Sorting

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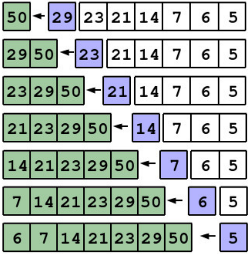


The Sorting Problem

- Sorting data is one of the most thoroughly explored algorithmic problems.

Problem *(Sort)*

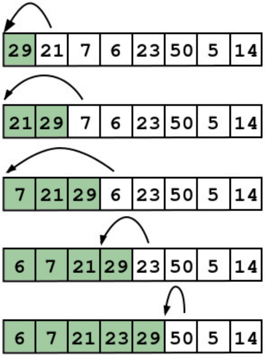
Input: a sequence A of values $\langle a_1, a_2, \ldots, a_N \rangle$

Output: a permutation (reordering) $\langle a'_1, a'_2 \ldots, a'_N \rangle$ of A such that $a'_1 \leq a'_2 \leq \cdots \leq a'_N$

- There are many solutions.
- We shall be exploring several of the best known ones.
- Sorting is an important problem. It is part of the solution to many other problems.
- Understanding the complexity of sorting algorithms helps design good solutions to these other problems.

Insertion Sort

- The Insertion Sort algorithm divides A into a sorted part, initially just $\langle a_1 \rangle$, and the remaining unsorted part
- Elements from the unsorted part are then inserted into the sorted part



Insertion Sort

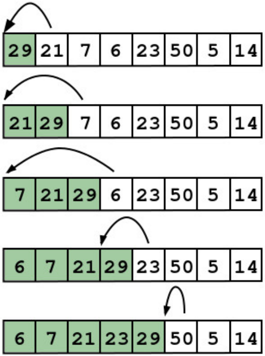
Insertion Sort(Input: sequence $A = \langle a_1, \ldots, a_N \rangle$)

- For each element a_i in $\langle a_2, \ldots, a_N \rangle$
    - $j = i$
    - While $j > 1$ and $a_j < a_{j-1}$
        - Exchange a_j and a_{j-1}
        - Decrement j
- HALT

- Insertion Sort is a so-called comparison sort algorithm
- To insert a_i in the correct position it must be compared with elements in the sorted part of the list
- Is Insertion Sort correct?

Insertion Sort

- The subsequence $\langle a_1, \ldots, a_{i-1} \rangle$ is sorted initially
- An insertion maintains this invariant
- Formalising this argument is the business of reasoning about programs
- Invariants are a tool we will use in designing correct algorithms



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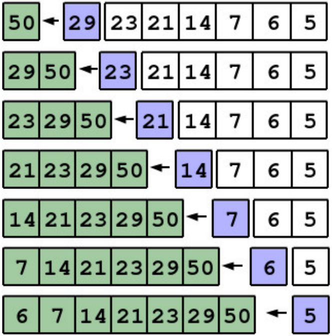
- What is the worst case input?
- What is the best case input?
- What is the time complexity in the best and worst cases?

Worst Case

- Running time of Insertion Sort has two dimensions:
  - Number of insertions
  - 'Size' of insertion



Insertion Sort: Java

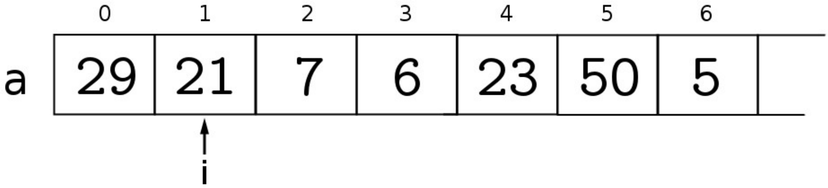
```java
public static void sort(int[] a) {
    int l = a.length;
    int j, t;

    for (int i = 1; i < l; i++) {
        t = a[i];
        j = i - 1;
        while(j >= 0 && t < a[j]) {
            a[j + 1] = a[j];
            j--;
        }
        a[j + 1] = t;
    }
}
```

- The sorted part of the array is everything before `a[i]`
- Swaps are not fully implemented
- The element being inserted is saved into temp variable `t`
- Sorted elements greater than `t` are each shuffled to the right

Insertion Sort

	0	1	2	3	4	5	6
a	29	21	7	6	23	50	5

Insertion Sort



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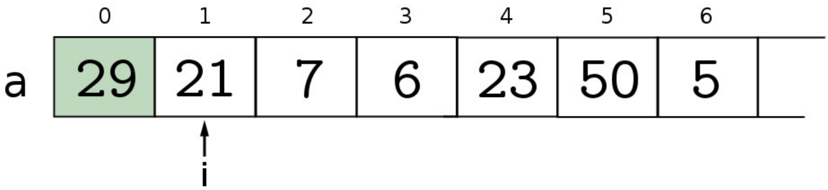

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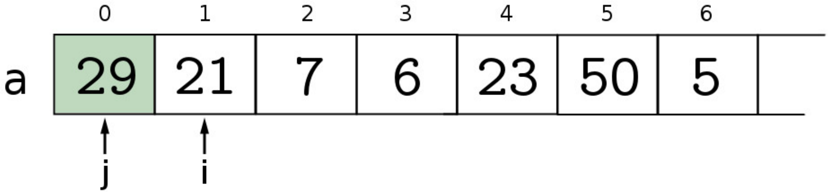

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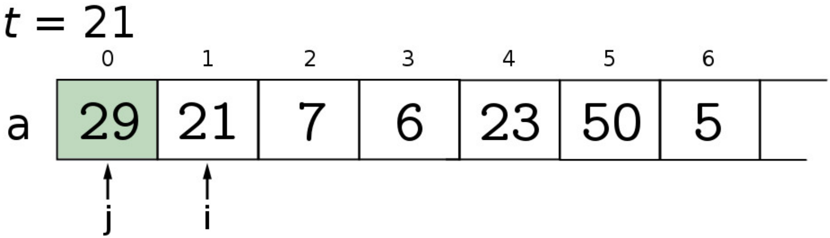

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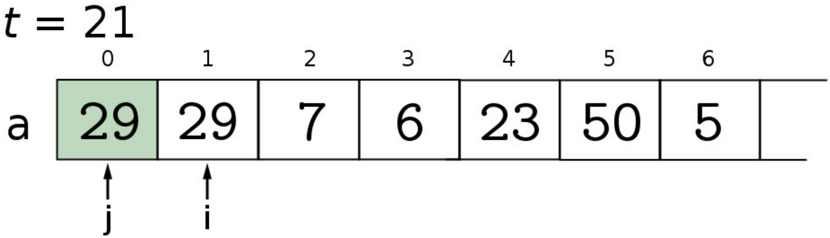

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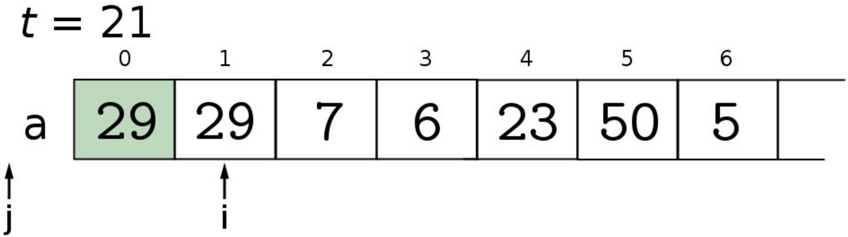

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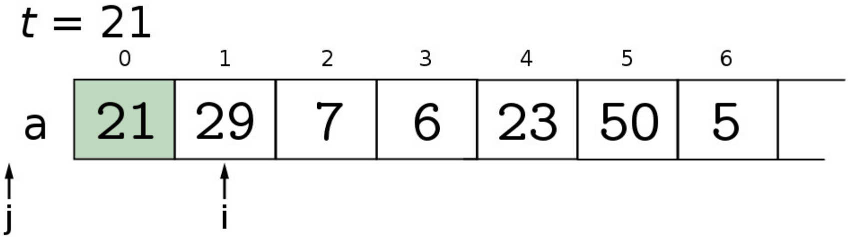

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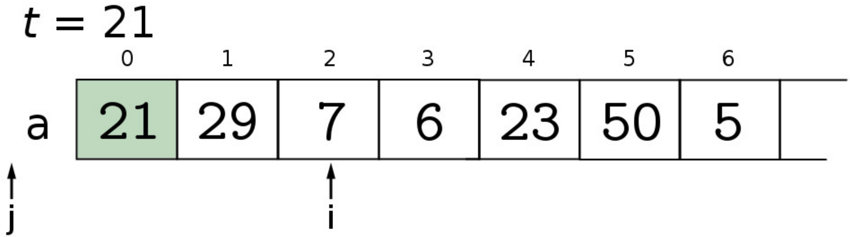

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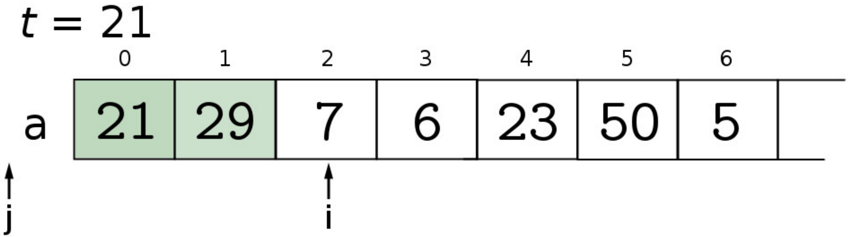

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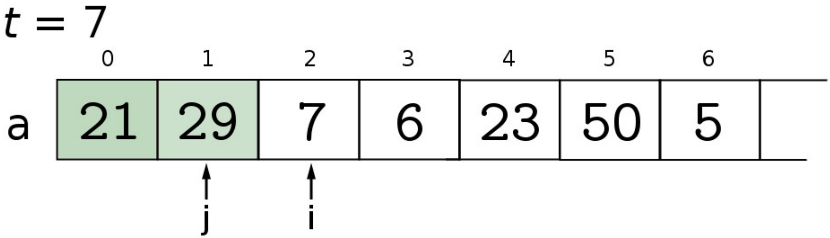
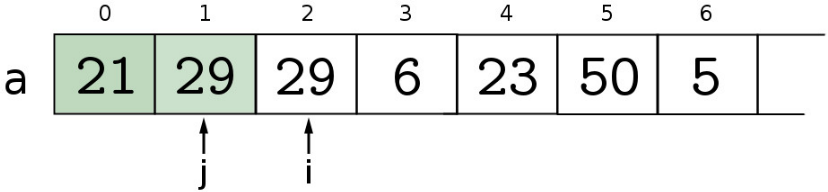


Insertion Sort

$t = 7$



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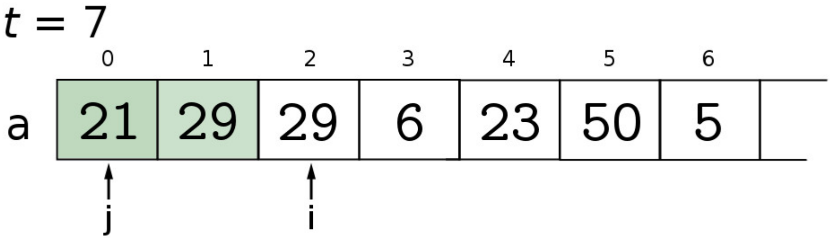

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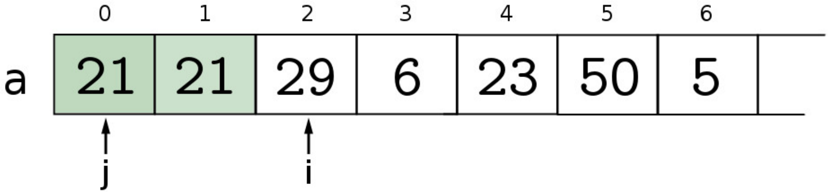

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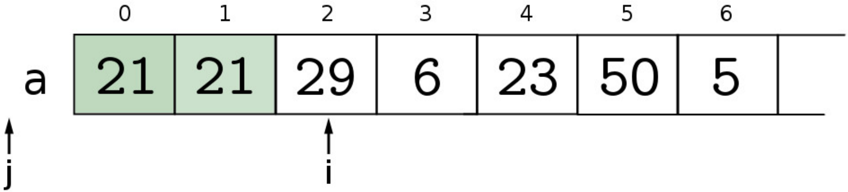

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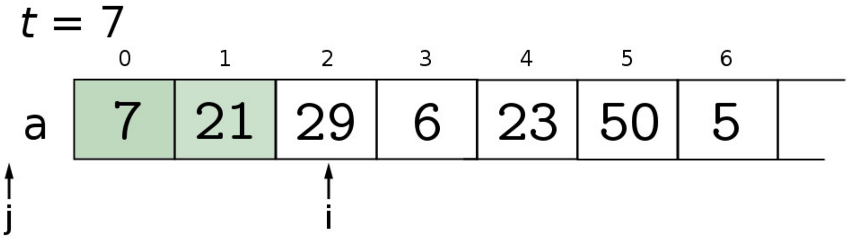

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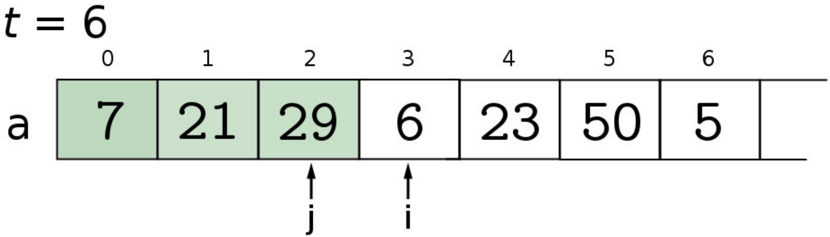

Insertion Sort

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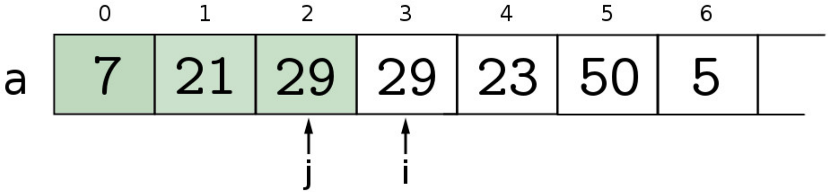

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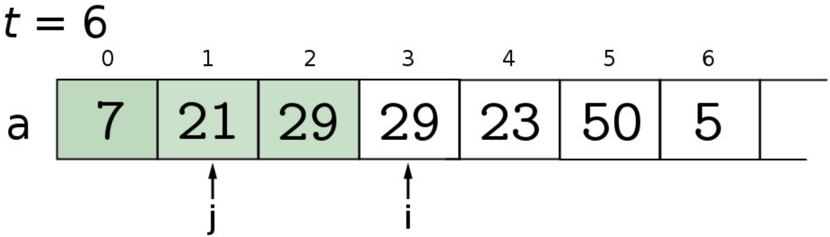

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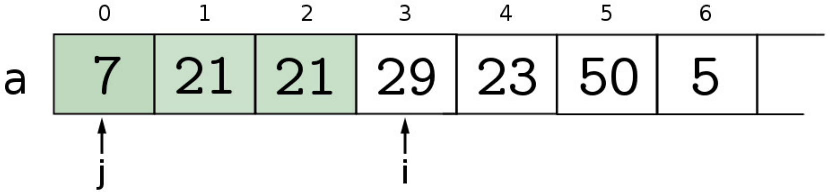

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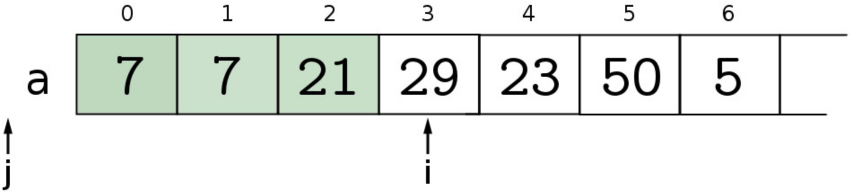

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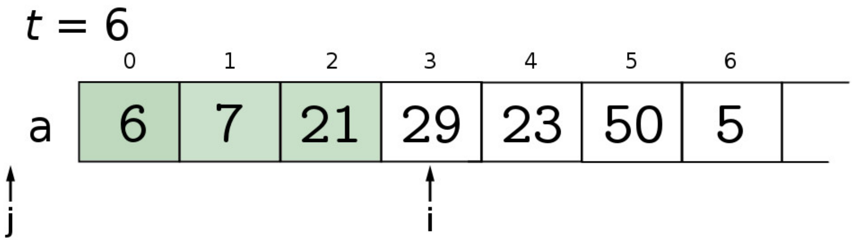

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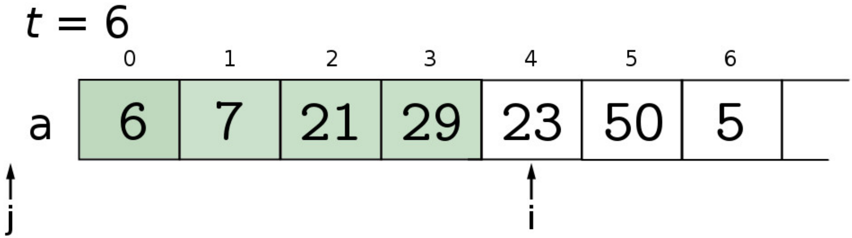

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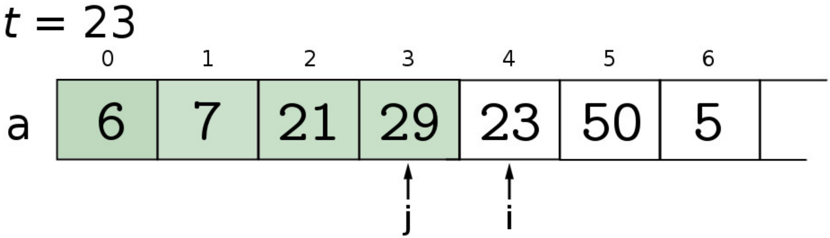

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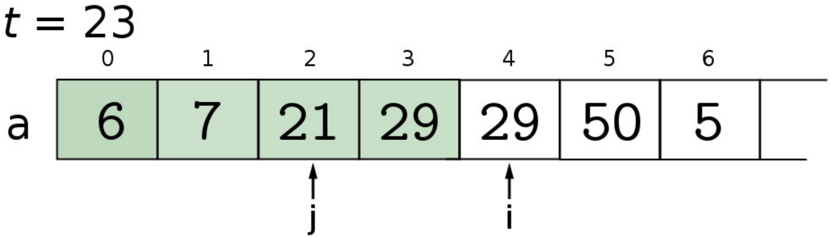
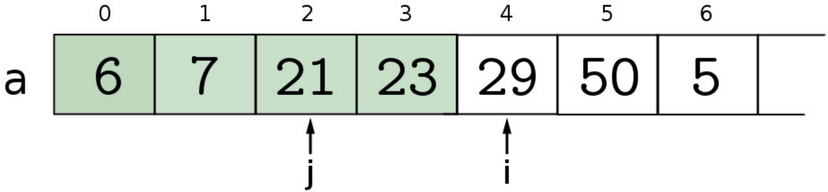


Insertion Sort

$t = 23$



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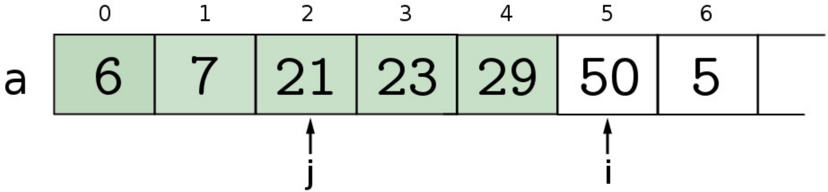

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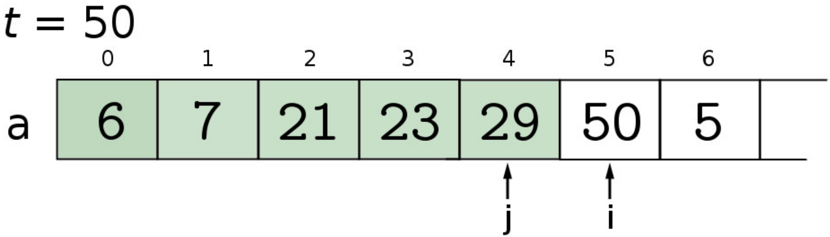

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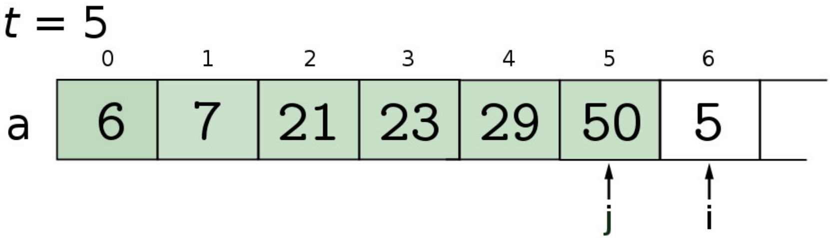

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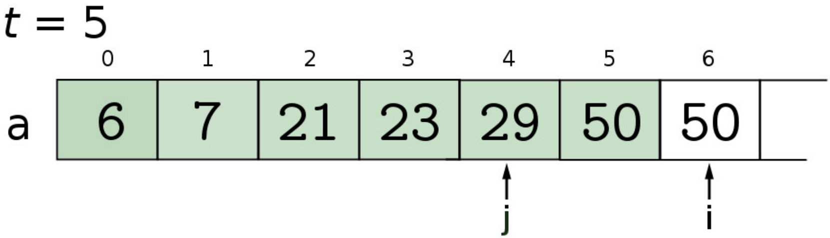

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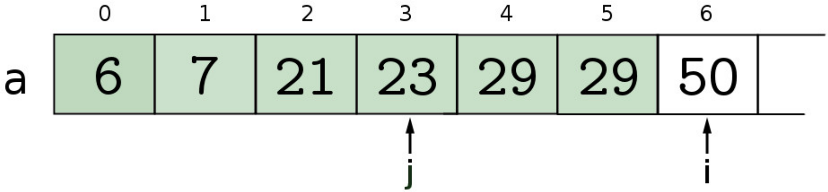

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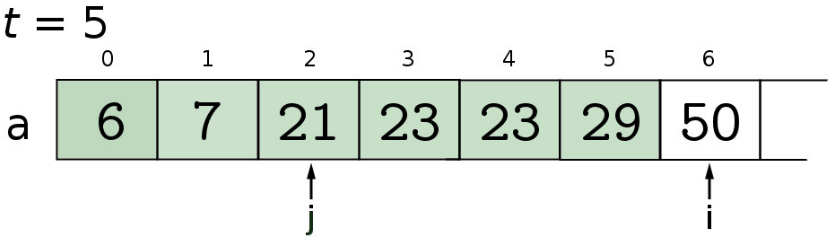

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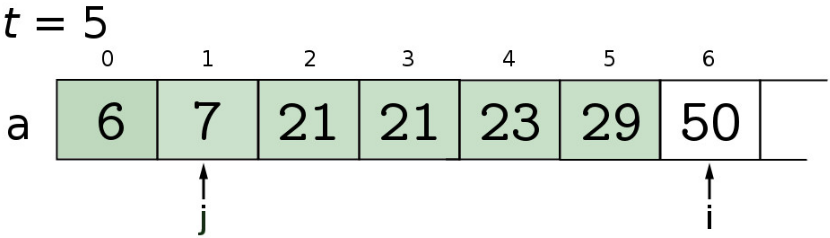

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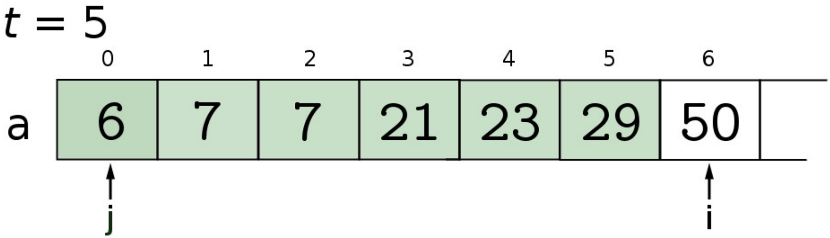

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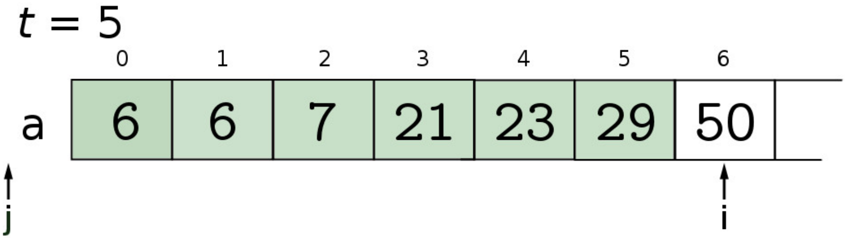

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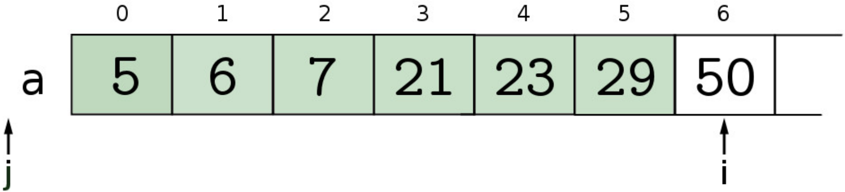

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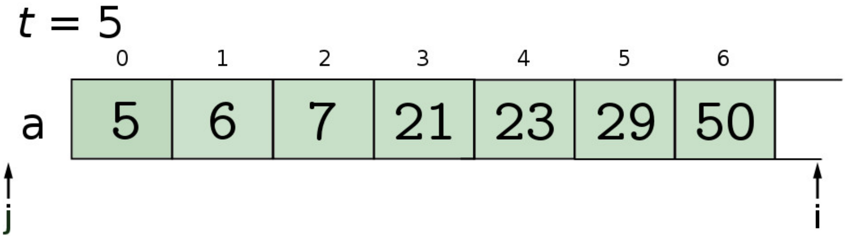

Properties of Insertion Sort

- Sorts the array in place – do not need to create a new array to hold the answer
- Best case input is already sorted array ($\Theta(N)$)
- Worst case input is reverse sorted array ($\Theta(N^2)$)
- Time complexity in general is $O(N^2)$
- What about the 'average' case?

Merge Sort

Will a divide and conquer approach work?

- Combining (merging) already sorted lists is fast ($\Theta(N)$).
- The list $\langle a \rangle$ is already sorted.
- So, we can sort $A = \langle a_1, \ldots, a_N \rangle$ by repeated merging of sublists.

29	21	7	6	23	50	5	14

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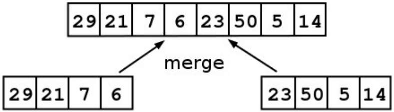

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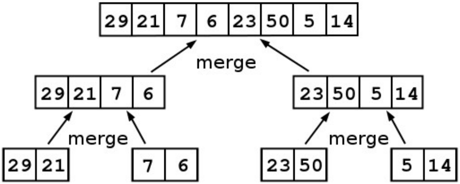

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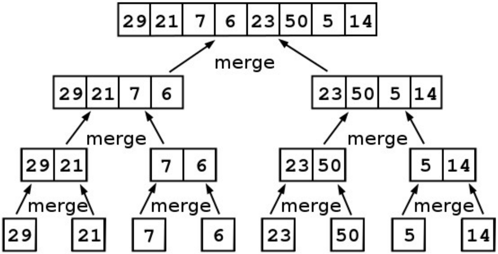

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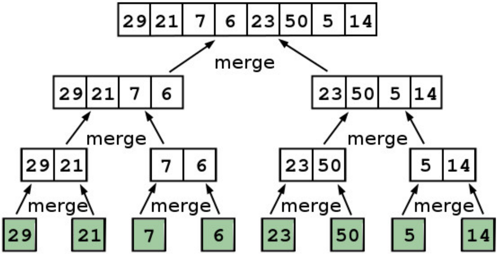

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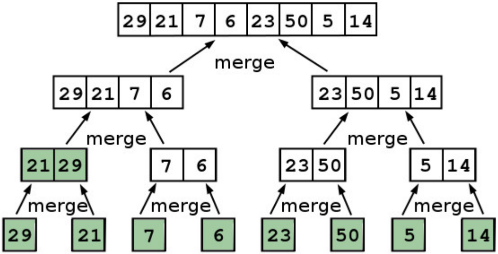

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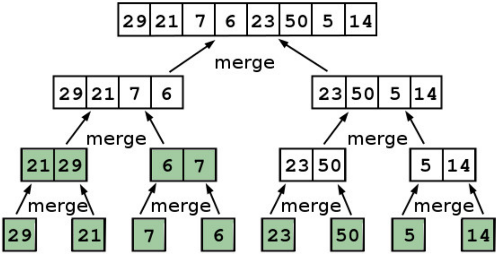

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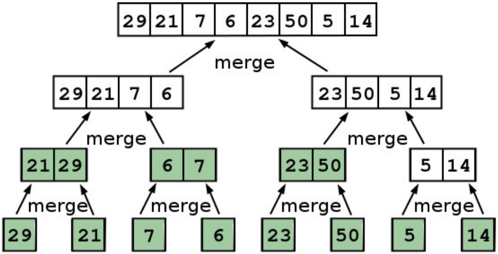

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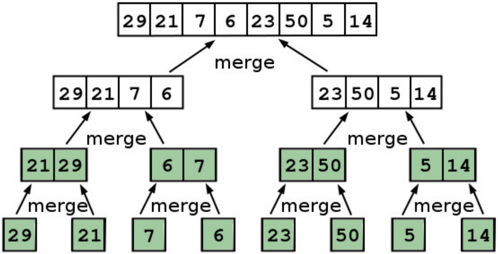

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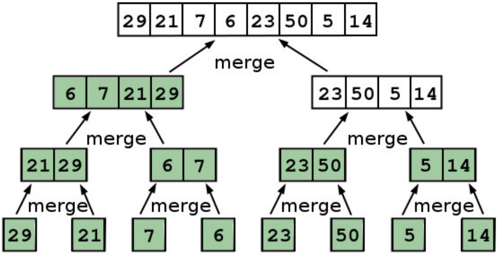

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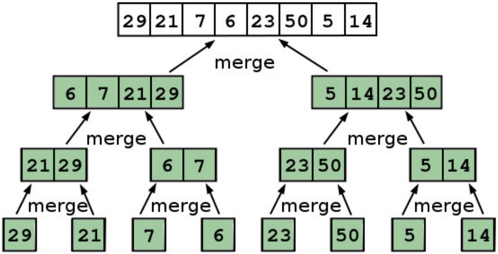

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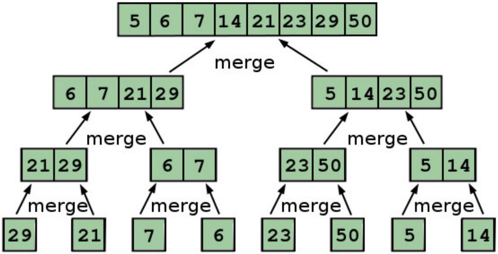


- What are the 'dimensions' of Merge Sort?

Merge Sort

Merge Sort (Input: sequence $A = \langle a_1, \ldots, a_N \rangle$, where $N \geq 1$)

- If A has one element
  - HALT
- Otherwise
  - Merge Sort $\langle a_1, \ldots, a_{\lfloor N/2 \rfloor} \rangle$
  - Merge Sort $\langle a_{\lfloor N/2 \rfloor + 1}, \ldots, a_N \rangle$
  - Merge $\langle a'_1, \ldots, a'_{\lfloor N/2 \rfloor} \rangle$ and $\langle a'_{\lfloor N/2 \rfloor + 1}, \ldots, a'_N \rangle$
  - HALT

- The sorting appears to be happening in place, but the list is copied during `Merge`

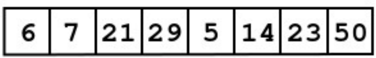
Merge Sort

- The Merge procedure takes two sublists of A and combines them
- The sublists $\langle a_L, \ldots, a_M \rangle$ and $\langle a_{M+1}, \ldots, a_N \rangle$ must be sorted
- The result is a globally sorted list $\langle a'_L, \ldots, a'_N \rangle$

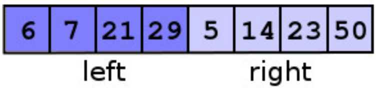
Merge (Input: sequence A, indices L, M and N, where $N > M \geq L$)

- Copy $\langle a_L, \ldots, a_M \rangle$ to a new list P and $\langle a_{M+1}, \ldots, a_N \rangle$ to a new list Q
- Copy the elements of P and Q back to form $\langle a'_L, \ldots, a'_N \rangle$ as follows:
    - p is the smallest element of P not yet used
    - q is the smallest element of Q not yet used
    - the next element of $\langle a'_L, \ldots, a'_N \rangle$ is the smaller of p and q

Merging Sublists



Merging Sublists



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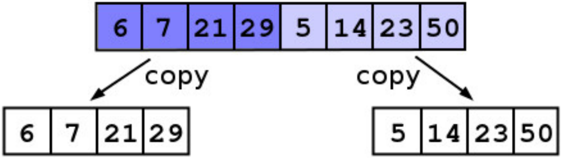

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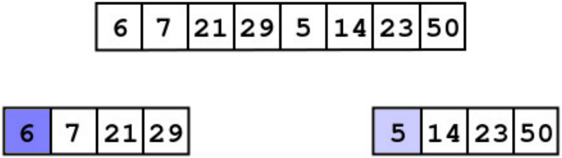

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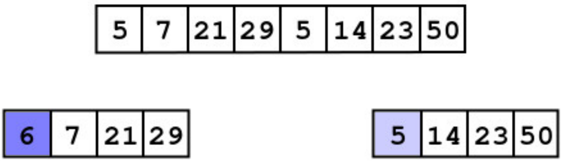

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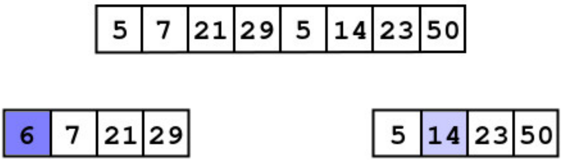

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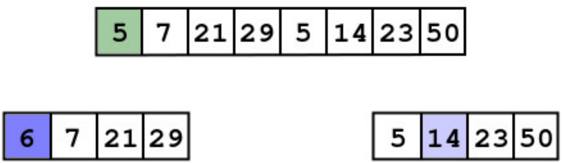

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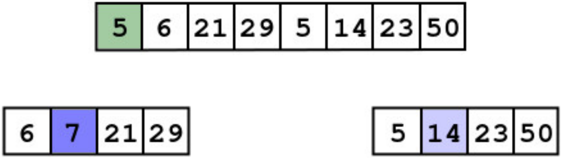

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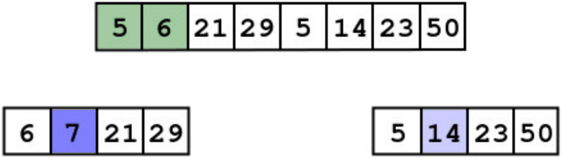

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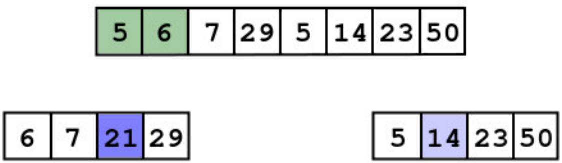

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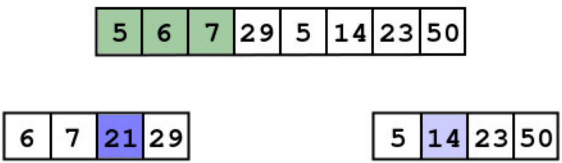

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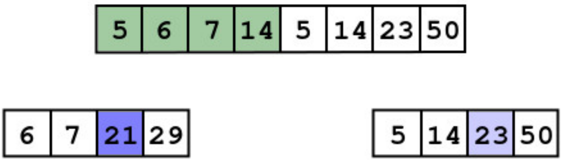

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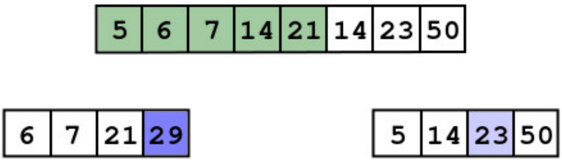

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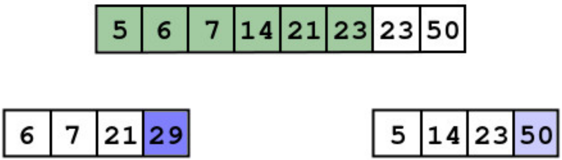
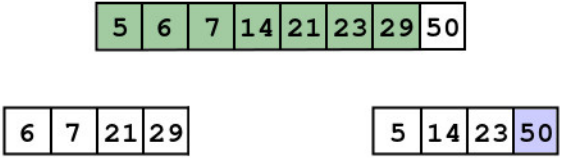


Merging Sublists



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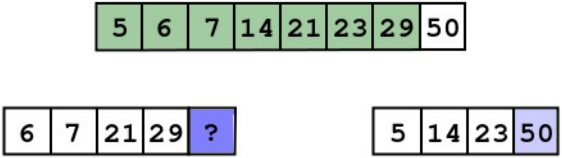
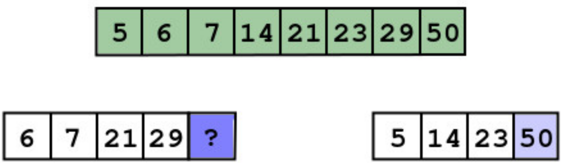


- We can add a sentinel value to the copied lists
- This avoids the need for extra conditions at the end of the lists
- What should the sentinel be?

Merging Sublists



- We can add a sentinel value to the copied lists
- This avoids the need for extra conditions at the end of the lists
- What should the sentinel be?

Merge Sort: Java

```
public static void sort(int[] a) {
  int l = a.length;
  sort(a, 0, l);
}

private static void sort(int[] a, int from, int to) {

  if (to - from <= 1) return;

  int mid = (from + to) / 2;
  sort(a, from, mid);
  sort(a, mid, to);
  merge(a, from, mid, to);
}
```

- The recursive method is called from a public ‘interface’ method
- The recursive method can sort any sub-array
- A sub-array including `a[i]...a[j]` is represented by `(a,i,j + 1)`

Merge Sort: Java

```java
private static void merge(int[] a, int from, int mid, int to) {

    int[]  left = copyOf(a, from, mid);
    int[] right = copyOf(a, mid, to);

    int i = 0, j = 0, k = from;
    while (k < to) {
        if (left[i] < right[j]) {
            a[k++] = left[i++];
        } else {
            a[k++] = right[j++];
        }
    }
}

private static int[] copyOf(int[] a, int from, int to) {
    int    len = to - from;              // length of range copied
    int[] copy = new int[len + 1];       // new array with sentinel
    int      i = 0;                      // index within copy

    for (int j = from; j < to; j++) {
        copy[i++] = a[j];
    }
    copy[len] = Integer.MAX_VALUE;
    return copy;
}
```

Merge Sort: Analysis

Generalised divide and conquer scheme:

$$T(N) = \begin{cases} \Theta(1) & \text{, if } N \leq c \\ aT(N/b) + D(N) + C(N) & \text{, otherwise} \end{cases}$$

where

- c is a small value of N (e.g. 0 or 1) corresponding to a base case that runs in constant time
- a is the number of subproblems
- N/b is the 'size' of each subproblem
- $D(N)$ is the cost of dividing up the original problem
- $C(N)$ is the cost of combining the subproblem solutions

Merge Sort: Analysis

Applying this scheme to Merge Sort:

- Use simplifying assumption that N is a power of 2
- (For $N > 1$) we get two subproblems of size $N/2$, so:

$$T(N) = \begin{cases} \Theta(1) & \text{, if } N \leq 1 \\ 2T(N/2) + D(N) + C(N) & \text{, otherwise} \end{cases}$$

- What are $D(N)$ and $C(N)$?

Merge Sort: Analysis

Applying this scheme to Merge Sort:

- Use simplifying assumption that N is a power of 2
- (For $N > 1$) we get two subproblems of size $N/2$, so:

$$T(N) = \begin{cases} \Theta(1) & \text{, if } N \leq 1 \\ 2T(N/2) + D(N) + C(N) & \text{, otherwise} \end{cases}$$

- What are $D(N)$ and $C(N)$?

$$T(N) = \begin{cases} \Theta(1) & \text{, if } N \leq 1 \\ 2T(N/2) + \Theta(1) + \Theta(N) & \text{, otherwise} \end{cases}$$

Evaluating Equations with Asymptotic Notation

In

$$T(N) = \begin{cases} \Theta(1) & \text{, if } N \leq 1 \\ 2T(N/2) + \Theta(1) + \Theta(N) & \text{, otherwise} \end{cases}$$

- $\Theta(1)$ is some function $f(N) = \Theta(1)$
- $\Theta(N)$ is some function $g(N) = \Theta(N)$
- $f(N) + g(N)$ will also be a linear function of N, for all f, g, so

$$T(N) = \begin{cases} \Theta(1) & \text{, if } N \leq 1 \\ 2T(N/2) + \Theta(N) & \text{, otherwise} \end{cases}$$

- Likewise, if $T(N/2)$ is
  - $\Theta(N^2)$ then $T(N)$ is $\Theta(N^2)$
  - $\Theta(1)$ then $T(N)$ is $\Theta(N)$

Merge Sort: Analysis

Let $T(1) = d$ and cost of merge per element be c

$$T(N) = \begin{cases} d & \text{, if } N \leq 1 \\ 2T(N/2) + cN & \text{, otherwise} \end{cases}$$

Substituting in the same way we saw for binary search:

$$\begin{aligned} T(N) &= cN + 2T(N/2) && (1) \\ &= cN + 2(cN/2 + 2T(N/4)) && (2) \\ &= cN + cN + 4T(N/4) && (3) \\ &= cN + cN + cN + 8T(N/8) && (4) \\ &= cN \times \log_2(N) + dN && (5) \end{aligned}$$

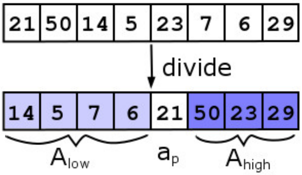
So, $T(N) = \Theta(N \log_2 N)$

Properties of Merge Sort

- Uses (extra) memory proportional to N
- Time complexity is $\Theta(N \log_2 N)$
- Faster than Insertion Sort for large, unsorted lists
- Slower than Insertion Sort if the list is already sorted
- Slower than Insertion Sort for small N

Quick Sort

- Merge Sort works bottom up – all the action happens when recombining the solutions to subproblems
- Quicksort is a top down divide and conquer sorting algorithm
- The input list is divided into three parts: A_{low}, $\langle a_p \rangle$, A_{high}
- a_p is called the pivot and the division ensures that $\forall a \in A_{low}(a < a_p)$ and $\forall a \in A_{high}(a \geq a_p)$



- We are left with the subproblems of sorting A_{low} and A_{high}

Quicksort

Quicksort (Input: sequence $A = \langle a_1, \ldots, a_N \rangle$)

- If A has more than one element
    - $p =$ Partition A
    - Quicksort $\langle a_1, \ldots, a_{p-1} \rangle$
    - Quicksort $\langle a_{p+1}, \ldots, a_N \rangle$
    - HALT
- Otherwise
    - HALT

- The Quicksort divide step is called partitioning
- The Partition procedure returns the final index of the pivot
- The base case must include $\langle\rangle$ since p might be 1 or N
- Subproblem solutions do not need to be combined

Quicksort

- The Partition procedure maintains two sublists which grow
- The sublists can both grow left-to-right (as below), or one left-to-right and one right-to-left

Partition (Given sequence $A = \langle a_1, \ldots, a_N \rangle$)

- Choose a pivot element $p \in A$
- Exchange p with a_N
- Let $storeIndex = 1$
- For each a in $\langle a_1, \ldots, a_{N-1} \rangle$
  - If $a < p$: exchange a with $a_{storeIndex}$ and increment $storeIndex$
- Exchange a_N with $a_{storeIndex}$
- Return $storeIndex$

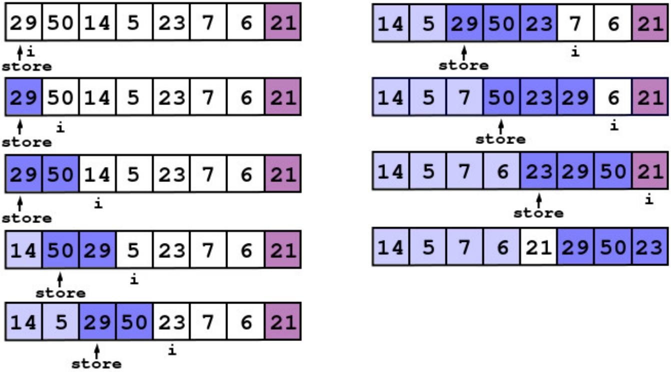
Quicksort in Java

```java
private static void sort(int[] a, int from, int to) {
    if (to - from > 1) {
        int pivot = partition(a, from, to);
        sort(a, from, pivot);
        sort(a, pivot + 1, to);
    }
}

public static int partition(int[] a, int from, int to) {
    int pivot = to - 1;
    int store = from;
    for (int i = from; i < pivot; i++) {
        if (a[i] < a[pivot]) { swap(a, i, store++); }
    }
    swap(a, pivot, store);
    return store;
}
```

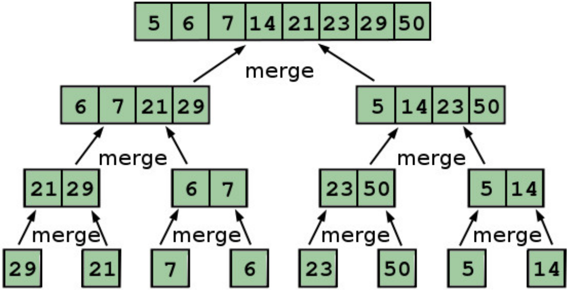
- Sub-array representation as for Mergesort
- Within `partition`, the final element (`a[to - 1]`) is the pivot
- Elements before `a[store]` are less than the pivot
- Elements between `a[store]` and `a[i]` are not less than the pivot

The partition method



Quicksort Performance

- The ‘dimensions’ of Quicksort are much like those of Merge Sort
- With balanced partitioning there will be a similar tree of subproblems

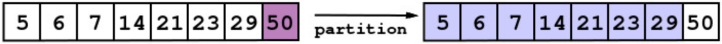


- How will partition behave? What cases are there?

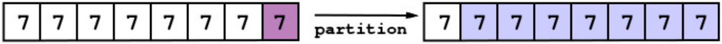
Quicksort Performance

Using our Java `partition` method:

- will only remove one element from sorted or reverse-sorted data



- will only remove one element from data with many duplicates



These types of input will produce $O(N^2)$ performance

Quicksort Performance

With non-worst-case inputs Quicksort usually outperforms Merge Sort

- The constant factors are smaller
- Quicksort common choice for library sort functions

Strategies for avoiding $O(N^2)$ performance include:

- Choose the pivot at random
- Choose the pivot as the median of three random elements [Sedgewick]
  - Both give expected $\Theta(N \log_2 N)$ performance
  - The worst case is still possible, but unlikely
- Modify `partition` to generate three partitions containing elements that are strictly less than, equal to and greater than the pivot