Predicate Logic

Fariba Sadri

Example: MSc regulations

pe:	pass exams
pc:	pass courseworks
pp:	pass projects
re:	retake exams
ce:	cheat in exams

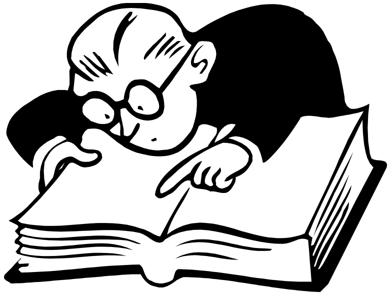
In propositional logic:

$$\mathbf{pe \land pc \land pp \rightarrow pm}$$

$$\mathbf{(\neg pc \lor ce) \rightarrow (\neg\, pm \land \neg\, re)}$$

Not expressive enough if we want to consider individual students, to check who has passed the MSc, and who has not, for example.

Example



John:

passes the coursework

cheats in exams

Mary:

passes the coursework

passes exams

passes projects

Who passes the MSc?

Increase the expressive power of the formal language by adding

- predicates
- variables
- quantification.

E.g.

For all individuals X:

$$pe(X) \wedge pc(X) \wedge pp(X) \rightarrow pm(X)$$

For all individuals X:

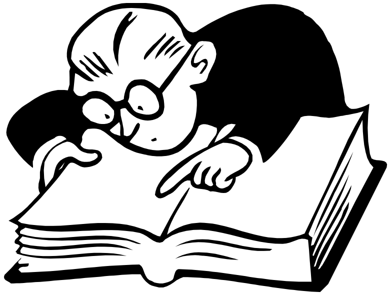
$$(\neg pc(X) \vee ce(X)) \rightarrow (\neg pm(X) \wedge \neg re(X))$$

Now given:

pc(john) **pc(mary)**

ce(john) **pe(mary)**

pp(mary)



We can conclude:

¬ pm(john) **pm(mary)**

¬ re(john)

More formal expression of the MSc regulations

$$\forall X\ (pe(X) \wedge pc(X) \wedge pp(X) \rightarrow pm(X))$$

$$\forall X((\neg pc(X) \vee ce(X)) \rightarrow (\neg\ pm(X) \wedge \neg\ re(X)))$$

$\forall$: Universal Quantifier

Another example

Every student has a tutor.

for all X

(if X is a student then

there is a Y such that Y is tutor of X)

$\forall X\ (student(X) \rightarrow \exists\ Y\ tutor(Y,X))$

$\exists$: **Existential Quantifier**

The Predicate Logic Language Alphabet:

- **Logical connectives (same as propositional logic):** $\wedge \vee \ \neg \ \rightarrow \ \leftrightarrow$
- **Predicate symbols (as opposed to propositional symbols):a set of symbols each with an associated arity>=0.**
- **A set of constant symbols.**

  E.g. mary, john, 101, 10a, peter_jones
- **Quantifiers** $\forall \ \exists$
- **A set of variable symbols. E.g. X, Y, X1, YZ.**

Arity

In the previous examples:

Predicate Symbol	Arity
student	1
tutor	2
pm	1
pp	1

A predicate symbol with

arity = 0 is called a **nullary predicate**,

arity = 1 is called a **unary predicate,**

arity = 2 is called a **binary predicate.**

A predicate symbol with arity=n (usually n>2) is called an **n-ary** predicate.

Definition:

A **Term** is any constant or variable symbol.

Syntax of a grammatically correct sentence (wff) in predicate logic

- **p(t1,..., tn) is a wff if p is an n-ary predicate symbol and the ti are terms.**
- **If W, W1, and W2 are wffs then so are the following:**

  $\neg W$ $\quad$ $W1 \wedge W2$ $\quad$ $W1 \vee W2$

  $W1 \rightarrow W2$ $\quad$ $W1 \leftrightarrow W2$

  $\forall X(W)$ $\quad$ $\exists X(W)$

  where X is a variable symbol.
- **There are no other wffs.**

From the description above you can see that propositional logic is a special case of predicate logic.

Convention used in most places in these notes:

- Predicate and constant symbols start with lower case letters.
- Variable symbols start with upper case letters.

Examples

The following are wffs:

1. $\neg$ **married(john)**

2. **$\forall$X($\neg$ married(X) $\rightarrow$ single(X) $\vee$ divorced(X) $\vee$ widowed(X))**

3. **$\exists$X (bird(X) $\wedge$ $\neg$ fly(X))**

The following are not wffs:

4. $\neg$ X

5. single(X) $\rightarrow \forall$Y

6. $\forall\exists$X (bird(X) $\rightarrow$ feathered(X))

Exercise
which of the following are wffs?

1. $\forall X\ p(X)$
2. $\forall X\ p(Y)$
3. $\forall X\ \exists Y\ p(Y)$
4. $q(X,Y,Z)$
5. $p(a) \rightarrow \exists q(a,X,b)$
6. $p(a) \vee p(a,b)$

7. $\neg \neg \forall X\ r(X)$
8. $\exists X\ \exists Y\ p(X,Y)$
9. $\exists X,Y\ p(X,Y)$
10. $\forall X\ (\neg\exists Y)$
11. $\forall x\ (\neg\exists Y\ p(x,Y))$

Exercise

Formalise the following in predicate logic using the following predicates (with their more or less obvious meaning):

lecTheatre/1, office/1, contains/2, lecturer/1, has/2, same/2, phd/1, supervises/2, happy/1, completePhd/1.

1. 311 is a lecture theatre and 447 is an office.
2. Every lecture theatre contains a projector.
3. Every office contains a telephone and either a desktop or a laptop computer.
4. Every lecturer has at least one office.
5. No lecturer has more than one office.

6. No lecturers share offices with anyone.
7. Some lecturers supervise PhD students and some do not.
8. Each PhD student has an office, but all PhD students share their office with at least one other PhD student.

9. A lecturer is happy if the PhD students he/she supervises successfully complete their PhD.
10. Not all PhD students complete their PhD.

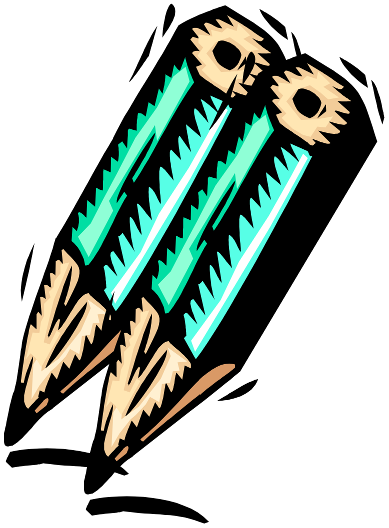
Note:

$\exists X\ p(X)$ states that there is **at least** one X such that p is true of X.

E.g. **$\exists X$ father(X, john)**

says John has **at least** one father (assuming *father(X,Y)* is to be read as X is father of Y).

Exercise



Assuming a predicate *same(X,Y)* that expresses that X and Y are the same individual, express the statement that John has exactly one father. You may also assume a binary predicate "father" as above.

Some useful equivalences

All propositional logic equivalences hold for predicate logic wffs.

E.g. $\neg(\mathbf{A} \wedge \mathbf{B}) \equiv \neg\, \mathbf{A} \vee \neg\, \mathbf{B}$

So

$\neg\, (\text{academic(john)} \wedge \text{rich(john)}) \equiv$

$\neg\, \text{academic(john)} \vee \neg\, \text{rich(john)}$

Another instance of the same equivalence:

$$\neg(A \wedge B) \equiv \neg A \vee \neg B$$

$$\neg(\forall X\,(\text{able\_to\_work}(X) \rightarrow \text{employed}(X)) \wedge \text{inflation}(\text{low})) \equiv$$

$$\neg\,(\forall X\,(\text{able\_to\_work}(X) \rightarrow \text{employed}(X))) \vee \neg\,\text{inflation}(\text{low})$$

Some other equivalences in predicate logic

- $\forall X p(X) \equiv \neg \exists X \neg p(X)$

  all true, none false

- $\forall X \neg p(X) \equiv \neg \exists X\, p(X)$

  all false - none true

- $\exists X p(X) \equiv \neg \forall X \neg p(X)$

  at least one true - not all false

- $\exists X \neg p(X) \equiv \neg \forall X\, p(X)$

  at least one false - not all true

Some other equivalences in predicate logic

Suppose W1, W2 are wffs.

If W1 can be transformed to W2 by a consistent renaming of variables, then W1 and W2 are equivalent.

E.g.

$\forall X\ p(X) \equiv \forall Y\ p(Y)$

Some other equivalences in predicate logic

$\forall X \ \exists Y \ (p(X,Y) \rightarrow q(Y, X)) \equiv$

$\forall Z \ \exists W \ (p(Z,W) \rightarrow q(W, Z)) \equiv$

$\forall Y \ \exists X \ (p(Y, X) \rightarrow q(X,Y))$

But

$\forall X \ \exists Y \ (likes(X,Y) \rightarrow likes(Y, X))$

$\forall Z \ \exists W \ (likes(Z,W) \rightarrow likes(Z, Z))$

are not equivalent.

Some other equivalences in predicate logic

If two wffs differ only in the order of two adjacent quantifiers of the same kind, then they are equivalent.

E.g.

$\forall X\ \forall Y\ p(X,Y) \equiv \forall Y\ \forall X\ p(X,Y)$

But

$\forall X\ \exists Y\ p(X,Y)$ **is not equivalent to**

$\exists Y\ \forall X\ p(X,Y)$

More Equivalences

$\forall X\,(A \wedge B) \equiv \forall X\, A \wedge \forall X\, B$

$\exists X(A \vee B) \equiv \exists XA \vee \exists XB$

Some notes on quantifiers

1. Free and Bound variables:

   An occurrence of a variable in a wff is bound if it is within the scope of a quantifier in that sentence. It is free if it is not within the scope of any quantifier in that wff.

$$\forall X\ (p(X) \rightarrow q(Y,X))$$

Both occurrences of X in the above sentence are bound (they are both within the scope of the $\forall$.)

The occurrence of Y is free (it is not within the scope of any quantifier.)

$$(\forall X\ p(X)) \wedge (\exists X q(X))$$

In the sentence above, both occurrences of X are bound, the first by the $\forall$, the second by the $\exists$.

$$(\forall X\ p(X)) \wedge (\exists Y q(X,Y))$$

In the sentence above, the first occurrence of X is bound, the second is free. The occurrence of Y is bound.

2. A particular occurrence of a variable is bound by the closest quantifier which can bind it.

E.g.

$$\forall X\ (p(X) \rightarrow \forall X\ q(X)) \equiv$$

$$\forall X\ (p(X) \rightarrow \forall Y\ q(Y))$$

3. Law of vacuous quantification

$\forall X\ W \equiv W$ if W (a wff) contains no free occurrences of X.

E.g.

$\forall X\ (p(a) \rightarrow q(a)) \equiv p(a) \rightarrow q(a)$

$\forall X\ \exists X\ p(X) \equiv \exists X\ p(X)$

$\forall X \forall X\ (p(X,X) \rightarrow q(X)) \equiv \forall X\ (p(X,X) \rightarrow q(X))$

More Equivalences

If x doesn't occur free in A, then

$\exists X(A \wedge B)$ is equivalent to $A \wedge \exists XB$, and

$\forall X(A \vee B)$ is equivalent to $A \vee \forall XB$.

If x does not occur free in A then

$\forall X(A \to B)$ is equivalent to $A \to \forall XB$, and

$\exists X(A \to B)$ is equivalent to $A \to \exists XB$.

More Equivalences

If x does not occur free in B then

$\forall X(A \rightarrow B)$ is equivalent to $\exists XA \rightarrow B$, and

$\exists X(A \rightarrow B)$ is equivalent to $\forall XA \rightarrow B$.

Be careful:

The quantifier changes.

Exercise

What about the following?

Are the pairs equivalent?

If not, what is the relationship between them?

$\forall X(A \rightarrow B)$ and $\forall XA \rightarrow \forall XB$

$\exists X(A \wedge B)$ and $\exists XA \wedge \exists XB$

$\forall XA \vee \forall XB$ and $\forall X (A \vee B)$

Warning: non-equivalences

The following are *NOT* logically equivalent (though always, the first $\models$ the second):

$\forall X(A \rightarrow B)$ and $\forall XA \rightarrow \forall XB$

$\exists X(A \wedge B)$ and $\exists XA \wedge \exists XB$

$\forall XA \vee \forall XB$ and $\forall X\,(A \vee B)$

Can you find a ‘counter-example’ for each one?

Counter-example for

$\forall X(p(X) \rightarrow q(X))$ and $\forall X p(X) \rightarrow \forall X q(X)$

Take

$p(a)$	$p(b)$	$\neg p(c)$
$q(a)$	$\neg q(b)$	

Then RHS is true, but LHS is not.

Definition.

If a wff contains no free occurrences of variables it is said to be **closed**, otherwise it is said to be **open**.

A wff with no free occurrences of variables is also called a **sentence**.