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- The idea is to map the feature vectors nonlinearly into another space and learn a linear classifier there.
- The linear classifier in this new space would be an appropriate nonlinear classifier in the original space.

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- Results in huge computational cost both for learning and and final operation of the classifier.
- We need to learn $O(m^p)$ parameters rather than $O(m)$ parameters. Hence may need much larger number of examples for achieving proper generalization.
- SVM offers an elegant solution to both.

Support Vector Machines

- Learning of **optimal** hyperplane.

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- We need solve only a quadratic optimization problem.
- Now we formulate the SVM method, first for linearly separable case.

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- To start with, assume training set is linearly separable. That is, exist $W \in \Re^m$ and $b \in \Re$ such that

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\begin{aligned}
W^T X_i + b &> 0, \quad \forall i \;\; s.t. \;\; y_i = +1 \\
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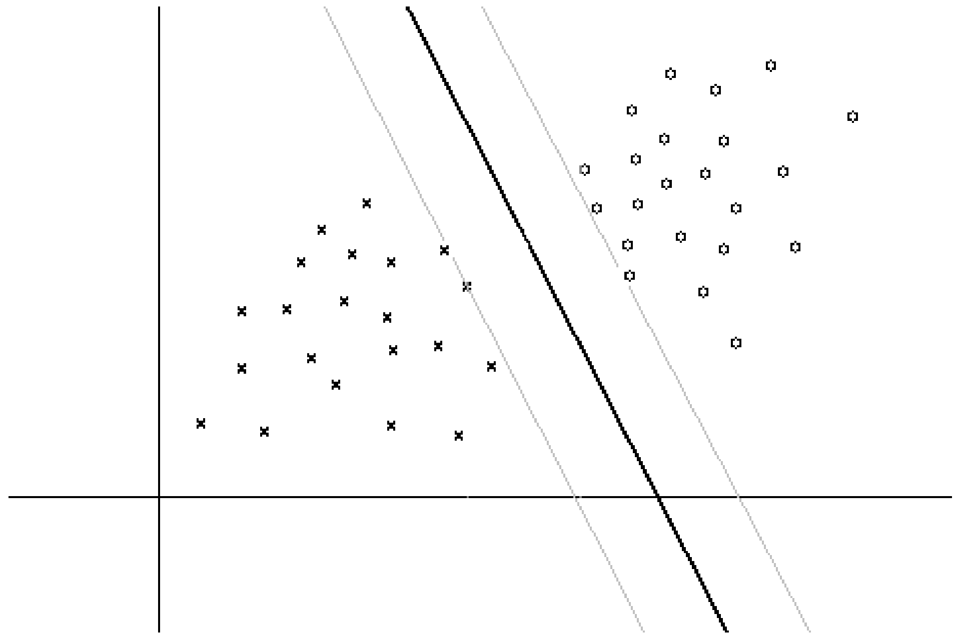
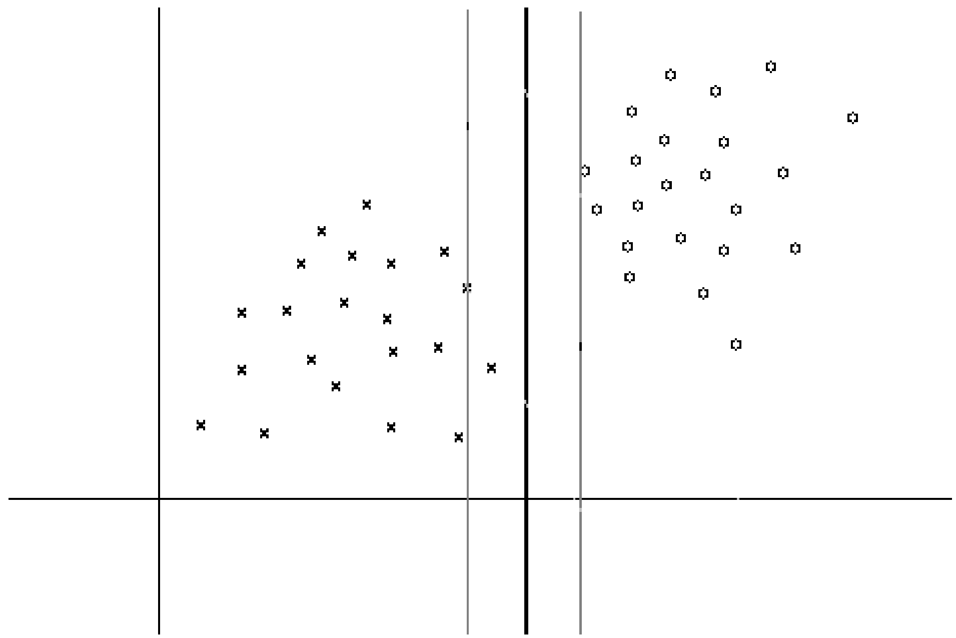
- Infinitely many separating hyperplanes exist.

A good separating hyperplane



Another separating hyperplane



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- Since the training set is finite, $\exists \epsilon > 0 \; s.t.$

$$
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W^T X_i + b &\geq \epsilon, \quad \forall i \;\; s.t. \;\; y_i = +1 \\
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- Hence, we can scale $W,\ b$ such that

$$\begin{aligned} W^T X_i + b &\geq +1 \text{ if } y_i = +1 \\ W^T X_i + b &\leq -1 \text{ if } y_i = -1 \end{aligned}$$

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or, equivalently

$$y_i(W^T X_i + b) \geq 1, \;\; \forall i.$$

(Recall that $y_i \in \{+1, \quad -1\}$)

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- Then there are no training patterns between the two parallel hyperplanes

$$W^T X + b = +1$$

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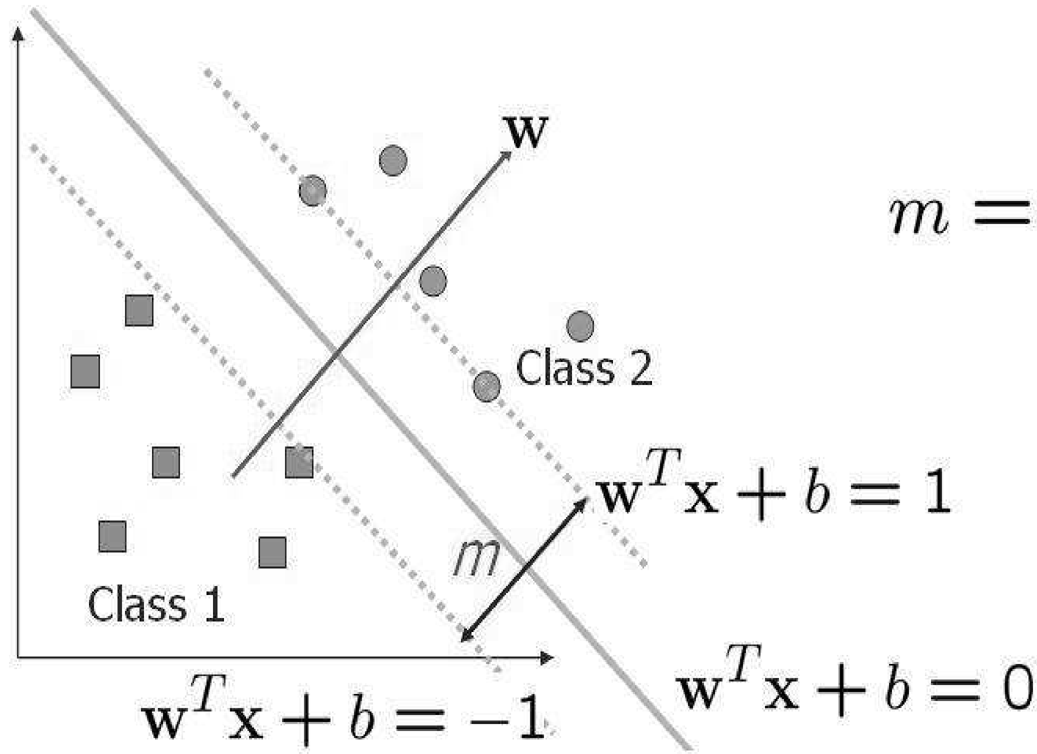
Optimal hyperplane

- Distance between these two hyperplanes is: $\frac{2}{||W||}$. Called **margin** of the separating hyperplane.

Margin of a hyperplane



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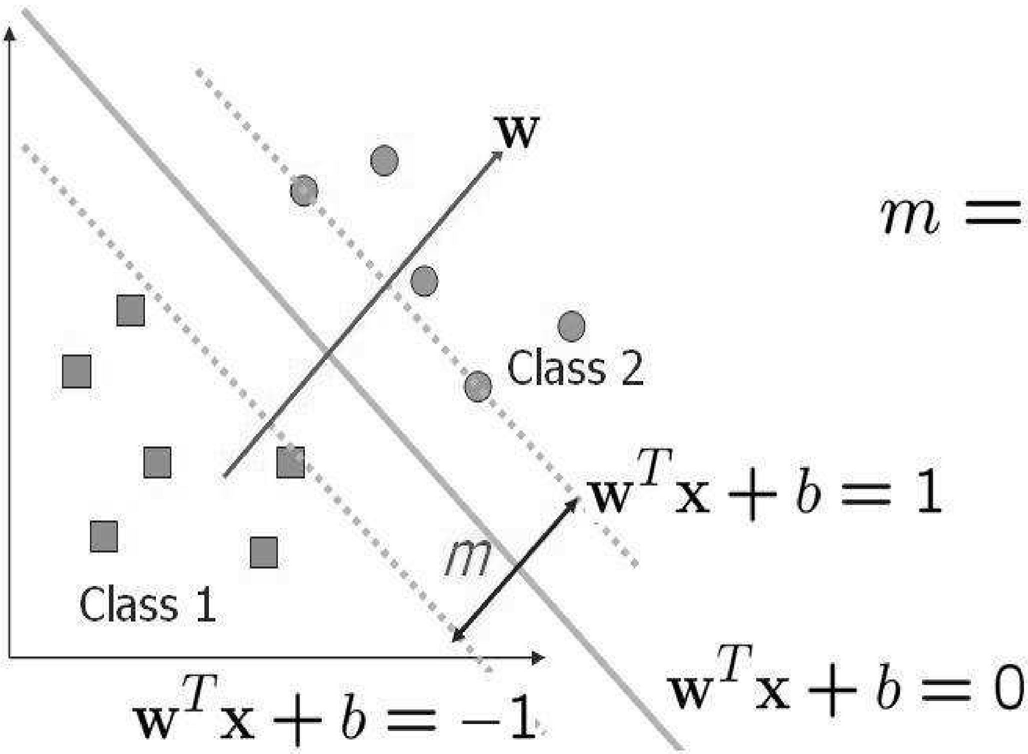

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- Hence distance between the hyperplane and the closest pattern is $\frac{1}{||W||}$.
- Intuitively, more the margin, better is the chance of correct classification of new patterns.
- **Optimal Hyperplane** – separating hyperplane with maximum margin.

The optimization problem

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- This is a constrained optimization problem with quadratic cost function and linear inequality constraints.

Constrained Optimization

- Consider the following optimization problem

$$\begin{array}{rl} \text{minimize} & f(\mathbf{x}) \\ \text{subject to} & \mathbf{a}_j^T \mathbf{x} + b_j \leq 0, \quad j = 1, \ldots, r \end{array}$$

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  $\mathbf{a}_j \in \Re^m$, $b_j \in \Re$, $j = 1, \cdots, r$.

- A point, $\mathbf{x} \in \Re^m$, is called a **feasible** point (for this problem) if $\mathbf{a}_j^T \mathbf{x} + b_j \leq 0$, $j = 1, \cdots, r$.

- Any $\mathbf{x}^* \in \Re^m$ is called a **local minimum** of the problem if $f(\mathbf{x}^*) \leq f(\mathbf{x})$ for all $\mathbf{x}$ that is feasible and is in a small neighbourhood of $\mathbf{x}^*$.

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- If $f(\mathbf{x}^*) \leq f(\mathbf{x}), \forall \mathbf{x} \in \Re^m$ and $\mathbf{x}$ feasible, then $\mathbf{x}^*$ is a **global minimum.**
- Unlike in unconstrained optimization, here we need to minimize only over the feasible set.

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- $f : \Re^m \longrightarrow \Re$ is said to be a convex function if for all $\mathbf{x}_1, \mathbf{x}_2 \in \Re^m$ and for all $\alpha \in (0,\ 1)$,

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- For example, $f(\mathbf{x}) = \mathbf{x}^T\mathbf{x}$ is a convex function.
- When f is convex, in our optimization problem, every local minimum is also a global minimum.

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- Essentially, the constrained optimization problem can be solved through unconstrained optimization of L.

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- These are the so called Kuhn-Tucker conditions for our optimization problem with convex cost function and linear constraints.

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- This is the approach taken in SVM algorithm.
- Duality is an important concept in optimization.
- Here we discuss only one way of formulating the dual which is useful when the objective function is convex and constraints are linear.

- Our optimization problem is

$$\begin{array}{rl} \text{minimize} & f(\mathbf{x}) \\ \text{subject to} & \mathbf{a}_j^T\mathbf{x} + b_j \leq 0, \quad j = 1, \ldots, r \end{array}$$

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- This is known as the **primal** problem.
- Here the optimization variables are $\mathbf{x} \in \Re^m$.

- Recall that the Lagrangian is

$$L(\mathbf{x}, \boldsymbol{\mu}) = f(\mathbf{x}) + \sum_{j=1}^{r} \mu_j(\mathbf{a}_j^T \mathbf{x} + b_j)$$

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- If for a particular $\boldsymbol{\mu}$, if the infimum is not attained then $q(\boldsymbol{\mu})$ would take value $-\infty$.

The Dual problem

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- There is a nice connection between the primal and dual problems.

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- We would be using the dual formulation for the optimization problem in SVM

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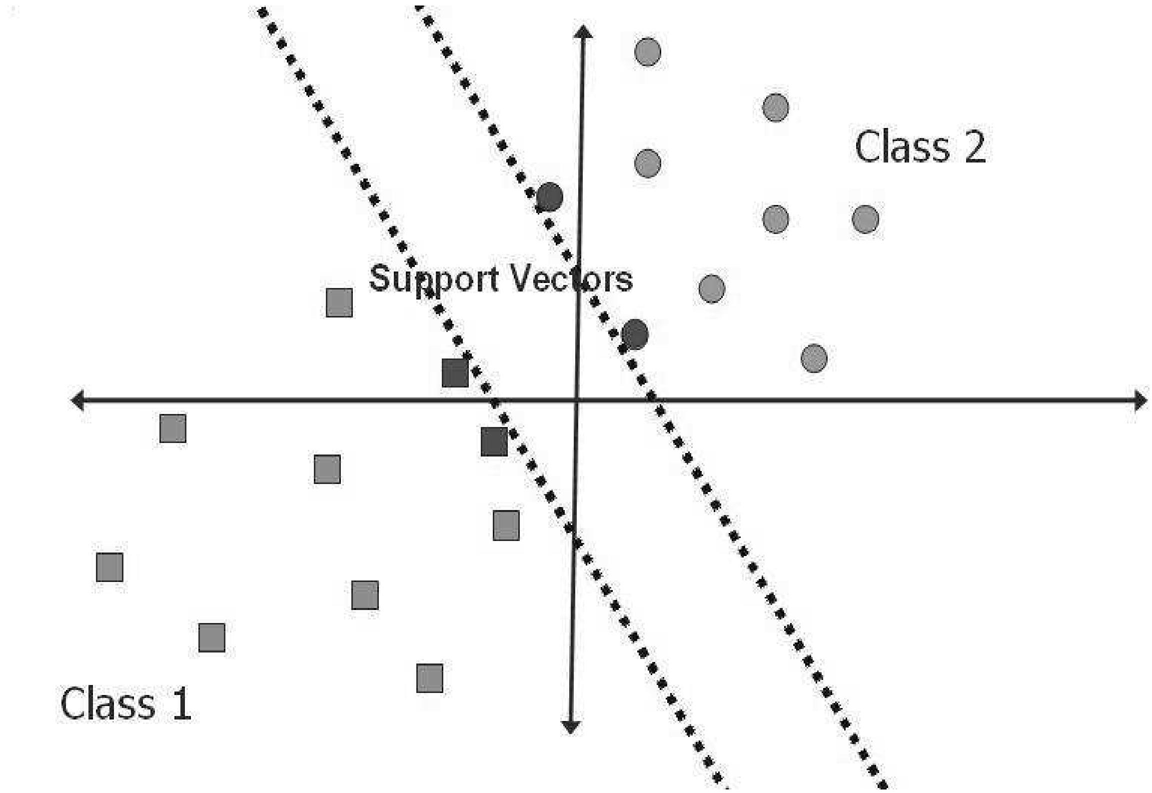
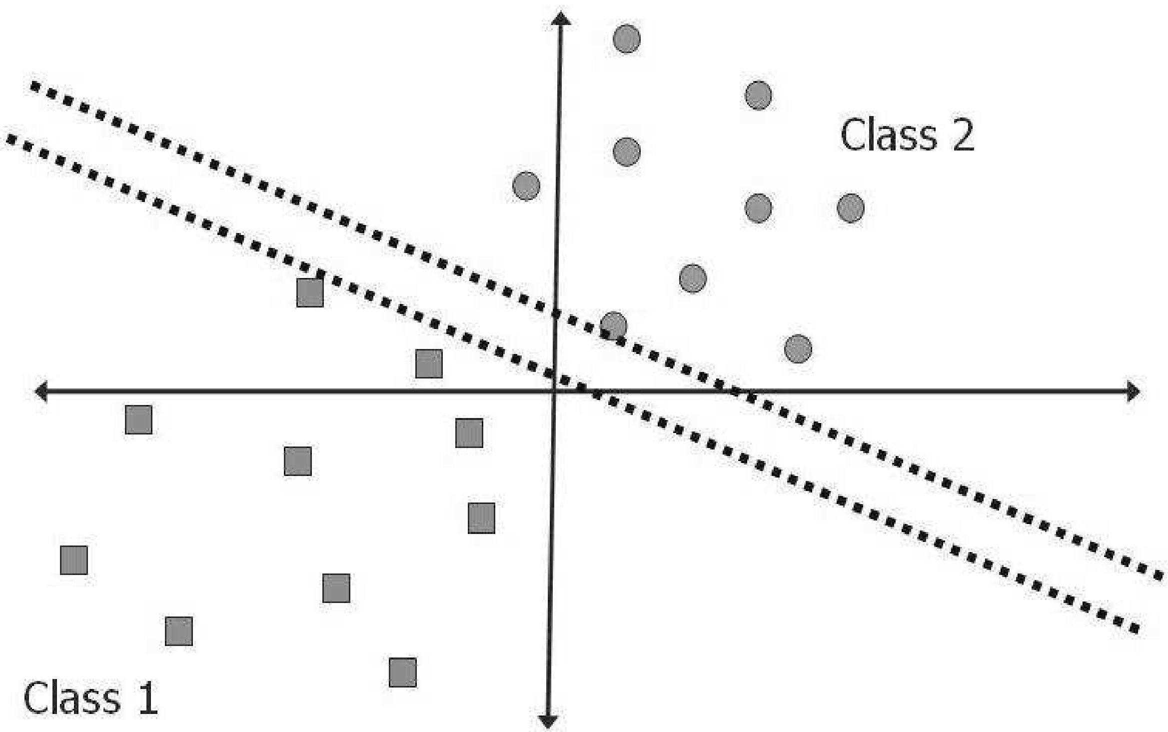


Non-optimal hyperplane



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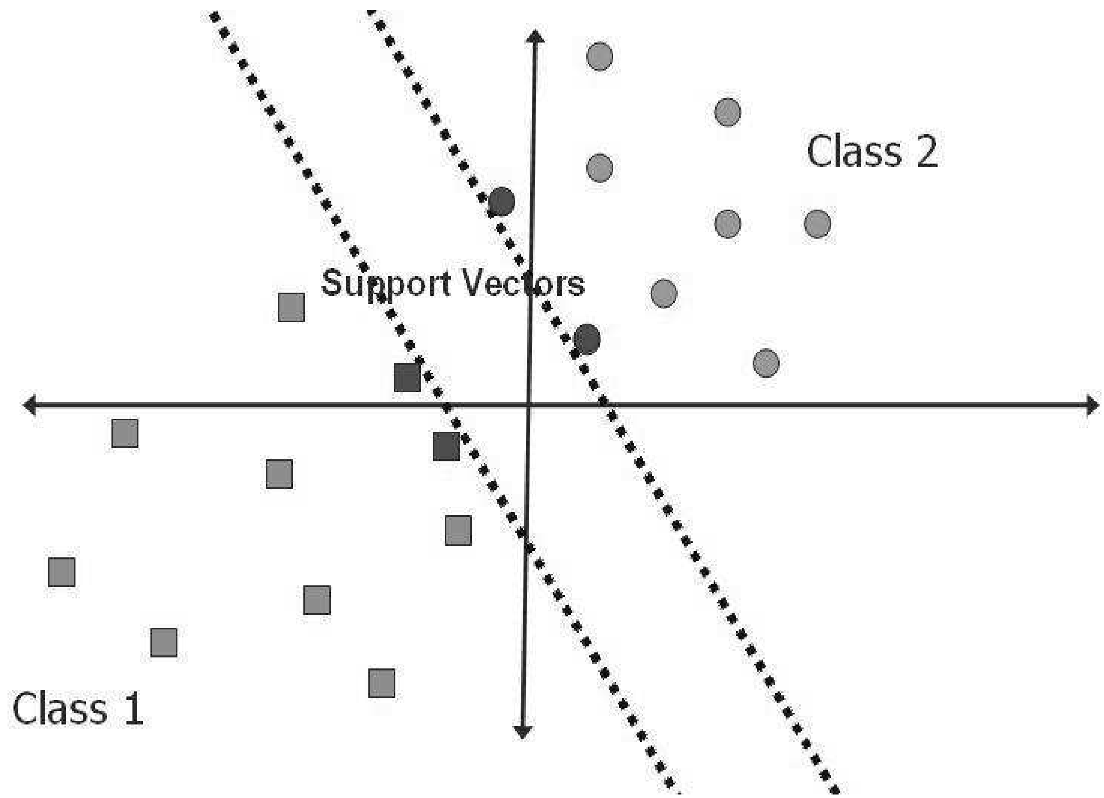
- Optimal W is a linear combination of Support vectors.
- Support vectors constitute a very useful output of the method.

Optimal hyperplane



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- Thus, W^*, b^* are determined by $\mu_i^*, \ \ i = 1, \ldots, n$.
- We can use the dual of the optimization problem to get μ_i^*.

Dual optimization problem for SVM

- The dual function is

$$q(\boldsymbol{\mu}) = \inf_{W,b} \left\{ \frac{1}{2} W^T W + \sum_{i=1}^{n} \mu_i [1 - y_i (W^T X_i + b)] \right\}$$

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- We obtain the dual by substituting $W = \sum \mu_i y_i X_i$ and imposing $\sum \mu_i y_i = 0$.

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- Optimization is over $\Re^n$ irrespective of the dimension of X_i.