XVI. *Functions of Positive and Negative Type, and their Connection with the Theory of Integral Equations.*

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Introduction.

The present memoir is the outcome of an attempt to obtain the conditions under which a given symmetric and continuous function $\kappa(s, t)$ is definite, in the sense of Hilbert.* At an early stage, however, it was found that the class of definite functions was too restricted to allow the determination of necessary and sufficient conditions in terms of the determinants of § 10. The discovery that this could be done for functions of positive or negative type, and the fact that almost all the theorems which are true of definite functions are, with slight modification, true of these, led finally to the abandonment of the original plan in favour of a discussion of the properties of functions belonging to the wider classes.

The first part of the memoir is devoted to the definition of various terms employed, and to the re-statement of the consequences which follow from Hilbert's theorem.

In the second part, keeping the theory of quadratic forms in view, the necessary and sufficient conditions, already alluded to, are obtained. These conditions are then applied to obtain certain general properties of functions of positive and negative type.

Part III. is chiefly devoted to the investigation of a particular class of functions of positive type. In addition, it includes a theorem which shows that, in general, from each function of positive type it is possible to deduce an infinite number of others of that type.

Lastly, in the fourth part, it is proved that when $\kappa(s, t)$ is of positive or negative type it may be expanded as a series of products of normal functions, and that this series converges both absolutely and uniformly.

* 'Gött. Nachr.' (1904), Heft I.

PART I.—DEFINITIONS AND DEDUCTIONS FROM HILBERT'S THEOREM.

§ 1. Let $\kappa(s, t)$ be a continuous symmetric function of the variables s, t which is defined in the closed square $a \leqq s \leqq b$, $a \leqq t \leqq b$; and let Θ be the class of all functions which are continuous in the closed interval (a, b). When the function θ ranges through the class Θ, there are three possible ways in which the double integral

$$\int_a^b \int_a^b \kappa(s, t)\, \theta(s)\, \theta(t)\, ds\, dt$$

may behave :—

(i) There may be two members of Θ, say θ_1 and θ_2, such that

$$\int_a^b \int_a^b \kappa(s, t)\, \theta_1(s)\, \theta_1(t)\, ds\, dt, \qquad \int_a^b \int_a^b \kappa(s, t)\, \theta_2(s)\, \theta_2(t)\, ds\, dt$$

have opposite signs ;

(ii) Each function θ may be such that

$$\int_a^b \int_a^b \kappa(s, t)\, \theta(s)\, \theta(t)\, ds\, dt \geqq 0 ;$$

(iii) Each function θ may be such that

$$\int_a^b \int_a^b \kappa(s, t)\, \theta(s)\, \theta(t)\, ds\, dt \leqq 0.$$

This suggests a classification of continuous symmetric functions defined in the closed square. We shall speak of those which have the property (i) as functions of *ambiguous type*, whilst the others will be said to be of *positive* or *negative type*, according as they satisfy (ii) or (iii).

§ 2. From the point of view of integral equations this classification is of considerable importance. HILBERT has proved* that

$$\int_a^b \int_a^b \kappa(s, t)\, \theta(s)\, \theta(t)\, ds\, dt = \sum_{n=1} \frac{1}{\lambda_n} \left[\int_a^b \psi_n(s)\, \theta(s)\, ds\right]^2,$$

where $\psi_1(s)$, $\psi_2(s)$, ..., $\psi_n(s)$, ..., are a complete system of normal functions relating to the characteristic function $\kappa(s, t)$ of the integral equation

$$f(s) = \phi(s) - \lambda \int_a^b \kappa(s, t)\, \phi(t)\, dt,$$

and $\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots$, respectively, are the corresponding singular values. It follows at once from this that, when the singular values are all positive, $\kappa(s, t)$ is of positive

* 'Gött. Nachr.' (1904), pp. 69–70. See also SCHMIDT, 'Math. Ann.,' Band 63, pp. 452, 453. We shall refer to the result given above as HILBERT'S *theorem*. The theorem stated by HILBERT on p. 70 of the paper referred to can be deduced by writing $\theta(s) = x(s) + y(s)$ in the equation written above.

type in accordance with the above definition. Conversely, we may prove that, for every function of positive type, the above integral equation has only positive singular values. For, if we multiply along the homogeneous equation

$$\psi_n(s) = \lambda_n \int_a^b \kappa(s,t)\,\psi_n(t)\,dt$$

by $\psi_n(s)$, and integrate with respect to s between the limits a and b, we obtain

$$\lambda_n \int_a^b \int_a^b \kappa(s,t)\,\psi_n(s)\,\psi_n(t)\,ds\,dt = 1.$$

Since the double integral on the left cannot be negative, and λ_n is a finite number, it appears that

$$\lambda_n > 0.$$

Thus *the necessary and sufficient condition that a continuous symmetric function should be of positive type is that the integral equation of the second kind of which it is the characteristic function should have all its singular values positive.**

In a similar manner it may be proved that this statement remains true when we replace the word *positive* by *negative*, in both places where it occurs.* Moreover, since a function must be of ambiguous type when it is of neither the positive nor the negative type, we conclude that *the necessary and sufficient condition for a continuous symmetric function to be of ambiguous type is the existence of both positive and negative singular values of the integral equation of the second kind of which it is the characteristic function.*

§ 3. It is easy to see that, corresponding to a function $\kappa(s,t)$ whose type is ambiguous, there exists a function $\theta(s)$ which is not zero in the whole interval (a, b), and satisfies the relation

$$\int_a^b \int_a^b \kappa(s,t)\,\theta(s)\,\theta(t)\,ds\,dt = 0. \quad \text{. . . . . (A)}$$

For, if we employ the notation of (i) above, and suppose that k is any real constant, we shall have

$$\int_a^b \int_a^b \kappa(s,t)\,[\theta_1(s)+k\theta_2(s)]\,[\theta_1(t)+k\theta_2(t)]\,ds\,dt = \int_a^b \int_a^b \kappa(s,t)\,\theta_1(s)\,\theta_1(t)\,ds\,dt$$
$$+k\int_a^b \int_a^b \kappa(s,t)\,[\theta_1(s)\,\theta_2(t)+\theta_2(s)\,\theta_1(t)]\,ds\,dt + k^2\int_a^b \int_a^b \kappa(s,t)\,\theta_2(s)\,\theta_2(t)\,ds\,dt.$$

* It follows from these results that, unless $\kappa(s,t)$ is identically zero, we cannot have

$$\int_a^b \kappa(s,t)\,\theta(s)\,\theta(t)\,ds\,dt = 0,$$

for all members of Θ. We shall prove this result in a different manner further on (§ 12), but it is useful to make the remark at this stage, since it shows conclusively that a function which is not identically zero cannot be both of positive and negative type.

The coefficient of k^2 on the right has a sign opposite to that of the term independent of k; accordingly, when we equate the right-hand member to zero, the resulting quadratic has its roots real. It follows that, if we suppose one of them to be α, the function

$$\theta(s) = \theta_1(s) + \alpha\theta_2(s)$$

will satisfy (A), and it cannot be identically zero, because this would imply that $\theta_1(s)$ is a constant multiple of $\theta_2(s)$, and hence that the two integrals mentioned in (i) have the same sign.

The converse of this theorem, however, is not true, for there are functions both of the positive and of the negative type which agree in this property with those of ambiguous type; these are known as the *semi-definite functions*. The remainder are called *definite functions*, and have the property that (A) can only be satisfied by a function $\theta(s)$ which is zero at each point of (a, b).

The two classes of functions we have just mentioned have distinctive properties in the theory of integral equations. For, if $\kappa(s, t)$ is of positive or negative type, it is evident from HILBERT'S theorem that (A) can only hold when

$$\int_a^b \psi_n(s)\,\theta(s)\,ds = 0 \qquad (n = 1, 2, \ldots).$$

By a known theorem* we must, therefore, have

$$\int_a^b \kappa(s, t)\,\theta(t)\,dt = 0 \qquad (a \leq s \leq b).$$

Thus *the necessary and sufficient condition that a function of positive or negative type should be definite is that it should be perfect.*

PART II.—THE NATURE OF FUNCTIONS OF POSITIVE AND NEGATIVE TYPE.

§ 4. The double integral

$$\int_a^b\int_a^b \kappa(s, t)\,\theta(s)\,\theta(t)\,ds\,dt, \quad \ldots\ldots\ldots \quad (1)$$

in which $\kappa(s, t)$ is an assigned symmetric and continuous function, and θ is any member of the class Θ, may be regarded as the limit of a certain set of quadratic expressions. For, let $a_1, a_2, \ldots, a_n$ be points of the interval (a, b), taken in such a way that the distances between consecutive members of the set of points consisting of a, b and these n are all equal. Then, by the theory of double integration, and in virtue of the symmetry of $\kappa(s, t)$, (1) is precisely equal to

$$(b-a)^2 \operatorname*{Lt}_{n\to\infty} \frac{[\kappa(a_1,a_1)\theta^2(a_1)+\kappa(a_2,a_2)\theta^2(a_2)+\ldots+\kappa(a_n,a_n)\theta^2(a_n)+2\kappa(a_1,a_2)\theta(a_1)\theta(a_2)+\ldots]}{n^2},$$

* *Cf.* SCHMIDT, *op. cit.*, pp. 451, 452.

The quantity inside the square brackets is evidently a particular value of a quadratic form whose coefficients are $\kappa(a_1, a_1)$, $\kappa(a_2, a_2)$, $\kappa(a_n, a_n)$, $2\kappa(a_1, a_2)$, ...; and, when θ ranges through the class Θ, the numbers $\theta(a_1)$, $\theta(a_2)$, ..., $\theta(a_n)$ will assume all possible real values.

It is thus suggested that we are to look upon the double integral (1), when θ ranges through Θ, as the limiting case of a quadratic form whose variables assume all possible real values. The function $\kappa(s, t)$ clearly takes the place of the coefficients of the form. Moreover, when $\kappa(s, t)$ is of positive type, the double integral (1) corresponds to a quadratic form which cannot take negative values for real values of the variable; and similarly in regard to the case when $\kappa(s, t)$ is of negative type.

Now the question, whether a quadratic form does, or does not, take both signs, as the variables assume all real values, has been shown to depend on the signs of certain determinants whose elements are coefficients of the form.* The considerations we have just indicated seem, therefore, to point to the existence of properties of the function $\kappa(s, t)$ which will decide its type, without directly considering the integral (1). It is the object of the present section to show that this is actually the case.

§ 5. Let us, for the present, confine our attention to a function $\kappa(s, t)$ of positive type, so that

$$\int_a^b \int_a^b \kappa(s, t)\, \theta(s)\, \theta(t)\, ds\, dt \geqq 0,$$

for all functions θ belonging to Θ.

We shall, in the first place, define a particular class of the functions Θ. Let s_1 be any point of the open interval (a, b), and suppose that ϵ and η are any two positive numbers which are so small that the points $s_1 \pm (\eta+\epsilon)$ also belong to the interval. Then the continuous function which is zero for $a \leqq s \leqq s_1-\eta-\epsilon$ and $s_1+\eta+\epsilon \leqq s \leqq b$, which is equal to unity for $s_1-\eta \leqq s \leqq s_1+\eta$, and which is a linear function of s in the intervals $(s_1-\eta-\epsilon, s_1-\eta)$, $(s_1+\eta, s_1+\eta+\epsilon)$, will be denoted by $\theta_{\epsilon,\eta}(s; s_1)$. The values of the function in these latter intervals will be given by

$$\frac{s-(s_1-\eta-\epsilon)}{\epsilon}, \quad \frac{(s_1+\eta+\epsilon)-s}{\epsilon}$$

respectively, and will evidently be positive numbers less than unity at interior points.

Consider now the values of the function

$$\theta_{\epsilon,\eta}(s; s_1)\, \theta_{\epsilon,\eta}(t; s_1)$$

at the various points of the square $a \leqq s \leqq b$, $a \leqq t \leqq b$ of the (s, t) plane. In the accompanying figure this large square, which we shall denote by Q, is intersected by

* See, for example, BROMWICH, 'Quadratic Forms and their Classification by means of Invariant Factors' (1907), chap. ii., where *necessary* conditions are obtained. It is not difficult to obtain conditions which are both *necessary* and *sufficient*.

two sets of four lines drawn parallel to the axes of s and t; these are the lines $t = s_1 \pm \eta$, $t = s_1 \pm (\eta + \epsilon)$; $s = s_1 \pm \eta$, $s = s_1 \pm (\eta + \epsilon)$ respectively, and they may be identified by observing that the number at the point where any one of them intersects an axis is the value of the corresponding variable which is constant along it. It will thus be seen that the square denoted by q_{11} is bounded by the four lines

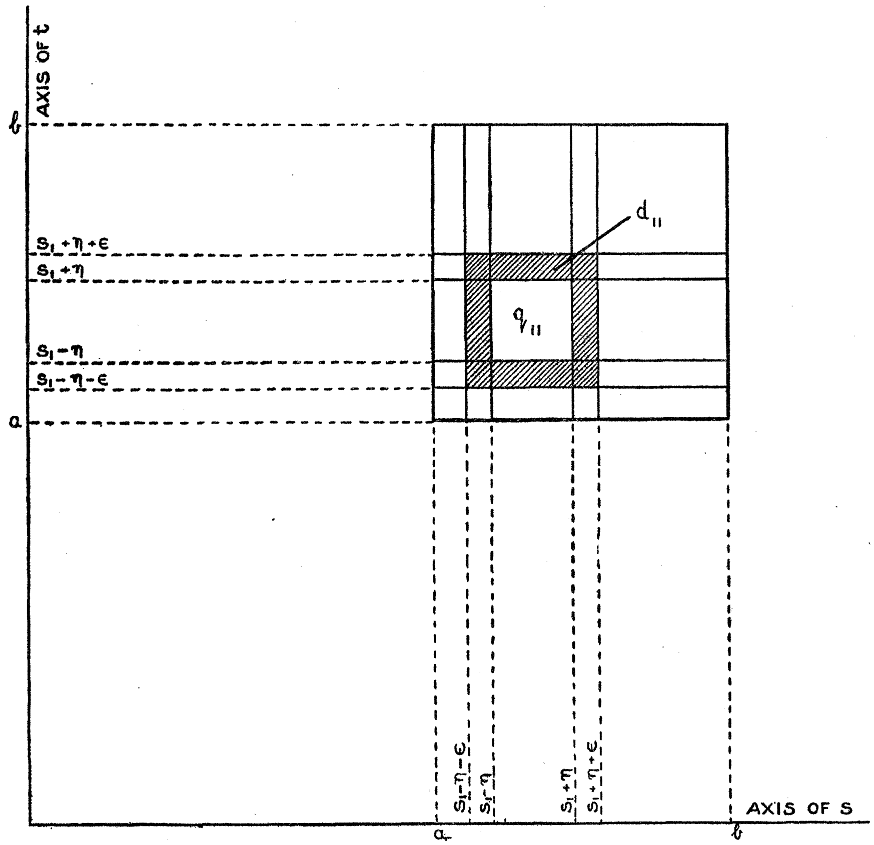

Fig. 1.

$s = s_1 \pm \eta$, $t = s_1 \pm \eta$; while the area d_{11}, which is shaded in the figure, and which will be referred to as the border of q_{11}, is the part of the square bounded by $s = s_1 \pm (\eta + \epsilon)$, $t = s_1 \pm (\eta + \epsilon)$ exterior to q_{11}. A little reflection will show that, at points of Q which do not belong either to q_{11} or to d_{11}, one or other of the functions $\theta_{\epsilon,\eta}(s\,;\,s_1)$, $\theta_{\epsilon,\eta}(t\,;\,s)$ is zero; that, at points of q_{11}, each of these functions is unity; and, finally, that in d_{11} neither function exceeds unity. It follows then that

$$\begin{aligned} \theta_{\epsilon,\eta}(s\,;\,s_1)\,\theta_{\epsilon,\eta}(t\,;\,s_1) &= 1 \text{ in } q_{11}, \\ &\leq 1 \text{ in } d_{11}, \\ &= 0 \text{ elsewhere.} \end{aligned}$$

§ 6. The integral

$$\int_a^b \int_a^b \kappa(s, t)\,\theta_{\epsilon,\eta}(s\,;\,s_1)\,\theta_{\epsilon,\eta}(t\,;\,s_1)\,ds\,dt$$

may be looked upon as $\int \kappa(s, t)\, \theta_{\epsilon,\eta}(s\,;\, s_1)\, \theta_{\epsilon,\eta}(t\,;\, s_1)\,(ds\, dt)$ taken over Q, or, as it is usually written,*

$$\int_{Q} \kappa(s\,;\, t)\, \theta_{\epsilon,\eta}(s\,;\, s_1)\, \theta_{\epsilon,\eta}(t\,;\, s_1)\,(ds\, dt)\,;$$

and, from what has been said in the preceding paragraph, that portion of the latter which arises from the part of Q exterior to d_{11} is zero, while that arising from q_{11} is simply

$$\int_{q_{11}} \kappa(s, t)\,(ds\, dt).$$

We have, therefore,

$$\int_a^b \int_a^b \kappa(s, t)\, \theta_{\epsilon,\eta}(s\,;\, s_1)\, \theta_{\epsilon,\eta}(t\,;\, s_1)\, ds\, dt$$
$$= \int_{q_{11}} \kappa(s, t)\,(ds\, dt) + \int_{d_{11}} \kappa(s, t)\, \theta_{\epsilon,\eta}(s\,;\, s_1)\, \theta_{\epsilon,\eta}(t\,;\, s_1)\,(ds\, dt). \quad . \quad . \quad (2)$$

Again the total area of d_{11} is $4\epsilon(2\eta+\epsilon)$, and so, it M is the maximum value of $|\kappa(s, t)|$ in Q, we have

$$\left|\int_{d_{11}} \kappa(s, t)\, \theta_{\epsilon,\eta}(s\,;\, s_1)\, \theta_{\epsilon,\eta}(t\,;\, s_1)\,(ds\, dt)\right| \leq 4\epsilon(2\eta+\epsilon)\,\mathrm{M}\,;$$

also the remaining integral on the right-hand side of (2) can be replaced by

$$\int_{s_1-\eta}^{s_1+\eta} \int_{s_1-\eta}^{s_1+\eta} \kappa(s, t)\, ds\, dt,$$

which is evidently equal to

$$\int_{-\eta}^{\eta} \int_{-\eta}^{\eta} \kappa(s_1+u,\, s_1+v)\, du\, dv.$$

Thus it follows from (2) that

$$\left|\int_a^b \int_a^b \kappa(s, t)\, \theta_{\epsilon,\eta}(s\,;\, s_1)\, \theta_{\epsilon,\eta}(t\,;\, s_1)\, ds\, dt - \int_{-\eta}^{\eta} \int_{-\eta}^{\eta} \kappa(s_1+u,\, s_1+v)\, du\, dv\right| \leq 4\epsilon(2\eta+\epsilon)\,\mathrm{M}. \quad . \quad (3)$$

Now let us suppose it possible for $\kappa(s_1, s_1)$ to have a negative value, say $-\alpha$; then, because $\kappa(s, t)$ is continuous, we can choose a value of η so small that

$$\kappa(s_1+u,\, s_1+v) < -\tfrac{1}{2}\alpha,$$

for all values of u and v whose moduli are not greater than η. We shall therefore have

$$-\int_{-\eta}^{\eta} \int_{-\eta}^{\eta} \kappa(s_1+u,\, s_1+v)\, du\, dv > 2\eta^2\alpha.$$

Recalling our hypothesis that $\kappa(s, t)$ is of positive type, it follows from this and (3) that

$$\eta^2\alpha \leq 2\epsilon(2\eta+\epsilon)\,\mathrm{M},$$

* *Cf.* HOBSON, 'The Theory of Functions of a Real Variable' (1907), p. 416.

for all values of ϵ which are less than a certain positive number (§ 5). But this is evidently impossible, because, when ϵ tends to zero, the right-hand side tends to zero, and we arrive at the contradiction that a fixed positive quantity (viz., $\eta^2 a$) is less than, or equal to, zero.

We conclude that $\kappa(s_1, s_1)$ cannot be negative when s_1 lies in the open interval (a, b); and hence, since $\kappa(s, s)$ is continuous in the same interval when regarded as closed, we have the result that *every function $\kappa(s, t)$ which is of positive type in the square $a \leqq s \leqq b$, $a \leqq t \leqq b$ satisfies the inequality*

$$\kappa(s_1, s_1) \geqq 0^* \qquad (a \leqq s_1 \leqq b).$$

§ 7. This is a first condition which must be satisfied by these functions, and we may obtain a second on similar lines. Let s_1 and s_2 be any two distinct points of the open interval (a, b), and, as before, let ϵ and η be two positive numbers; the latter will now be supposed so small that the intervals $[s_1-(\eta+\epsilon), s_1+(\eta+\epsilon)]$, $[s_2-(\eta+\epsilon), s_2+(\eta+\epsilon)]$ are both contained within (a, b) and do not overlap. We now propose to consider the values of the function

$$[x_1\theta_{\epsilon,\eta}(s\,;\,s_1) + x_2\theta_{\epsilon,\eta}(s\,;\,s_2)]\,[x_1\theta_{\epsilon,\eta}(t\,;\,s_1) + x_2\theta_{\epsilon,\eta}(t\,;\,s_2)]$$

at points interior to Q, when x_1 and x_2 are any real constants. For this purpose we may make use of a diagram (fig. 2) which is an obvious extension of the one employed in the previous paragraph. The square Q is divided in this case not by eight, but by sixteen lines, viz., those whose equations are $s = s_\alpha \pm \eta$, $s = s_\alpha \pm (\eta+\epsilon)$; $t = s_\beta \pm \eta$, $t = s_\beta \pm (\eta+\epsilon)$ $(\alpha, \beta = 1, 2)$. By giving α and β all possible values in the equations just written, it will be seen that we obtain four sets of eight, for each of which we can distinguish a square $q_{\alpha\beta}$ bounded by the lines $s = s_\alpha \pm \eta$, $t = s_\beta \pm \eta$; moreover, these squares will evidently have borders $d_{\alpha\beta}$ of width ϵ. It is not difficult to see that, in those parts of Q which are exterior to the borders $d_{\alpha\beta}$ $(\alpha, \beta = 1, 2)$, we have either

$$\theta_{\epsilon,\eta}(s\,;\,s_1) = \theta_{\epsilon,\eta}(s\,;\,s_2) = 0,$$

or

$$\theta_{\epsilon,\eta}(t\,;\,s_1) = \theta_{\epsilon,\eta}(t\,;\,s_2) = 0\,;\dagger$$

that in the square $q_{\alpha\beta}$ we have

$$\theta_{\epsilon,\eta}(s\,;\,s_\alpha) = \theta_{\epsilon,\eta}(t\,;\,s_\beta) = 1,$$

$$\theta_{\epsilon,\eta}(s\,;\,s_{3-\alpha}) = \theta_{\epsilon,\eta}(t\,;\,s_{3-\beta}) = 0\,;$$

* The reader may compare this with the fact that, when we have a quadratic form which only assumes non-negative values, and we put all the variables save one (say x_1) equal to zero, we deduce that the coefficient of x_1^2 must be $\geqq 0$.

† Both these pairs of equalities will hold in certain parts of the square, but we only require that at least one of them should be true.

and that in the border $d_{\alpha\beta}$ the last pair of equations still hold, but $\theta_{\epsilon,\eta}(s\,;\,s_\alpha)$, $\theta_{\epsilon,\eta}(t\,;\,s_\beta)$ are each less than, or equal to, unity. From this it appears that the function

$$\begin{aligned}[x_1\theta_{\epsilon,\eta}(s\,;\,s_1) + x_2\theta_{\epsilon,\eta}(s\,;\,s_2)]\,[x_1\theta_{\epsilon,\eta}(t\,;\,s_1) + x_2\theta_{\epsilon,\eta}(t\,;\,s_2)] &= x_\alpha x_\beta \text{ in } q_{\alpha\beta}\,(\alpha,\beta = 1, 2),\\ &= 0 \text{ outside the borders } d_{\alpha\beta},\end{aligned}$$

and that in the border $d_{\alpha\beta}$ its modulus is $\leq |x_\alpha x_\beta|$.

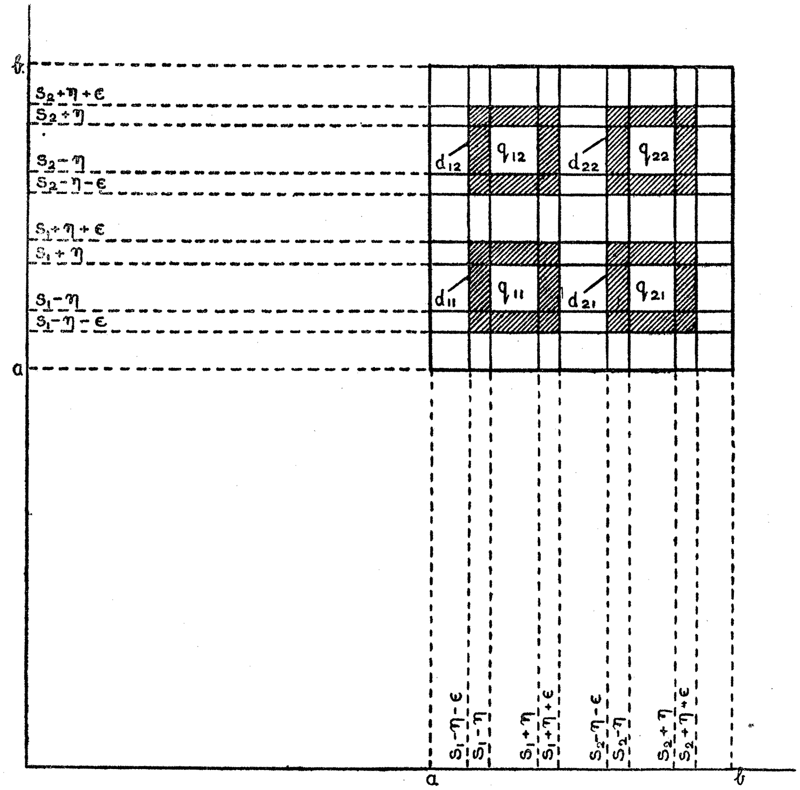

Fig. 2.

§ 8. Let us now write

$$\theta(s) = x_1\theta_{\epsilon,\eta}(s\,;\,s_1) + x_2\theta_{\epsilon,\eta}(s\,;\,s_2),$$

for the sake of brevity. It follows from the remarks of the preceding paragraph that

$$\begin{aligned}\int_a^b\int_a^b \kappa(s,t)\,\theta(s)\,\theta(t)\,ds\,dt &= \sum_{\alpha=1}^{2}\sum_{\beta=1}^{2} x_\alpha x_\beta \int_{q_{\alpha\beta}} \kappa(s,t)\,(ds\,dt)\\ &+ \sum_{\alpha=1}^{2}\sum_{\beta=1}^{2}\int_{d_{\alpha\beta}} \kappa(s,t)\,\theta(s)\,\theta(t)\,(ds\,dt). \quad . \quad . \quad (4)\end{aligned}$$

Now the area of each of the borders $d_{\alpha\beta}$ is $4\epsilon(2\eta+\epsilon)$, and so we have

$$\left|\sum_{\alpha=1}^{2}\sum_{\beta=1}^{2}\int_{d_{\alpha\beta}} \kappa(s,t)\,\theta(s)\,\theta(t)\,(ds\,dt)\right| \leq 4\epsilon(2\eta+\epsilon)\,(|x_1| + |x_2|)^2\,\mathrm{M}\,;$$

moreover, it is easily proved that, in virtue of the symmetry of $\kappa(s, t)$,

$$\sum_{\alpha=1}^{2} \sum_{\beta=1}^{2} x_\alpha x_\beta \int_{q_{\alpha\beta}} \kappa(s, t)\,(ds\,dt)$$

can be written as

$$\int_{-\eta}^{\eta} \int_{-\eta}^{\eta} [x_1^2 \kappa(s_1+u, s_1+v) + 2x_1x_2\kappa(s_1+u, s_2+v) + x_2^2\kappa(s_2+u, s_2+v)]\,du\,dv. \quad (5)$$

From this and the equation (4) we finally obtain the inequality

$$\left| \int_a^b \int_a^b \kappa(s, t)\,\theta(s)\,\theta(t)\,ds\,dt - \int_{-\eta}^{\eta} \int_{-\eta}^{\eta} \mathrm{F}_2(u, v)\,du\,dv \right| \leq 4\epsilon(2\eta+\epsilon)(|x_1|+|x_2|)^2\,\mathrm{M},$$

where $\mathrm{F}_2(u, v)$ is the integrand of (5).

The function $\mathrm{F}_2(u, v)$ is, of course, dependent on the real constants x_1 and x_2; let us suppose it possible to choose them in such a way that

$$\mathrm{F}_2(0, 0) = x_1^2\kappa(s_1, s_1) + 2x_1x_2\kappa(s_1, s_2) + x_2^2\kappa(s_2, s_2)$$

takes a negative value, say $-\alpha$. Owing to the fact that $\kappa(s, t)$ is continuous, it is then clear that we can chose η so small that

$$\mathrm{F}_2(u, v) < -\tfrac{1}{2}\alpha,$$

for $|u| \leq \eta$, $|v| \leq \eta$. From this we deduce the inequality

$$\eta^2\alpha \leq 2\epsilon(2\eta+\epsilon)(|x_1|+|x_2|)^2\,\mathrm{M},$$

as in the corresponding place in § 6; and hence, as this is impossible for sufficiently small values of ϵ, it follows that, when s_1 and s_2 lie in the open interval (a, b), and x_1 and x_2 are real, $\mathrm{F}_2(0, 0)$ is not negative. Accordingly, since $\kappa(s, t)$ is continuous, it is easily seen that *every function $\kappa(s, t)$ which is of positive type in the square $a \leq s \leq b$, $a \leq t \leq b$ is such that, when x_1 and x_2 are any real numbers,*

$$x_1^2\kappa(s_1, s_1) + 2x_1x_2\kappa(s_1, s_2) + x_2^2\kappa(s_2, s_2) \geq 0 \begin{pmatrix} a \leq s_1 \leq b \\ a \leq s_2 \leq b \end{pmatrix}.$$

§ 9. The reader will now be prepared for a general theorem of which those already considered are particular cases. After having been through the latter in detail it will be sufficient to sketch the general proof.

Take any n distinct point $s_1, s_2, \ldots, s_n$ in the open interval (a, b), and suppose that ϵ and η are so small that the intervals $[s_\alpha-(\eta+\epsilon), s_\alpha+(\eta+\epsilon)]$ $(\alpha = 1, 2, \ldots, n)$ form a non-overlapping set contained within (a, b). Now let

$$\theta(s) = \sum_{\alpha=1}^{n} x_\alpha \theta_{\epsilon,\eta}(s\,;\,s_\alpha),$$

where $x_1, x_2, \ldots, x_n$ are any real constants; and consider the values of the function $\theta(s)\,\theta(t)$ in Q. It will be seen on consideration that in this general case Q must be regarded as divided by $8n$ lines, and that there are n^2 squares $q_{\alpha\beta}$, each having a border $d_{\alpha\beta}$ $(\alpha, \beta = 1, 2, \ldots, n)$. It will also be seen that

$$\begin{aligned}\theta(s)\,\theta(t) &= x_\alpha x_\beta \text{ in } q_{\alpha\beta}\ (\alpha, \beta = 1, 2, \ldots, n),\\ &= 0 \text{ outside the borders } d_{\alpha\beta},\end{aligned}$$

and that in the border $d_{\alpha\beta}$ we have

$$|\theta(s)\,\theta(t)| \leqq |x_\alpha x_\beta|.$$

Proceeding then as in the case $n = 2$, we obtain the inequality

$$\left|\int_a^b\int_a^b \kappa(s, t)\,\theta(s)\,\theta(t)\,ds\,dt - \int_{-\eta}^{\eta}\int_{-\eta}^{\eta} \mathrm{F}_n(u, v)\,du\,dv\right| \leqq 4\epsilon\,(2\eta+\epsilon)\left(\sum_{\alpha=1}^{n}|x_\alpha|\right)^2 \mathrm{M},$$

where

$$\begin{aligned}\mathrm{F}_n(u, v) = x_1^2\kappa(s_1+u, s_1+v) + x_2^2\kappa(s_2+u, s_2+v) + \ldots + x_n^2\kappa(s_n+u, s_n+v)\\ + 2x_1x_2\kappa(s_1+u, s_2+v) + \ldots;\end{aligned}$$

and hence we establish that $\mathrm{F}_n(0, 0)$ is always $\geqq 0$. Eventually we obtain the general theorem :—

Every function $\kappa(s, t)$ *which is of positive type in the square* $a \leqq s \leqq b$, $a \leqq t \leqq b$ *must be such that, when* $s_1, s_2, \ldots, s_n$ *are any points of the closed interval* (a, b), *we have*

$$x_1^2\kappa(s_1, s_1) + x_2^2\kappa(s_2, s_2) + \ldots + x_n^2\kappa(s_n, s_n) + 2x_1x_2\kappa(s_1, s_2) + \ldots \geqq 0,$$

for all real values of $x_1, x_2, \ldots, x_n$.

§ 10. In accordance with the notation employed by FREDHOLM, let

$$\kappa\begin{pmatrix} s_1, s_2, \ldots, s_n \\ s_1, s_2, \ldots, s_n \end{pmatrix} = \begin{vmatrix} \kappa(s_1, s_1) & \kappa(s_1, s_2) & \ldots & \kappa(s_1, s_n) \\ \kappa(s_2, s_1) & \kappa(s_2, s_2) & \ldots & \kappa(s_2, s_n) \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \kappa(s_n, s_1) & \kappa(s_n, s_2) & \ldots & \kappa(s_n, s_n) \end{vmatrix}.$$

Then, by the theory of quadratic forms, it is known that, in virtue of the inequality which has just been obtained, we must have*

$$\kappa\begin{pmatrix} s_1, s_2, \ldots, s_n \\ s_1, s_2, \ldots, s_n \end{pmatrix} \geqq 0; \quad \ldots \ldots \ldots \ldots \quad (6)$$

* *Vide* BROMWICH, 'Quadratic Forms and their Classification by means of Invariant Factors' (1906), pp. 19, 20.

and this is true independently of the number of points, $s_1, s_2, \ldots, s_n$ and their situation in the interval (a, b).

Conversely, by an appeal to the theory of integral equations, we may prove that any continuous symmetric function $\kappa(s, t)$ defined in Q, which satisfies this condition, is of positive type. For it will be remembered that, according to FREDHOLM'S theory,* the singular values of the equation

$$f(s) = \phi(s) - \lambda \int_a^b \kappa(s, t)\, \phi(t)\, dt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

are the zeros of the integral function

$$D(\lambda) = 1 - \lambda \int_a^b \kappa(s_1, s_1)\, ds_1 + \frac{\lambda^2}{2\,!} \int_a^b \int_a^b \kappa\begin{pmatrix} s_1, s_2 \\ s_1, s_2 \end{pmatrix} ds_1 ds_2 \ldots$$
$$+ \frac{(-\lambda)^n}{n\,!} \int_a^b \cdots \int_a^b \int_a^b \kappa\begin{pmatrix} s_1, s_2, \ldots, s_n \\ s_1, s_2, \ldots, s_n \end{pmatrix} ds_1\, ds_2 \ldots ds_n + \ldots.$$

Applying our hypothesis that (6) holds for all values of $s_1, s_2, \ldots, s_n$, it appears that the coefficient of $\frac{(-\lambda)^n}{n\,!}$ in the series on the right cannot be negative; moreover, HILBERT has proved that every continuous symmetric function has its singular values all real. It follows, therefore, that, if λ_r is any one of the zeros of $D(\lambda)$, we shall have

$$\lambda_r \left[\int_a^b \kappa(s_1, s_1)\, ds_1 + \frac{\lambda_r^{\,2}}{3\,!} \int_a^b \int_a^b \int_a^b \kappa\begin{pmatrix} s_1, s_2, s_3 \\ s_1, s_2, s_3 \end{pmatrix} ds_1\, ds_2\, ds_3 + \ldots \right.$$
$$\left. + \frac{\lambda_r^{\,2n}}{2n+1\,!} \int_a^b \cdots \int_a^b \int_a^b \kappa\begin{pmatrix} s_1, s_2, \ldots, s_{2n+1} \\ s_1, s_2, \ldots, s_{2n+1} \end{pmatrix} ds_1\, ds_2 \ldots ds_{2n+1} + \ldots \right]$$
$$= 1 + \frac{\lambda_r^{\,2}}{2\,!} \int_a^b \int_a^b \kappa\begin{pmatrix} s_1, s_2 \\ s_1, s_2 \end{pmatrix} ds_1\, ds_2 + \ldots + \frac{\lambda_r^{\,2n}}{2n\,!} \int_a^b \cdots \int_a^b \int_a^b \kappa\begin{pmatrix} s_1, s_2, \ldots, s_{2n} \\ s_1, s_2, \ldots, s_{2n} \end{pmatrix} ds_1\, ds_2 \ldots ds_{2n}$$
$$+ \ldots,$$

where the series in the square brackets on the left is not negative and that on the right is positive; and hence, that λ_r must be positive. Since we have seen that, for $\kappa(s, t)$ to be of positive type, it is sufficient that all the singular values of (7) should be positive, we may now state the following theorem:—

In order that a continuous symmetric function $\kappa(s, t)$ *defined in the square* $a \leqq s \leqq b$, $a \leqq t \leqq b$ *may be of positive type, it is necessary and sufficient that the functions*

$$\kappa(s_1, s_1),\ \kappa\begin{pmatrix} s_1, s_2 \\ s_1, s_2 \end{pmatrix},\ \ldots,\ \kappa\begin{pmatrix} s_1, s_2, \ldots, s_n \\ s_1, s_2, \ldots, s_n \end{pmatrix}, \ldots \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

should never take negative values when the variables $s_1, s_2, \ldots, s_n \ldots$ *each range over the closed interval* (a, b).

* *Vide* 'Acta Mathematica,' XXVII (1903).

It may be remarked that, as a corollary of this theorem, we have the notable fact that, if any continuous symmetric function is such that the integrals

$$\int_a^b \kappa(s_1, s_1)\, ds_1, \int_a^b\int_a^b \kappa\begin{pmatrix} s_1, s_2 \\ s_1, s_2 \end{pmatrix} ds_1\, ds_2, \ldots, \int_a^b \ldots \int_a^b\int_a^b \kappa\begin{pmatrix} s_1, s_2, \ldots, s_n \\ s_1, s_2, \ldots, s_n \end{pmatrix} ds_1\, ds_2 \ldots ds_n, \ldots$$

are none of them negative, then the functions (8) have the same property.

§ 11. The properties of the determinants (8) may be used to obtain some idea of the nature of functions of positive type. Let us suppose, in the first place, that there is a point (a_1, a_1) belonging to Q at which one of these functions $\kappa(s, t)$ vanishes. The determinant $\kappa\begin{pmatrix} s, a_1 \\ s, a_1 \end{pmatrix}$ evidently reduces to $-[\kappa(s, a_1)]^2$; hence, because it can never be negative,

$$\kappa(s, a_1) = \kappa(a_1, s) = 0.$$

In other words, if we draw the square Q and the diagonal $s = t$, the existence of a point (a_1, a_1) on this diagonal at which $\kappa(s, t)$ vanishes involves the fact that $\kappa(s, t)$ vanishes everywhere on the lines drawn through this point parallel to the axes of s and t. In particular, we deduce from this that *a function $\kappa(s, t)$ which is of positive type, and is not zero everywhere in* Q, *cannot vanish everywhere on the diagonal $s = t$.*

More generally, let us suppose that there are points $a_1, a_1, \ldots, a_n$ of the interval (a, b) such that

$$\kappa\begin{pmatrix} a_1, a_2, \ldots, a_n \\ a_1, a_2, \ldots, a_n \end{pmatrix} = 0. \qquad (9)$$

By considering the determinant whose elements are the first minors of the four elements belonging to the first two rows and columns of

$$\kappa\begin{pmatrix} s, a_1, a_2, \ldots, a_n \\ s, a_1, a_2, \ldots, a_n \end{pmatrix}, \qquad (10)$$

we obtain the equation*

$$\kappa\begin{pmatrix} s, a_1, a_2, \ldots, a_n \\ s, a_1, a_2, \ldots, a_n \end{pmatrix} \kappa\begin{pmatrix} a_2, a_3, \ldots, a_n \\ a_2, a_3, \ldots, a_n \end{pmatrix}$$
$$= \kappa\begin{pmatrix} s, a_2, \ldots, a_n \\ s, a_2, \ldots, a_n \end{pmatrix} \kappa\begin{pmatrix} a_1, a_2, \ldots, a_n \\ a_1, a_2, \ldots, a_n \end{pmatrix} - \left[\kappa\begin{pmatrix} s, a_2, \ldots, a_n \\ a_1, a_2, \ldots, a_n \end{pmatrix}\right]^2.$$

Recalling that the first term on the right vanishes in virtue of our hypothesis, and that neither of the terms in the product on the left can be negative, it is clear that we have

$$\kappa\begin{pmatrix} s, a_2, \ldots, a_n \\ a_1, a_2, \ldots, a_n \end{pmatrix} = 0$$

* *Vide* SCOTT and MATHEWS, 'Theory of Determinants' (1904), p. 62.

at each point of the interval (a, b); and it can be proved in a similar way that the remainder of the functions

$$\kappa\begin{pmatrix} a_1, & a_2, & \dots, & a_{r-1}, & s, & a_{r+1}, & \dots, & a_n \\ a_1, & a_2, & \dots, & a_{r-1}, & a_r, & a_{r+1}, & \dots, & a_n \end{pmatrix} (r = 1, 2, \dots, n) \quad . \quad . \quad . \quad . \quad (11)$$

have the same property.

Again, because the determinant (9) and the functions (11) all vanish, it is easily seen that the function (10) vanishes identically. Accordingly, if any one of the functions (8) vanishes for all values of the variables, so must all those which follow it. It appears, therefore, that, when $\kappa(s, t)$ is of positive type, the determinant of the integral equation (7) is either an infinite power series in λ whose coefficients are alternately positive and negative numbers, or else it is a polynomial whose coefficients obey the same law.

Another property which is worth noticing is that, if L is the upper limit of the function $\kappa(s, s)$ in the interval (a, b), then

$$-\mathrm{L} \leqq \kappa(s, t) \leqq \mathrm{L}$$

in the whole of the square Q. This follows immediately from the fact that, since

$$\kappa\begin{pmatrix} s, & t \\ s, & t \end{pmatrix} \geqq 0,$$

we have

$$\mathrm{L}^2 \geqq \kappa(s, s)\, \kappa(t, t) \geqq [\kappa(s, t)]^2.$$

§ 12. We have so far confined ourselves to the consideration of functions of positive type, but the reader will easily perceive that the results obtained for these functions may be made applicable to those of negative type by a simple device. In fact, if $\kappa(s, t)$ is of negative type in the square Q, and we suppose that

$$\kappa'(s, t) = -\kappa(s, t),$$

it is evident that $\kappa'(s, t)$ is of positive type in Q. Applying then what we have said about functions of positive type to $\kappa'(s, t)$, we may deduce the analogous properties of $\kappa(s, t)$; for instance, *the necessary and sufficient condition that a continuous symmetric function $\kappa(s, t)$ defined in the square $a \leqq s \leqq b$, $a \leqq t \leqq b$ may be of negative type is that the functions*

$$-\kappa(s_1, s_1),\ \kappa\begin{pmatrix} s_1, & s_2 \\ s_1, & s_2 \end{pmatrix},\ \dots,\ (-1)^n\,\kappa\begin{pmatrix} s_1, & s_2, & \dots, & s_n \\ s_1, & s_2, & \dots, & s_n \end{pmatrix},\ \dots,$$

should never be negative when the variables $s_1, s_2, \dots, s_n, \dots$ each range over the closed interval (a, b).

We may remark that this result and that of § 10 prove the classes of functions of positive and negative types to be mutually exclusive, save for the trivial case when $\kappa(s, t)$ vanishes everywhere. For, if $\kappa(s, t)$ belongs to both classes, we must have

$$\kappa(s_1, s_1) \geqq 0, \quad -\kappa(s_1, s_1) \geqq 0$$

for all points of the interval (a, b); and hence $\kappa(s_1, s_1)$ must be zero everywhere in this interval. It follows, then, from a remark made in § 11, that $\kappa(s, t)$ is zero in the whole square Q.

Part III.—Certain Functions of Positive Type.

§ 13. In the present section we propose to investigate certain species of functions which are of positive type. The remark made at the end of the previous section (§ 12) will make it plain that there is no loss in thus limiting ourselves, since the corresponding results for functions of negative type may be at once deduced by the device there explained.

Let us again consider the square Q of the (s, t) plane which is bounded by the lines $s = a$, $s = b$, $t = a$, $t = b$; and let us suppose that it is divided into two triangles by the diagonal whose equation is $s = t$. The most direct method of defining a continuous symmetric function in Q is, evidently, to define a continuous function in one of the triangles, say that in which $s \leqq t$; and then to suppose this continued into the remaining portion of the square by defining its value at a point for which $s > t$ to be that at its image by reflection in the diagonal. For example, if $\theta(s)$ is a continuous function of s in the interval (a, b), and we define $\kappa(s, t)$ to be equal to $\theta(s)$ in the triangle $s \leqq t$, then the continuation of this function into the triangle $s > t$ is evidently $\theta(t)$.

The theorem of § 10 may be applied to the function we have just defined, and hence the condition that it should be of positive type deduced. Instead of doing this, however, we shall consider the more general function*

$$\begin{aligned} \kappa(s, t) &= \theta(s)\,\phi(t) \quad (s \leqq t) \\ &= \phi(s)\,\theta(t) \quad (s \geqq t), \end{aligned}$$

where $\theta(s)$ and $\phi(s)$ are both continuous in the interval (a, b). It will be remembered that functions of this kind occur as Green's functions of certain linear differential equations of the second order, and that it is therefore of some interest to know when they are of positive type. Accordingly we shall seek necessary and sufficient conditions which will ensure that this is so.

§ 14. In the first place, let us suppose that $\theta(s)$ and $\phi(s)$ are any continuous functions whatever; and let Σ be the set of points belonging to (a, b) at which neither of them vanish. This set will evidently be dense in itself in virtue of the

* *Cf.* Bateman, 'Messenger of Mathematics,' New Series, 1907, p. 93.

continuity of the functions; but it cannot be closed, unless it contains every point of the interval. Moreover, it can be proved that α and β, its lower and upper limits respectively, do not belong to the set, unless they coincide with the end points of the interval.

At each point of the set Σ the quotient

$$\theta(s)/\phi(s)$$

will have a definite value, because $\phi(s)$ is never zero. We may therefore define a single-valued function $f(s)$, whose domain is Σ, and whose value at any point is that of this quotient. It will appear in the sequel that the properties of $\kappa(s, t)$ depend very largely on the nature of $f(s)$, and accordingly, in anticipation of this, we shall speak of it as the *discriminator of* $\kappa(s, t)$. The discriminator will evidently be continuous in its domain, but it will never have the value zero.

§ 15. Let us now suppose that $\kappa(s, t)$ is of positive type, and is not zero everywhere in the square Q. We have proved (§ 11) that, under these circumstances, the function $\kappa(s_1, s_1)$, which in the present case is simply $\theta(s_1)\phi(s_1)$, cannot be zero in the whole of (a, b); also, at points where it does not vanish, we know that $\kappa(s_1, s_1)$ is positive (§§ 6, 10). It follows that, for a function of positive type, the set Σ certainly exists, and that in it the discriminator only takes positive values.

Again, when s_1 and s_2 are any two points of Σ, and $s_2 > s_1$, we have

$$\kappa\begin{pmatrix} s_1, s_2 \\ s_1, s_2 \end{pmatrix} = [\phi(s_1)\phi(s_2)]^2 f(s_1)[f(s_2)-f(s_1)];$$

hence, since $f(s_1)$ is a positive number, it follows by the theorem of § 10 that

$$f(s_2) \geqq f(s_1).$$

This result may be combined with the previous one in the statement that the discriminator of $\kappa(s, t)$ is a non-decreasing function whose values are all positive.

We have next to consider the points of (a, b) at which one or both of the functions $\theta(s)$, $\phi(s)$ vanish. These fall naturally into three sets, according as they belong to (1) the closed interval (a, α), (2) the closed interval (β, b), or (3) the open interval (α, β). As regards (1), it is not difficult to show that $\theta(s)$ vanishes in the whole interval. For, if a_1 is any point of (a, α), one at least of the numbers $\theta(a_1)$, $\phi(a_1)$ must be zero; and hence, since $\kappa(a_1, a_1)$ is zero, the function $\kappa(s, a_1)$ is zero at each point of (a, b) (§ 11).

Now when $s > a_1$ we have

$$\kappa(s, a_1) = \theta(a_1)\phi(s),$$

and, at points of Σ, $\phi(s)$ does not vanish; we must therefore have $\theta(a_1) = 0$. It can be proved in a similar manner that $\phi(s)$ vanishes everywhere in the interval (β, b).

Finally, we can show that, at points of the open interval (α, β) which do not belong to Σ, *both* $\theta(s)$ and $\phi(s)$ vanish. In fact, if a_1 is any one of these points, there are clearly points of Σ both on its right and on its left. The argument we have just employed will then establish that, by reason of the former, $\theta(\alpha_1)$ is zero, and that, by reason of the latter, $\phi(\alpha_1)$ is zero.

§16. Conversely, let us suppose that $\kappa(s, t)$ is defined in terms of continuous functions $\theta(s)$, $\phi(s)$ which have the properties mentioned in the preceding paragraph; and let us consider the function

$$\kappa\begin{pmatrix} s_1, & s_2, & \ldots, & s_n \\ s_1, & s_2, & \ldots, & s_n \end{pmatrix}, \quad \ldots \ldots \ldots \quad (12)$$

where s_1, s_2, ..., s_n are variables each confined to the interval (a, b). We may remark that, as this function is symmetric, it will take all possible values in the domain $s_1 \leq s_2 \leq s_3 \leq \ldots \leq s_n$. Thus, since we are only concerned with the sign of the function, we may always suppose the variables to satisfy these inequalities. Firstly, let us suppose that one of the variables has a value not belonging to the domain of the discriminator of $\kappa(s, t)$. If such a value belongs to (a, α), the point s_1 must evidently lie in this interval; hence, since

$$\kappa(s_1, s_r) = \theta(s_1)\,\phi(s_r)\ (r = 1, 2, \ldots, n),$$

and $\theta(s_1)$ vanishes by our hypothesis, it is evident that all the elements of the first row of (12) are zero. In a similar manner it may be proved that, when one of the variables has a value belonging to the interval (b, β), all the elements of the last row vanish. Again, if one of the variables, say s_m, has a value belonging to the open interval (α, β), but not to Σ, we shall have

$$\theta(s_m) = \phi(s_m) = 0$$

by our hypothesis. It is thus easily seen that the elements of the m^{th} row of (12) all vanish. Summing up our results so far, we conclude that the function (12) can only take values different from zero when the variables $s_1, s_2, \ldots, s_n$ are each confined to the set Σ.

§17. Let us next consider the case when the variables are restricted in this manner. The function (12), when expressed in terms of the functions θ and ϕ, is

$$\begin{vmatrix} \theta(s_1)\,\phi(s_1), & \theta(s_1)\,\phi(s_2), & \ldots, & \theta(s_1)\,\phi(s_n) \\ \theta(s_1)\,\phi(s_2), & \theta(s_2)\,\phi(s_2), & \ldots, & \theta(s_2)\,\phi(s_n) \\ \theta(s_1)\,\phi(s_3), & \theta(s_2)\,\phi(s_3), & \ldots, & \theta(s_3)\,\phi(s_n) \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \theta(s_1)\,\phi(s_n), & \theta(s_2)\,\phi(s_n), & \ldots, & \theta(s_n)\,\phi(s_n) \end{vmatrix} \quad (s_1 \leq s_2 \leq \quad \leq s_n),$$

hence, by dividing through both the r^{th} row and the r^{th} column of this determinant by $\phi(s_r)$ $(r = 1, 2, \ldots, n)$, its value is seen to be

$$[\phi(s_1)\,\phi(s_2)\ldots\phi(s_n)]^2 \begin{vmatrix} f(s_1), & f(s_1), & \ldots, & f(s_1) \\ f(s_1), & f(s_2), & \ldots, & f(s_2) \\ f(s_1), & f(s_2), & \ldots, & f(s_3) \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ f(s_1), & f(s_2), & \ldots, & f(s_n) \end{vmatrix}.$$

The determinant just written can be evaluated without difficulty, and thus we find that (12) is

$$[\phi(s_1)\,\phi(s_2)\ldots\phi(s_n)]^2 f(s_1)\,[f(s_2)-f(s_1)]\,[f(s_3)-f(s_2)]\ldots[f(s_n)-f(s_{n-1})].$$

Now, according to our hypothesis, $f(s_1)$ is positive and each of the factors $[f(s_n)-f(s_{n-1})]$ is positive or zero. It follows, then, that (12) cannot take negative values when the variables are each restricted to the set Σ. Taking this in conjunction with what was said in the previous paragraph, we see that the functions

$$\kappa(s_1, s_1),\ \kappa\begin{pmatrix} s_1, s_2 \\ s_1, s_2 \end{pmatrix},\ \ldots,\ \kappa\begin{pmatrix} s_1, s_2, \ldots, s_n \\ s_1, s_2, \ldots, s_n \end{pmatrix},\ \ldots$$

can never take negative values, when the variables $s_1, s_2, \ldots, s_n, \ldots$ each range over the interval (a, b), and hence, by the theorem of § 10, that $\kappa(s, t)$ is of positive type. We may, therefore, state our results in the following theorem :—

If $\theta(s)$ *and* $\phi(s)$ *are each continuous functions defined in the interval* (a, b), *the necessary and sufficient conditions that the function*

$$\begin{aligned} \kappa(s, t) &= \theta(s)\,\phi(t) \quad (s \leqq t) \\ &= \phi(s)\,\theta(t) \quad (s \geqq t), \end{aligned}$$

should be of positive type are (1) *that the discriminator of the function should be positive and non-decreasing in its domain* Σ, *and* (2) *that, if* α *and* β *are the lower and upper limits of* Σ, $\theta(s)$ *should be zero in the interval* (a, α), $\phi(s)$ *zero in the interval* (β, b), *and both* $\theta(s)$ *and* $\phi(s)$ *zero at points of the open interval* (α, β) *which do not belong to* Σ.

As a corollary of this, by supposing that $\phi(s) = 1$ $(a \leqq s \leqq b)$, the reader may deduce the corresponding conditions for the function defined in § 13.

§ 18. Let us now investigate under what circumstances a function $\kappa(s, t)$, which satisfies the conditions stated in the enunciation of the theorem of § 17, is definite. If the domain of its discriminator is not dense everywhere, it will be possible to find an interval (c, d), lying within (a, b), such that at each of its points the function $\theta(s_1)\phi(s_1)$ is zero. We shall, therefore, have (§ 11)

$$\begin{aligned}\kappa(s, t) &= 0 \quad (c \leq s \leq d, \quad a \leq t \leq b) \\ &= 0 \quad (c \leq t \leq d, \quad a \leq s \leq b);\end{aligned}$$

in particular, $\kappa(s,t)$ will vanish everywhere in the square $c \leq s \leq d$, $c \leq t \leq d$. Now, if $\chi(s)$ is any continuous function of s defined in the interval (a, b), which is zero in the intervals $a \leq s \leq c$, $d \leq s \leq b$, but does not vanish everywhere in (c, d), we shall have

$$\begin{aligned}\int_a^b \int_a^b \kappa(s, t)\chi(s)\chi(t)\,ds\,dt &= \int_c^d \int_c^d \kappa(s, t)\chi(s)\chi(t)\,ds\,dt \\ &= 0,\end{aligned}$$

by the properties of $\chi(s)$ and $\kappa(s, t)$. It follows from this that, if $\kappa(s, t)$ is definite, the domain of its discriminator must be dense everywhere in (a, b).

Again, let us suppose that the discriminator of $\kappa(s, t)$ has a constant value p throughout a certain interval (c, d). It will then be seen that within the square $c \leq s \leq d$, $c \leq t \leq d$

$$\kappa(s, t) = p\phi(s)\phi(t);$$

and hence, if $\chi(s)$ is defined as before, that

$$\int_a^b \int_a^b \kappa(s, t)\chi(s)\chi(t)\,ds\,dt = p\left[\int_c^d \phi(s)\chi(s)\,ds\right]^2.$$

It may be proved without difficulty that there exists a function $\chi(s)$ which is not everywhere zero, and is such that

$$\int_c^d \phi(s)\chi(s)\,ds = 0. \qquad (13)$$

For, let $\chi_1(s)$ and $\chi_2(s)$ be any two functions which are not mere multiples of one another, and which satisfy the conditions imposed on $\chi(s)$. Then, if either of the integrals

$$\int_c^d \phi(s)\chi_1(s)\,ds, \quad \int_c^d \phi(s)\chi_2(s)\,ds$$

is zero, we shall have an obvious solution of (13). On the other hand, if their respective values μ_1, μ_2 be different from zero, it is easily seen that

$$\chi(s) = \frac{\chi_1(s)}{\mu_1} - \frac{\chi_2(s)}{\mu_2}$$

satisfies (13); and, in virtue of our hypothesis, $\chi(s)$ is not zero everywhere in (a, b). We conclude, therefore, that we can always find a function $\chi(s)$ which is such that

$$\int_a^b \int_a^b \kappa(s, t)\chi(s)\chi(t)\,ds\,dt = 0.$$

It thus appears that the discriminator of a definite function of positive type cannot be constant throughout any interval.

§ 19. Conversely, we may show that every function of positive type, whose discriminator (1) has a domain which is dense everywhere in (a, b), and (2) has not a constant value in the whole of any interval, is definite. For, if this were not so, we would be able to find a continuous function $\psi(s)$ other than zero, such that

$$\int_a^b \kappa(s, t)\,\psi(t)\,dt = 0 \quad (a \leq s \leq b).*$$

Supplying in the value of $\kappa(s, t)$, this equation may be written

$$\phi(s)\int_a^s \theta(t)\,\psi(t)\,dt + \theta(s)\int_s^b \phi(t)\,\psi(t)\,dt = 0 \quad (a \leq s \leq b). \quad . \quad . \quad . \quad (14)$$

Now, as $\psi(s)$ is continuous, and is not zero everywhere, we can find an interval (c, d) of (a, b) within which it does not vanish; also, as the domain of the discriminator is dense everywhere, it will be possible to find a point, and, therefore, a whole interval (γ, δ), belonging both to (c, d) and the domain. The interval (γ, δ) will thus be such that in it the functions $\psi(s)$, $\theta(s)$, $\phi(s)$ do not vanish. It follows that in this interval the function of s

$$\int_s^b \phi(t)\,\psi(t)\,dt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

has a derivative which does not vanish; and hence, by a well-known theorem of the differential calculus, that this function cannot be zero more than once in (γ, δ). It is, therefore, evident that by contracting (γ, δ) sufficiently we can ensure for it the additional property that (15) vanishes at no point belonging to it.

Returning now to the equation (14), and supposing that s is confined to the interval (γ, δ), we see that

$$f(s) = -\int_a^s \theta(t)\,\psi(t)\,dt \Big/ \int_s^b \phi(t)\,\psi(t)\,dt.$$

Hence, since both the numerator and the denominator on the right are differentiable, and the latter does not vanish in (γ, δ), the function $f(s)$ is differentiable in this interval. In fact, by applying the ordinary rules, we obtain

$$f'(s) = 0 \quad (\gamma \leq s \leq \delta).$$

But this is impossible, because by our hypothesis $f(s)$ cannot be constant in any interval. We conclude, therefore, that $\kappa(s, t)$ is a definite function.

§ 20. It may be remarked that the conditions (1) and (2) of the preceding paragraph may be stated in another and more convenient form. For, if a discrimi-

* *Vide* § 3.

nator satisfying these conditions had the same value at two distinct points, it would necessarily have that value at all points of its domain which lie between them (§ 15). Thus, since the condition (1) and the continuity of $\theta(s)$, $\phi(s)$ assure us of an interval of the domain which lies between these points, the condition (2) would be violated. Hence a discriminator of this kind must be a steadily increasing function; and, conversely, a steadily increasing discriminator satisfies (2). We may, therefore, combine the results of the two preceding paragraphs in the theorem :—

The necessary and sufficient condition, that a function $\kappa(s, t)$, *satisfying the requirements of the theorem of* § 17, *should be definite, is that its discriminator should be a steadily increasing function whose domain is dense everywhere in* (a, b).

As an application of this theorem we may consider the function*

$$\begin{aligned}\kappa(s, t) &= (s-a)(b-t) \quad (s \leq t),\\ &= (t-a)(b-s) \quad (s \geq t).\end{aligned}$$

The discriminator has the open interval (a, b) for its domain, and its value at any point is

$$(s-a)/(b-s),$$

which steadily increases with s. It follows from § 17 and the theorem just stated that $\kappa(s, t)$ is a definite function of positive type.

§ 21. Leaving the particular class of functions with which we have been dealing, let us now suppose that $\kappa(s, t)$ is *any* function of positive type defined in the square $a \leq s \leq b$, $a \leq t \leq b$. Let $a_1, a_2, \ldots, a_m$ be any m points of the interval (a, b) which are such that

$$\kappa\begin{pmatrix}a_1, a_2, \ldots, a_m\\ a_1, a_2, \ldots, a_m\end{pmatrix} \neq 0.$$

Then the function

$$h(s, t) = \kappa\begin{pmatrix}s, a_1, a_2, \ldots, a_m\\ t, a_1, a_2, \ldots, a_m\end{pmatrix}\Big/\kappa\begin{pmatrix}a_1, a_2, \ldots, a_m\\ a_1, a_2, \ldots, a_m\end{pmatrix}$$

will evidently be symmetric and continuous in the square $a \leq s \leq b$, $a \leq t \leq b$.

Again, when the function

$$\kappa\begin{pmatrix}s_1, s_2, \ldots, s_n, a_1, a_2, \ldots, a_m\\ s_1, s_2, \ldots, s_n, a_1, a_2, \ldots, a_m\end{pmatrix}$$

is expressed as a determinant, it is easy to see that the minor obtained by suppressing all but the i^{th} of the first n rows and all but the j^{th} of the first n columns is

$$\kappa\begin{pmatrix}s_i, a_1, a_2, \ldots, a_m\\ s_j, a_1, a_2, \ldots, a_m\end{pmatrix}.$$

* This is the generalised form of HILBERT'S classical function, *vide* 'Gött. Nachr.,' p. 227 (1904).

The determinant of n rows and columns, whose elements are these minors, will therefore be

$$\left[\kappa\begin{pmatrix}a_1, & a_2, & \dots, & a_m\\ a_1, & a_2, & \dots, & a_m\end{pmatrix}\right]^n h\begin{pmatrix}s_1, & s_2, & \dots, & s_n\\ s_1, & s_2, & \dots, & s_n\end{pmatrix}.$$

But, by the theory of determinants, we also know that it is equal to*

$$\left[\kappa\begin{pmatrix}a_1, & a_2, & \dots, & a_m\\ a_2, & a_2, & \dots, & a_m\end{pmatrix}\right]^{n-1} \kappa\begin{pmatrix}s_1, & s_2, & \dots, & s_n, & a_1, & \dots, & a_m\\ s_1, & s_2, & \dots, & s_n, & a_1, & \dots, & a_m\end{pmatrix}.$$

Thus, equating these two values, we find

$$h\begin{pmatrix}s_1, & s_2, & \dots, & s_n\\ s_1, & s_2, & \dots, & s_n\end{pmatrix} = \kappa\begin{pmatrix}s_1, & s_2, & \dots, & s_n, & a_1, & \dots, & a_m\\ s_1, & s_2, & \dots, & s_n, & a_1, & \dots, & a_m\end{pmatrix} \Big/ \kappa\begin{pmatrix}a_1, & a_2, & \dots, & a_m\\ a_1, & a_2, & \dots, & a_m\end{pmatrix}. \quad . \quad . \quad (16)$$

Now, in virtue of our hypothesis that $\kappa(s, t)$ is of positive type, it follows from § 10 that the quotient on the right-hand side of this equation has a denominator which is positive and a numerator which is not negative. Hence we have

$$h\begin{pmatrix}s_1, & s_2, & \dots, & s_n\\ s_1, & s_2, & \dots, & s_n\end{pmatrix} \geqq 0;$$

and thus, as this is true for all values of n, the theorem of § 10 shows that *$h(s, t)$ is of positive type.*

§ 22. In the light of this result, it appears that each function of positive type can be used to generate an infinite series of such functions. We might, therefore, expect to obtain other species of functions of positive type by taking $\kappa(s, t)$ to be of the kind considered in §§ 14–20.

For simplicity, let us consider the function

$$h(s, t) = \kappa\begin{pmatrix}s, & a_1\\ t, & a_1\end{pmatrix} \Big/ \kappa(a_1, a_1),$$

where

$$\kappa(a_1, a_1) = \theta(a_1)\,\phi(a_1) \neq 0.$$

Confining our attention to the triangle $s \leqq t$, it will be seen that the variables s and t can be related to the constant a_1 by either of the inequalities:—

$$\text{(i)} \quad s \leqq t \leqq a_1,$$

$$\text{(ii)} \quad s \leqq a_1 \leqq t,$$

$$\text{(iii)} \quad a_1 \leqq s \leqq t.$$

The reader may find it convenient to refer to the accompanying diagram, in which

* *Vide* SCOTT and MATHEWS, *op. cit.*, pp. 67, 68.

the square $a \leqq s \leqq b$, $a \leqq t \leqq b$ is drawn, and the portion of the triangle $s \leqq t$ which corresponds to each set of inequalities is marked with its number.

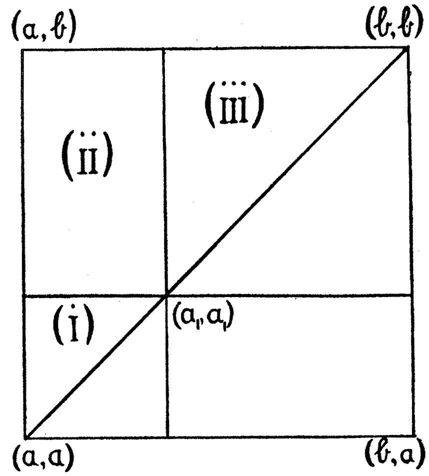

Fig. 3.

By expressing $h(s, t)$ in terms of the functions θ and ϕ, it is easily seen that at each point of the region (i)

$$h(s, t) = \phi(a_1)\,\theta(s)\left[\frac{\phi(t)}{\phi(a_1)} - \frac{\theta(t)}{\theta(a_1)}\right],$$

that in (ii) $h(s, t)$ is everywhere zero, and that in (iii)

$$h(s, t) = \theta(a_1)\,\phi(t)\left[\frac{\theta(s)}{\theta(a_1)} - \frac{\phi(s)}{\phi(a_1)}\right].$$

In a similar way, or by a mere interchange of the variables s and t, the values of $h(s, t)$ in the corresponding divisions of the triangle $s > t$ can be obtained.

Now, let $\theta_1(s)$, $\phi_1(s)$ be continuous functions defined by

$$\begin{aligned} \theta_1(s) &= \phi(a_1)\,\theta(s) && (a \leqq s \leqq b); \\ \phi_1(s) &= \frac{\phi(s)}{\phi(a_1)} - \frac{\theta(s)}{\theta(a_1)} && (a \leqq s \leqq a_1), \\ &= 0 && (a_1 \leqq s \leqq b); \end{aligned}$$

also let $\theta_2(s)$, $\phi_2(s)$ be two others defined by

$$\begin{aligned} \theta_2(s) &= 0 && (a \leqq s \leqq a_1), \\ &= \frac{\theta(s)}{\theta(a_1)} - \frac{\phi(s)}{\phi(a_1)} && (a_1 \leqq s \leqq b); \\ \phi_2(s) &= \theta(a_1)\,\phi(s) && (a \leqq s \leqq b); \end{aligned}$$

and, finally, let two functions $h_r(s, t)$ $(r = 1, 2)$ be defined in the square $a \leqq s \leqq b$, $a \leqq t \leqq b$ by

$$\begin{aligned} h_r(s, t) &= \theta_r(s)\,\phi_r(t) \quad (s \leqq t), \\ &= \phi_r(s)\,\theta_r(t) \quad (s \geqq t). \end{aligned}$$

On comparing these latter functions with $h(s, t)$, it will be seen that we have

$$h_1(s, t) = h(s, t) \quad a \leqq s \leqq a_1, \; a \leqq t \leqq a_1,$$
$$= 0 \text{ elsewhere};$$

and

$$h_2(s, t) = h(s, t) \quad a_1 \leqq s \leqq b, \; a_1 \leqq t \leqq b,$$
$$= 0 \text{ elsewhere.}$$

It follows from this that we have

$$h(s, t) = h_1(s, t) + h_2(s, t)$$

at each point of the square in which these functions are defined. But it is easily seen that, as $\kappa(s, t)$ is of positive type, the functions $h_r(s, t)$ satisfy the requirements of the theorem enunciated in § 17. Thus $h(s, t)$ is merely the sum of two functions of the same nature as $\kappa(s, t)$, and hence, as it is obvious *à priori* that the sum of any number of functions of positive type is a function of positive type, it appears that we do not in this way obtain any new species of these functions.

The reader may convince himself in a similar manner that the same conclusion holds in regard to the more general function considered in the preceding paragraph.

§ 23. Although the result of § 21 proves to be so barren in this respect, it may be applied to obtain an interesting property of the symmetrical minors of the determinant of the integral equation

$$f(s) = \phi(s) - \lambda \int_a^b \kappa(s, t)\, \phi(t)\, dt, \quad \ldots \ldots \ldots \quad (7)$$

when $\kappa(s, t)$ is of positive type. Adopting the notation and hypothesis of the paragraph referred to, let $\Delta(\lambda)$ be the determinant of the above integral equation when $h(s, t)$ replaces $\kappa(s, t)$. Then, since

$$\Delta(\lambda) = 1 - \lambda \int_a^b h(s_1, s_1)\, ds_1 + \frac{\lambda^2}{2!} \int_a^b \int_a^b h\begin{pmatrix} s_1, s_2 \\ s_1, s_2 \end{pmatrix} ds_1\, ds_2 - \ldots$$
$$+ \frac{(-\lambda)^n}{n!} \int_a^b \ldots \int_a^b \int_a^b h\begin{pmatrix} s_1, s_2, \ldots, s_n \\ s_1, s_2, \ldots, s_n \end{pmatrix} ds_1\, ds_2 \ldots ds_n + \ldots,$$

it is easily seen from (16) that

$$\Delta(\lambda) = \mathrm{D}\left(\lambda; \begin{matrix} a_1, a_2, \ldots, a_m \\ a_1, a_2, \ldots, a_m \end{matrix}\right) \Big/ \kappa\begin{pmatrix} a_1, a_2, \ldots, a_m \\ a_1, a_2, \ldots, a_m \end{pmatrix}, \quad \ldots \ldots \quad (17)$$

where

$$\mathrm{D}\left(\lambda; \begin{matrix} a_1, a_2, \ldots, a_m \\ a_1, a_2, \ldots, a_m \end{matrix}\right) = \kappa\begin{pmatrix} a_1, a_2, \ldots, a_m \\ a_1, a_2, \ldots, a_m \end{pmatrix} - \lambda \int_a^b \kappa\begin{pmatrix} s_1, a_1, a_2, \ldots, a_m \\ s_1, a_1, a_2, \ldots, a_m \end{pmatrix} ds_1 + \ldots$$

and is, therefore, a symmetrical m^{th} minor of $\mathrm{D}(\lambda)$, the determinant of (7), in accordance with FREDHOLM's definition. But, as we have shown that $h(s, t)$ is of positive type, the function $\Delta(\lambda)$ has its zeros all real and positive. It follows, therefore, from (17) that all the zeros of the minor

$$D\left(\lambda\,;\begin{matrix}a_1, a_2, \ldots, a_m\\ a_1, a_2, \ldots, a_m\end{matrix}\right)$$

are real and positive. Since the minor must be identically zero if

$$\kappa\begin{pmatrix}a_1, & a_2, & \ldots, & a_m\\ a_1, & a_2, & \ldots, & a_m\end{pmatrix} = 0$$

(*cf.* § 11), we have thus proved the theorem :—

The zeros of all symmetrical minors of the determinant of an integral equation of the second kind, whose characteristic function is of positive type, are all real and positive.

In particular, as $K_\lambda(s, t)$, the solving function of (7), is defined by

$$K_\lambda(s, t) = D(\lambda\,;\ s, t)/D(\lambda),$$

where

$$D(\lambda\,;\ s, t) = \kappa(s, t) - \lambda\int_a^b \kappa\begin{pmatrix}s, & s_1\\ t, & s_1\end{pmatrix} ds_1 + \frac{\lambda^2}{2\,!}\int_a^b\int_a^b \kappa\begin{pmatrix}s, & s_1, & s_2\\ t, & s_1, & s_2\end{pmatrix} ds_1\, ds_2 + \ldots,$$

it appears that, when $s = t$, the solving function only vanishes for positive values of λ.

PART IV.—THE EXPANSION OF FUNCTIONS OF POSITIVE AND NEGATIVE TYPE.

§ 24. It is to be remarked that HILBERT and SCHMIDT have been able to give very little information about the expansion of a given symmetric characteristic function in a series of products of normal functions. HILBERT* has indeed shown incidentally that, if the number of singular values is finite,

$$\kappa(s, t) = \sum_{n=1} \frac{\psi_n(s)\,\psi_n(t)}{\lambda_n}; \quad \ldots\ldots\ldots \quad (18)$$

and SCHMIDT† in his dissertation has established that this equation remains valid when the series on the right is uniformly convergent. The latter theorem is, of course, much wider than the former as regards its generality; but it has the defect that the uniform convergence, which it postulates, is not connected with any other of the properties of $\kappa(s, t)$. In the present section we shall attempt to remedy this in some measure by proving that the equality (18) certainly holds when $\kappa(s, t)$ is of positive or negative type.

* 'Gött. Nachr.,' 1904, p. 73.

† Printed with additions in 'Math. Ann.,' Band LXIII. The theorem referred to will be found on pp. 449, 450. From a remark made on p. 453 I gather that it is originally due to HILBERT.

§ 25. In the paper referred to above, SCHMIDT* has proved that, if $\kappa(s, t)$ is any continuous symmetric function, the solution of

$$f(s) = \phi(s) - \lambda \int_a^b \kappa(s, t)\, \phi(t)\, dt$$

is given by

$$\phi(s) = f(s) + \Sigma \frac{\lambda \psi_n(s)}{\lambda_n - \lambda} \int_a^b f(x)\, \psi_n(x)\, dx,$$

provided that λ is not one of the singular values $\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots$;† moreover, the convergence of the series on the right is both absolute and uniform. Now, when we take

$$f(s) = \kappa(s, t),$$

it is known that, in virtue of one of the characteristic relations,

$$\phi(s) = \mathrm{K}_\lambda(s, t).$$

It follows, therefore, from the above expansion and the homogeneous equations

$$\psi_n(t) = \lambda_n \int_a^b \psi_n(x)\, \kappa(x, t)\, dx \quad (n = 1, 2, \ldots),$$

that

$$\mathrm{K}_\lambda(s, t) = \kappa(s, t) + \sum_{n=1} \frac{\lambda \psi_n(s)\, \psi_n(t)}{\lambda_n(\lambda_n - \lambda)}. \qquad (19)$$

It should be remarked that SCHMIDT's theorem only allows us to assume that the series on the right of (19) is uniformly convergent with respect to s $(a \leq s \leq b)$, for each assigned value of t; and hence, by symmetry, that it is uniformly convergent with respect to t $(a \leq t \leq b)$, for each assigned value of s. When $\kappa(s, t)$ is of positive type, we may establish the uniform convergence of the series in the whole of the square $a \leq s \leq b$, $a \leq t \leq b$, as follows. If we write $t = s$ in (19), it is clear that the terms of the series on the right become functions of s, which, with the possible exception of a finite number, are all of the same sign as λ; accordingly, by DINI's theorem,‡ this series is uniformly convergent in the interval $a \leq s \leq b$. But, in virtue of the inequality

$$2\,|\psi_n(s)\, \psi_n(t)| \leq \psi_n^2(s) + \psi_n^2(t),$$

the terms of the series on the right of (19) are never greater in absolute value than those of

$$\tfrac{1}{2} \sum_{n=1} \frac{\lambda\left[\psi_n^2(s) + \psi_n^2(t)\right]}{\lambda_n(\lambda_n - \lambda)}.$$

* Pp. 453, 454.

† We shall always suppose this to be the case in what follows.

‡ DINI, "Fondamenti per la teoria delle funzioni di variabili reali" (Pisa, 1878), § 99. See also YOUNG, "On Monotone Sequences of Continuous Functions," 'Proc. Camb. Phil. Soc.,' vol. XIV., pp. 520–3.

Hence, as the latter converges uniformly for $a \leqq s \leqq b$, $a \leqq t \leqq b$ by what has just been said, the result follows.

§ 26. Let us denote the sum of the first m terms of the series on the right of (19) by $S_m(\lambda; s, t)$, and the remainder after these terms by $R_m(\lambda; s, t)$. We have

$$S_m(\lambda; s, t) = \sum_{n=1}^{m} \frac{\psi_n(s)\,\psi_n(t)}{\lambda_n - \lambda} - \sum_{n=1}^{m} \frac{\psi_n(s)\,\psi_n(t)}{\lambda_n};$$

and hence, keeping m fixed,

$$\operatorname*{Lt}_{\lambda \to \infty} S_m(\lambda; s, t) = -\sum_{n=1}^{m} \frac{\psi_n(s)\,\psi_n(t)}{\lambda_n},$$

$$\operatorname*{Lt}_{\lambda \to -\infty} S_m(\lambda; s, t) = -\sum_{n=1}^{m} \frac{\psi_n(s)\,\psi_n(t)}{\lambda_n}.$$

Thus, since (19) can be written

$$K_\lambda(s, t) - R_m(\lambda; s, t) = \kappa(s, t) + S_m(\lambda; s, t),$$

we obtain the equations

$$\operatorname*{Lt}_{\lambda \to \infty} [K_\lambda(s, t) - R_m(\lambda; s, t)] = \operatorname*{Lt}_{\lambda \to -\infty} [K_\lambda(s, t) - R_m(\lambda; s, t)] = \kappa(s, t) - \sum_{n=1}^{m} \frac{\psi_n(s)\,\psi_n(t)}{\lambda_n}. \quad (20)$$

This relation holds for any continuous function $\kappa(s, t)$, but we now add the further limitation that the function shall be of positive type. Then, since

$$R_m(\lambda; s, s) = \sum_{n=m+1} \frac{\lambda\,[\psi_n(s)]^2}{\lambda_n(\lambda_n - \lambda)},$$

we shall have

$$R_m(\lambda; s, s) < 0, \quad (21)$$

for each negative value of λ.

Let us, in the next place, investigate the values of $K_\lambda(s, s)$ for negative values of λ, it being supposed, as above, that $\kappa(s, t)$ is of positive type. If $\theta(s)$ is any continuous function defined in the interval (a, b), it follows from (19) and the theorem proved at the end of the preceding paragraph that

$$\int_a^b\!\!\int_a^b K_\lambda(s, t)\,\theta(s)\,\theta(t)\,ds\,dt = \int_a^b\!\!\int_a^b \kappa(s, t)\,\theta(s)\,\theta(t)\,ds\,dt + \sum_{n=1} \frac{\lambda}{\lambda_n(\lambda_n - \lambda)} \left[\int_a^b \psi_n(s)\,\theta(s)\,ds\right]^2.$$

Recalling HILBERT'S theorem, it will be seen without difficulty that this reduces to

$$\int_a^b\!\!\int_a^b K_\lambda(s, t)\,\theta(s)\,\theta(t)\,ds\,dt = \sum_{n=1} \frac{1}{\lambda_n - \lambda} \left[\int_a^b \psi_n(s)\,\theta(s)\,ds\right]^2.$$

Now, when λ is negative, the terms of the series on the right must be either zero or positive. We conclude, therefore, that for all functions of the class Θ

$$\int_a^b \int_a^b \mathrm{K}_\lambda(s, t)\,\theta(s)\,\theta(t)\,ds\,dt \geqq 0 \qquad (\lambda < 0).$$

In other words, $\mathrm{K}_\lambda(s, t)$ is of positive type for these values of λ. Applying, then, the theorem proved above (§§ 6, 10), we see that

$$\mathrm{K}_\lambda(s, s) \geqq 0 \qquad (a \leqq s \leqq b), \quad (\lambda < 0). \qquad (22)$$

§ 27. Returning to the formula (20) and writing $s = t$, we obtain

$$\underset{\lambda \to -\infty}{\mathrm{L}t}\,[\mathrm{K}_\lambda(s, s) - \mathrm{R}_m(\lambda\,;\; s,\, s)] = \kappa(s, s) - \sum_{n=1}^{m} \frac{[\psi_n(s)]^2}{\lambda_n}.$$

Accordingly, from (21) and (22), it follows that

$$\kappa(s, s) > \sum_{n=1}^{m} \frac{[\psi_n(s)]^2}{\lambda_n}. \qquad (23)$$

This is true, of course, for all values of m which are sufficiently great; and, further, when we increase m we only add positive terms to the right-hand side. By a well-known theorem of the elementary theory of series, we thus see that

$$\sum_{n=1} \frac{[\psi_n(s)]^2}{\lambda_n}$$

converges for each value of s in the interval (a, b); and hence, since

$$2\,|\,\psi_n(s)\,\psi_n(t)\,| \leqq [\psi_n(s)]^2 + [\psi_n(t)]^2,$$

that the series

$$\sum_{n=1} \frac{\psi_n(s)\,\psi_n(t)}{\lambda_n}$$

converges absolutely for each pair of values of the variables satisfying the inequalities $a \leqq s \leqq b$, $a \leqq t \leqq b$. From this last result it follows that the function

$$f(s, t) = \kappa(s, t) - \sum_{n=1} \frac{\psi_n(s)\,\psi_n(t)}{\lambda_n} \qquad (24)$$

has a definite finite value when the variables are restricted in the manner just mentioned. In the paragraphs which follow we shall consider the properties of $f(s, t)$, and eventually prove that it is everywhere zero. It may be remarked that the inequality (23) proves the relation

$$0 \leqq f(s, s) \leqq \kappa(s, s).$$

§ 28. If ϵ is any arbitrarily assigned positive quantity, it follows from the absolute convergence of the series on the right of (24) that we can choose m great enough to ensure the inequality

$$\sum_{n=m+1} \frac{|\psi_n(s)\psi_n(t)|}{\lambda_n} < \frac{\epsilon}{3}. \qquad (25)$$

And, when this is done, it is easily seen that, since

$$\frac{|\psi_n(s)\psi_n(t)|}{\lambda_n} > \frac{|\lambda\psi_n(s)\psi_n(t)|}{\lambda_n(\lambda_n-\lambda)} \quad \begin{matrix}(n>m)\\(\lambda<0)\end{matrix},$$

we have

$$|\mathrm{R}_m(\lambda;\ s,t)| < \frac{\epsilon}{3} \quad (\lambda < 0).$$

Again, from (20), we see that a negative number L′ can be chosen with so great an absolute value that, when $\lambda < \mathrm{L}'$,

$$\left|\mathrm{K}_\lambda(s,t) - \mathrm{R}_m(\lambda;\ s,t) - \left[\kappa(s,t) - \sum_{n=1}^{m} \frac{\psi_n(s)\psi_n(t)}{\lambda_n}\right]\right| < \frac{\epsilon}{3};$$

while, from (24) and (25), we deduce

$$\left|\left[\kappa(s,t) - \sum_{n=1}^{m} \frac{\psi_n(s)\psi_n(t)}{\lambda_n}\right] - f(s,t)\right| < \frac{\epsilon}{3}.$$

Adding the three inequalities just written, we obtain

$$|\mathrm{K}_\lambda(s,t) - f(s,t)| < \epsilon \quad (\lambda < \mathrm{L}').$$

In other words, we have proved the theorem

$$\mathop{\mathrm{L}t}_{\lambda \to -\infty} \mathrm{K}_\lambda(s,t) = f(s,t) \quad (a \leqq s \leqq b,\ a \leqq t \leqq b).$$

§ 29. It may be proved* that, if c is any constant and a_1 any point of the interval (a, b), then the solving function corresponding to the characteristic function

$$h(s,t) = \kappa(s,t) - \frac{\kappa(a_1,s)\,\kappa(a_1,t)}{c}$$

is

$$\mathrm{H}_\lambda(s,t) = \mathrm{K}_\lambda(s,t) - \frac{\mathrm{K}_\lambda(a_1,s)\,\mathrm{K}_\lambda(a_1,t)}{c+\mathrm{K}_\lambda(a_1,a_1) - \kappa(a_1,a_1)}, \qquad (26)$$

whilst the corresponding determinant is easily seen to be

$$\Delta(\lambda) = \mathrm{D}(\lambda)\left[1 + \frac{\mathrm{K}_\lambda(a_1,a_1) - \kappa(a_1,a_1)}{c}\right].$$

* *Cf.* BATEMAN, 'Messenger of Mathematics' (1908), p. 184. The result in question follows from equations (24), (25), and (26), by writing $f(s) = \dfrac{\kappa(a_1,s)}{c}$, $g(t) = \kappa(a_1,t)$ and observing that $\phi(s) = \dfrac{\mathrm{K}_\lambda(a_1,s)}{c}$, $\chi(t) = \mathrm{K}_\lambda(a_1,t)$, $\lambda\tau_{11} = \dfrac{\mathrm{K}_\lambda(a_1,a_1) - \kappa(a_1,a_1)}{c}$.

Now, if we write $s = t = a_1$ in (19) and (24), it is easy to see that

$$\mathrm{K}_\lambda(a_1, a_1) = f(a_1, a_1) + \sum_{n=1} \frac{[\psi_n(a_1)]^2}{\lambda_n - \lambda},$$

and hence that $\mathrm{K}_\lambda(a_1, a_1)$ constantly increases with λ, so long as the latter is negative. Consequently, when ϵ is any positive quantity, and we take c to be

$$\kappa(a_1, a_1) - f(a_1, a_1) + \epsilon,$$

it follows from the theorem of the preceding paragraph that

$$\mathrm{K}_\lambda(a_1, a_1) - \kappa(a_1, a_1) + c$$

can only vanish for positive values of λ. Thus, as $\mathrm{D}(\lambda)$ has no negative roots, $h(s, t)$ is of positive type,* and, therefore, in virtue of the remark at the end of § 27,

$$\underset{\lambda \to -\infty}{\mathrm{L}t}\, \mathrm{H}_\lambda(s, s) \geqq 0 \quad (a \leqq s \leqq b).$$

Using the formula (26), it will be seen that this becomes

$$f(s, s) - \frac{[f(a_1, s)]^2}{\epsilon} \geqq 0 \quad (a \leqq s \leqq b).$$

But, as ϵ may be taken as small as we please, this is evidently impossible unless $f(a_1, s)$ vanishes. It follows that, as a_1 and s may each have any assigned values belonging to (a, b), we must have

$$f(s, t) = 0 \quad (a \leqq s \leqq b,\ a \leqq t \leqq b).$$

We have thus shown that, in the case of a function of positive type, the series

$$\sum_{n=1} \frac{\psi_n(s)\,\psi_n(t)}{\lambda_n} \quad \ldots \ldots \ldots \ldots \quad (27)$$

has $\kappa(s, t)$ for its sum-function. It was shown in § 27 that the convergence of this series is absolute, and, by an application of DINI's theorem, it may be shown that the convergence is also uniform in the square $a \leqq s \leqq b$, $a \leqq t \leqq b$. Hence, *if* $\psi_1(s)$, $\psi_2(s)$, ..., $\psi_n(s)$, ... *are a complete system of normal functions relating to a function* $\kappa(s, t)$ *of positive type and* $\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots$ *are the corresponding singular values, then the series*

$$\sum_{n=1} \frac{\psi_n(s)\,\psi_n(t)}{\lambda_n}$$

converges both absolutely and uniformly, and its sum-function is $\kappa(s, t)$.

* Owing to the fact that $\Delta(\lambda)$ has only positive roots.

§ 30. From this theorem several interesting results may be deduced. For example, replacing $\kappa(s, t)$ by the series (27) in (19), we obtain

$$K_\lambda(s, t) = \sum_{n=1} \frac{\psi_n(s)\psi_n(t)}{\lambda_n - \lambda}, \quad \ldots \quad (28)$$

where the series on the right is uniformly convergent. Again, if we write $s = t$ in (28), and integrate with respect to s between the limits a and b, we obtain

$$\int_a^b K_\lambda(s, s)\, ds = \sum_{n=1} \frac{1}{\lambda_n - \lambda}. \quad \ldots \quad (29)$$

Provided that λ is not positive, the terms of the series on the right are all positive and less than those of the series

$$\sum_{n=1} \frac{1}{\lambda_n},$$

which, by writing $\lambda = 0$ in (29), is seen to converge. It thus follows that, for $\lambda \leqq 0$, the former series is uniformly convergent. Integrating (29) between the limits 0 and λ, where the latter is negative, and recollecting FREDHOLM's formula

$$-\frac{d}{d\lambda}[\log D(\lambda)] = \int_a^b K_\lambda(s, s)\, ds,$$

it is easily seen that

$$D(\lambda) = \prod_{n=1} \left(1 - \frac{\lambda}{\lambda_n}\right) \quad (\lambda \leqq 0),$$

since $D(0) = 1$. It now follows that, as the right-hand member of this equation is an integral function of λ, we may drop the restriction $\lambda \leqq 0$. We have thus expressed $D(\lambda)$ as an infinite product.

Finally, we may remark that if $|\lambda|$ is less than the least of the numbers $\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots$ the right-hand side of (29) may be expressed as a power series in which the coefficient of λ^m is

$$\sum_{n=1} \frac{1}{\lambda_n^{m+1}}.$$

Also, by employing NEUMANN's expansion for $K_\lambda(s, t)$, it is easily seen that the coefficient of λ^m on the left is

$$\int_a^b \kappa_{m+1}(s, s)\, ds,$$

where in the usual notation

$$\kappa_{m+1}(s, t) = \int_a^b \ldots \int_a^b \int_a^b \kappa(s, s_1)\,\kappa(s_1, s_2) \ldots \kappa(s_m, t)\, ds_1\, ds_2 \ldots ds_m \quad (m \geqq 1),$$

and

$$\kappa_1(s, t) = \kappa(s, t).$$

It follows that

$$\int_a^b \kappa_m(s, s)\, ds = \sum_{n=1} \frac{1}{\lambda_n{}^m} \quad (m = 1, 2, \ldots).$$

§ 31. In conclusion, it may be pointed out that the theorem of § 29 holds also when $\kappa(s, t)$ is of negative type. This may be deduced from the theorem mentioned by employing the usual device, or it may be proved directly by commencing with the equation

$$\underset{\lambda \to \infty}{\mathrm{Lt}} \left[K_\lambda(s, s) - R_m(\lambda; s, s)\right] = \kappa(s, s) - \sum_{n=1}^{m} \frac{[\psi_n(s)]^2}{\lambda_n}$$

instead of that at the beginning of § 27, and proceeding by a method similar to that which we have used above.

It may also be of interest to remark that by a very slight modification of these proofs we may show that (27) represents $\kappa(s, t)$ when the latter has only a finite number of singular values of one sign, but an unrestricted number of the other.