 0, 0)^T$, and the viewpoint is given by $\mathbf{C} = (20, -5, 10)$. Given that the view is in the horizontal plane ($\boldsymbol{v} = (0, 1, 0)^T$) find the value of $\boldsymbol{u}$.

Transformations

The reference coordinate system is assumed to be left-handed. The viewing coordinate system, $\{\, \mathbf{u}, \mathbf{v}, \mathbf{w} \,\}$, *is also left-handed so we have the cross-product identities:*

$$\mathbf{u} \times \mathbf{v} = \mathbf{w}$$
$$\mathbf{w} \times \mathbf{u} = \mathbf{v}$$
$$\mathbf{v} \times \mathbf{w} = \mathbf{u}$$

The third identity can be used to find the answer:

$$\mathbf{u} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$$

Transformations

e Hence, or otherwise, find the viewing transformation matrix.

The viewing matrix has the form

$$\begin{pmatrix} u_x & u_y & u_z & -\mathbf{C}\cdot\mathbf{u} \\ v_x & v_y & v_z & -\mathbf{C}\cdot\mathbf{v} \\ w_x & w_y & w_z & -\mathbf{C}\cdot\mathbf{w} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

u, **v** *and* **w** *are known from the previous question.*

$$-\mathbf{C}\cdot\mathbf{u} = -\begin{pmatrix} 20 \\ -5 \\ 10 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} = 10$$

$$-\mathbf{C}\cdot\mathbf{v} = -\begin{pmatrix} 20 \\ -5 \\ 10 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 5$$

$$-\mathbf{C}\cdot\mathbf{w} = -\begin{pmatrix} 20 \\ -5 \\ 10 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} = 20$$

Putting these together gives the viewing matrix as

$$\begin{pmatrix} 0 & 0 & -1 & 10 \\ 0 & 1 & 0 & 5 \\ -1 & 0 & 0 & 20 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$